GEOMETRY OF POSITIVE OPERATORS AND UHLMANN'S APPROACH TO THE GEOMETRIC PHASE

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Dedicated to Carlos Segovia, with affection and admiration.

In Uhlmann's description of the differential geometry of the space Ω of density operators, a relevant role is played by the parallel condition $\omega^*\omega = \dot{\omega}^*\omega$, where ω is a lifting of a curve γ in Ω, i.e. $\omega(t)\omega(t)^* = \gamma(t)$ for all t. In this paper we get a principal bundle with a natural connection over the space $\mathbf{G}^+$ of all positive invertible elements of a C^*-algebra such that the parallel transport is ruled by Uhlmann's parallel equation.

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1. Introduction

After the discovery by M. V. Berry [1] of the geometric phase (sometimes also called "Berry phase"), B. Simon [12] interpreted it as a holonomy of a natural connection which rules the parallel transport of pure states in a Hilbert space $\mathcal{H}$. Later, A. Uhlmann [14–16] extended this approach to mixed states by studying some geometric properties of the map $\pi(A) = A^*A$ restricted to the set Ω of normalized density operators, i.e. positive operators with trace 1. The main problem with this study consists in the lack of smoothness of Ω: it is not a manifold with a boundary, but a stratification. In order to avoid this obstruction, L. Dąbrowski and A. Jadczyk [6] performed Uhlmann's programme on a dense subset Ω_0 of Ω, namely the set of $A \in \Omega$ with positive eigenvalues. On Ω_0 they obtained a principal bundle with a connection such that the related parallel transport coincides with what Uhlmann calls "parallel amplitudes". On the other hand, in a series of papers [3–5], H. Porta, L. Recht and the authors of this paper have studied a natural connection on the principal bundle defined by the action

$$\begin{aligned} \mathbf{G} \times \mathbf{G}^+ &\to \mathbf{G}^+ \\ (g, a) &\mapsto gag^* \end{aligned} \tag{1}$$

(here $\mathbf{G}$ is the group of invertible elements of a unital C^*-algebra A and $\mathbf{G}^+$ is the set of positive invertible elements of the algebra).

This paper is devoted to the study of a different principal fibre bundle over $\mathbf{G}^+$ with a connection such that the corresponding parallel transport coincides with Uhlmann's, in the sense that both are ruled by the same equations. The contents of the paper is the following. Section 2 contains some results about the existence and formulae of solutions of the equation $AX - XB = Y$, for $A, B, Y \in L(\mathcal{H})$, which are needed later. Section 3 contains a survey of the differential structure of the fibre bundle $(\mathbf{G}, \mathbf{G}^+)$ as a homogeneous space, with a natural connection, geodesics and a Finsler metric, as studied in [5] and [3]. In Section 4 we give a brief description of some of Uhlmann's results about the geometry of density operators, such as the parallel condition for a lift and the transport equation for a given curve γ of density operators. Section 5 contains the description of the fibre bundle $(\mathbf{G}, \mathbf{G}^+)$, as seen in Section 3, but considering another connection which gives rise to the corresponding transport equation, covariant derivative and geodesics. This bundle is named Uhlmann's fibration because the equations derived from it coincide with those given by Uhlmann for density operators. Finally, in Section 6 both approaches are compared.

2. Preliminaries

Let $\mathcal{H}$ be a Hilbert space and $L(\mathcal{H})$ the algebra of linear bounded operators on $\mathcal{H}$. Given $A, B, Y \in L(\mathcal{H})$, consider the equation $AX - XB = Y$. This equation has been first studied by Sylvester [13] in the finite dimensional setting and later in its full generality by M. G. Krein [8], J. Daleckii [7], M. Rosenblum [11], and others. The reader is referred to [2] for a nice survey on the subject. We shall need the following facts:

1) if $\sigma(A) \cap \sigma(B) = \emptyset$ then the equation has a unique solution for every Y,
2) if Re $\lambda > 0$, Re $\mu < 0$ for all $\lambda \in \sigma(A)$, $\mu \in \sigma(B)$, then the unique solution X has the form
$$X = \int_0^\infty e^{-tA} Y e^{tB} dt.$$

In particular, if $\sigma(A) \subset \mathbb{R}^+ = \{t \in \mathbb{R} : t > 0\}$ then the equation $AX + XA = Z$ has a unique solution for every Z, namely $X = \int_0^\infty e^{-tA} Z e^{-tA} dt$. Observe that if $A \in GL(\mathcal{H})^+$ and $Z^* = Z$ then $X^* = X$.

More generally if A is a unital C^*-algebra, $a \in A$ and $C_a = R_a + L_a$, where R_a and L_a are respectively the operators of the right and left multiplication by a, then $\sigma(C_a) \subset \sigma(a) + \sigma(a)$ so that if $\sigma(a) \subset \mathbb{R}^+$, C_a is invertible and then the equation

$$C_a(x) = ax + xa = z$$

has a unique solution, namely $C_a^{-1}(z)$. Also

$$x = C_a^{-1}(z) = \int_0^\infty e^{-at} z e^{-at} dt.$$

3. Differential geometry of $\mathbf{G}^+$

A short description of $\mathbf{G}^+$ as a Finsler manifold and as a homogeneous space of $\mathbf{G}$ is presented in this section. Most of the results we mention here are contained in [3–5]. Throughout this paper A denotes a unital C^*-algebra represented in a Hilbert space $\mathcal{H}$, $\mathbf{G}$ is the group of invertible elements of A, $\mathcal{U}$ is the group of unitaries, A_h is the (real) Banach space of hermitian (i.e. self-adjoint) elements of A, A_{ah} consists of all antihermitian elements of A, $A^+ = \{X \in A_h : X \geq 0\}$ and $\mathbf{G}^+ = \mathbf{G} \cap A^+$. Every $a \in \mathbf{G}^+$ defines an equivalent scalar product on $\mathcal{H}$ by

$$\langle x, y \rangle_a = \langle ax, y \rangle, \qquad x, y \in \mathcal{H}.$$

If ^{*_a} denotes the adjoint with respect to the scalar product $\langle\ ,\ \rangle_a$, then it is easy to see that

$$T^{*_a} = a^{-1} T^* a.$$

Denote by A_h^a the hermitian elements of A with respect to $\langle\ ,\ \rangle_a$, i.e. $A_h^a = \{X : a^{-1} X^* a = X\}$, A_{ah}^a the a-antihermitian elements of A, and $\mathcal{U}^a$ the a-unitary elements of A. The reader may suppose that A is the algebra $L(\mathcal{H})$ of all bounded linear operators on a Hilbert space $\mathcal{H}$. Because $\mathbf{G}$ is an open subset of A, $\mathbf{G}^+$ is an open subset of A_h, so it has a natural structure of an open submanifold of A_h. There is also a natural action of $\mathbf{G}$ over $\mathbf{G}^+$, namely $L : \mathbf{G} \times \mathbf{G}^+ \to \mathbf{G}^+$, $L(g, a) = gag^*$. L is differentiable and transitive: if $a, b \in \mathbf{G}^+$ then $L(g, a) = b$ for $g = b^{1/2} a^{-1/2}$ (for $c \in \mathbf{G}^+$, $c^{-1/2}$ denotes the inverse of the positive square root of c). Thus the map $p_a : \mathbf{G} \to \mathbf{G}^+$, $p_a(g) = gag^*$ is surjective and $s : \mathbf{G}^+ \to \mathbf{G}$, $s(b) = b^{1/2} a^{-1/2}$, is a global differentiable section of p_a, i.e. $p_a \circ s = \mathrm{id}_{\mathbf{G}^+}$. As an open submanifold of A_h, the tangent of $\mathbf{G}^+$ at $a \in \mathbf{G}^+$ naturally identifies to A_h : $(T\mathbf{G}^+)_a = A_h$ ($a \in \mathbf{G}^+$). Then the tangent map of $p_a : \mathbf{G} \to \mathbf{G}^+$, at 1, $(T_{p_a})_1 : A \to A_h$ is $(T_{p_a})_1(X) = Xa + aX^*$. The isotropy group of a is the subgroup I_a of $\mathbf{G}$ of all $g \in \mathbf{G}$ such that $p_a(g) = gag^* = a$. I_a coincides with the a^{-1}-unitary group $\mathcal{U}^{a^{-1}} = \{g \in \mathbf{G} : ag^*a^{-1} = g^{-1}\}$ which acts freely on the fibres by right multiplication. Observe that $(TI_a)_1 = \{X \in A : Xa + aX^* = 0\} = \{X \in A : X^{*_{a^{-1}}} = -X\}$, the a^{-1}-antihermitian elements of A. In particular, for $a = 1$, $(T\mathcal{U})_1 = \{X \in A : X + X^* = 0\} = A_{ah}$.

