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Local Bishop–Phelps–Bollobás properties ☆

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A B S T R A C T

In this paper we introduce some local versions of Bishop–Phelps–Bollobás type property for operators. That is, the function η which appears in their definitions depends not only on a given $\varepsilon > 0$, but also on either a fixed norm-one operator T or a fixed norm-one vector x. We investigate those properties and show differences between local and uniform versions.

1. Introduction

It is well-known that Bishop and Phelps proved in [5] the denseness of the set of all norm attaining functionals in X^*. They also asked if this result remains valid for bounded linear operators between any Banach spaces X and Y. Nevertheless, Lindenstrauss [22] proved that this is not true in general, by showing that there is a strictly convex Banach space $\mathcal{Z}$ such that the set of norm attaining bounded linear operators from c_0 into $\mathcal{Z}$ is not dense in the whole space of bounded linear operators from c_0 into $\mathcal{Z}$. Moreover, he started a systematic study of the conditions on the involved Banach spaces that guarantees an operator version of the Bishop–Phelps theorem. In 1970, Bollobás [6] improved the theorem of Bishop and Phelps by showing that, whenever we take a norm-one functional x^* and a norm-one point x satisfying that

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$x^*(x)$ is sufficiently close to 1, it is possible to find a new norm-one functional y^* and a new norm-one point y such that y^* attains its norm at y, y is close to x and y^* is close to x^*. This theorem is known nowadays as the Bishop–Phelps–Bollobás theorem. Motivated by Lindenstrauss' results, there has been an effort of many authors to study some geometric conditions of the Banach spaces X and Y in order to get a Bishop–Phelps–Bollobás type theorem for bounded linear operators from X into Y. The first one was the seminal work [2] due to M. Acosta, R. Aron, D. García and M. Maestre, where the Bishop–Phelps–Bollobás property for a pair of Banach spaces (X, Y) was introduced and studied. Essentially, a pair (X, Y) has the Bishop–Phelps–Bollobás property if a Bishop–Phelps–Bollobás type theorem holds for bounded linear operators from X into Y. They proved, among other results, that finite dimensional Banach spaces satisfy it and that, whenever Y has the Lindenstrauss property β, the pair (X, Y) has the Bishop–Phelps–Bollobás property for all Banach spaces X. A characterization of those Banach spaces Y such that the pair (ℓ_1, Y) has the Bishop–Phelps–Bollobás property was also given. After the mentioned article [2] in 2008, a lot of attention was given to this topic and many interesting problems related to this property were discussed. For more information the reader can refer, for example, to [1,4,3,8,9,21].

To make the article entirely accessible, we present usual notations and necessary preliminaries. We work with Banach spaces X over the field $\mathbb{K}$, which can be either the set of real numbers $\mathbb{R}$, or the set of complex numbers $\mathbb{C}$. We denote by S_X, B_X and X^* the unit sphere, the unit ball and the topological dual of X, respectively, and by $\mathcal{L}(X, Y)$ the set of all bounded linear operators from X into Y. We say that $T \in \mathcal{L}(X, Y)$ is norm attaining whenever $\|T\| = \sup_{x \in S_X} \|T(x)\| = \|T(x_0)\|$ for some $x_0 \in S_X$ and we denote by $\mathrm{NA}(X, Y)$ the set of all norm attaining operators.

Following [2, Definition 1.1], we say that the pair (X, Y) has the Bishop–Phelps–Bollobás property (BPBp, for short) if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy

$$\|T(x_0)\| > 1 - \eta(\varepsilon), \tag{1}$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1, \quad \|x_1 - x_0\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

It is clear that the pair $(X, \mathbb{K})$ has the BPBp for all Banach spaces X by the Bishop–Phelps–Bollobás theorem. Very recently, a stronger property, called the Bishop–Phelps–Bollobás *point* property, was defined and studied in [12] (see also [11]). The authors added the word "point" in the middle since, in this new property, we fix the point x_0 in the definition of BPBp and move the operator; that is, the new operator S, which is close to T, attains its norm at the same point that T almost attains its norm. Precisely, we say that the pair (X, Y) has the Bishop–Phelps–Bollobás point property (BPBpp, for short) if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy (1), there is $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

$$\|S(x_0)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

It is immediate that the BPBpp implies the BPBp but the opposite implication does not hold. Indeed, the pair (ℓ_1, Y) has the BPBp for every Banach space having the geometric AHSP property (see [2, Theorem 4.1]) but (ℓ_1, Y) cannot have the BPBpp for any Banch space Y (see [12, Proposition 2.3]).

Inspired by a result which characterizes uniformly convex Banach spaces (see [21, Theorem 2.1]), another stronger property than the BPBp was studied in [10]. In this property, which we call here as the Bishop–Phelps–Bollobás *operator* property, one fixes the operator and moves the point instead of fixing the point and moving the operator as in the BPBpp. Specifically, we say that the pair (X, Y) has the Bishop–

Phelps–Bollobás operator property (BPBop, for short) if given $\varepsilon > 0$ there is $\eta(\varepsilon) > 0$ such that, whenever $T \in \mathcal{L}(X,Y)$ with $\|T\| = 1$ and $x_0 \in S_X$ satisfy (1), there is $x_1 \in S_X$ such that

$$\|T(x_1)\| = 1 \quad \text{and} \quad \|x_1 - x_0\| < \varepsilon.$$

It is worth mentioning that the BPBop is actually the dual property of the BPBpp in the sense that $(X,\mathbb{K})$ has the BPBop if and only if $(X^*,\mathbb{K})$ has the BPBpp. This follows from the characterizations of uniformly convex and uniformly smooth Banach spaces obtained in [21, Theorem 2.1] and [12, Proposition 2.1]. In [11] it was proved that if X and Y are real Banach spaces of dimension greater than or equal to 2, then the pair (X,Y) fails the BPBop. Hence, the BPBop holds only for the pairs $(\mathbb{K},Y)$ for every Banach space Y and $(X,\mathbb{K})$ when X is uniformly convex. However, a local version of property BPBop (where the function η in the definition depends not only on ε but also on the operator T) was addressed in [10], obtaining some positive results that put in evidence the difference between the BPBop and its local version.

Our aim in this article, is to follow the research line of local versions of Bishop–Phelps–Bollobás type properties. In Section 2, we study two local Bishop–Phelps–Bollobás type properties which we call $\mathbf{L}_{p,p}$ and $\mathbf{L}_{o,o}$. The first one is the local version of property BPBpp, where the function η depends not only on ε but also on a fixed point $x_0 \in S_X$. The second is the local version of the BPBop addressed in [10]. We note that strongly subdifferentiability (SSD, for short) of the norms of X and X^* characterizes properties $\mathbf{L}_{p,p}$ and $\mathbf{L}_{o,o}$ of $(X,\mathbb{K})$ respectively. This establishes the first difference between these local properties and the uniform properties BPBpp and BPBop. For instance, we have the following.

- If X is c_0 or the predual of Lorentz sequence space $d_*(w,1)$ or the space VMO (which is the predual of the Hardy space H^1) or the finite dimensional spaces ℓ_1^N, ℓ_∞^N when $N \geqslant 2$, then $(X,\mathbb{K})$ has the $\mathbf{L}_{p,p}$ but does not have the BPBpp.
- If X is ℓ_1^N or ℓ_∞^N when $N \geqslant 2$ or the space $\left(\bigoplus_{k=1}^{\infty} \ell_\infty^k\right)_{\ell_2}$, then $(X,\mathbb{K})$ has the $\mathbf{L}_{o,o}$ but does not have the BPBop.

From this characterization of $\mathbf{L}_{p,p}$ and $\mathbf{L}_{o,o}$ in terms of strong subdifferentiability, we get some consequences. For example, if X is smooth (Gâteaux differentiable) and the pair $(X,\mathbb{K})$ satisfies the $\mathbf{L}_{p,p}$, then the norm of X is Fréchet differentiable and the converse is also true. We also prove that if X^* has the w^*-Kadec–Klee property, then the norm of X is SSD or, equivalently, $(X,\mathbb{K})$ has the $\mathbf{L}_{p,p}$. Our main results in Section 2 concern the vector-valued case of property $\mathbf{L}_{p,p}$. We show that if the pair (X,Y) satisfies the $\mathbf{L}_{p,p}$, then the norm of X must be SSD. In particular, we see that (ℓ_1,Y) fails the $\mathbf{L}_{p,p}$ for all Banach spaces Y. However, we are able to prove that the pair (ℓ_1^N,X) has the $\mathbf{L}_{p,p}$ when X is uniformly convex, while it is known that it fails the BPBpp. We also show that if X and Y are finite dimensional, then the pair (X,Y) satisfies the $\mathbf{L}_{p,p}$. This establishes another difference with the BPBpp, since there exist finite dimensional spaces X_0, Y_0 such that (X_0,Y_0) fails the BPBpp (see [12, Example 2.10]). Finally, we prove that both (c_0,c_0) and (c_0,X) has the $\mathbf{L}_{p,p}$ when X is a (complex) uniformly convex Banach space. Again, these are examples of pairs satisfying the $\mathbf{L}_{p,p}$ but failing the BPBpp.

In Section 3, we study local Bishop–Phelps–Bollobás properties, that is, the BPBp when the function η depends on a fixed norm-one point x or on a fixed norm-one operator T. We call them properties $\mathbf{L}_p$ and $\mathbf{L}_o$, respectively. We prove that if Y is strictly convex and either (ℓ_1^2,Y) has the $\mathbf{L}_p$ or (ℓ_1,Y) has the $\mathbf{L}_o$, then Y must be uniformly convex. This is useful to get some counterexamples. We also show that if every norm-one point $x \in S_X$ is strongly exposed, then the pair (X,Y) has the $\mathbf{L}_p$ for all Banach spaces Y whenever $\mathrm{NA}(X,Y)$ is dense in $\mathcal{L}(X,Y)$. This last result provides us examples of Banach spaces X for which the pair (X,Y) has the $\mathbf{L}_p$ for all Banach spaces Y. These examples are stated in Section 4, where we investigate stability results and we deal with universal spaces for the local properties. Actually, we prove that both universal BPBp range space and universal $\mathbf{L}_o$ range space are equivalent properties. Also, if X

is finite dimensional, we prove that X is universal BPBp domain space if and only if it is a universal $\mathbf{L}_p$ domain space.

