A non-Markovian approach for two dissipative quantum walks

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(Dated:)

We study the non-Markovian evolution of two free spinless distinguishable particles in a 1D lattice using a completely positive map. The Renewal theory is used to introduce, in a phenomenological way, the concept of disorder in the random time-interventions of the environment. If the waiting-time of the Renewal approach is exponential we recover a semigroup description for the density matrix. Introducing a non-Poissonian random-time bath-interventions a non-Markovian evolution for the density matrix has been worked out. Under this scenario we have studied the time evolution of the Quantum Coherence and Negativity measures for local and non-local initial conditions. We show the relevance of the (weak) non-Markovian evolution in calculating short-time correlations, while in the long-time regime a renormalized Markovian evolution appears.

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I. INTRODUCTION

A quantum system interacting with the degree of freedom of the environment is the key ingredient to describe its decoherence and fluctuations, and the so called Markov completely positive (CP) dynamics are the starting point for considering fluctuations and dissipation in quantum mechanics [1]. A natural generalization of *the* Markov dynamics is the so called memory-kernel (non-Markov) Quantum Master Equation [2, 3]. These types of evolutions have been used in different research areas, and the problem arising from the indiscriminately use of memory-kernels was pointed out in [4]. One important contribution in that direction was presented introducing stochastic interventions in the quantum dynamics, then it was shown that if the environment interventions occur a random times and if these random times follows a renewal stochastic structure the non-Markov dynamics for the density matrix turns to be CP [5]. A much wider class of time memory-kernel CP evolution was also written in terms of piecewise dynamics with microscopic interaction embedded in a continuous-time description [6], as well as in the context of the random measurement interventions [7]. It is interesting to point out that same extra mathematical constraints have recently been presented in order to assure a CP dynamics [2] and sufficient conditions to have a memory-kernel CP structure [3].

Given a memory-kernel CP dynamics, and due to the complexity in solving this type of non-local operational evolution, an important point to consider is to introduce a systematic perturbation approach to be able to find some approximation for the non-Markov evolution of the density matrix. In order to tackle this issue it is convenient to consider a simple but non-trivial model to test the perturbation approach and its consequence on the quantum correlations. So in the present work we will work out a tight-binding system interacting with a phonon bath, this is an interesting model which has been used to study transport in thermally activated electrons [8, 9], and tunneling in one-band conductivity [10]. This model is a prototype of open quantum system which is now called dissipative Quantum Walk (QW) and it is suitable for study quantum correlations [11, 12]. An important point to be addressed in the context of Markov QW, is the analysis of the quantum to classical transition. It has been shown that considering *two* particles this transition is drastically changed [13] with respect to the *one* particle model [14]. Then, it would be important to study the inference of a non-Markov dynamics on the quantum correlations between many QWs.

QW models are expected to be useful for designing quantum algorithms and modeling coherent transport on complex networks [15], as well as in describing nano-experiments [16, 17]. The important point in a QW model is the analysis of the evolution of the coherent superposition of states (something impossible in the classical random walk framework). QW on random environment is a promising approach which has been studied with diagonal disorder using unitary evolutions [8, 9]. A *dissipative* quantum walk (DQW) is a related process but takes into account the explicit interaction with a phonon thermal bath. In the case of a dissipative evolution the simplest dynamics is dictated by a quantum

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semigroup [18]. Nevertheless, it is well known that a Markov QW process cannot always describe complex relaxation processes [15, 17, 19]. For this reason, it would be important to work out a generalized (many-body) non-Markov DQW model [20].

We noted that even when the description of a DQW in a *random* environment is a very important issue, up to now it is still not completely understood, mainly because in general the dynamics of the Green mean-value over quenched disorder will not be a semigroup. In the present paper we will not directly tackle a non-unitary *random* model, here we are going to construct a well-defined quantum non-Markov approach borrowing renewal ideas from the classical random jump problem [21–23]. Then, from the present random quantum jump model we will get information concerning the evolution of coherence and quantum correlations in the framework of a non-Markov dynamics.

We use the concept of a CP map [1] to present a Markov chain into quantum mechanics. Then, the Renewal theory is used to introduce a CP infinitesimal generator for the density matrix, allowing to build up a non-Markov CP approach in the QW framework. In this context we point out that the waiting-time functions of the Renewal theory are probability densities used to model *random* disruptive interventions of the bath into the system of interest. The Renewal approach is also presented in the context of *subordination* processes [24], in this way it is possible to consider a transformation from an internal-time into the physical-time [5], which may helps to work-out the non-Markov evolution in some special cases. Here we are going to solve, in a perturbative way, a particular non-Markov bath intervention acting over two free QWs. Then, the time evolution of a two-body density matrix will be used as our main tool to study its inference in the short-time decoherence and correlations evolution. Conclusions on the mechanism of destruction of quantum correlations within a non-Markov approach will be presented.

II. DISSIPATIVE QUANTUM EVOLUTIONS

A. Discrete-time: Quantum Markov chains

In quantum mechanics the concept of a Markov chain can be stated in terms of a CP map. That is, the most general map acting on the density matrix preserving Hermitian, trace = 1, and positive definite [1]. This map has the Kraus form:

$$\mathcal{E}\left[\bullet\right]=\sum_{j}V_{j}\left[\bullet\right]V_{j}^{\dagger},\ \text{with}\ \ \sum_{j}V_{j}V_{j}^{\dagger}=\mathbf{I},\tag{1}$$

where $\{V_j\}$ is any set of operators determining the irreversible dissipative evolution of the density matrix, and $\mathbf{I}$ the identity operator. Therefore in order to represent the intervention of a bath $\mathcal{B}$ on a system $\mathcal{S}$, we assume a discrete-time model in terms of the CP map $\mathcal{E}\left[\bullet\right]$, similar to a discrete recurrence relation for a classical Markov chain [23].

In the *interaction representation* with respect to the Hamiltonian of the system $H_{\mathcal{S}}$, we propose a quantum Markov chain acting on the density matrix σ in the form:

$$\sigma\left(n+1\right)=\mathcal{E}\left[\sigma\left(n\right)\right],\text{with initial condition }\sigma\left(0\right).\tag{2}$$

The solution of this map can be found using the generating function technique (the z-transform of $\sigma\left(n\right)$)

$$\sigma\left(z\right)=\sum_{n=0}^{\infty}z^{n}\sigma\left(n\right)=\left(\mathbf{I}-z\mathcal{E}\left[\bullet\right]\right)^{-1}\sigma\left(0\right).\tag{3}$$

Let σ be the density matrix of a given system $\mathcal{S}$, then the solution (3) gives the evolution in discrete-time considering the disruptive interventions of $\mathcal{B}$ on $\mathcal{S}$.

B. Continuous-times: non-Markovian evolutions

To consider a continuous time random intervention of the bath on $\mathcal{S}$, we now introduce a *Renewal* process characterizing the random number of events n in a continuous-time representation [23, 25]. The continuous-time evolution of $\sigma(t)$ can be studied in terms of the probability density $\psi_{(n)}\left(t\right)$ for the n-th event-intervention:

$$Q\left(t\right)=\sum_{n=0}^{\infty}\psi_{(n)}\left(t\right)\sigma\left(n\right),\tag{4}$$

here the quantity $Q(t)\,dt$ gives the density matrix *just* at time t and $t+dt$ for a given realization of the random events. The *Renewal* process is characterized by the waiting-time probability distribution function (pdf) $\psi(t) \geq 0, \forall t \geq 0$ and the hierarchy of distributions $\psi_{(n)}(t)$:

$$\begin{aligned} \psi_{(n)}(t) &= \int_0^t \psi_{(n-1)}(\tau)\,\psi(t-\tau)\,d\tau, \quad (n \in \mathcal{N}) \\ \psi_{(0)}(t) &= \delta\left(t-0^+\right). \end{aligned}$$

From (4), going to Laplace's representation we get

$$Q(u) = (\mathbf{I} - \psi(u)\,\mathcal{E}[\bullet])^{-1}\,\sigma(0). \tag{5}$$

In order to calculate the density matrix $\sigma(t)$ at time $t > 0$ we have to consider also the possibility of remaining without any disruptive intervention from the bath. This effect is taken into account considering the probability of no-intervention during the time interval $[0, t]$:

$$\phi(t) = 1 - \int_0^t \psi(\tau)\,d\tau, \quad \text{with} \quad \int_0^\infty \psi(\tau)\,d\tau = 1. \tag{6}$$

Going back to the time-representation of $Q(t)$ (from solution (5)) and using the probability of no-intervention (6) we can write the density matrix $\sigma(t)$ in continuous-time in the form:

$$\sigma(t) = \int_0^t \phi(t-\tau)\,Q(\tau)\,d\tau, \quad t > 0. \tag{7}$$

After some algebra [20], we can write an *exact* evolution equation for the density matrix in the *interaction representation* $\sigma(t) = e^{-t\mathcal{L}_0}\rho(t)$ as:

$$\partial_t \sigma(t) = \int_0^t \Phi(t-\tau)\,e^{-\tau\mathcal{L}_0}\;\left(\mathcal{E}[\bullet] - \mathbf{I}\right)\;e^{+\tau\mathcal{L}_0}\sigma(\tau)\,d\tau, \tag{8}$$

where $\mathcal{L}_0[\bullet] = -i[H_{\mathcal{S}}, \bullet]$ is the von Neumann superoperator, $\mathbf{I}$ the identity operator, and $\Phi(t)$ the kernel of the evolution, which in the Laplace representation is given by

$$\Phi(u) = \frac{u\psi(u)}{1-\psi(u)}. \tag{9}$$

The function $\Phi(t)$ is the memory of the *generalized* Kossakowski-Lindblad (KL) infinitesimal generator [1]

$$\mathbf{KL}[\bullet] \equiv \mathcal{E}[\bullet] - \mathbf{I}. \tag{10}$$

When $\psi(t) = \lambda^{-1}e^{-t/\lambda}$ (exponential waiting-time model) we recover a semigroup because $\Phi(t-\tau) = \delta(t-\tau)\,\lambda$. In this particular case the KL evolution is characterized by a Poissonian statistics for the disruptive interventions of $\mathcal{B}$. In the general case $\Phi(t)$ is a correlation function and therefore it need not to be a positive function, but the area of $\Phi(t)$ indeed characterizes a dissipative constant [23].