From now on we consider the case $a = 1$ and we shall be concerned with the map $p = p_1 : \mathbf{G} \to \mathbf{G}^+$, $p(g) = gg^*$, with the tangent map at 1, $T : A \to A_h$, $T(X) = X + X^*$. Observe the decomposition $A = \ker T \oplus R(T) = A_{ah} \oplus A_h$. The projection of A onto A_h with the kernel A_{ah} is $\frac{1}{2}T$. The triple $(\mathbf{G}, \mathbf{G}^+, \mathcal{U})$ is a principal fibre bundle with the base $\mathbf{G}^+$ and structural group $\mathcal{U}$. For $g \in \mathbf{G}$ define $H_g = gA_h$. This is a natural connection, i.e. a smooth distribution of subspaces such that (i) $A = H_g \oplus V_g$ if $V_g = \{X \in A : Xg^* + gX = 0\}$; (ii) $uH_1u^* = H_1$; (iii) $H_g u = H_{gu}$ for all $g \in \mathbf{G}$, $u \in \mathcal{U}$. Here, *smooth* means that the map $g \mapsto P_g$, which assigns to each g the unique projection over A with the kernel V_g and range H_g, is differentiable. The subspaces H_g are called *horizontal*. It is well known that any smooth curve in the base space of a principal bundle with a connection admits a horizontal lift. In our case we have the following theorem.

THEOREM 3.1. *If $\gamma : [0,1] \to \mathbf{G}^+$ is a C^∞ curve and $\gamma(0) = p(g)$ for some $g \in \mathbf{G}$ then the solution of the problem*

$$\begin{cases} \dot\Gamma(t) = \frac{1}{2}\dot\gamma(t)\gamma(t)^{-1}\Gamma(t), \\ \Gamma(0) = g, \end{cases} \tag{2}$$

is the unique horizontal lift $\Gamma : [0,1] \to \mathbf{G}$ (i.e. $p(\Gamma(t)) = \gamma(t)$, $\dot\Gamma(t) \in H_{\Gamma(t)}$ for all $t \in [0,1]$ such that $\Gamma(0) = g$. Moreover, a C^∞ lift Γ of γ is horizontal if and only if its unitary part u satisfies the differential equation $2\dot u u^{-1} = \left(\gamma^{1/2}\right)^{\cdot} \gamma^{-1/2} - \gamma^{-1/2}\left(\gamma^{1/2}\right)^{\cdot}$.