Finally, we finish this paper with Section 5 which our aim is to compare all those properties with each other.

2. The $\mathbf{L}_{p,p}$ and the $\mathbf{L}_{o,o}$

In this section we study two local properties which are dual from each other in the reflexive case. Before we start, recall that a stronger property than the BPBp, which we call BPBop, was studied in [10] (see also [25,26]). As we already mention in the Introduction, it is known that there is no vector-valued version for the BPBop (see last section of [11]) and then it only makes sense to study it in a local sense.

Definition 2.1. (a) A pair (X, Y) has the $\mathbf{L}_{p,p}$ if given $\varepsilon > 0$ and $x \in S_X$, there is $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, x),$$

there is $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ such that

$$\|S(x)\| = 1 \quad \text{and} \quad \|S - T\| < \varepsilon.$$

(b) ([10, Definition 2.2]) A pair (X, Y) has the $\mathbf{L}_{o,o}$ if given $\varepsilon > 0$ and $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there is $x_0 \in S_X$ such that

$$\|T(x_0)\| = 1 \quad \text{and} \quad \|x_0 - x\| < \varepsilon.$$

It is worth mentioning that if (X, Y) has the $\mathbf{L}_{o,o}$, then every linear operator from X into Y attains its norm. By using James theorem it is possible to construct an operator that never attain its norm in the non-reflexive case. So, in order to get positive results about this property, the domain space X should be reflexive.

2.1. Scalar-valued case

We first focus on $\mathbf{L}_{p,p}$ and $\mathbf{L}_{o,o}$ for bounded linear functionals. We start with the following straightforward observation but, for the sake of completeness, we give its proof for one implication.

Proposition 2.2. *Let X be a reflexive Banach space. Then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only if $(X^*, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.*

Proof. Let $\varepsilon > 0$ and $x^{**} \in S_{X^{**}}$ be given. Since X is reflexive, there is $x \in S_X$ such that $\hat{x} = x^{**}$ where $\hat{\cdot}$ is the canonical inclusion. Consider $\eta(\varepsilon, x) > 0$ the $\mathbf{L}_{p,p}$ function for the pair $(X, \mathbb{K})$. Suppose that $x^* \in S_{X^*}$ satisfies

$$|x^{**}(x^*)| > 1 - \eta(\varepsilon, x).$$

Then $|x^*(x)| > 1 - \eta(\varepsilon, x)$ and so there is $y^* \in S_{X^*}$ such that

$$|y^*(x)| = 1 \quad \text{and} \quad \|y^* - x^*\| < \varepsilon.$$

Since $|x^{**}(y^*)| = |y^*(x)| = 1$, the pair $(X^*, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ with $\eta(\varepsilon, x^{**}) := \eta(\varepsilon, x) > 0$. The converse is completely analogous. □

If the one-side limit $\lim_{t\to 0^+} \frac{\|x+th\|-\|x\|}{t}$ exists uniformly for $h \in B_X$, we say that the norm of X is *strongly subdifferentiable* (SSD, for short) at x. When this happens for all $x \in S_X$, we say that the norm of X is SSD. When the $\lim_{t\to 0} \frac{\|x+th\|-\|x\|}{t}$ exists, then we say that the norm of X is Gâteaux differentiable at x and, finally, if this last limit exists uniformly for all $h \in B_X$, then the norm of X is said to be Fréchet differentiable at x. It turns out that SSD characterizes the pairs $(X, \mathbb{K})$ to have the $\mathbf{L}_{p,p}$. This was observed by G. Godefroy, V. Montesinos and V. Zizler in [19] as a consequence of a characterization of strong subdifferentiability due to C. Franchetti and R. Payá [17].

Theorem 2.3. *([17, Theorem 1.2] and [19]) Let X be a Banach space.*

(a) *Then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ if and only if the norm of X is SSD.*
(b) *Then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ if and only if X is reflexive and the norm of X^* is SSD.*

Note that the norm of X is Fréchet differentiable at x if and only if it is Gâteaux differentiable and SSD at x. We rewrite this equivalence using Theorem 2.3.(a).

Theorem 2.4. *Let X be a Banach space and suppose that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$. The norm of X is Gâteaux differentiable if and only if it is Fréchet differentiable.*

Next, we introduce three well known rotundities which are stronger than strict convexity.

(1) We say that X is *locally uniformly rotund* (LUR, for short) if for all $x, x_n \in S_X$ satisfying $\lim_n \|x_n + x\| = 2$, we have that $\lim_n \|x_n - x\| = 0$.
(2) We say that X is *weakly locally uniformly rotund* (w-LUR, for short) if $\lim_{n\to\infty} \|x_n + x_0\| = 2$ with $x_n, x_0 \in S_X$ implies $\lim_{n\to\infty} x_0^*(x_n) = 1$ whenever $x_0^* \in S_{X^*}$ and $x_0^*(x_0) = 1$.
(3) We say that X is a *midpoint locally uniformly rotund* (MLUR, for short) space if whenever $(x_n), (y_n) \subset S_X$ are norm-one sequences in S_X with $\frac{1}{2}(x_n + y_n)$ converging to some $x_0 \in S_X$, we have that $\|x_n - y_n\| \longrightarrow 0$.

It is known that

$$\text{LUR} \Rightarrow w\text{-LUR} \Rightarrow \text{strict convexity} \quad \text{and} \quad \text{LUR} \Rightarrow \text{MLUR} \Rightarrow \text{strict convexity}$$

and that none of them are equivalent. Another well known fact is that the norm of X is Fréchet (respectively, Gâteaux) differentiable if X^* is LUR (respectively, strictly convex), see for instance [14, Fact 8.12 and 8.18]. Hence, we see that if X^* is LUR, then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$. Also, we have that if X^* is Fréchet differentiable, then $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$, since such X is reflexive (see [14, Fact 8.6]). In particular, if X is LUR and reflexive then $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$. Moreover, we have the following equivalence of those rotundities.

Theorem 2.5. *Let X be a Banach space and suppose that the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$.*

(a) *A Banach space X is strictly convex if and only if X is MLUR.*
(b) *A Banach space X is strictly convex if and only if X^* is Fréchet differentiable.*
(c) *A Banach space X is w-LUR if and only if it is LUR.*

Proof. We only need to prove directions from left to right.

Proof of (a): Suppose that X is strictly convex but not MLUR. Then there are $x_0 \in S_X$, sequences $(x_n), (y_n) \subset S_X$ and some $\delta \in (0,1)$ such that

$$\lim_n \left\| \left(\frac{x_n + y_n}{2} \right) - x_0 \right\| = 0 \quad \text{but} \quad \|x_n - y_n\| \geqslant \delta, \ \forall\, n \in \mathbb{N}. \tag{2}$$

Take $x_0^* \in S_{X^*}$ to be such that $\operatorname{Re} x_0^*(x_0) = 1$, then $\lim_n \operatorname{Re} x_0^* \left(\frac{x_n+y_n}{2}\right) = 1$. This implies that $\lim_n \operatorname{Re} x_0^*(x_n) = \lim_n \operatorname{Re} x_0^*(y_n) = 1$. So there is $n_0 \in \mathbb{N}$ such that

$$\operatorname{Re} x_0^*(x_n) > 1 - \min\left\{ \eta\left(\frac{\delta^2}{64}, x_0^*\right), \frac{\delta^2}{64} \right\} \quad \text{and} \quad \operatorname{Re} x_0^*(y_n) > 1 - \min\left\{ \eta\left(\frac{\delta^2}{64}, x_0^*\right), \frac{\delta^2}{64} \right\}$$

for all $n \geqslant n_0$ where $\eta(\,\cdot\,, x_0^*) > 0$ is the $\mathbf{L}_{o,o}$ function for the pair $(X, \mathbb{K})$. Then there are $u_n, v_n \in S_X$ such that

$$|x_0^*(u_n)| = |x_0^*(v_n)| = 1, \quad \|u_n - x_n\| < \frac{\delta^2}{64} \ \text{ and } \ \|v_n - y_n\| < \frac{\delta^2}{64}, \ \forall\, n \geqslant n_0.$$

For each $n \geqslant n_0$, set

$$x_0^*(u_n) = \lambda_n |x_0^*(u_n)| = \lambda_n \ \text{ and } \ x_0^*(v_n) = \mu_n |x_0^*(v_n)| = \mu_n$$

for $|\lambda_n| = |\mu_n| = 1$. So $x_0^*(\lambda_n^{-1} u_n) = 1 = x_0^*(\mu_n^{-1} v_n)$ and since X is strictly convex, we have that $\lambda_n^{-1} u_n = \mu_n^{-1} v_n$ for all $n \geqslant n_0$. Now note that

$$\begin{aligned} 1 - \operatorname{Re} \lambda_n = 1 - \operatorname{Re} x_0^*(u_n) &= 1 + \operatorname{Re} x_0^*(x_n - u_n) - \operatorname{Re} x_0^*(x_n) \\ &\leqslant 1 - \operatorname{Re} x_0^*(x_n) + \|x_n - u_n\| \\ &< \frac{\delta^2}{32}. \end{aligned}$$

Since $|\lambda_n| = 1$, we have that

$$|1 - \lambda_n|^2 = 2(1 - \operatorname{Re} \lambda_n) < \frac{\delta^2}{16}.$$

So $|1 - \lambda_n| < \frac{\delta}{4}$. Analogously, $|1 - \mu_n| < \frac{\delta}{4}$. Then, for $n \geqslant n_0$, we have that

$$\begin{aligned} \|x_n - y_n\| &\leqslant \|x_n - \lambda_n^{-1} u_n\| + \|\mu_n^{-1} v_n - y_n\| \\ &\leqslant \|x_n - u_n\| + |1 - \lambda_n| + |1 - \mu_n| + \|v_n - y_n\| < \delta \end{aligned}$$

which contradicts (2).