In the *interaction representation* a picture for the dissipative quantum evolution can be thought in terms of quantum jumps any time the intervention of the environment applies. Then a unitary evolution results during a random waiting-time interval $[0, \tau]$, until another CP map is applied again, and so on time and again in continuous-time. The resulting evolution operator is a generalized CP infinitesimal generator that has a time-convolution structure reminiscent of the classical CTRW theory [21]. Short and long times asymptotic regimes can analytically be studied in terms of the waiting-time model that we may have used [23]. The important point is that introducing a "random" time intervention of the bath $\mathcal{B}$ produces an evolution equation which is a generalized (non-Markov) CP infinitesimal generator.

Going to the *original representation* $\rho(t) = e^{+t\mathcal{L}_0}\sigma(t)$, from (8) we get:

$$\partial_t \rho(t) = \mathcal{L}_0[\rho(t)] + \int_0^t \Phi(t-\tau)\,e^{(t-\tau)\mathcal{L}_0}\mathbf{KL}[\bullet]\,\rho(\tau)\,d\tau, \tag{11}$$

which is the non-local in time evolution equation to be solved. This equation can be worked out by perturbations on the intensity of the bath interaction $\mathbf{KL}[\bullet]$, or alternatively on the non-Markovian characteristic time-scale associated to the Kernel $\Phi(t)$.

C. Exact non-Markov solution in the interaction representation

In the particular case when von Neumann's superoperator $\mathcal{L}_0[\bullet]$ commutes with the CP map $\mathcal{E}[\bullet]$, the non-Markovian evolution equation (8) reduces to

$$\partial_t \sigma(t) = \int_0^t \Phi(t-\tau)\, \mathbf{KL}[\bullet]\, \sigma(\tau)\, d\tau. \tag{12}$$

This equation can formally be solved in the Laplace representation in the form

$$\sigma(u) = \frac{1}{u - \Phi(u)\, \mathbf{KL}[\bullet]} \sigma(0). \tag{13}$$

Therefore, going back to the original representation we get

$$\begin{aligned} \rho(t) &= e^{+t\mathcal{L}_0} \sigma(t) \\ &= e^{+t\mathcal{L}_0} \mathcal{L}_u^{-1} \left[\frac{1}{u - \Phi(u)\, \mathbf{KL}[\bullet]} \right] \sigma(0). \end{aligned} \tag{14}$$

Note that even if there exist a representation where $\mathcal{L}_0[\bullet]$ and the superoperator $\mathbf{KL}[\bullet]$ were diagonal, the difficult part in expression (14) is its Laplace inversion. So we have to deal with some perturbation theory in order to find an approximated solution for the density matrix in the non-Markovian case.

Alternatively, the solution of (12), $\sigma(t)$, can be written in an integral form with the help of an *escort* Markovian density matrix: $\sigma^M(\varsigma)$. After some algebra [5], the final expression turns to be

$$\sigma(t) = \int_0^\infty \mathcal{P}(t,\varsigma)\, \sigma^M(\varsigma)\, d\varsigma, \tag{15}$$

where $\sigma^M(\varsigma)$ is the solution of the *subordinated* Markovian evolution [24]

$$\partial_\varsigma \sigma^M(\varsigma) = \mathbf{KL}\left[\sigma^M(\varsigma)\right], \quad \sigma^M(0) = \sigma(0). \tag{16}$$

Here the real positive function $\mathcal{P}(t,\varsigma)$ defines a pdf for the *dimensionless* time ς. In the Laplace representation this pdf is given by

$$\mathcal{P}(u,\varsigma) = \frac{1}{\Phi(u)} \exp\left[\frac{-\varsigma u}{\Phi(u)}\right], \tag{17}$$

fulfilling conditions: $\int_0^\infty \mathcal{P}(t,\varsigma)\, d\varsigma = 1$ and $\mathcal{P}(t,\varsigma) \geq 0, \forall\{t,\varsigma\} \geq 0$. Then, solution (15) allows to interpret the non-Markovian evolution of $\sigma(t)$ in terms of a subordination random process [24], where the mapping from an internal-time ς to the physical-time t is characterized by the pdf $\mathcal{P}(t,\varsigma)$.

Notably, a particular model of non-Markovian DQW can be solved analytically in the context of fractional dynamics. That is the pdf (17) can be inverted in Laplace allowing to handle (15), and therefore the expression (14) can be worked out analytically, more details on this subject will be presented in a future work.

D. Perturbative expansion in the intensity of bath interventions

If the map $\mathcal{E}[\bullet]$ commutes with the von Neumann superoperator $\mathcal{L}_0[\bullet]$ it is possible to simplify the non-Markovian evolution equation (11), then we can write

$$\frac{d}{dt}\rho(t) = \mathcal{L}_0[\rho(t)] + \int_0^t \Phi(t-\tau)\, \mathbf{KL}[\bullet]\, e^{(t-\tau)\mathcal{L}_0} \rho(\tau)\, d\tau. \tag{18}$$

For an arbitrary memory kernel $\Phi(t)$ this evolution equation can be studied by perturbations in the *intensity* of the superoperator $\mathbf{KL}[\bullet]$. In particular we can write a series expansion in $\mathcal{O}(\mathbf{KL}[\bullet]^m)$. To first order in the intensity of the interventions of the CP map we can approximate $e^{(t-\tau)\mathcal{L}_0}\rho(\tau) \simeq \rho(t)$ into the integrand, and so we can get an approximated non-Markovian evolution dictated by the equation (see appendix A):

$$\frac{d}{dt}\rho(t) = \mathcal{L}_0[\rho(t)] + \left(\int_0^t \Phi(t-\tau)\, d\tau\right) \mathbf{KL}[\rho(t)] + \mathcal{O}\left(\mathbf{KL}[\rho(t)]^2\right), \tag{19}$$

where $\mathbf{KL}[\bullet]$ is given by (10) and $\mathcal{L}_0[\bullet]$ corresponds to the unitary evolution with a Hamiltonian $H_{\mathcal{S}}$. Eq. (19) is an evolution approximation in the intensity of the bath intervention, not in the non-Markovian character of the evolution equation (18). Equation (19) can easily be solved calling an auxiliary function:

$$\chi(t) \equiv \int_0^t \Phi(t-\tau)\, d\tau = \int_0^t \Phi(t')\, dt' \geq 0, \quad \forall t \geq 0. \tag{20}$$

We see that in a first order approximation the non-Markovian structure has a global multiplicative character in the infinitesimal generator of the density matrix. We note that in the presence of a waiting-time $\psi(\tau)$ with a long-tail (strong-disordered disruptive interventions of the bath) we would get $\chi(t \to \infty) \to 0$ requiring that the correlation $\Phi(t)$ is not always positive, as we mentioned before this fact does not posse any restriction of the CP character of the evolution.

Calling

$$D(t) \equiv \int_0^t \chi(\tau)\, d\tau > 0,$$

we can write the formal solution of (19), to $\mathcal{O}\left(\mathbf{KL}[\bullet]^2\right)$, in the form:

$$\rho(t) \simeq \exp\left(t\mathcal{L}_0[\bullet] + D(t)\,\mathbf{KL}[\bullet]\right)\rho(0), \tag{21}$$

where $D(t)$ is a time-dependent (positive) dissipative coefficient, the nature of this function can be studied in terms of the waiting-time model $\psi(t)$ that we may use (see appendix B). We note that in the Markovian limit $\Phi(t) \to \delta(t)/\lambda$, so the function $D(t)$ is linear in time: $D(t) \to t/\lambda$.

A weak-disordered model for the random intervention of the CP map can be considered using the Gamma waiting-time function, in this case (see appendix B) the expression for the function $D(t)$ is given, in the Laplace representation, as

$$D(u) = \frac{\Phi(u)}{u^2} = \frac{u^{-1}}{(1+\lambda u)^b - 1}, \tag{22}$$

then, short and long time asymptotic regimes can analytically be obtained:

$$D(t) \simeq \left(\frac{t}{\lambda}\right)^b + \cdots, \qquad b > 0,\ t \ll \lambda,$$

and

$$D(t) \simeq \frac{t}{\lambda b} + \cdots, \qquad b > 0,\ t \gg \lambda.$$

Therefore, using the Gamma pdf at long time we recover a renormalized Markovian behavior. We note that in the present work we will be interested in the case $b \geq 1$, see B4 and B5 in Appendix B.