Given a curve γ in $\mathbf{G}^+$ and a tangent field X along γ we define the covariant derivative

$$\begin{aligned} \frac{DX}{dt} &= \Gamma(t)\frac{d}{dt}\left(\left(TL_{\Gamma(t)^{-1}}\right)_{\gamma(t)} X(t)\right)\Gamma(t)^* \\ &= \dot X - \frac{1}{2}\left(X\gamma^{-1}\dot\gamma + \dot\gamma\gamma^{-1}X\right). \end{aligned}$$

A field X is called *parallel* along γ if $\frac{DX}{dt} = 0$. A curve γ is a *geodesic* (of the connection) if $\dot\gamma$ is parallel along γ if and only if $\ddot\gamma = \dot\gamma\gamma^{-1}\dot\gamma$. The unique geodesic γ such that $\gamma(0) = a$ and $\dot\gamma(0) = X \in (T\mathbf{G}^+)_a$ is $\gamma(t) = a^{1/2}e^{ta^{-1/2}Xa^{-1/2}}a^{1/2}$ ($t \in [0,1]$). Given $a, b \in \mathbf{G}^+$ there is a unique geodesic $\gamma_{a,b}$ such that $\gamma_{a,b}(0) = a$ and $\gamma_{a,b}(1) = b$, namely $\gamma_{a,b}(t) = a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^t a^{1/2}$. Observe also that if γ is a geodesic and $g \in \mathbf{G}$ then $g\gamma g^*$ is also a geodesic.

Let us introduce a Finsler structure on $\mathbf{G}^+$, i.e. define a norm $\|\cdot\|_a$ on each $(T\mathbf{G}^+)_a$. Define $\|X\|_a = \|a^{-1/2}Xa^{-1/2}\|$ ($X \in (T\mathbf{G}^+)_a$, $a \in \mathbf{G}^+$) and the *length* of a curve γ in $\mathbf{G}^+$, $L(\gamma) = \int_0^1 \|\dot\gamma(t)\|_{\gamma(t)}dt$. One can also prove that

$$\|gXg^*\|_{gag^*} = \|X\|_a \tag{3}$$

for all $g \in \mathbf{G}$. As a consequence, $L(g\gamma g^*) = L(\gamma)$ for all curves γ in $\mathbf{G}^+$ and $g \in \mathbf{G}$. An easy computation shows that $L(\gamma_{a,b}) = \| \log\left(a^{-1/2}ba^{-1/2}\right) \|$. In fact,

$$\begin{aligned}
\|\dot{\gamma}_{a,b}(t)\|_{\gamma_{a,b}(t)} &= \|a^{1/2}\log\left(a^{-1/2}ba^{-1/2}\right)\left(a^{-1/2}ba^{-1/2}\right)^t a^{1/2}\|_{a^{1/2}\left(a^{-1/2}ba^{-1/2}\right)^t a^{1/2}} \\
&= \| \log\left(a^{-1/2}ba^{-1/2}\right)\left(a^{-1/2}ba^{-1/2}\right)^t \|_{\left(a^{-1/2}ba^{-1/2}\right)^t} \\
&= \| \left(a^{-1/2}ba^{-1/2}\right)^{-t/2}\log\left(a^{-1/2}ba^{-1/2}\right)\left(a^{-1/2}ba^{-1/2}\right)^{t/2} \| \\
&= \| \log\left(a^{-1/2}ba^{-1/2}\right) \|,
\end{aligned}$$

where the second equality holds by (3) and the third one from functional calculus.

The main result in this section is the following theorem.

THEOREM 3.2. *The geodesic $\gamma_{a,b}$ is a shortest curve among all curves γ in $\mathbf{G}^+$ such that $\gamma(0) = a$ and $\gamma(1) = b$.*

It should be noticed that there exist infinite many C^∞ curves joining a, b which are the shortest (Nussbaum [10]). The remarkable fact here is the conjunction of being a geodesic and being a shortest curve in a highly non-Riemannian context.