Proof of (b): This has been observed in [26], but we give details briefly. Since the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$, then X is reflexive and X^* is SSD by Theorem 2.3.(b). Since X is strictly convex, the norm of X^* is Gâteaux differentiable. Hence it is Fréchet differentiable by Theorem 2.4.

Proof of (c): Since X is reflexive, the pair $(X^*, \mathbb{K})$ has the $\mathbf{L}_{p,p}$. Also, w-LUR implies strict convexity for X and so X^* is Fréchet differentiable by (b). Now, we apply [23, Theorem 2.4] to get that X is LUR. □

A Banach space X has the *Kadec–Klee property* if weak and norm topologies coincide on S_X. Also, we say that X^* has the *w^*-Kadec–Klee property* if the weak* and norm topologies coincide in S_{X^*}. It is well-known that if X is LUR (respectively, X^* is LUR), then it satisfies the Kadec–Klee property (respectively, the w^*-Kadec–Klee property).

Proposition 2.6. *Let X be a Banach space. If X^* has the w^*-Kadec–Klee property, then the norm of X is SSD or, equivalently, the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.*

Proof. Otherwise, there are $\varepsilon_0 > 0$ and $x_0 \in S_X$ such that for each $n \in \mathbb{N}$, there is $x_n^* \in S_{X^*}$ such that

$$1 \geqslant |x_n^*(x_0)| \geqslant 1 - \frac{1}{n}$$

and whenever $x^* \in S_{X^*}$ satisfies $\|x^* - x_n^*\| < \varepsilon_0$, we have $|x^*(x_0)| < 1$. By the Banach–Alaoglu theorem, there is a subnet of (x_n^*), which we denote again by (x_n^*), such that $x_n^* \xrightarrow{w^*} x_0^*$ for some $x_0^* \in B_{X^*}$. Then $x_n^*(x_0) \longrightarrow x_0^*(x_0)$ and since $|x_n^*(x_0)| \longrightarrow 1$, we get that $|x_0^*(x_0)| = 1$ and then $x_0^* \in S_{X^*}$. By hypothesis, $x_n^* \longrightarrow x_0^*$ in norm and since $|x_0^*(x_0)| = 1$, we have a contradiction. □

As an immediate consequence of the previous proposition and Proposition 2.2 we see that if X is a reflexive space which satisfies the Kadec–Klee property, then the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$. It is worth mentioning that this is a particular case of [25, Theorem 2.12] where it was proved that, under the same assumption on X, the pair (X, Y) has the $\mathbf{L}_{o,o}$ for compact operators for all Banach spaces Y.

As was mentioned in the Introduction, the results in this section establish the first differences between the local properties $\mathbf{L}_{p,p}$ and $\mathbf{L}_{o,o}$ with respect with their uniform versions BPBpp and BPBop. For instance, the norm of the space c_0 is SSD (see, for instance, [16]) and, hence, the pair $(c_0, \mathbb{K})$ has property $\mathbf{L}_{p,p}$. On the other hand, the pair $(c_0, \mathbb{K})$ fails the BPBpp, since c_0 is not uniformly smooth (it is not even reflexive). There are many other examples of non-reflexive spaces X with a SSD norm and, consequently, such that $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ but fails the BPBpp. For example, it is known that the Hardy space H^1 of analytic functions on the ball, the Lorentz spaces $L_{p,1}(\mu)$ and the trace class $\mathcal{C}_1$ are non-reflexive dual spaces that have the w^*-Kadec–Klee property. Indeed, they have a stronger property called the w^*-uniform Kadec–Klee property (see [13] and references therein). As a consequence, if X is the predual of any of those spaces, then X^* has the w^*-Kadec–Klee property and, by Proposition 2.6, the pair $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ (but it fails the BPBpp). We can also mention examples of reflexive space X such that $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$ and it fails the BPBpp. For example, if $X = \ell_1^N$ or ℓ_∞^N with $N \geqslant 2$ then the norm of X is SSD (indeed, every finite dimensional spaces is SSD, see [17]) but these spaces are not uniformly smooth and then, $(X, \mathbb{K})$ fails the BPBpp. On the other hand, it is clear that if $X = \ell_1^N$ or ℓ_∞^N with $N \geqslant 2$ then $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$ and it fails the BPBop. Also, by [17, Theorem 2.4], we know that the reflexive space $X = \left(\bigoplus_{k=1}^\infty \ell_\infty^k\right)_{\ell_2}$ is such that the norm of X^* is SSD and, hence, $(X, \mathbb{K})$ has the $\mathbf{L}_{o,o}$. But this space is not uniformly convex (see [15, Chapter 9.2]) and, consequently, $(X, \mathbb{K})$ fails the BPBop.

2.2. Vector-valued case

In this subsection we focus on property $\mathbf{L}_{p,p}$. For the vector-valued case of property $\mathbf{L}_{o,o}$, we suggest the references [10], [25] and [26]. To start, we get the following observation which shows that if the $\mathbf{L}_{p,p}$ holds for operators from X into Y, then the $\mathbf{L}_{p,p}$ holds for the pair $(X, \mathbb{K})$. The proof follows the same lines as [12, Proposition 2.3]. Consequently, by Proposition 2.3, we have that the norm of X must be SSD whenever the pair (X, Y) has the $\mathbf{L}_{p,p}$ for some Y.

Proposition 2.7. *Let X and Y be Banach spaces. If the pair (X, Y) has the $\mathbf{L}_{p,p}$, then $(X, \mathbb{K})$ has the $\mathbf{L}_{p,p}$.*

Thus, in order to get positive results for the $\mathbf{L}_{p,p}$ in the vector-valued case, we have to assume that the domain space is SSD. For that reason (ℓ_1, Y) fails the $\mathbf{L}_{p,p}$ for all Banach spaces Y, since the norm of ℓ_1 is SSD only at the points in the unit sphere which are sequences with finitely many nonzero terms (see, for example, [16]). In Section 3 (see Remark 3.3 below) we show that the converse of Proposition 2.7 is not true in general. However, this converse holds if we consider Banach spaces Y satisfying the property β defined by Lindenstrauss in [22]. Typical examples of Banach spaces satisfying this property are c_0 and ℓ_∞. In [12, Proposition 2.4] it was proved that the pair (X,Y) has the BPBpp whenever $(X,\mathbb{K})$ has the BPBpp and Y has property β, and the same proof can be applied for the $\mathbf{L}_{p,p}$. Since the norm of c_0 is SSD and it has property β, we also get a particular case.

Proposition 2.8. *Let X and Y be Banach spaces. Assume that Y has property β. If the pair $(X,\mathbb{K})$ has the $\mathbf{L}_{p,p}$, then the pair (X,Y) has the $\mathbf{L}_{p,p}$. In particular, the pair (c_0,c_0) satisfies the $\mathbf{L}_{p,p}$.*

Since every finite dimensional Banach space is SSD, whenever X is finite dimensional and Y satisfies property β, the pair (X,Y) has the $\mathbf{L}_{p,p}$. On the other hand, it was proved in [10] that if X is a finite dimensional Banach space, then (X,Y) has the $\mathbf{L}_{o,o}$ for all Banach spaces Y. This last statement is not true for the $\mathbf{L}_{p,p}$ (see Remark 3.3) but if we restrict the range space to be finite dimensional, we get a positive result.

Proposition 2.9. *Let X and Y be finite dimensional spaces. Then the pair (X,Y) has the $\mathbf{L}_{p,p}$.*

Proof. Otherwise, there are $\varepsilon_0>0$ and $x_0\in S_X$ such that for each $k\in\mathbb{N}$, there is $T_k\in\mathcal{L}(X,Y)$ with $\|T_k\|=1$ and

$$1\geqslant\|T_k(x_0)\|\geqslant 1-\frac{1}{k} \tag{3}$$

such that whenever $S\in\mathcal{L}(X,Y)$ with $\|S\|=1$ satisfies $\|T_k-S\|<\varepsilon_0$, we have $\|S(x_0)\|<1$. By compactness, we may assume that $\{T_k\}$ is convergent and let $T\in B_{\mathcal{L}(X,Y)}$ be the limit of $\{T_k\}$. So there is $k_0\in\mathbb{N}$ such that $\|T_k-T\|<\varepsilon_0$ for all $k\geqslant k_0$. By (3) we have that $\|T\|=\|T(x_0)\|=1$ which is a contradiction. So the pair (X,Y) has the $\mathbf{L}_{p,p}$. $\square$

Note that the previous proposition is not true for the property BPBpp. In fact, it is shown in [12, Example 2.10] that there exist finite dimensional Banach spaces X_0 and Y_0 such that (X_0,Y_0) fails the BPBpp.

Recall that the *modulus of convexity* of a Banach space Z is defined for each $\varepsilon\in(0,2]$ by

$$\delta(\varepsilon):=\inf\left\{1-\left\|\frac{z_1+z_2}{2}\right\| \ :\ z_1,z_2\in B_X, \|z_1-z_2\|\geqslant\varepsilon\right\}$$

and Z is said to be *uniformly convex* if $\delta(\varepsilon)>0$ for $\varepsilon\in(0,2]$. Next we prove that if the range space is uniformly convex, we get a positive result for the $\mathbf{L}_{p,p}$. Actually, it seems to be interesting since in our result the domain is ℓ_1^N and, as we already commented before, the pair (ℓ_1,Y) fails the $\mathbf{L}_{p,p}$ for all Banach spaces Y.