E. On the Kossakowski-Lindblad infinitesimal generator

It is well known that starting from a CP Kraus' map the KL infinitesimal generator is well defined and it is given by Eq. (10), while the inverse is in general not true [1]. In the particular case when the KL infinitesimal generator is given, and if we would like to use the present non-Markov Renewal approach, what we can do is to introduce an approximation with some control parameter to study the time evolution of the system $\mathcal{S}$.

To be specific let us assume that the *dimensionless* KL infinitesimal generator has the form

$$\eta\mathcal{KL}[\bullet] = \frac{\eta}{2}\sum_{j,m} a_{jm}\left(\left[V_j\bullet, V_m^\dagger\right] + \left[V_j, \bullet V_m^\dagger\right]\right)$$

where η is a small parameter and $\{a_{jm}\}$ is a positive definite matrix. Following the theory of one-parameter contracting semigroup on Banach spaces [1], it is possible to associate a CP map $\mathcal{E}[\bullet]$ in the following way:

$$\mathcal{E}[\bullet] - \mathbf{I} = \left(e^{\eta\mathcal{KL}[\bullet]} - \mathbf{I}\right).$$

Therefore, if the pdf $\psi(t)$ of the Renewal approach accepts a dense point process parametrized, for example, by some very large mean number of events by unit of time, we can introduce a double simultaneous limit in order to write down a non-Markov evolution equation for the density matrix [5]. In terms of the Kernel $\Phi(t)$ of Eq. (11) it means that there is a common factor $\langle t\rangle^{-1}$, such that the limit $\langle t\rangle \to 0$ represents an infinitely dense point process in the Renewal approach. Then, we can take a double simultaneous limit in such a way that $\lim_{\eta,\langle t\rangle\to 0}\eta/\langle t\rangle \to$constant, thus the evolution equation (11) can written in the form (denoting $\Phi(t)=\tilde{\Phi}(t)/\langle t\rangle$):

$$\partial_t\rho(t)=\mathcal{L}_0[\rho(t)]+\text{constant}\int_0^t\tilde{\Phi}(t-t')\,e^{(t-t')\mathcal{L}_0}\mathcal{KL}[\bullet]\,\rho(t')\,dt'+\mathcal{O}(\eta)^2\,. \tag{23}$$

Equation (23) is the starting point to analyze, in a phenomenological way, a non-Markovian evolution when the KL infinitesimal generator is given. In an analogous way, in the interaction representation, and when $\mathcal{L}_0[\bullet]$ and $\mathcal{KL}[\bullet]$ commute we can write (8) taking the double simultaneous $\lim_{\eta,\langle t\rangle\to 0}\eta/\langle t\rangle \to$constant, in the form

$$\partial_t\sigma(t)\simeq\text{constant}\int_0^t\tilde{\Phi}(t-t')\,\mathcal{KL}[\bullet]\,\sigma(t')\,dt'+\mathcal{O}(\eta)^2\,. \tag{24}$$

In conclusion, we can use the evolution (23) when the intervention of the bath is modeled from an infinitesimal generator $\eta\mathcal{KL}[\bullet]$ rather than by the action of the CP map $\mathcal{E}[\bullet]$, iff the set of random time $\{t_1<t_2<\cdots<t_n\}$ (characterized by the pdf $\psi(t)$) that set-in the disruptive bath's interventions admits to take the limit when the number of events by unit of time go to infinite.

Many models of waiting-time functions $\psi(t)$ allow to take that mentioned limit preserving the characteristic of a well defined positive pdf $\psi(t)$ [5]. We will exemplify this procedure in the present paper by considering *weak* disordered random time events characterized by a Gamma pdf [20]. In this sense we said that the disorder is *weak* because, asymptotically at long-time the set of random times $\{t_i\}_{i=1}^{i=n}$ gets a Poissonian characteristics, then leading (at long-time) to a renormalized Markovian evolution, see appendix B.

III. TWO DISSIPATIVE QUANTUM WALKS

A. The Born-Markov approximation

A model of two free *distinguishable* particles coupled to a common bath $\mathcal{B}$ can be written using the Wannier base [12]. A similar analysis in the Fock representation for N *indistinguishable* particles can also be introduced [13]. Let the total Hamiltonian be $H_{\mathcal{T}}=H_{\mathcal{S}}+H_{\mathcal{B}}+H_{\mathcal{SB}}$, here $H_{\mathcal{S}}$ is the *free tight-binding* Hamiltonian (our system $\mathcal{S}$), which can be written in the form:

$$H_{\mathcal{S}}=2E_0\mathbf{I}-\frac{\Omega}{2}\left(a_{12}^{\dagger}+a_{12}\right), \tag{25}$$

here $\{a_{12}^{\dagger},a_{12}\}$ are shift Wannier's operators for particles labeled 1 and 2, $\mathbf{I}=\sum_{s,s'}|s,s'\rangle\langle s,s'|$ is the identity in the Wannier base, then:

$$a_{12}^{\dagger}|s_j,s_l\rangle=|s_j+1,s_l\rangle+|s_j,s_l+1\rangle \tag{26}$$

$$a_{12}|s_j,s_l\rangle=|s_j-1,s_l\rangle+|s_j,s_l-1\rangle. \tag{27}$$

We note that due to distinguishable character of particles the "shift operator" translates each particle individually. Here we have used a "pair-ordered" bra-ket $|s_j,s_l\rangle$ representing the particle "1" at site s_j and particle "2" at site s_l. From Eqs. (26)-(27) it is simple to see that $\left[a_{12}^{\dagger},a_{12}\right]=0$, and the fact that

$$a_{12}a_{12}^{\dagger}|s_j,s_l\rangle=2|s_j,s_l\rangle+|s_j-1,s_l+1\rangle+|s_j+1,s_l-1\rangle.$$

We adopt $H_{\mathcal{B}}$ as the phonon bath Hamiltonian $H_{\mathcal{B}}=\sum_n\hbar\omega_n\mathcal{B}_n^{\dagger}\mathcal{B}_n$, thus $\{\mathcal{B}_n^{\dagger},\mathcal{B}_n\}$ are bosonic operator characterizing the thermal bath at equilibrium. The term $H_{\mathcal{SB}}$ in the total Hamiltonian represents the interaction term between $\mathcal{S}$ and $\mathcal{B}$, here we use a linear interaction between $\mathcal{S}$ and the bath operators. Our model is a many-body generalization of the van Kampen approach used to address the nature of a physical dissipative particle interacting with a boson bath [18]. Because shift operators a_{12} and $a_{12}^{\dagger}$ commute with $H_{\mathcal{S}}$ (do not evolve in time), any bath interaction with these shift operators will lead to a CP infinitesimal generator. Thus, for two *distinguishable* particles we propose the interaction term in the form $H_{\mathcal{SB}}=\hbar\Gamma\left(a_{12}\otimes\sum_n v_n\mathcal{B}_n+a_{12}^{\dagger}\otimes\sum_n v_n^*\mathcal{B}_n^{\dagger}\right)$, where v_n represents the spectral intensity weight function from the phonon bath at thermal equilibrium, and Γ is a dimensionless interaction parameter. We have chosen this interaction model in order to recover the classical master equation for two independent random walks in the case when $\Omega=0$ (that is when the von Neumann evolution disappears), an extended discussion on the issue of one particle can be seen in [14].

In order to study the non-equilibrium evolution of $\mathcal{S}$ we derive from $H_{\mathcal{T}}$, eliminating the bath variables an infinitesimal generator $\mathcal{KL}[\bullet]$. Tracing out bath variables in the Ohmic approximation and assuming as initial state of the total system a density matrix in the form of a product $\rho_T(0)=\rho(0)\otimes e^{-H_{\mathcal{B}}/k_BT}/Z$, where $Z=\mathrm{Tr}\left(e^{-H_{\mathcal{B}}/k_BT}\right)$, we can write in a second order Born-Markov approximation the Quantum Master Equation (QME) [23, 26]:

$$\begin{aligned}\dot{\rho} &= \frac{-i}{\hbar}\left[H_{eff},\rho\right]+\frac{\Gamma^2k_BT}{2\hbar}\left(2a_{12}\rho a_{12}^{\dagger}-a_{12}^{\dagger}a_{12}\rho-\rho a_{12}a_{12}^{\dagger}\right)\\ &+ \frac{\Gamma^2k_BT}{2\hbar}\left(2a_{12}^{\dagger}\rho a_{12}-a_{12}a_{12}^{\dagger}\rho-\rho a_{12}^{\dagger}a_{12}\right),\end{aligned}\tag{28}$$

here T is the temperature of the bath $\mathcal{B}$ and Γ is the small dimensionless interaction parameter, we note here that $\hbar/k_BT$ is a macroscopic characteristic time scale. Adding $-2E_0+\Omega$ to $H_{\mathcal{T}}$ the effective Hamiltonian turns to be: $H_{eff}=\Omega\left(\mathbf{I}-\frac{a_{12}^{\dagger}+a_{12}}{2}\right)-\hbar\omega_c a_{12}a_{12}^{\dagger}$, where $\omega_c\equiv 2\tilde{\omega}_c\Gamma^2$ is related to the frequency cut-off $\tilde{\omega}_c$ in the Ohmic approximation (linear) [1, 18, 26]. It can be seen from the *strength function* $g(\omega)$ of thermal oscillators (defined by $g(\omega)\Delta\omega\leftrightarrow\left[\sum_n v_n^2\right]_{\{\omega<\omega_n<\omega+\Delta\omega\}}$) that the high-frequency oscillators (beyond $\tilde{\omega}_c$) only modify the effective Hamiltonian. This von Neumann dynamics can be defused by going to the interaction representation. However, here we will be interested in studying the non-equilibrium evolution of $\mathcal{S}$ as a function of the rate of energies Ω and k_BT. Then, in order to simplify the analysis of the QME (28) we will drop-out the term $\hbar\omega_c a_{12}a_{12}^{\dagger}$ in the effective Hamiltonian, which only produces additional reversible coherence. Under the assumption that $\hbar\tilde{\omega}_c/k_BT\ll 1$, the dissipative coefficient appearing in (28) comes from the strength function, $g(\omega)$, and the thermal bath correlation function. This terms only involves bath oscillators in the low-frequency region. It is also possible to see that the Markov approximation used to get (28) involves a *coarse-grained* time scale such that $\tilde{\omega}_c\Delta t\gg 1$ in addition to the second order weak interaction approach [18].