As a corollary, the *geodesic distance* $d(a,b) = \inf L(\gamma)$ (the infimum is taken over all C^∞ curves joining a and b) can be explicitely computed: $d(a,b) = \| \log\left(a^{-1/2}ba^{-1/2}\right) \|$. Observe that, at a first sight, the right-side expression does not seem to be symmetric or satisfy the triangle inequality. The theorem shows that the equality holds and then

$$\| \log\left(a^{-1/2}ba^{-1/2}\right) \| = \| \log\left(b^{-1/2}ab^{-1/2}\right) \|$$

and

$$\| \log\left(a^{-1/2}ba^{-1/2}\right) \| \leq \| \log\left(a^{-1/2}ca^{-1/2}\right) \| + \| \log\left(c^{-1/2}bc^{-1/2}\right) \|$$

for all $a, b, c \in \mathbf{G}^+$.

4. Geometry of density operators

In this section we give a brief description of some of the results obtained by A. Uhlmann and others [14–16, 6] in the study of the differential geometry of the set of density operators as the base space of the bundle of Hilbert–Schmidt operators.

For a Hilbert space $\mathcal{H}$, consider the set Ω of density operators on $\mathcal{H}$, i.e. the set of positive operators in $L(\mathcal{H})$ of trace 1, and the *extended space* $\mathcal{H}^{\text{ext}}$ of Hilbert–Schmidt operators in $L(\mathcal{H})$, $\mathcal{H}^{\text{ext}} = \{\omega \in L(\mathcal{H}) : \operatorname{tr}(\omega\omega^*) < \infty\}$, with the scalar product given by $\langle\omega_1, \omega_2\rangle = \operatorname{tr}(\omega_1^*\omega_2)$.

Consider the projection $\pi : \mathcal{H}^{\text{ext}} \to \Omega$ given by $\pi(\omega) = \frac{\omega\omega^*}{\operatorname{tr}(\omega\omega^*)}$. Any ω in the fibre of $b \in \Omega$, i.e. $\omega \in \pi^{-1}(\{b\})$, is called a *purification* of b and ω is called a *standard purification* if $\operatorname{tr}(\omega\omega^*) = 1$. In this case $b = \pi(\omega) = \omega\omega^*$.

Consider a smooth curve of density operators $\gamma : [0,1] \to \Omega$ and a curve $\omega : [0,1] \to \mathcal{H}^{\text{ext}}$ such that $\pi \circ \omega = \gamma$ and $\text{tr}(\omega\omega^*) = 1$, ω is called a parallel lift if ω satisfies

$$\omega^*\dot{\omega} = \dot{\omega}^*\omega. \tag{4}$$

This parallel condition can be solved by considering the solution ω of $\dot{\omega} = g\omega$ with $g = g^*$. Inserting this solution into $\gamma = \omega\omega^*$ one gets g as a solution of the equation

$$\dot{\gamma} = g\gamma + \gamma g. \tag{5}$$

Also, the Euler equation of the variational problem for

$$\inf_{\omega} \int_0^1 \langle \dot{\omega}, \dot{\omega} \rangle^{1/2} dt$$

(where the infimum is taken over all standard purifications ω of γ) gives again the parallel condition (4).

5. Uhlmann's fibration

For a fixed $a \in \mathbf{G}^+$, consider the fibre bundle $\left(\mathbf{G}^+, \mathbf{G}, \mathcal{U}^{a^{-1}}\right)$ of Section 3. As we have already seen, if $p_a : \mathbf{G} \to \mathbf{G}^+$ is the map defined by $p_a(g) = L_g a = gag^*$, $g \in \mathbf{G}$, its tangent map at 1 is $(Tp_a)_1 : A \to A_h$, $(Tp_a)_1(X) = Xa + aX^*$, so that $\ker (Tp_a)_1 = \{X \in A : aX^*a^{-1} = -X\} = A_{ah}^{a^{-1}}$ and $R((Tp_a)_1) = A_h$. The isotropy group I_a is the subgroup of $\mathbf{G}$ of a^{-1}-unitary elements of $\mathbf{G}$, $\mathcal{U}^{a^{-1}}$, i.e. $I_a = \{g \in \mathbf{G} : g^{*a} = g^{-1}\} = \mathcal{U}^{a^{-1}}$, with $(TI_a)_1 = \ker(Tp_a)_1$.