Proposition 2.10. *Let X be a uniformly convex Banach space and let $N\in\mathbb{N}$. Then, the pair (ℓ_1^N,X) has the $\mathbf{L}_{p,p}$.*

Proof. Let $\varepsilon\in(0,1)$ and $x=(x_1,\ldots,x_N)\in S_{\ell_1^N}$ be given. By composing with an isometry if necessary, we may assume that $x_j\geqslant 0$ for all $j=1,\ldots,N$. Set $A_x:=\{i\in\{1,...,n\}: x_i\neq 0\}$ and $K_x:=\min\{x_j : j\in A_x\}>0$. Set

$$\eta(\varepsilon,x):=K_x\delta_X(\varepsilon)>0,$$

where $\delta_X(\cdot)$ is the modulus of convexity of X. Let $T : \ell_1^N \longrightarrow X$ with $\|T\| = 1$ be such that

$$\|T(x)\| > 1 - \eta(\varepsilon, x)$$

and consider $x_0^* \in S_{X^*}$ satisfying $\operatorname{Re} x_0^*(T(x)) > 1 - \eta(\varepsilon, x)$. Write $x = \sum_{j\in A_x} x_j e_j$ with $\sum_{j\in A_x} x_j = 1$. Then for all $i \in A_x$, we have that

$$\begin{aligned}
1 - K_x\delta_X(\varepsilon) = 1 - \eta(\varepsilon, x) &< \operatorname{Re} x_0^*(T(x)) \\
&= \operatorname{Re} \sum_{j\in A_x} x_j x_0^*(T(e_j)) \\
&\leqslant \sum_{j\in A_x\setminus\{i\}} x_j + x_i \operatorname{Re} x_0^*(T(e_i)) \\
&= 1 - x_i + x_i \operatorname{Re} x_0^*(T(e_i)) = 1 + x_i(\operatorname{Re} x_0^*(T(e_i)) - 1)
\end{aligned}$$

which implies that

$$1 - \operatorname{Re} x_0^*(T(e_i)) < \frac{K_x}{x_i}\delta_X(\varepsilon) \leqslant \delta_X(\varepsilon),$$

and so we get that $\operatorname{Re} x_0^*(T(e_i)) > 1 - \delta_X(\varepsilon)$ for all $i \in A_x$. Since X is reflexive, there exists $x_0 \in S_X$ such that $x_0^*(x_0) = 1$. Define $S : \ell_1^N \longrightarrow X$ by $S(e_i) := x_0$ for all $i \in A_x$ and $S(e_i) = T(e_i)$, otherwise. Then $\|S\| = 1$ and

$$\|S(x)\| = \left\|\sum_{j\in A_x} x_j S(e_j)\right\| = \sum_{j\in A_x} x_j = 1.$$

Finally, for $i \in A_x$, since $\operatorname{Re} x_0^*(S(e_i)) = x_0^*(x_0) = 1$ and

$$\left\|\frac{S(e_i) + T(e_i)}{2}\right\| \geqslant \frac{\operatorname{Re} x_0^*(S(e_i)) + \operatorname{Re} x_0^*(T(e_i))}{2} > 1 - \delta_X(\varepsilon),$$

we get that $\|S(e_i) - T(e_i)\| < \varepsilon$ for all $i \in A_x$ and then $\|S - T\| < \varepsilon$ since $S(e_i) = T(e_i)$ if $i \notin A_x$. This proves that the pair (ℓ_1^N, X) has the $\mathbf{L}_{p,p}$ as desired. □

We know that the pair (c_0, c_0) has the $\mathbf{L}_{p,p}$, so it is natural to ask if the pair (c_0, X) has the $\mathbf{L}_{p,p}$ for others Banach spaces X different from c_0. In what follows, we show that this pair satisfies that property whenever X is a (complex) uniformly convex Banach space. Before we do that, let us give some preliminaries about complex uniformly convex Banach spaces.

For a complex Banach space Z, the $\mathbb{C}$-*modulus of convexity* $\delta_{\mathbb{C}}$ is defined for every $\varepsilon > 0$ by

$$\delta_{\mathbb{C}}(\varepsilon) := \inf\big\{\sup\{\|z_1 + \lambda\varepsilon z_2\| - 1 :\ \lambda \in \mathbb{C},\ |\lambda| = 1\} :\ z_1, z_2 \in S_Z\big\}.$$

The Banach space Z is called $\mathbb{C}$-*uniformly convex* if $\delta_{\mathbb{C}}(\varepsilon) > 0$ for every $\varepsilon > 0$ (see, for example, [18]). Every uniformly convex complex space is $\mathbb{C}$-uniformly convex space and the converse is not true. Also, it was proved in [18] that the complex $L_1(\mu)$-space is $\mathbb{C}$-uniformly convex.

To prove the next theorem, we need the following lemma. Its proof is similar to [1, Lemma 2.3] and we omit it. For a given $A \subset \mathbb{N}$, we define $P_A : c_0 \to c_0$ by $P_A(x) := \sum_{n\in A} x(n)e_n$, where $(e_j)_j$ is the canonical basis of c_0.

Lemma 2.11. *Let X be a $\mathbb{C}$-uniformly convex Banach space with modulus $\delta_{\mathbb{C}}$. Let $\varepsilon > 0$ be given. If $T \in \mathcal{L}(c_0, X)$ with $\|T\| = 1$ and $A \subset \mathbb{N}$ has the property that $\|TP_A\| > 1 - \frac{\delta_{\mathbb{C}}}{1+\delta_{\mathbb{C}}}$, then $\|T(I - P_A)\| \leqslant \varepsilon$, where I is the identity mapping on c_0.*

Besides that, we are using that the pair (c_0, X) has the BPBp whenever X is a $\mathbb{C}$-uniformly convex Banach space (see [1,20]). Note also that in the definition of the BPBp we can consider the initial operator in the ball of the space instead of the sphere by doing an easy change of parameters. We are using this observation in the next theorem.

Theorem 2.12. *Let X be a $\mathbb{C}$-uniformly convex Banach space. Then, the pair (c_0, X) has the $\mathbf{L}_{p,p}$.*

Proof. Let $\varepsilon \in (0, 1)$ and $z \in S_{c_0}$ be given. Set

$$A_z := \{i \in \mathbb{N} : |z(i)| = 1\} \quad \text{and} \quad K_z := \min\left\{1 - \max_{i \in A_z^c} |z(i)|,\ \varepsilon\right\} > 0.$$

We know that the pair (c_0, X) has the BPBp for some function $\tilde{\eta}(\varepsilon) > 0$. By [4, Theorem 2.1], the pair $(\ell_\infty(A), X)$ has BPBp for any finite subset $A \subset \mathbb{N}$ with the same $\tilde{\eta}(\varepsilon)$. If $\delta_{\mathbb{C}}(\varepsilon)$ is the $\mathbb{C}$-modulus of convexity of X, we set

$$\eta(\varepsilon) := \min\left\{\tilde{\eta}(\varepsilon), \frac{\delta_{\mathbb{C}}}{1 + \delta_{\mathbb{C}}}\right\} > 0.$$

Let $T \in \mathcal{L}(c_0, X)$ with $\|T\| = 1$ be such that

$$\|T(z)\| > 1 - K_z \eta(\varepsilon)$$

and consider $x^* \in S_{X^*}$ to be such that $\operatorname{Re} x^*(T(z)) > 1 - K_z \eta(\varepsilon)$. Then

$$1 - K_z \eta(\varepsilon) < \operatorname{Re} x^*(T(z)) = \operatorname{Re}(T^* x^*)(z) = \operatorname{Re} \sum_{i \in \mathbb{N}} z(i)(T^* x^*)(i). \tag{4}$$

By composing with an isometry if necessary, we may assume that $(T^* x^*)(i) = \operatorname{Re}(T^* x^*)(i) \geqslant 0$ for every i. Now set

$$C_z := \{i \in \mathbb{N} : \operatorname{Re} z(i) > 1 - K_z\}.$$

Note that $C_z \subset A_z$. Also, note that

$$\begin{aligned}
\operatorname{Re} \sum_{i \in \mathbb{N}} z(i)(T^* x^*)(i) &= \sum_{i \in C_z} \operatorname{Re} z(i)(T^* x^*)(i) + \sum_{i \in C_z^c} \operatorname{Re} z(i)(T^* x^*)(i) \\
&\leqslant \sum_{i \in C_z} T^*(x^*)(i) + (1 - K_z) \sum_{i \in C_z^c} T^*(x^*)(i) \\
&= \sum_{i \in C_z} T^*(x^*)(i) - (1 - K_z) \sum_{i \in C_z} T^*(x^*)(i) + (1 - K_z) \sum_{i \in \mathbb{N}} T^*(x^*)(i) \\
&\leqslant K_z \sum_{i \in C_z} (T^* x^*)(i) + 1 - K_z \\
&= K_z \left(-1 + \sum_{i \in C_z} (T^* x^*)(i)\right) + 1.
\end{aligned}$$

By using this and (4), we get that

$$x^* \left(TQ_{C_z}(1_{C_z})\right) = (T^*x^*)(Q_{C_z}(1_{C_z})) = \sum_{i \in C_z} (T^*x^*)(i) > 1 - \eta(\varepsilon) \tag{5}$$

where Q_{C_z} is the canonical extension from $\ell_\infty(C_z)$ to c_0 and 1_{C_z} is the element whose components are 1 in C_z and 0 otherwise. It is clear that C_z is nonempty. Let P_{C_z} be the canonical projection from c_0 into $\ell_\infty(C_z)$ and note that $Q_{C_z}P_{C_z}(x) = \sum_{n \in C_z} x(n)e_n$. By (5), we have that $\|TQ_{C_z}P_{C_z}\| > 1 - \frac{\delta_{\mathbb{C}}}{1+\delta_{\mathbb{C}}}$ and by Lemma 2.11, we get

$$\|T(I - Q_{C_z}P_{C_z})\| \leqslant \varepsilon. \tag{6}$$

Now set $T_{C_z} := TQ_{C_z}$, the restriction of T to $\ell_\infty(C_z)$. Then $\|T_{C_z}\| \leqslant 1$ and by (5),

$$\|T_{C_z}(1_{C_z})\| \geqslant x^* \left(TQ_{C_z}(1_{C_z})\right) > 1 - \tilde{\eta}(\varepsilon).$$