From (28) and calling $\eta=\Gamma^2$ it is simple to extract the *dimensionless* KL infinitesimal generator in the form

$$\mathcal{KL}[\bullet]=\left(a_{12}\bullet a_{12}^{\dagger}+a_{12}^{\dagger}\bullet a_{12}-a_{12}^{\dagger}a_{12}\bullet-\bullet a_{12}a_{12}^{\dagger}\right),\tag{29}$$

where we have used that $a_{12}^{\dagger}a_{12}=a_{12}a_{12}^{\dagger}\neq\mathbf{I}$. In the rest of the work we are going to use the evolution equation (23) with $\psi(t)$ given by (B1), and $\mathbf{KL}[\bullet]=\mathcal{KL}[\bullet]$ from (29).

B. Non-Markovian evolution for two dissipative quantum walks

As we commented before, a CP non-Markov evolution can be modeled considering quantum jumps inserted between the unitary evolution of the system, the elapsed time for the occurrence of these disruptive CP map interventions is dictated by the waiting-time function of the Renewal theory [20]. As a result of this dynamics the density matrix is governed by a generalized QME (not a semigroup). Then, the exact evolution equation is (11), which fulfills the CP condition.

Noting that $\mathcal{L}_0[\bullet]=i[H_S,\bullet]$ with H_S given in (25) and $\mathbf{KL}[\bullet]$ given by (29), it is possible to show that both superoperators commute. Then we can use the evolution equation (18), and we can introduce a perturbation approach in the intensity of the disruptive intervention to work out its solution (21), see appendix A. Then, we get

$$\rho(t)=\exp\left(t\mathcal{L}_0[\bullet]+D(t)\,\mathbf{KL}[\bullet]\right)\rho(0)+\mathcal{O}\left(\mathbf{KL}[\bullet]^2\right),\tag{30}$$

with $D(t)$ characterized by a waiting-time function $\psi(t)$. Expression (30) can explicitly be worked out in a particular basis.

In the present paper we will use a weak-disordered model for the disruptive interventions characterized by the Gamma pdf $\psi(t)$ (see appendix B). Then, in the Laplace representation $D(t)$ is given by (22). In figure 1 we show

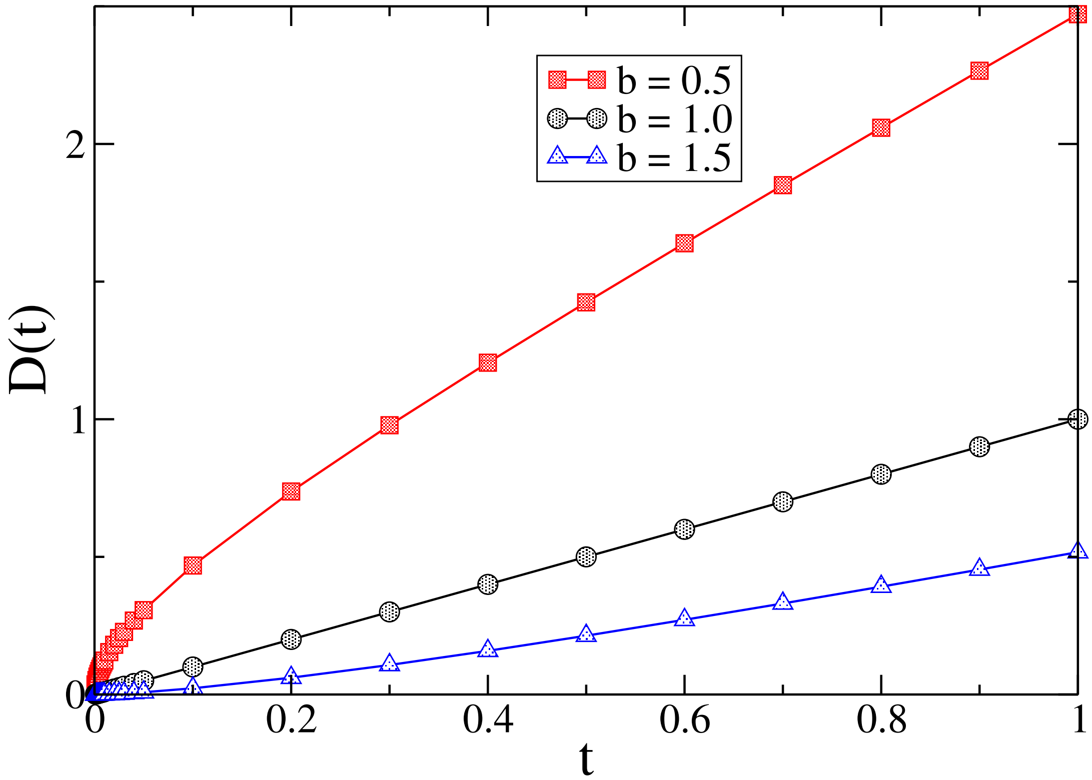

FIG. 1: Time-dependent dissipative coefficient $D(t)$ using a numerical inverse Laplace transform of (22) as function of dimensionless time $t \to t/\lambda$ for several values of b. At short-time we get the expected analytical behavior (B4). In the asymptotic long-time limit we get $D(t) \sim t/b\lambda$ leading to a renormalized Markov regime (B5).

$D(t)$ performing a numerical evaluation of the inverse Laplace transform, as can be seen the expected analytical results for short and long times are obtained.

An explicit solution for the density matrix can be written in the Wannier representation, now we will use the localized Initial Condition (IC) in the lattice:

$$\rho(0) = |s_1, s_2\rangle\langle s_1, s_2| = |\vec{0}\rangle\langle\vec{0}|. \tag{31}$$

From the formal solution (30) we calculate the elements of $\rho(t)$ in the discrete Fourier representation, noting that a Fourier "*bra-ket*" is defined in terms of a Wannier basis for *two distinguishable particles* in the form, using

$$|k_1, k_2\rangle = \frac{1}{2\pi} \sum_{s_1, s_2 \in \mathcal{Z}} e^{ik_1 s_1} e^{ik_2 s_2} |s_1, s_2\rangle,$$

with $k_j \in (-\pi, \pi)$ and $s_1, s_2 \in$ integers. Therefore

$$\langle k_1, k_2 |\rho(t)| k_1', k_2'\rangle = \exp\left(-\mathcal{F}(k_1, k_1', k_2, k_2')\ t\right) \langle k_1, k_2 |\rho(0)| k_1', k_2'\rangle, \tag{32}$$

where

$$\mathcal{F}(k_1, k_1', k_2, k_2') \equiv \{\mathcal{F}^{(1)}(k_1, k_1') + \mathcal{F}^{(1)}(k_2, k_2') + 2D(t)\left[\mathbf{C}(k_1, k_2') + \mathbf{C}(k_2, k_1') - \mathbf{C}(k_1, k_2) - \mathbf{C}(k_1', k_2')\right]\},$$

and

$$\mathcal{F}^{(1)}(k, k') \equiv \left[\frac{-i}{\hbar}(\mathcal{E}_k - \mathcal{E}_{k'}) + 2D(t)\left(\mathbf{C}(k, k') - 1\right)\right],$$

is the one-particle infinitesimal generator in the Fourier representation [13], with $\mathcal{E}_k \equiv \Omega\{1 - \cos k\}$ and $\mathbf{C}(k_1, k_2) \equiv \cos(k_1 - k_2)$. Note that $\mathcal{F}(k_1, k_1, k_2, k_2) = 0$ leading to a momentum-like conservation law: $\langle k_1, k_2|\frac{d\rho(t)}{dt}|k_1, k_2\rangle = 0$.