Consider the set of a-hermitian elements of A, $A_h^a = \{X \in A : X^{*a} = X\} = \{X \in A : X^*a = aX\}$. Then

$$A = A_{ah}^{a^{-1}} \oplus A_h^a,$$

because if $X \in A_{ah}^{a^{-1}} \cap A_h^a$ then $Xa^2 + a^2X = 0$, so that $X = 0$ and then $A_{ah}^{a^{-1}} \cap A_h^a = \{0\}$. Given $X \in A$ consider the unique solution X_1 of the equation

$$X_1a^2 + a^2X_1 = Xa^2 + aX^*a.$$

Then $X_1 \in A_h^a$ because, as the left-hand side of the equation is a-hermitian, X_1^{*a} is also a solution and then $X_1^{*a} = X_1$. If $X_2 = X - X_1$ then $X_2a + aX_2^* = 0$, so that $X_2 \in A_{ah}^{a^{-1}}$ and $X = X_1 + X_2$, with $X_1 \in A_h^a$ and $X_2 \in A_{ah}^{a^{-1}}$.

LEMMA 5.1. *$(Tp_a)_1\big|_{A_h^a} : A_h^a \to A_h$ is an isomorphism and the inverse map $K_a = \left((Tp_a)_1\big|_{A_h^a}\right)^{-1}$, $K_a : A_h \to A_h^a$ is given by $K_a(Y) = X$ if X is the unique solution of $Xa^2 + a^2X = Ya$.*

Proof: For $X \in A_h^a$ $(Tp_a)_1(X) = Xa + aX^* = Xa + a^2Xa^{-1}$. Then if $(Tp_a)_1(X) = 0$, $Xa + a^2Xa^{-1} = 0$ or, equivalently, $Xa^2 + a^2X = 0$, so that $X = 0$ and $(Tp_a)_1\big|_{A_h^a}$ is injective. Given $Y \in A_h$, consider the unique solution X of $Xa^2 + a^2X = Ya$, then $X \in A_h^a$ and $(Tp_a)_1(X) = Xa + a^2Xa^{-1} = Y$, and $(Tp_a)_1\big|_{A_h^a}$ is surjective. Then if $\left((Tp_a)_1\big|_{A_h^a}\right)^{-1} = K_a$, $K_a : A_h \to A_h^a$, we have that $K_a(Y) = X$ if $Xa + a^2Xa^{-1} = Y$ or X is the unique solution of $Xa^2 + a^2X = Ya$. □

We are in the position of defining a connection as follows. Denote by $V_1 = \ker(Tp_a)_1$, $H_1 = A_h^a$ and, for each $g \in \mathbf{G}$, $V_g = gV_1$, $H_g = H_1g$.

LEMMA 5.2. i) *For each $g \in \mathbf{G}$ $V_g \oplus H_g = A$.*

ii) *For all $u \in \mathcal{U}^{a^{-1}}$, $u^*H_1u^{*^{-1}} = H_1$ (or equivalently, for all $v \in \mathcal{U}^a$, $vH_1v^{-1} = H_1$).*

iii) $V_g = \ker(Tp_a)_g$.

Proof: i) If $X \in V_g \cap H_g$ for $g \in \mathbf{G}$, then Xg^{-1} is a solution of $gag^*aW + Wgag^*a = 0$. But $gag^* \in \mathbf{G}^+$ and $\sigma(gag^*a) = \sigma(a^{1/2}gag^*a^{1/2})$ lies in $\mathbb{R}^+$ (where $\sigma(c)$ denotes the spectrum of $c \in A$). Then the equation above admits the unique solution $W = 0$, so that $X = 0$.