Hence, there are $S \in \mathcal{L}(\ell_\infty(C_z), X)$ with $\|S\| = 1$ and $z^1 \in S_{\ell_\infty(C_z)}$ such that

$$\|S(z^1)\| = 1, \quad \|1_{C_z} - z^1\| < \varepsilon \quad \text{and} \quad \|S - T_{C_z}\| < \varepsilon.$$

Let $S' := SP_{C_z}$, the natural extension of S on c_0. Define $z^2 \in S_{\ell_\infty(C_z)}$ by $z^2(i) := \frac{z^1(i)}{|z^1(i)|}$ for each $i \in C_z$. Then it is easy to see that

$$\|Sz^2\| = 1 \quad \text{and} \quad \|1_{C_z} - z^2\| < 2\varepsilon.$$

Finally, define $U \in \mathcal{L}(c_0, c_0)$ by $Ue_i = \frac{z^2(i)}{z(i)}e_i$ for $i \in C_z$ and $Ue_i = e_i$, otherwise. Then U is an isometry such that $\|S'Uz\| = 1$. So $\|S'U\| = 1$ and it attains its norm at z. It remains to prove that $\|S'U - T\|$ is small. Indeed, note first that for each $i \in C_z \subset A_z$, we have

$$\operatorname{Re} z(i) + \varepsilon \geqslant \operatorname{Re} z(i) + K_z > 1$$

which implies that $1 - \operatorname{Re} z(i) \leqslant \varepsilon$. So since $|z(i)| = 1$ for $i \in C_z$ we have $\operatorname{Im} z(i)^2 = 1 - \operatorname{Re} z(i)^2$. Then

$$|1 - z(i)|^2 = (1 - \operatorname{Re} z(i))^2 + \operatorname{Im} z(i)^2 = 2(1 - \operatorname{Re} z(i)) \leqslant 2\varepsilon,$$

and so

$$\begin{aligned}\left|\frac{z^2(i)}{z(i)} - 1\right| &= |z^2(i) - z(i)| \leqslant |z^2(i) - z^1(i)| + |z^1(i) - 1| + |1 - z(i)| \\ &< 2|1 - z^1(i)| + \sqrt{2\varepsilon} < 2\varepsilon + \sqrt{2\varepsilon}.\end{aligned}$$

Because of this and using (6), we get that

$$\begin{aligned}\|S'U - T\| &\leqslant \|S'U - S'\| + \|S' - TQ_{C_z}P_{C_z}\| + \|TQ_{C_z}P_{C_z} - T\| \\ &\leqslant \|U - I\| + \|S - TQ_{C_z}\| + \varepsilon \\ &\leqslant \max_{i \in C_z}\left\{\left|\frac{z^2(i)}{z(i)} - 1\right|\right\} + 2\varepsilon \leqslant 4\varepsilon + \sqrt{2\varepsilon}. \quad \square\end{aligned}$$

Since the complex spaces $L_p(\mu)$ with $1 \leqslant p < \infty$ are $\mathbb{C}$-uniformly convex for every positive measure μ, we get the following consequence.

Corollary 2.13. *Let μ be a positive measure and $1 \leqslant p < \infty$. In the complex case, the pairs $(c_0, L_p(\mu))$ has the $\mathbf{L}_{p,p}$.*

It is worth mentioning that the pairs (ℓ_1^N, X) and (c_0, X) (for X uniformly convex and $\mathbb{C}$-uniformly convex, respectively), which were shown to satisfy property $\mathbf{L}_{p,p}$, fail the BPBpp. This is a simple consequence of [12, Proposition 2.3].

3. The $\mathbf{L}_p$ and the $\mathbf{L}_o$

In this section we study the BPBp in the local sense as we did with the BPBpp and the BPBop.

Definition 3.1. (a) A pair (X, Y) has the $\mathbf{L}_p$ if given $\varepsilon > 0$ and $x \in S_X$, there is $\eta(\varepsilon, x) > 0$ such that whenever $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, x),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

(b) A pair (X, Y) has the $\mathbf{L}_o$ if given $\varepsilon > 0$ and $T \in S_{\mathcal{L}(X,Y)}$, there is $\eta(\varepsilon, T) > 0$ such that whenever $x \in S_X$ satisfies

$$\|T(x)\| > 1 - \eta(\varepsilon, T),$$

there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_0 \in S_X$ such that

$$\|S(x_0)\| = 1, \quad \|x_0 - x\| < \varepsilon \quad \text{and} \quad \|S - T\| < \varepsilon.$$

It is immediate that the BPBp implies both $\mathbf{L}_p$ and $\mathbf{L}_o$. Also the $\mathbf{L}_{p,p}$ implies the $\mathbf{L}_p$ and the $\mathbf{L}_{o,o}$ implies the $\mathbf{L}_o$. We give useful results to get counterexamples for reverse implications.

Proposition 3.2. *Let Y be a strictly convex Banach space.*

(a) *If the pair (ℓ_1^2, Y) has the $\mathbf{L}_p$, then Y is uniformly convex.*
(b) *If the pair (ℓ_1, Y) has the $\mathbf{L}_o$, then Y is uniformly convex.*

Proof. Suppose that Y is not uniformly convex. Then there is $\varepsilon_0 \in (0, 1)$ such that for each $k \in \mathbb{N}$, there are $y_1^k, y_2^k \in S_Y$ with $\|y_1^k - y_2^k\| > \varepsilon_0$ and

$$\left\| \frac{y_1^k + y_2^k}{2} \right\| > 1 - \frac{1}{k}. \tag{7}$$

We start with (a). Let $n \in \mathbb{N}$ be such that

$$\frac{1}{n} < \eta\left(\frac{\varepsilon_0}{2}, \frac{e_1 + e_2}{2}\right).$$

Define $T_n : \ell_1^2 \longrightarrow Y$ by $T_n(e_1) = y_1^n$ and $T_n(e_2) = y_2^n$. For $x = (x_1, x_2) \in S_{\ell_1^2}$, we have $\|T_n(x)\| \leqslant 1$. Since $\|T_n(e_1)\| = \|y_1^n\| = 1$, we get $\|T_n\| = 1$. Now

$$\left\|T_n\left(\frac{e_1+e_2}{2}\right)\right\| = \left\|\frac{y_1^n+y_2^n}{2}\right\| > 1-\frac{1}{n} > 1-\eta\left(\frac{\varepsilon_0}{2}, \frac{e_1+e_2}{2}\right)$$

and so there are $S_n \in \mathcal{L}(\ell_1^2, Y)$ with $\|S_n\| = 1$ and $z = (z_1, z_2) \in S_{\ell_1^2}$ such that

$$\|S_n(z)\| = 1, \quad \left\|z - \left(\frac{e_1+e_2}{2}\right)\right\| < \frac{\varepsilon_0}{2} < \frac{1}{2} \quad \text{and} \quad \|S_n - T_n\| < \frac{\varepsilon_0}{2}.$$

Since $\left\|z - \left(\frac{e_1+e_2}{2}\right)\right\| < \frac{1}{2}$, we have that z_1 and z_2 are nonzero. Assume that $z_1, z_2 \geqslant 0$. Since $1 = \|S_n(z)\| = \|z_1 S_n(e_1) + z_2 S_n(e_2)\| \leqslant 1$ and Y is strictly convex, we have that $S_n(e_1) = S_n(e_2)$. So

$$\|y_1^n - y_2^n\| = \|T_n(e_1) - T_n(e_2)\| \leqslant \|T_n(e_1) - S_n(e_1)\| + \|S_n(e_2) - T_n(e_2)\| < \varepsilon_0.$$

This contradicts (7).

Now we prove (b). For each $n \in \mathbb{N}$, define $T : \ell_1 \longrightarrow Y$ by $T(e_n) := y_1^n$ and $T(e_{n+1}) := y_2^n$. We have $\|T\| = 1$ and, by hypothesis, there is $\eta\left(\frac{\varepsilon_0}{2}, T\right) > 0$. Assume that $\frac{1}{n} < \eta\left(\frac{\varepsilon_0}{2}, T\right)$. Since

$$\left\|T\left(\frac{e_n+e_{n+1}}{2}\right)\right\| = \left\|\frac{1}{2}T(e_n) + \frac{1}{2}T(e_{n+1})\right\| = \left\|\frac{1}{2}y_1^n + \frac{1}{2}y_2^n\right\| > 1-\frac{1}{n} > 1-\eta\left(\frac{\varepsilon_0}{2}, T\right),$$

there are $S \in \mathcal{L}(\ell_1, Y)$ with $\|S\| = 1$ and $z \in S_{\ell_1}$ such that

$$\|S(z)\| = 1, \quad \left\|z - \left(\frac{1}{2}e_n + \frac{1}{2}e_{n+1}\right)\right\|_1 < \frac{\varepsilon_0}{2} \quad \text{and} \quad \|S - T\| < \frac{\varepsilon_0}{2}.$$

If $z = \sum_{i=1}^{\infty} z_i e_i \in S_{\ell_1}$, then $z_i = 0$ for all $i \neq n, n+1$. Indeed, if there is $i_0 \in \mathbb{N}$ with $i_0 \neq n, n+1$ and $z_{i_0} \neq 0$, we have that

$$\begin{aligned} 1 = \|S(z)\| = \left\|\sum_{i=1}^{\infty} z_i S(e_i)\right\| &= \left\|\sum_{i \neq i_0} z_i S(e_i) + z_{i_0} S(e_{i_0})\right\| \\ &\leqslant \sum_{i \neq i_0} |z_i| + |z_{i_0}|\|S(e_{i_0})\| \\ &= \sum_{i \neq i_0} |z_i| + |z_{i_0}|\|S(e_{i_0}) - T(e_{i_0})\| \\ &< \sum_{i \neq i_0} |z_i| + |z_{i_0}| \cdot \frac{\varepsilon_0}{2} < \sum_{i \neq i_0} |z_i| + |z_{i_0}| = 1, \end{aligned}$$

which is a contradiction. Since $\left\|z - \left(\frac{e_n+e_{n+1}}{2}\right)\right\| < \frac{1}{2}$, we have z_n and z_{n+1} are nonzero. So we may assume that $z_n, z_{n+1} \geqslant 0$. Since $1 = \|S_n(z)\| = \|z_n S(e_n) + z_{n+1} S(e_{n+1})\| \leqslant 1$ and Y is strictly convex, we have that $S(e_n) = S(e_{n+1})$. So

$$\|y_1^n - y_2^n\| = \|T(e_n) - T(e_{n+1})\| \leqslant \|T(e_n) - S(e_{n+1})\| + \|S(e_{n+1}) - T_n(e_{n+1})\| < \varepsilon_0.$$