Elements of $\rho(t)$ can be calculated in the Wannier basis by simple inversion

$$|s_1, s_2\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} dk_1 dk_2\ e^{-ik_1 s_1} e^{-ik_2 s_2} |k_1, k_2\rangle.$$

After some algebra and using Bessel's function properties we can write an analytical formula for $\rho(t)$ in Wannier representation $\langle s_1, s_2|\rho(t)|s'_{1,}s'_2\rangle$. To simplify the notation we use the definitions:

$$t_\Omega \equiv \frac{\Omega t}{\hbar}, \tag{33}$$

$$t_D \equiv 2D\,(t)\,, \tag{34}$$

with

$$D\,(t) \equiv \int_0^t d\tau \int_0^\tau \Phi\,(\tau - t')\,dt'., \tag{35}$$

whenever it is necessary. Here the function $\Phi\,(t)$ is characterized by the pdf $\psi(t)$, see appendix B.

From (32) the exact solution in the Wannier representation can be written in the form

$$\begin{aligned}\langle s_1, s_2|\rho(t)|s'_1, s'_2\rangle = {} & i^{(s_1 - s'_1 + s_2 - s'_2)} e^{-2t_D} \sum_{\{n_1,n_2,n_3,n_4,n_5,n_6\}\in\mathcal{Z}} (-1)^{n_4+n_5} \\ & \times J_{s_1+n_1+n_2+n_5}(t_\Omega)\, J_{s'_1+n_1+n_3+n_4}(t_\Omega) \\ & \times J_{s_2+n_3-n_5+n_6}(t_\Omega)\, J_{s'_2+n_2-n_4+n_6}(t_\Omega) \prod_{n_i=1}^{6} I_{n_i}(t_D)\,,\ \{s_j, s'_l\} \in \mathcal{Z},\end{aligned} \tag{36}$$

and J_n and I_n are Bessel's functions of integer order $n \in \mathcal{Z}$. These functions satisfy that

$$J_{-n}(t) = (-1)^n J_n(t),\ \ J_n(-t) = (-1)^n J_n(t),$$

and

$$I_{-n}(t) = I_n(t),\ \ I_n(-t) = (-1)^n I_n(t).$$

This solution is symmetric under the exchange of particles (preserving the symmetry of the IC), is Hermitian and fulfills normalization in the lattice $\text{Tr}[\rho(t)] = \sum_{\{s_1,s_2\}\in\mathcal{Z}} \langle s_1, s_2|\rho(t)|s_1, s_2\rangle = 1,\ \forall t$. The positivity is assured because of the CP structure of the infinitesimal generator. The probability of finding one particle in site s_1 and another in s_2 is given by the profile: $P_{s_1,s_2}(t) \equiv \langle s_1, s_2|\rho(t)|s_1, s_2\rangle$ and shows the expected reflection symmetry in the plane: $s_1 - s_2 = 0$.

The behavior of $D(t)$ controls the dynamics of the evolution of the DQW, and the waiting-time time-scale is $\lambda = \hbar/k_B T$ (see appendix B). Therefore we can check that in the case $b = 1$ we recover the Markovian limit [12, 13]. In what follows we are going to use b as the non-Markov control parameter.

For a quantum closed system, without dissipation, we recover the solution for *two* free QWs:

$$\langle s_1, s_2|\rho(t)|s'_1, s'_2\rangle_{D=0} = \prod_{j=1}^{2} i^{(s_j - s'_j)} J_{s_j}(t_\Omega) J_{s'_j}(t_\Omega)\,, \tag{37}$$

this is the well known tight-binding result, this means that from an uncorrelated initial condition $\rho(0)$ the solution $\rho(t \geq 0)$ is written as the direct product of two independent particles. As we mentioned before a *classical* random walk regime [23] cannot be recovered. Calling $\tilde{D} = 1/\lambda$ we see that for $\tilde{D} \gg \Omega/\hbar$ the two-body density matrix $\rho(t) \neq \rho_1(t) \otimes \rho_2(t)$. From Eq.(36) it can be proved that when $\tilde{D} \gg \Omega/\hbar$ we get

$$\lim_{\tilde{D}\gg\Omega/\hbar} P_{s_1,s_2}(t) \neq P_{s_1}(t) \times P_{s_2}(t) = e^{-2t_D}\ I_{s_1}(t_D)\, I_{s_2}(t_D)\,,$$

here P_{s_j} is the classical probability profile for each particle. So a classical regime [for $t \to \infty$] cannot be obtained. This means that the profile for two DQWs will not be a Gaussian bell-shape in 2D. In addition, we note that there exist an important competition between building correlations vs inducing dissipative decoherence, which can now be studied in terms of the non-Markov control parameter b.

The one-particle density matrix is recovered tracing-out the degrees of freedom of the second one, say $j = 2$:

$$\rho^{(1)}(t) \equiv \mathrm{Tr}_2[\rho(t)],$$

then

$$\langle s_1|\rho^{(1)}(t)|s_1'\rangle = i^{(s_1-s_1')}e^{-t_D}\sum_{n\in\mathcal{Z}} J_{s_1+n}(t_\Omega)\, J_{s_1'+n}(t_\Omega)\, I_n(t_D)\,,$$

solution that indeed shows asymptotically a random walk behavior [14]. In addition we note that for $b = 1$ the classical random walk solution is recover $P_t(s) = e^{-2\tilde{D}t}\ I_s\left(2\tilde{D}t\right)$, and from this expression it is simple to get the Gaussian profile in the lattice continuous limit $s \to x$ [23].

IV. QUANTUM COHERENCE WITHIN A NON-MARKOVIAN DESCRIPTION

A. Quantum coherence (cross terms of the two-body $\rho(t)$)

To quantify (indirectly) the building of bath-induced correlations between particles, we can calculate the total coherence contribution from the cross-terms of the two-body density matrix. This object is defined as

$$\mathcal{G}\,(t) = \sum_{(s_1\neq s_1')(s_2\neq s_2')} |\langle s_1, s_2|\rho(t)|s_1', s_2'\rangle|\,,$$

here we have taken a localized initial condition for $\rho\,(0)$, see (31). This measure $\mathcal{G}\,(t)$ has recently been used to quantify the quantum coherence (QC) [27, 28].

In order to calculate elements of $\rho(t)$ we now adopts the Gamma waiting-time model, and to characterize the disorder we use $b \geq 1$, see Appendix B.

In figure 2 we chose as free parameters $\langle t\rangle = b\lambda$ and $b \geq 1$, then we maintain the mean waiting-time $\langle t\rangle = 1.5$ fixed and varies the non-Markovian parameter b. We can see an inflexion point at time $t_c \sim 0.7$ showing that building of correlations are delayed with respect to the Markovian case ($b = 1$, as in reference [13]), after this point the behavior of QC is almost linear. This is so because $D\,(t)$ is in the regime where $D\,(t) \sim t/\lambda b$. For $t > t_c$ off-diagonal elements of $\rho\,(t)$ are dominated by the J-Bessel functions.

B. Negativity (mixed IC for the two-body $\rho(t)$)

Another measure to quantify bath-induced correlations between particles can be achieved calculating the Negativity $N(\rho)$ defined in terms of the partial transpose of the density matrix [29]. This correlation can easily be calculated for bipartite systems $\rho_{AB} \equiv \rho$, where the partial transpose of ρ in the space A or B is required. Partial transpose (in space A) of the two-body density matrix ρ is calculated as:

$$\langle i_A, j_B|\rho^{T_A}|i_A', j_B'\rangle \equiv \langle i_A', j_B|\rho|i_A, j_B'\rangle, \tag{38}$$

where the orthonormal product basis $|i_A, j_B\rangle \equiv |i_A\rangle \otimes |j_B\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ is used, with $\mathcal{H}_{A(B)}$ representing the Hilbert space in $A(B)$ respectively. Using (38) the Negativity function $N(\rho)$ gets:

$$N(\rho) = \frac{\|\rho^{T_A}\| - 1}{2}, \tag{39}$$

where $\|.\|$ represents the trace norm. From (39) the negativity can also be expressed in the form:

$$N(\rho) = \sum_i |\mu_i|, \tag{40}$$

where μ_i is any negative eigenvalue of ρ^{T_A}. This is a simpler procedure to obtain a measure of quantum correlations for a bipartite mixed state.