Given $X \in A$, consider the unique solution W of $gag^*aW + Wgag^*a = Xag^*a + gaX^*a$ and $X_1 = Wg$. Then, as gag^*a and $Xag^*a + gaX^*a \in A_h^a$, $W \in A_h^a$, so that $X_1 \in H_1g = H_g$. If $X_2 = X - X_1$ then $g^{-1}X_2 \in A_{ah}^{a^{-1}} = V_1$, so $X_2 \in gV_1 = V_g$.

ii) Notice that if $v \in \mathcal{U}^a$ and $X \in H_1$ then $a^{-1}(vXv^{-1})^*a = va^{-1}X^*av^{-1} = vXv^{-1}$, as we have already seen, in the case $a = 1$, so that $vXv^{-1} \in H_1$ and $vH_1v^{-1} = H_1$. But $v \in \mathcal{U}^a$ if and only if $v^* \in \mathcal{U}^{a^{-1}}$.

iii) Differentiating at g the relation $p_a = \ell_g \circ p_a \circ \ell_{g^{-1}}$, where ℓ_g is the left multiplication by g, we get $(Tp_a)_g = (T\ell_g)_a(Tp_a)_1\ell_{g^{-1}}$ and then $\ker(Tp_a)_g = g\ker(Tp_a)_1 = gV_1 = V_g$. □

Given a differentiable curve $\gamma : [0,1] \to \mathbf{G}^+$, we look for a horizontal lift ω of γ, i.e. a curve $\omega : [0,1] \to \mathbf{G}$ such that $p_a(\omega(t)) = L_{\omega(t)}a = \gamma(t)$ and $\dot\omega(t) \in H_{\omega(t)}$, $t \in [0,1]$.

THEOREM 5.3. *For every differentiable curve $\gamma : [0,1] \to \mathbf{G}^+$ such that $\gamma(0) = b$ and any $g \in \mathbf{G}$ such that $L_ga = b$, there is a unique horizontal lift $\omega : [0,1] \to \mathbf{G}$ such that $\omega(0) = g$, namely the solution of the differential equation*

$$\begin{cases} \gamma a\dot\omega\omega^{-1} + \dot\omega\omega^{-1}\gamma a = \dot\gamma a, \\ \omega(0) = g. \end{cases} \tag{6}$$

This equation is called the transport equation associated to γ.

Proof: Suppose, first, that there exists a curve ω in $\mathbf{G}$ such that $p_a(\omega) = \gamma$ and $\dot\omega(t) \in H_{\omega(t)}$, $t \in [0, 1]$. Then $\omega a \omega^* = \gamma$ and, differentiating at each $t \in [0, 1]$, $\dot\omega a \omega^* + \omega a \dot\omega^* = \dot\gamma$ or $\dot\omega\omega^{-1}\gamma + \gamma\omega^{*^{-1}}\dot\omega^* = \dot\gamma$.

Using the fact that ω is horizontal, i.e. $\dot\omega\omega^{-1} \in H_1$ or $(\dot\omega\omega^{-1})^* = a\dot\omega\omega^{-1}a^{-1}$, we get

$$\dot\omega\omega^{-1}\gamma + \gamma a\dot\omega\omega^{-1}a^{-1} = \dot\gamma$$

or

$$\dot\omega\omega^{-1}\gamma a + \gamma a\dot\omega\omega^{-1} = \dot\gamma a.$$

If ω is a horizontal lift of γ then $\dot\omega\omega^{-1}$ is a solution of Eq. (6). Observe that $\gamma(0) = \omega(0)a\omega^*(0) = L_g a = b$.

Conversely, observe that the equation

$$X\gamma a + \gamma a X = \dot\gamma a \tag{7}$$

admits a unique solution $X(t)$ for each $t \in [0, 1]$ because for γ and $a \in \mathbf{G}^+$ it holds $\sigma(\gamma a) \subset \mathbb{R}^+$. Besides, $\dot\gamma a$ and $\gamma a \in A_h^a$ then, taking ^{*_a} in (4.6), $X^{*_a}\gamma a + \gamma a X^{*_a} = \dot\gamma a$. Hence X^{*_a} is also a solution, but then $X^{*_a} = X$. Then (7) admits a unique solution, and this solution is a-hermitian.