This contradicts (7) again. $\square$

Remark 3.3. There is a Banach space X which is SSD but the pair (X, Y) fails the $\mathbf{L}_{p,p}$ for some Y. Indeed, Propositions 2.10 and 3.2.(b) show that for a strictly convex Banach space Y, the pair (ℓ_1^2, Y) has the $\mathbf{L}_p$ if and only if Y is uniformly convex. Now if we take a Banach space Y_0 which is strictly but not uniformly convex, then the pair (ℓ_1^2, Y_0) fails the $\mathbf{L}_p$. Since the $\mathbf{L}_{p,p}$ implies the $\mathbf{L}_p$, we get the desired counterexample.

To finish this section, we obtain some conditions under which the pair (X, Y) has the $\mathbf{L}_p$. We will return to this result in Section 4.

Proposition 3.4. *Let X and Y be Banach spaces. Suppose that the set $\mathrm{NA}(X, Y)$ is dense in $\mathcal{L}(X, Y)$. If every point $x \in S_X$ is strongly exposed, then the pair (X, Y) has the $\mathbf{L}_p$.*

Proof. Let $\varepsilon \in (0, 1)$ and $x \in S_X$ be given. Since x is strongly exposed, there are $\delta(\varepsilon, x) > 0$ and $x_0^* \in S_{X^*}$ with $\operatorname{Re} x_0^*(x) = 1$ such that whenever $z \in S_X$ satisfies $\operatorname{Re} x_0^*(z) > 1 - \delta(\varepsilon, x)$, we have $\|z - x\| < \varepsilon$. Let $\varepsilon_2 := \varepsilon_2(\varepsilon, x) > 0$ be such that

$$1 - 2\varepsilon_2 - 2\varepsilon_2^2 + \varepsilon_2^3 + e_2^4 > 1 - \delta(\varepsilon, x) \quad \text{and} \quad \varepsilon_2^2 + 2\varepsilon_2 < \varepsilon.$$

Set $\eta(\varepsilon, x) := \varepsilon_2^2 > 0$. Let $T \in \mathcal{L}(X, Y)$ with $\|T\| = 1$ be such that

$$\|T(x)\| > 1 - \eta(\varepsilon, x)$$

and define $\widetilde{T}_1 \in \mathcal{L}(X, Y)$ by

$$\widetilde{T}_1(z) := T(z) + \varepsilon_2 x_0^*(z) T(x) \qquad (z \in X).$$

Then $\|\widetilde{T}_1 - T\| < \varepsilon_2$ and

$$1 + \varepsilon_2 \geqslant \|\widetilde{T}_1\| \geqslant \|\widetilde{T}_1(x)\| = \|T(x) + \varepsilon_2 T(x)\| > (1 - \varepsilon_2^2)(1 + \varepsilon_2) > 0.$$

So $\widetilde{T}_1 \neq 0$ and $\left|1 - \|\widetilde{T}_1\|\right| < \varepsilon_2$. Let $T_1 := \frac{\widetilde{T}_1}{\|\widetilde{T}_1\|} \in \mathcal{L}(X, Y)$. Since $\overline{\mathrm{NA}(X, Y)} = \mathcal{L}(X, Y)$, there are $S \in \mathcal{L}(X, Y)$ with $\|S\| = 1$ and $x_1 \in S_X$ such that

$$\|S(x_1)\| = 1 \quad \text{and} \quad \|S - T_1\| < \varepsilon_2^2.$$

Since $\|S(x_1) - T_1(x_1)\| \leqslant \|S - T_1\| < \varepsilon_2^2$, we get that $\|T_1(x_1)\| > 1 - \varepsilon_2^2$. Note that we can take x_1 to be such that $x_0^*(x_1) \geqslant 0$. So

$$\begin{aligned} 1 + \varepsilon_2 x_0^*(x_1) \geqslant \|T(x_1) + \varepsilon_2 x_0^*(x_1) T(x_1)\| &= \|\widetilde{T}_1(x_1)\| \\ &\geqslant (1 - \varepsilon_2^2)\|\widetilde{T}_1\| \\ &> (1 - \varepsilon_2^2)(1 - \varepsilon_2^2)(1 + \varepsilon_2) \\ &= 1 + \varepsilon_2 - 2\varepsilon_2^2 - 2\varepsilon_2^3 + \varepsilon_2^4 + e_2^5 \end{aligned}$$

which implies that

$$x_0^*(x_1) > 1 - 2\varepsilon_2 - 2\varepsilon_2^2 + \varepsilon_2^3 + e_2^4 > 1 - \delta(\varepsilon, x).$$

So $\|x_1 - x\| < \varepsilon$. Moreover, we have that

$$\|S - T\| \leqslant \|S - T_1\| + \|T_1 - \widetilde{T}_1\| + \|\widetilde{T}_1 - T\| < \varepsilon_2^2 + 2\varepsilon_2 < \varepsilon.$$

This proves that the pair (X, Y) has the $\mathbf{L}_p$ as desired. $\square$

4. Stability results and universal spaces

In his seminal paper [22], Lindenstrauss considered two properties which are called property A and property B. A Banach space X has property A if the set of norm attaining operators from X into Y is dense for arbitrary Y and Y has property B if the set of norm attaining operators from X into Y is dense for arbitrary X. He found some Banach spaces which have properties A or B in geometric terms. We study analogous spaces for the local properties.

Definition 4.1. Let X and Y be Banach spaces.

(a) Let P be one of the following properties: $\mathbf{L}_o$, $\mathbf{L}_p$, $\mathbf{L}_{p,p}$, $\mathbf{L}_{o,o}$ or BPBp. We say that X is a *universal* P *domain space* if for every Banach space Z, the pair (X, Z) has the P.
(b) Let P be one of the following properties: $\mathbf{L}_o$, $\mathbf{L}_p$ or BPBp. We say that Y is a *universal* P *range space* if for every Banach space Z, the pair (Z, Y) has the P.
(c) We say that Y is a *universal* $\mathbf{L}_{p,p}$ *range space* if for every SSD space Z, the pair (Z, Y) has the $\mathbf{L}_{p,p}$.
(d) We say that Y is a *universal* $\mathbf{L}_{o,o}$ *range space* if for every reflexive space Z whose dual is SSD, the pair (Z, Y) has the $\mathbf{L}_{o,o}$.

As a consequence of Proposition 3.4, we obtain some examples of universal $\mathbf{L}_p$ domain spaces.

Corollary 4.2. *If X is LUR and reflexive or $X = \ell_1$ with the equivalent norm defined by $|||x|||^2 = \|x\|_1^2 + \|x\|_2^2$ (where $\|\cdot\|_i$ denotes the canonical norm on ℓ_i, $i = 1, 2$), then X is a universal $\mathbf{L}_p$ domain space.*

Proof. On one hand, it is well-known that if X is LUR then every point $x \in S_X$ is strongly exposed and that reflexive spaces are universal Bishop–Phelps domain spaces (see [22, Theorem 1]). Then, by Proposition 3.4 we have that the pair (X, Y) has the $\mathbf{L}_p$ for every Banach space Y. On the other hand, the stated renorming of ℓ_1 is LUR (see [15, Lemma 13.26]) and, by a classical result of Bourgain [7], every renorming of ℓ_1 is a universal Bishop–Phelps domain space. Then, Proposition 3.4 gives once again the desired result. $\square$

In Section 5 we will prove that the space $\left(\bigoplus_{k=2}^{\infty} \ell_k^2\right)_{\ell_2}$ is another example of universal $\mathbf{L}_p$ domain space. Moreover, we will show that this space fails to be a universal BPBp domain space.

In the study of universal BPBp domain and range spaces, their stability plays an important role. For more details, we refer the reader to [4]. For a family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, we denote the c_0-sum (respectively, ℓ_1-sum, ℓ_∞-sum) of this family by $\left[\bigoplus_{i\in\Lambda} X_i\right]_{c_0}$ (respectively, $\left[\bigoplus_{i\in\Lambda} X_i\right]_{\ell_1}$, $\left[\bigoplus_{\lambda\in\Lambda} X_i\right]_{\ell_\infty}$).

Similarly to [4, Theorem 2.1], we get the following stability results for direct sums. The proof is almost the same and we omit it.