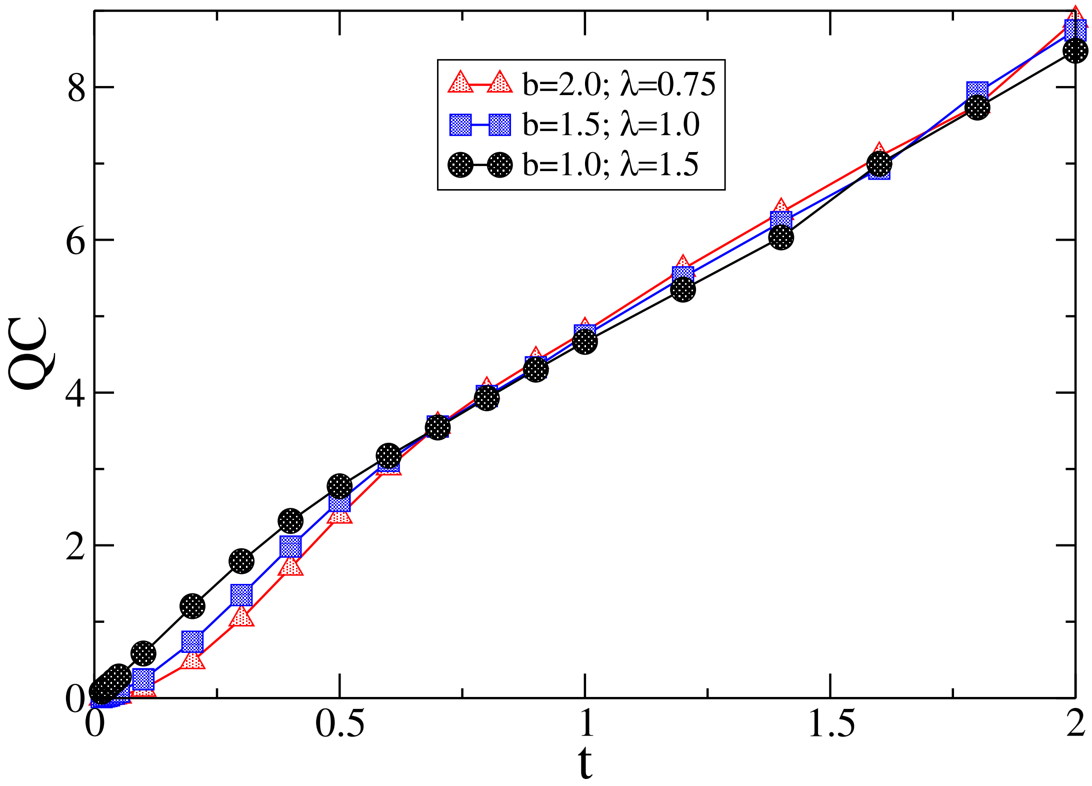

FIG. 2: Quantum coherence $\mathcal{G}(t)$ as function of the dimensionless time $t \to t_\Omega$, for several values of $b \geq 1$ and fixed value of mean waiting-time $\langle t \rangle = b\lambda = 1.5$.

In this section we are going to introduce an IC characterized by a mixed initial state, so we can investigate how the non-Markov effect changes the temporal behavior of the Negativity. We chose as the IC of our two-body density matrix:

$$\rho(0) = \left(\frac{4a-1}{3}\right) |\Psi^-\rangle\langle\Psi^-| + \left(\frac{1-a}{3}\right) \mathbf{I}, \tag{41}$$

where $\mathbf{I}$ is the identity matrix and

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} \left[|s_0, -s_0\rangle - |-s_0, s_0\rangle \right].$$

Note that for $t = 0$ the Negativity is a function of the parameter a, in the present study we chose $a = 0.9$, so the Negativity at $t = 0$ is $N(\rho) = 0.4$. In the figure (3) we show $N(\rho)$ of our bipartite system for a mixed IC in the Wannier basis (41). Here, we have plotted $N(\rho(t))$ as a function of the dimensionless time $t = t_\Omega$ and for several values of the non-Markov control parameter $b (= 0.5, 1, 2)$ for fixed $\lambda = 1$. We conclude that Negativity is more preserved (in time) for higher values of the non-Markovian control parameter $b > 1$. Note that in the time-asymptotic regime, $t \gg \lambda$, and for larger values of $b > 1$ the dissipation factor $D(t)$ decreases, see figure 1, then the rate of the lost of coherence decreases too, justifying the behavior of the Negativity as a function of b.

In figure (3) we also show the case $b < 0.5$ which can be associated to an anomalous subdiffusive regime [22] (see appendix B), this case has been plotted just for comparison. In the present paper we are only interested in the weak disorder case which corresponds to $b > 1$.

V. CONCLUSIONS

We have introduced a non-Markov approach to analyze two free particles (in a lattice) interacting with a thermal bath, the method is founded in the concept of random interventions of the environment. This method is based on the idea of the applications of a CP map $\mathcal{E}[\bullet]$ at discrete times n. Then using the Renewal theory we extend this model

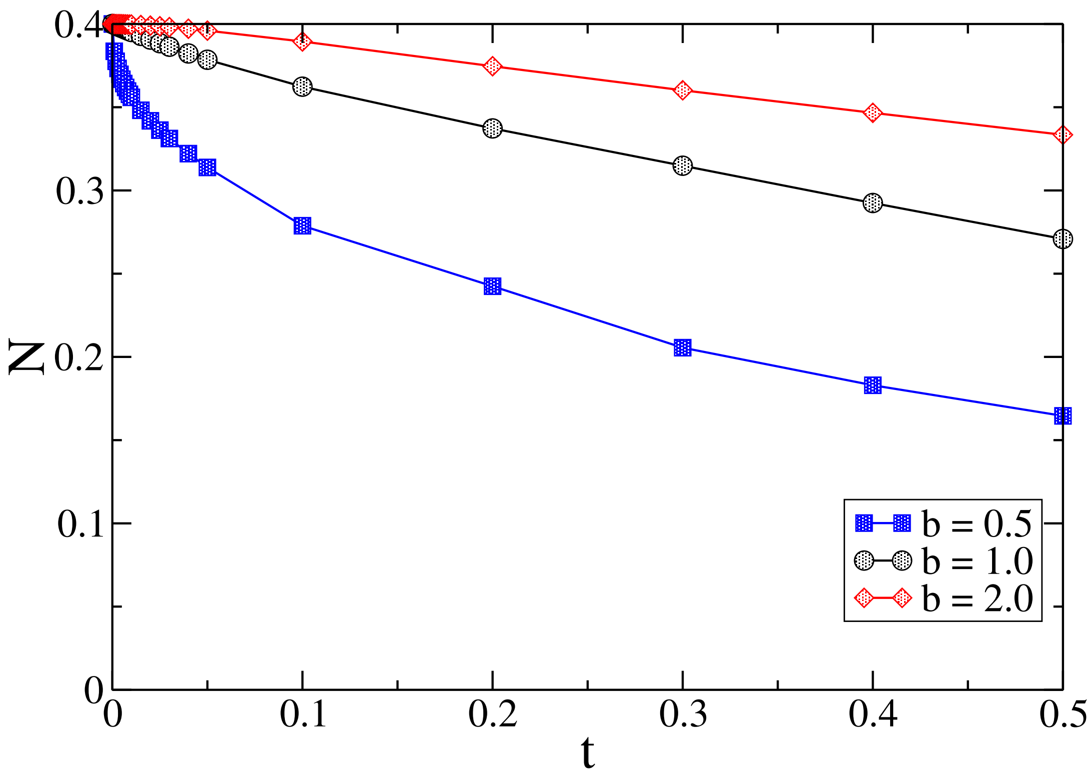

FIG. 3: Quantum correlation (Negativity) calculated from Eq. (39) as function of dimensionless time $t \to t_\Omega$, and several values of b and for fixed $\lambda = 1$.

to the continuous-time representation, where the key element is the random time elapsed between each application of $\mathcal{E}[\bullet]$, i.e., the waiting-time function $\psi(t)$. The evolution equation for the density matrix $\rho(t)$ is a generalization of the classical CTRW approach [21]. Our general result (8) or (11) is not restrictive of the dissipative tight-binding model that we have solved in the present paper.

The evolution equation for the two-body density matrix $\rho(t)$ is characterized by a non-local in time completely positive KL infinitesimal generator which has not counterpart in the classical two random walk problem. In the *interaction* representation the non-Markov evolution equation for density matrix $\sigma(t)$ can easily be solved when von Neumann's superoperator $\mathcal{L}_0[\bullet]$ commutes with the completely positive map $\mathcal{E}[\bullet]$. To solve the evolution in the *original* representation: $\rho(t) = e^{+t\mathcal{L}_0}\sigma(t)$, we introduced a perturbation expansion assuring the positive condition on the density matrix at all times. The present application (DQW) gives support to our perturbative analysis for solving, in general, memory-kernel CP dynamics.

As in the classical approach, here the non-Markovian character of the evolution of $\rho(t)$ has been modeled introducing a non-exponential waiting-time function: the Gamma pdf. This model for $\psi(t)$ allow us to study the evolution of $\rho(t)$ at all time. Wannier elements of the two-body density matrix have been obtained, in particular its short and long time regimes have been analytically solved. We have shown that for a *weak* non-Markovian model, $\rho(t)$ has an important time-structure in its short-time evolution, while in the long-time regime the behavior of $\rho(t)$ is Markovian with a renormalized dissipative coefficient. The elements of $\rho(t)$ have been used to analyze correlation functions against non-Markovian effects. In particular we have studied the behavior of the Quantum Coherence and the Negativity as a function of time. We show (for a localized IC of $\rho(0)$) that for a fixed mean waiting-time $\langle t \rangle = b\lambda$ of the random interventions of $\mathcal{B}$, the bath-induced Quantum Coherence is delayed until a critical time t_c with respect to the Markovian case. For the analysis of the Negativity we have used a mixed IC of $\rho(0)$, then we can conclude that Negativity is more preserved in time for longer values b with respect to the Markovian case $b = 1$; that is, initial entanglement is less efficiently destroyed under a non-Markovian dynamics.

The present approach can also be used to analyze intermittent bath interventions, leading to a strong non-Markovian structure in the evolution of the density matrix work in that direction is in progress.

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Author contribution statement

Both authors contributed equally to this paper.