Theorem 4.3. *Let $\{X_i : i \in I\}$ and $\{Y_j : j \in J\}$ be families of Banach spaces. Let X be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{X_i\}$ and let Y be the c_0-, ℓ_1-, or ℓ_∞-sum of $\{Y_j\}$. For every $i \in I$ and $j \in J$, denote by E_i and F_j the natural inclusion maps from X_i into X and from Y_j into Y, respectively. Also, P_i is for the natural projection from X onto X_i.*

(a) *If the pair (X, Y) has the $\mathbf{L}_{p,p}$ (resp. $\mathbf{L}_p$) with $\eta(\varepsilon, x) > 0$ at $x \in S_X$, then for every $i \in I$ and $j \in J$ the pair (X_i, Y_j) has the $\mathbf{L}_{p,p}$ (resp. $\mathbf{L}_p$) with $\eta(\varepsilon, E_i z) > 0$ at $z \in S_{X_i}$.*
(b) *If the pair (X, Y) has the $\mathbf{L}_{o,o}$ (resp. $\mathbf{L}_o$) with $\eta(\varepsilon, T) > 0$ at $T \in S_{\mathcal{L}(X,Y)}$, then for every $i \in I$ and $j \in J$ the pair (X_i, Y_j) has the $\mathbf{L}_{o,o}$ (resp. $\mathbf{L}_o$) with $\eta(\varepsilon, F_j S P_i)$ at $S \in S_{\mathcal{L}(X_i,Y_j)}$.*

This theorem shows that for $\mathbf{L}_p$ and $\mathbf{L}_{p,p}$ universal domain spaces, there exist universal functions. Indeed, for example, assume that (X, Y) has the $\mathbf{L}_{p,p}$ for every Y but there is no such universal function at some point x. Then, for some $\varepsilon > 0$, there exists a sequence of Banach spaces $\{Y_n\}$ such that all the possible choices of $\eta_n(\varepsilon, x)$, the function appears in the definition of $\mathbf{L}_{p,p}$ of (X, Y_n), converges to 0 when $n \to \infty$. But if we consider the space $Y = \left[\bigoplus_{n\in\mathbb{N}} Y_j\right]_{c_0}$, then (X, Y) has the $\mathbf{L}_{p,p}$ with $\eta(\cdot, z) > 0$ at $z \in S_X$ by the assumption and Theorem 4.3 gives that $\eta(\varepsilon, x)$ can be choosen for $\eta_n(\varepsilon, x)$ which is a contradiction. The case of $\mathbf{L}_p$ can be shown by using similar arguments.

As in [11, Proposition 2.1], we see the following stability result for $\mathbf{L}_{o,o}$ and $\mathbf{L}_{p,p}$.

Proposition 4.4. *Let X be Banach space and $\tilde{X}$ and $\tilde{Y}$ be one-complemented subspaces of X and Y, respectively.*

(a) *If (X, Y) has the $\mathbf{L}_{o,o}$, then $(\tilde{X}, \tilde{Y})$ has the $\mathbf{L}_{o,o}$.*
(b) *If (X, Y) has the $\mathbf{L}_{p,p}$, then $(\tilde{X}, Y)$ has the $\mathbf{L}_{p,p}$.*

We do not know whether the pair (X, Y) has the $\mathbf{L}_{p,p}$, then $(X, \tilde{Y})$ has the $\mathbf{L}_{p,p}$ for one-complemented subspace $\tilde{Y}$ of Y. Also, it is not known whether the same results holds for the BPBp or $\mathbf{L}_o$ or $\mathbf{L}_p$.

In what follows, we study relations on universal spaces. It is clear that $\mathbf{L}_{o,o}$ gives that every operator attains its norm. It is known however that for each infinite dimensional space X there exists an operator in $\mathcal{L}(X, c_0)$ which does not attain its norm (see [24, Lemma 2.2]). Hence, finite dimensional spaces are the only universal $\mathbf{L}_{o,o}$ domain spaces. Since uniformly convex space is universal BPBp domain space [21], we see that universal BPBp domain space is not the same as universal $\mathbf{L}_{o,o}$ domain space. On the other hand, it is known that ℓ_1^2 is not a universal BPBp domain space [4, Example 4.1], so universal BPB domain space is not the same as universal $\mathbf{L}_o$ domain space since $\mathbf{L}_{o,o}$ implies $\mathbf{L}_o$.

We prove that universality of BPBp and $\mathbf{L}_o$ on range spaces are equivalent properties although it is not true for domain spaces.

Proposition 4.5. *The Banach space Y is a universal BPBp range space if and only if it is a universal $\mathbf{L}_o$ range space.*

Proof. Since the BPBp implies the $\mathbf{L}_o$, it is clear the first direction. Suppose now that Y is a universal $\mathbf{L}_o$ range space. If Y is *not* a universal BPBp range space, there is a Banach space X such that the pair (X, Y) fails the BPBp. So there is $\varepsilon_0 > 0$ such that for all $n \in \mathbb{N}$, there are $T_n \in S_{\mathcal{L}(X,Y)}$ with $\|T_n\| = 1$ and $x_n \in S_X$ satisfying

$$1 \geqslant \|T_n(x_n)\| > 1 - \frac{1}{n}$$

such that for all $R \in S_{\mathcal{L}(X,Y)}$ $u \in S_X$ satisfying both $\|T_n - R\| < \varepsilon_0$ and $\|u - x_n\| < \varepsilon_0$, we have $\|R(u)\| < 1$.

Define the operator $T : \ell_1(X) \longrightarrow Y$ by

$$T(z) := \sum_{i=1}^{\infty} T_i(z_i) \quad (z = (z_i)_i \in \ell_1(X)).$$

Then $\|T\| = 1$ since $\|T_i\| = 1$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ be such that $\frac{1}{n_0} < \eta\left(\frac{\varepsilon_0}{2}, T\right)$ where η is the function appears in the definition of $\mathbf{L}_o$ for the pair $(\ell_1(X), Y)$ and take the embedding $Q_{n_0} : X \hookrightarrow \ell_1(X)$ in the n_0-th coordinate. So $Q_{n_0}(x_{n_0}) \in S_{\ell_1(X)}$ and

$$\|T(Q_{n_0}(x_{n_0}))\| = \|T_{n_0}(x_{n_0})\| > 1 - \frac{1}{n_0} > 1 - \eta\left(\frac{\varepsilon_0}{2}, T\right).$$

So there are $S \in \mathcal{L}(\ell_1(X), Y)$ with $\|S\| = 1$ and $u = (u_i)_i \in S_{\ell_1(X)}$ such that

$$\|S(u)\| = 1, \quad \|u - Q_{n_0}(x_{n_0})\| < \frac{\varepsilon_0}{2} \quad \text{and} \quad \|S - T\| < \varepsilon_0.$$

Since $\|u - Q_{n_0}(x_{n_0})\| < \frac{\varepsilon_0}{2}$, we have that $\|u_{n_0} - x_{n_0}\| < \frac{\varepsilon_0}{2}$ and so

$$\left\| x_{n_0} - \frac{u_{n_0}}{\|u_{n_0}\|} \right\| \leqslant \|x_{n_0} - u_{n_0}\| + |1 - \|u_{n_0}\|| = \|x_{n_0} - u_{n_0}\| + |\|x_{n_0}\| - \|u_{n_0}\|| < \varepsilon_0.$$

Consider the operator $SQ_{n_0} \in \mathcal{L}(X, Y)$. Then $\|SQ_{n_0}\| \leqslant 1$ and

$$\|SQ_{n_0} - T_{n_0}\| = \|SQ_{n_0} - TQ_{n_0}\| \leqslant \|S - T\| < \varepsilon_0.$$

To get a contradiction, we will prove that $\left\| SQ_{n_0}\left(\frac{u_{n_0}}{\|u_{n_0}\|}\right)\right\| = 1$. Indeed, let $y^* \in S_{Y^*}$ be such that $y^*(S(u)) = 1$. Then

$$1 = y^*(S(u)) = \sum_{i=1}^{\infty} y^*(SQ_i(u_i)) = \sum_{i \neq i_0} y^*(SQ_i(u_i)) + y^*(SQ_{n_0}(u_{n_0})) \leqslant \sum_{i \neq i_0} \|u_i\| + \|u_{n_0}\| = 1.$$

So $y^*(SQ_{n_0}(u_{n_0})) = \|u_{n_0}\|$ and then $\left\| SQ_{n_0}\left(\frac{u_{n_0}}{\|u_{n_0}\|}\right)\right\| = 1$ as desired. $\square$

We do not have an analogous result for the $\mathbf{L}_p$, but if X is finite dimensional, we have a similar result on domain spaces. It is worth noting that this result is not valid for infinite dimensional Banach spaces, as we will see in the next section.

Proposition 4.6. *Let X be a finite dimensional Banach space. Then X is a universal BPBp domain space if and only if it is a universal $\mathbf{L}_p$ domain space.*

Proof. Assume that there exists a finite dimensional space X which is a universal $\mathbf{L}_p$ domain space but not a universal BPBp domain space. Fix Y such that (X, Y) does not have BPBp, then for some $\varepsilon > 0$, there exist sequences $x_n \in S_X$ and $T_n \in S_{\mathcal{L}(X,Y)}$ such that $\|T_n x_n\|$ converges to 1 and $\|Sz\| < 1$ whenever $\|S - T_n\|, \|z - x_n\| < \varepsilon$ for some n. We assume that x_n converges to some $x \in S_X$.

Using the function $\eta(\cdot, x)$ which appears in the definition of $\mathbf{L}_p$ at x, we choose n_0 so that $\|x_{n_0} - x\| < \varepsilon/2$, $\|T_{n_0} x\| > 1 - \eta(\varepsilon/2, x)$. Therefore, we have an operator $U \in S_{\mathcal{L}(X,Y)}$ and a vector $z \in S_X$ such that $\|Uz\| = 1$ and $\|T_{n_0} - U\|, \|z - x_{n_0}\| < \varepsilon$ which give the desired contradiction. $\square$

5. Relations between the properties

In this final section we show the relations between the properties that we had studied in this paper. We consider the following diagram. In the picture, $\overline{\mathrm{NA}} = \mathcal{L}$ means that the set of norm attaining operators is dense.

$$\begin{tikzcd}
\text{BPBpp} \arrow[r] \arrow[dr] & \mathbf{L}_{p,p} \arrow[r] & \mathbf{L}_p & \\
& \text{BPBp} \arrow[ur] \arrow[dr] & & \overline{\mathrm{NA}} = \mathcal{L} \\
\text{BPBop} \arrow[r] \arrow[ur] & \mathbf{L}_{o,o} \arrow[r] & \mathbf{L}_o \arrow[ur] &
\end{tikzcd}$$

Our main aim is to see that all the converses implications are not true.

Let us first consider the pairs $(X, \mathbb{K})$. By the Bishop–Phelps–Bollobás theorem, this pair always satisfies the BPBp and, as we had seen, it is known that the BPBpp, the BPBop, the $\mathbf{L}_{o,o}$ and the $\mathbf{L}_{p,p}$ characterize uniform smoothness, uniform convexity, reflexivity and SSD of the dual X^* and SSD of X, respectively. Therefore, a finite dimensional space which is not uniformly convex is a counterexample for directions from $\mathbf{L}_{p,p}$ to BPBpp and from $\mathbf{L}_{o,o}$ to BPBop (see Proposition 2.9 and [10, Theorem 2.4]). Also, $(\ell_1, \mathbb{K})$ can be a counterexample for the implications from $\mathbf{L}_o$ to $\mathbf{L}_{o,o}$ and $\mathbf{L}_p$ to $\mathbf{L}_{p,p}$ (see Theorem 2.3).

It remains to check the opposite directions from $\mathbf{L}_o$ or $\mathbf{L}_p$ to BPBp and from $\overline{\mathrm{NA}} = \mathcal{L}$ to $\mathbf{L}_o$. In order to see that these are not true in general, we need to consider the vector-valued case.

To see that the $\mathbf{L}_p$ does not imply the BPBp, we recall a definition of uniformly strongly exposed family. We say that a family $\{x_\alpha\}_\alpha \subset S_X$ is *uniformly strongly exposed* with respect to a family $\{f_\alpha\}_\alpha \subset S_{X^*}$ if there is a function $\varepsilon \in (0,1) \longmapsto \delta(\varepsilon) > 0$ such that

$$f_\alpha(x_\alpha) = 1 \ \forall \alpha, \text{ and } \operatorname{Re} f_\alpha(x) > 1 - \delta(\varepsilon) \text{ implies } \|x - x_\alpha\| < \varepsilon \text{ whenever } x \in B_X.$$

And fixed $\varepsilon_0 \in (0,1)$ we say that this family $\{x_\alpha\}_\alpha$ is an ε_0-*dense uniformly strongly exposed family* if for each $x \in S_X$ there exist some x_{α_0} such that $\|x - x_{\alpha_0}\| < \varepsilon_0$.

It is known that if X is a superreflexive universal BPBp domain space, then for every $\varepsilon_0 \in (0,1)$ there is a ε_0-dense uniformly strongly exposed family of S_X [4, Corollary 3.6]. We consider a superreflexive space $\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}$ where ℓ_p^n is the n dimensional ℓ_p space. Then $\mathrm{NA}\left(\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}, Y\right)$ is dense in $\mathcal{L}\left(\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}, Y\right)$ for every Banach space Y [22]. Since every norm-one element in $\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}$ is strongly exposed, by Proposition 3.4, the pair $\left(\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}, Y\right)$ has the $\mathbf{L}_p$ for any Banach space Y. On the other hand, we show that there exists a number $\varepsilon_0 > 0$ such that there is no ε_0-dense uniformly strongly exposed family on the unit sphere of $\left(\bigoplus_{k=2}^\infty \ell_k^2\right)_{\ell_2}$.

Lemma 5.1. *Let Y, Z be Banach spaces and $X = Y \bigoplus_2 Z$. Suppose that there exists a ε_0-dense uniformly strongly exposed family of S_X with a function $\delta(\cdot)$, where $0 < \varepsilon_0 < 1/2$. Then, there exists a $2\varepsilon_0$-dense uniformly strongly exposed family of S_Y with function $\delta(\cdot/2)$.*

Proof. Let $\{x_\alpha\}_{\alpha\in\Lambda} \subset S_X$ be the ε_0-uniformly strongly exposed family with respect to $\{f_\alpha\}_{\alpha\in\Lambda} \subset S_{X^*}$ and, for convenience, write $x_\alpha = (y_\alpha, z_\alpha)$ and $f_\alpha = (y_\alpha^*, z_\alpha^*)$. Note that

$$1 = f_\alpha(x_\alpha) = y_\alpha^*(y_\alpha) + z_\alpha^*(z_\alpha) = \|y_\alpha^*\|\|y_\alpha\| + \|z_\alpha^*\|\|z_\alpha\|.$$

Fix $y \in S_Y$ and choose $\alpha_0 \in \Lambda$ such that $\|x_{\alpha_0} - (y,0)\| < \varepsilon_0$. Then we have $\|y_{\alpha_0}\| > 1-\varepsilon_0$ and, consequently, $\left\|y - \frac{y_{\alpha_0}}{\|y_{\alpha_0}\|}\right\| < 2\varepsilon_0$. It is enough to show that $\hat{y} = \frac{y_{\alpha_0}}{\|y_{\alpha_0}\|}$ is a strongly exposed point with respect to $\hat{y}^* = \frac{y^*_{\alpha_0}}{\|y^*_{\alpha_0}\|}$ and that $\delta(\cdot/2)$ is the modulus. If it not were the case, there would exist a positive number ε and a point $y_0 \in S_Y$ such that $\operatorname{Re} \hat{y}^*(y_0) > 1 - \delta(\varepsilon/2)$ and $\|y_0 - \hat{y}\| > \varepsilon$. Then, we would have

$$\operatorname{Re} f_{\alpha_0}(\|y_{\alpha_0}\|y_0, z_{\alpha_0}) = \|y^*_{\alpha_0}\|\|y_{\alpha_0}\| \operatorname{Re} \hat{y}^*(y_0) + \|z^*_{\alpha_0}\|\|z_{\alpha_0}\| > 1 - \delta(\varepsilon/2)$$

which gives $\|(\|y_{\alpha_0}\|y_0, z_{\alpha_0}) - x_{\alpha_0}\| < \varepsilon/2$. Hence, we obtain $\|y_0 - \hat{y}\| < \varepsilon$ which is the desired contradiction. $\square$

Proposition 5.2. *For some $\varepsilon_0 > 0$, there is no ε_0-dense uniformly strongly exposed family on the unit sphere of $\left(\bigoplus_{k=2}^{\infty} \ell_k^2\right)_{\ell_2}$.*

Proof. Assume that $\varepsilon_0 \in (0, 1/2)$ and there is ε_0-dense uniformly strongly exposed family with modulus $\delta(\cdot)$ on the unit sphere of $\left(\bigoplus_{k=2}^{\infty} \ell_k^2\right)_{\ell_2}$. Then from Lemma 5.1, every unit sphere of ℓ_k^2 $(k = 2, 3, 4, \ldots)$ has $2\varepsilon_0$-dense uniformly strongly exposed family $\mathcal{F}_k$ with a modulus $\delta(\cdot/2)$.

Fix $0 < \varepsilon < 1/2$. Choose $k \in \{2, 3, 4...\}$ so that $(1/2)^k < \delta(\varepsilon/2)$ and define sets $S_1 = \{(t_1, t_2) \in S_{\ell_k^2} \ : \ t_1 > 0, 0 \leqslant t_2 \leqslant 1/2\}$ and $S_2 = \{(t_1, t_2) \in S_{\ell_k^2} \ : \ t_1 > 0, -1/2 \leqslant t_2 \leqslant 0\}$. We see that every point in S_i can not be in $\mathcal{F}_k$. Indeed, the diameter of S_i is bigger than $1/2$, and for $(x_i, y_i) \in S_i$ whose exposing functional is $(x_i^{k-1}, |y_i|^{k-1}\operatorname{sign}(y_i)) \in \left(\ell_k^2\right)^*$, we have

$$(x_i^{k-1}, |y_i|^{k-1}\operatorname{sign}(y_i))(s_i, t_i) \geqslant x_i^{k-1} s_i \geqslant 1 - (1/2)^k > 1 - \delta(\varepsilon/2)$$

whenever $(s_i, t_i) \in S_i$. Since the diameter of $S_1 \cup S_2$ is bigger or equal to 1, we get the contradiction that $\mathcal{F}_k$ is not ε_0-dense. $\square$

To see that the $\mathbf{L}_o$ does not imply the BPBp, we consider the fact that if X is finite dimensional, then the pair (X, Y) satisfies the $\mathbf{L}_{o,o}$ for all Banach spaces Y and so does the $\mathbf{L}_o$. Nevertheless, there is a Banach space Y_0 such that the pair (ℓ_1^2, Y_0) fails the BPBp (see [4, Example 4.1]).

To check that denseness of norm attaining operators does not imply the $\mathbf{L}_o$, note that $\overline{\operatorname{NA}(\ell_1, Y)} = \mathcal{L}(\ell_1, Y)$ for all Banach spaces Y but if we take some strictly convex Banach space Y which is not uniformly convex, the pair (ℓ_1, Y) cannot have the $\mathbf{L}_o$ by Proposition 3.2.(b).

Moreover, there is no relation between the properties $\mathbf{L}_{o,o}$ and $\mathbf{L}_{p,p}$. Indeed, it is clear that the $\mathbf{L}_{p,p}$ implies the $\mathbf{L}_p$. Now take Y a infinite-dimensional strictly convex Banach space which is not uniformly convex. By Proposition 3.2.(a), the pair (ℓ_1^2, Y) fails the $\mathbf{L}_p$ and so the $\mathbf{L}_{p,p}$. On the other hand, the pair (ℓ_1^2, Y) has the $\mathbf{L}_{o,o}$ (see [10, Theorem 2.4]). Also, the $\mathbf{L}_{p,p}$ does not imply the $\mathbf{L}_{o,o}$. The pair (ℓ_2, Y) always has the $\mathbf{L}_{p,p}$ since it has the BPBpp (see [12, Theorem 2.5]) but (ℓ_p, ℓ_q) fails the $\mathbf{L}_{o,o}$ for $1 < p \leqslant q < \infty$ (see [10, Theorem 2.21.(b)]). As final remarks, we comment that it is not known whether $\mathbf{L}_p$ implies the denseness of norm attaining operators and the pair (ℓ_p^2, ℓ_q) for $1 < p, q < \infty$ has $\mathbf{L}_{p,p}$.

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