Appendix A: Systematic perturbation in $\mathbf{KL}[\bullet]$

Now we introduce a systematic perturbation in the intensity of the environment interventions $\mathcal{B}$. In the particular case when $\mathbf{KL}[\bullet]$ commutes with $\mathcal{L}_0[\bullet]$, from (18) we can write a Dyson perturbation, starting with the $\mathcal{O}(0)$ we have

$$\partial_t \rho(t) = \mathcal{L}_0[\rho(t)] + \cdots . \tag{A1}$$

Then to $\mathcal{O}(1)$ we get

$$\begin{aligned} \partial_t \rho(t) &\simeq \mathcal{L}_0[\rho(t)] + \left(\int_0^t \Phi(t-\tau)\, d\tau\right) \mathbf{KL}[\bullet]\, \rho(t) \\ &= \mathcal{L}_0[\rho(t)] + \chi(t)\, \mathbf{KL}[\bullet]\, \rho(t) + \cdots . \end{aligned} \tag{A2}$$

Thus to $\mathcal{O}(2)$ we can write

$$\begin{aligned} \partial_t \rho(t) &\simeq \left\{\mathcal{L}_0[\bullet] + \int_0^t \Phi(t-\tau)\, \mathbf{KL}[\bullet] \exp\left(D(\tau)\, \mathbf{KL}[\bullet]\right) d\tau\right\} \rho(t) \\ &\simeq \mathcal{L}_0[\rho(t)] + \chi(t)\, \mathbf{KL}[\bullet]\, \rho(t) + \left(\int_0^t \Phi(t-\tau)\, D(\tau)\, d\tau\right) \mathbf{KL}[\bullet]\, \mathbf{KL}[\bullet]\, \rho(t) \\ &\simeq \left\{\mathcal{L}_0[\bullet] + \chi(t)\, \mathbf{KL}[\bullet] + \beta(t)\, \mathbf{KL}[\bullet]\, \mathbf{KL}[\bullet] + \cdots\right\} \rho(t), \end{aligned} \tag{A3}$$

where $\beta(t) \equiv \int_0^t \Phi(t-\tau)\, D(\tau)\, d\tau$. Therefore calling

$$W(t) \equiv \int_0^t \beta(\tau)\, d\tau \quad \text{and} \quad D(t) \equiv \int_0^t \chi(\tau)\, d\tau$$

we can write the formal solution to $\mathcal{O}(2)$ in the form:

$$\rho(t) \simeq \exp\left(t\mathcal{L}_0[\bullet] + D(t)\, \mathbf{KL}[\bullet] + W(t)\, \mathbf{KL}[\bullet]^2 + \cdots\right) \rho(0), \tag{A4}$$

where $D(t)$ is a time-dependent (positive) dissipative coefficient. The nature of the function $W(t)$ can be studied in terms of the waiting-time model that we may have used. Introducing the Laplace transform of the function $\chi(t)$, we write $\chi(u) = \Phi(u)/u$, then we can get a general expression for $D(t)$ and $W(t)$ in the from:

$$D(u) = \frac{\chi(u)}{u} = \frac{\Phi(u)}{u^2} \tag{A5}$$

$$W(u) = \frac{\Phi(u)^2}{u^3}. \tag{A6}$$

Up to second order in the intensity of the map $\mathcal{E}[\bullet]$ the non-Markovian evolution is dictated by an equation which involves the superoperator $\mathbf{KL}[\bullet]^2$, producing a complex dynamics in the elements of the density matrix. In the $\mathcal{O}(2)$ approximation a second dissipative function $\beta(t)$ appears to characterize the evolution of the density matrix, which is now dictated by the equation (A3). We note that in the Markovian limit: $\Phi(t) \to \delta(t)/\lambda$, so functions $D(t)$ and $W(t)$ are powers of time: $D(t) \to t/\lambda, W(t) \to t^2/2\lambda$.

Appendix B: Weak-disordered bath disruptive interventions

A suitable model controlling the random interventions of the bath can be modeled using the Gamma pdf:

$$\psi(t) = \frac{1}{\lambda}\left(\frac{t}{\lambda}\right)^{b-1} \frac{\exp(-t/\lambda)}{\Gamma(b)}, \quad \lambda > 0, b > 0, t \geq 0. \tag{B1}$$

For values $b < 1$ the waiting-time (B1) diverges in the limit $t \to 0$, while for $b > 1$ goes to zero. All moments of this Gamma waiting-time pdf are well defined in particular: $\langle t \rangle = \lambda b$ and $\langle t^2 \rangle = (1+b)\, b\lambda^2$, indicating that the dispersion of the "elapsed times" between disruptive bath interventions is $\sigma_t^2 \equiv \langle t^2 \rangle - \langle t \rangle^2 = b\lambda^2$. For $b > 1$, the most likely "elapsed-time" is the mean waiting-time $\langle t\rangle$ and the shape of the $\psi(t)$ is unimodal and skewed. Note that we can use λ as the control parameter for the limit of a dense point Renewal process, as we mention in Section 1.5.

The Laplace transform of the Gamma waiting-time function is

$$\psi(u)=\frac{1}{(1+\lambda u)^{b}}. \tag{B2}$$

Normalization can be checked noting that $\psi(u=0)=1$, distinctive types of random-time pdf (bath interventions) can be obtained by changing the free parameters b,λ. The parameter b measures the withdrawal from the exponential behavior, which corresponds to the case characterizing the Markovian limit ($b=1$); λ is a scale of time. For $\lambda=$fixed and in the case $0<b\ll 1$ the Gamma pdf $\psi(t)$ has an integrable divergence at the origin indicating a strong departure from the exponential function, this singular behavior leads to the fact that its dispersion σ_t^2 gets smaller and smaller when $b\to 0$, then $\psi(t)$ becomes sharp. In our work, for the present application, we only will be interested in values $b\geq 1$.

We note that from a classical point of view this Gamma waiting-time represents a *weak disorder* model with a strength that depends on the quantity $\langle t\rangle$ [21]. In the limit $\{b\to\infty$ and $\lambda\to 0\}$ with $\langle t\rangle=\lambda b\to$constant the waiting-time pdf becomes singular $\psi(t)\to\delta(t-\lambda b)$ (the random interventions occurs at fixed regular intervals $\tau=n\langle t\rangle$, then a resonance frequency $\omega=2\pi n/\langle t\rangle$ appears in the model). On the other hand, for fixed λ and if $b>1$ the mean waiting-time $\langle t\rangle=\lambda b$ turns to be the inverse of a renormalized diffusion coefficient, so for $b\gg 1$ the diffusion coefficient get smaller than in the Markovian case. For fixed $\langle t\rangle$ and in the limit $\langle t\rangle/\lambda=b\to 0$ the waiting-time pdf $\psi(t)$ turns to be very wide and this pdf can be used for modeling anomalous diffusion [22].

As we mention before for $b=1$ we recover the Markovian description for the QME [20, 23], in this case the Renewal process is characterized by a Poisson number of events during a given time interval. That is, the probability of having $n-$events during the time interval $[0,t]$ is given by $\mathcal{P}(n,t,1)=e^{t/\lambda}\frac{1}{n!}\left(\frac{t}{\lambda}\right)^{n}$. In general for $b\neq 1$ the expression for $\mathcal{P}(n,t,b)$ can be written in the Laplace representation as:

$$\begin{aligned}\mathcal{P}(n,u,b) &= \int_0^\infty \mathcal{P}(n,t,b)\,e^{-ut}dt\\ &= \frac{1-\psi(u)}{u}\psi(u)^n.\end{aligned}$$

Then, all moments for the number of events can analytically be calculated: $\langle n(u)^q\rangle=\sum_{n=0}^{\infty}n^q\,\mathcal{P}(n,u,b)$, for example:

$$\begin{aligned}\langle n(u)\rangle &= \frac{\psi(u)}{u(1-\psi(u))}=\frac{1/u}{(1+\lambda u)^b-1}\\ \left\langle n(u)^2\right\rangle &= \frac{\psi(u)(1+\psi(u))}{u(1-\psi(u))^2}=\frac{1}{u}\frac{(1+\lambda u)^b+1}{\left((1+\lambda u)^b-1\right)^2},\end{aligned}$$

from which (after Laplace inversion) the dispersion for the numbers of disruptive bath interventions $\sigma_n^2(t)\equiv\left\langle n(t)^2\right\rangle-\langle n(t)\rangle^2$, can be studied in time for several values of the non-Markov parameter b. In the same way, and in general for any $b>0$, all cumulants: $\langle\langle n(u)^p\rangle\rangle$ can be calculated showing the departure from the linear behavior for the Poisson case: $\langle\langle n(t)^p\rangle\rangle|_{b=1}=t/\lambda,\ \forall p\geq 1$. In Figure (4) we have plotted $\mathcal{P}(n,t,b)$ as a function of n for several values of b. In the inset we show $\sigma_n^2(t)$ as a function of dimensionless time $t\to t/\lambda$ for the same values of parameter b.

Going back to the non-Markov QME evolution, the memory kernel of the KL infinitesimal generator (9) is using (B2) given by:

$$\Phi(u)=\frac{u}{(1+\lambda u)^b-1}. \tag{B3}$$

From this expression it is also possible to see that λ is a macroscopic characteristic time scale and b the non-Markov control parameter. At short-time the behavior of the dissipative function (A5) is controlled by the value of b, for example we get the asymptotic limits

$$D(u)\simeq\left\{\begin{array}{ll}\frac{1}{\lambda^b u^{b+1}}\left(\frac{1}{1-(1/\lambda u)^b}\right) & ,\ \text{if}\ \ 0<b<1\\ \frac{1}{\lambda^b u^{b+1}}\left(\frac{1}{1+b(1/\lambda u)}\right) & ,\ \text{if}\ \ 1<b<2\end{array}\right., \qquad \lambda u\gg 1,$$

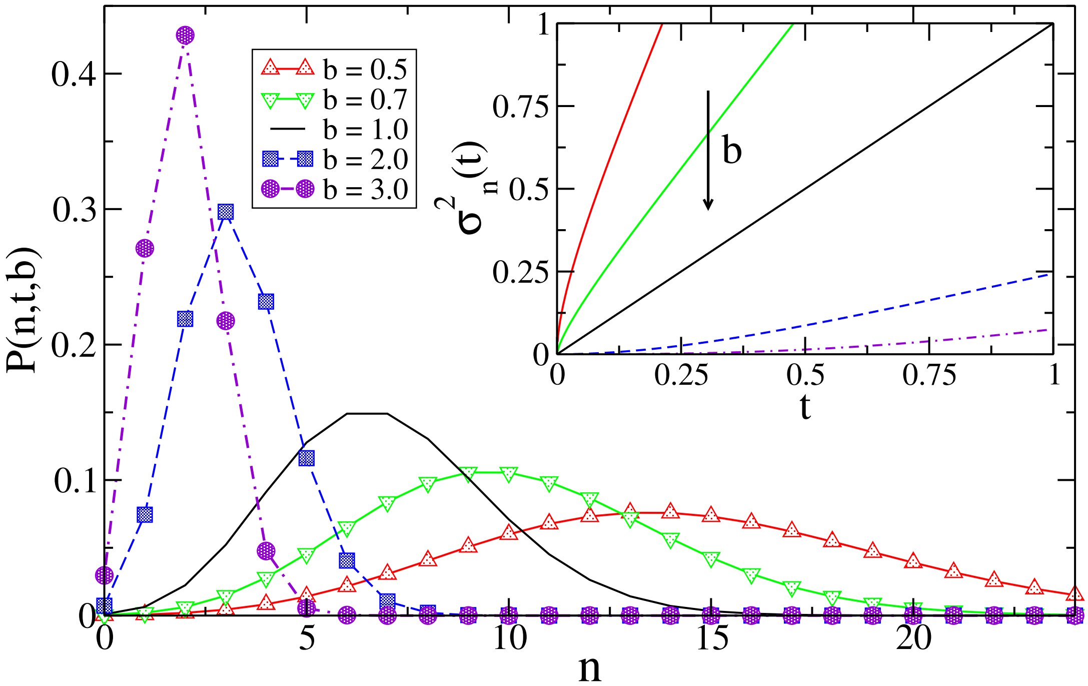

FIG. 4: (a) Probability $\mathcal{P}(n,t,b)$ as a function of the number of events n for dimentionless time $t \to t/\lambda = 7$ and for several values of b. The inset shows $\sigma_n^2(t)$ as a function of dimentionless time as in (a) the arrow indicates the increase of the value of b.

therefore, at very short-time we get the nonlinear behavior

$$D(t) \simeq \left(\frac{t}{\lambda}\right)^b + \cdots, \qquad b > 0,\ t \ll \lambda. \tag{B4}$$

While at the long-time limit $D(t)$ is characterized by a linear regime

$$D(t) \simeq t/(b\lambda), \quad t \gg \lambda \tag{B5}$$

We note that in the present paper we will be interested in the case $b \geq 1$ in order to consider that the dispersion ($\sigma_t^2 = b\lambda^2$) of the "elapsed times" between disruptive bath interventions gets larger if the non-Markov dynamics is enlarged $b \gg 1$.

Other models for the random disruptive interventions of the CP map can be worked out in a similar way. The main difference being that for *weak* disorder models, at long-time the behavior can always be renormalized to a Markovian one [20]. While in the *strong* disordered case, for example from a fractional dynamics or for intermittent bath interventions, at any time (short and long regimes) the evolution character of the Quantum Master Equation remains always non-Markovian [20, 23].

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[1] R. Alicki and K. Lendi, *"Quantum Dynamical Semigroups and Applications"* (Springer Verlag), Lecture Notes in Physics, V.286, Berlin, (1987).

[2] B. Vacchini, Generalized Master Equations Leading to Completely Positive Dynamics, Phy. Rev. Lett. **117**, 230401 (2016).

[3] D. Chruscinski and A. Kossakowski, Sufficient conditions for a memory-kernel master equation, Phys. Rev. **A 94**, 020103(R) (2016).

[4] S. M. Barnett and S. Stenholm, Hazards of reservoir memory, Phys. Rev. **A 64**, 033808 (2001); and reference therein.

[5] A.A. Budini, Stochastic representation of a class of non-Markovian completely positive evolutions, Phys. Rev. A, **69**, 042107 (2004); A.A. Budini, Random Lindblad equations from complex environments, Phy. Rev. **E 72**, 056106 (2005).

[6] B. Vacchini, Non-Markovian master equations from piecewise dynamics, Phy. Rev. **A 87**, 030101R (2013).

[7] A.A. Budini, Non-Markovian quantum jumps from measurements in bipartite Markovian dynamics, Phys. Rev. **A 88**, 012124 (2013); A.A. Budini, Embedding non-Markovian quantum collisional models into bipartite Markovian dynamics, Phys. Rev. **A 88**, 032115 (2013).

[8] A.K. Chattah and M.O. Cáceres, Effective eigenstates in a dynamical disordered tight-binding model, Physica D, **168**, 258 (2002).
[9] Y. Yin, D.E. Katsanos, & S.N. Evangelou, Quantum walks on a random environment, Phys. Rev. A **77**, 022302 (2008).
[10] M. Esposito and P. Gaspard, Emergence of diffusion in finite quantum systems, Phys. Rev. B 71, 214302 (2005).
[11] J. P. Keating, N. Linden, J. C. F. Matthews, and A. Winter, Localization and its consequences for quantum walk algorithms and quantum communication, Phys. Rev. A **76**, 012315 (2007).
[12] M. Nizama and M. O. Cáceres, Bath-induced correlations in an infinite-dimensional Hilbert space, Eur. Phys. J. B **90**, 164 (2017).
[13] M. Nizama and M. O. Cáceres, Quantum correlations and coherence between two particles interacting with a thermal bath, J. Phys. A: Math. Theor. 50, 215208 (2017).
[14] M. Nizama and M. O. Cáceres, Non-equilibrium transition from dissipative quantum walk to classical random walk, J. Phys. A: Math. Theor. **45**, 335303 (2012).
[15] O. Mülken and A. Blumen, in *"Continuous-time quantum walks: Models for coherent transport on complex networks"*, Phys. Rep. **502**, 37 (2011).
[16] (a) W. Dur, R. Raussendorf, V.M. Kendon and H-L Briegel, Quantum walks in optical lattices, Phys. Rev. A, 66, 052319 (2002); (b) P. Xue, R. Zhang, H. Qin, X. Zhan, Z. H. Bian, J. Li, & B.C. Sanders, Experimental Quantum-Walk Revival with a Time-Dependent Coin, Phys. Rev. Lett. **114**, 140502 (2015).
[17] A. E. Allahverdyan, R. Balian and T. M. Nieuwenhuizen, Understanding quantum measurement from the solution of dynamical models, Phys. Report **525**, 1-166, (2013).
[18] N.G. van Kampen, A soluble model for quantum mechanical dissipation, J. Stat. Phys. **78**, 299 (1995).
[19] Y. Goychuk, Rate processes with non-Markovian dynamical disorder, J. of Chem. Phys. 122, 164506 (2005); Y. Jung, E. Barkai, & R.J. Silbey, Lineshape theory and photon counting statistics for blinking quantum dots: a Lévy walk process, Chem. Phys. **284**, 181 (2002).
[20] M.O. Cáceres, Eur. Phys. J.B. **90**, 74 (2017).
[21] E.W. Montroll and G.H. Weiss, Random Walks on Lattices II, J. Math. Phys. **6**, 167, (1965).
[22] H. Scher and M. Lax, Stochastic Transport in a Disordered Solid. I. Theory, Phys. Rev. B, **7**, 4491 (1973).
[23] M.O. Cáceres, "*Non-equilibrium Statistical Physics with Application to Disordered Systems*", ISBN 978-3-319-51552-6, Springer (2017). (2011).
[24] W. Feller, *"An Introduction to Probability Theory and its Applycations"*, Vol. 2, J. Wiley and Sons Inc., Third Edition, (1967).
[25] C.R. Cox, *"Renewal Process"*, Monographs on Statistic and Applied Probability. Gen. Eds. C.R. Cox and D.V. Hinkley. Chapman and Hall. London 1962 (reprinted 1982).
[26] M. O. Cáceres and M. Nizama, The quantum Lévy walk, J. Phys. A: Math. Theor. **43** 455306 (2010).
[27] T. Baumgratz, M. Cramer, and M. B. Plenio, Quantifying Coherence, Phys. Rev. Lett. **113**, 140401 (2014).
[28] Mauro B. Pozzobom and Jonas Maziero (arxiv.org), Environment-induced quantum coherence spreading of a qubit, quant-ph/1605.04746 (2016).
[29] G. Vidal, and R. F. Werner, Computable measure of entanglement, Phys. Rev. A **65**, 032314 (2002).