Now suppose that ω is a solution of (6), then $\dot\omega\omega^{-1} \in A_h^a = H_1$ or $\dot\omega \in H_1\omega = H_\omega$. An easy computation shows that $\left(\omega^{-1}\gamma\omega^{*^{-1}}\right)^{\cdot} = 0$, and thus for all $t \in [0, 1]$, $\left(\omega^{-1}\gamma\omega^{*^{-1}}\right)(t) = \omega(0)^{-1}\gamma(0)\omega^{*^{-1}}(0) = a$. Then $\gamma(t) = p_a(\omega(t))$, $t \in [0, 1]$, and ω is a lift of γ. □

REMARK 5.4. i) In order to see that (6) admits a unique solution ω observe that $X\gamma a + \gamma a X = \dot\gamma a$ admits a unique solution $X = X(t)$, $t \in [0, 1]$. Then we look for the unique solution of

$$\begin{cases} \dot\omega = X\omega, \\ \omega(0) = g, \end{cases} \qquad t \in [0, 1],$$

and it suffices to show that $\omega(t) \in \mathbf{G}$ $(t \in [0, 1])$.

ii) Consider any lift ω of γ, i.e. $\gamma = \omega a \omega^*$. Then from the first part of the proof of the theorem we see that ω is such that $\dot\omega\omega^{-1}$ is a solution of

$$\beta\gamma + \gamma\beta^* = \dot\gamma, \tag{8}$$

and the horizontal lift corresponds to the unique a-self-adjoint solution.

iii) Using ii), if $\beta\gamma = R + I$ with $R^* = R$, $I^* = -I$, then $\beta = \frac{1}{2}\dot\gamma\gamma^{-1} + I\gamma^{-1}$. The horizontal lift corresponds to $\beta = \frac{1}{2}\dot\gamma\gamma^{-1} + I\gamma^{-1}$, where I is the solution of

$$\gamma a I + I a \gamma = \frac{1}{2}(\dot\gamma a \gamma - \gamma a \dot\gamma)$$

(or $a\gamma a I + a I a\gamma = \frac{1}{2}a(\dot\gamma a\gamma - \gamma a\dot\gamma)$) which admits a unique solution of the form

$$I(t) = \frac{1}{2}a^{-1}\int_0^\infty e^{-a\gamma(t)s}a(\dot\gamma(t)a\gamma(t) - \gamma(t)a\dot\gamma(t))e^{-a\gamma(t)s}ds.$$

iv) The transport equation associated to γ, studied in [5], is

$$\begin{cases} \dot\omega = \frac{1}{2}\dot\gamma\gamma^{-1}\omega, \\ \omega(0) = g, \end{cases}$$

and corresponds to the case $I = 0$ in iii).

Every $g \in \mathbf{G}$ admits a unique decomposition as $g = \lambda u$, with λ positive and u unitary. In fact $\lambda = |g^*| = (gg^*)^{1/2}$. This decomposition is called the polar decomposition of g; λ is the positive part of g and u the unitary part.

The horizontal lift ω of γ can be also characterized by means of the unitary part of $\omega a^{1/2}$ as follows. If ω is a lift of γ, taking the positive square root of $\gamma = \omega a\omega^*$ we get $\gamma^{1/2} = |a^{1/2}\omega^*|$. Then $\omega a^{1/2} = \gamma^{1/2}u$, where u is unitary and the right-hand side is the polar decomposition of $\omega a^{1/2}$. Then for any lift ω of γ, $\omega = \gamma^{1/2}ua^{-1/2}$, where u is a unitary curve.

PROPOSITION 5.5. *Let $\gamma : [0,1] \to \mathbf{G}^+$ be a smooth curve, with $\gamma(0) = a$. If u is the solution of the differential equation*

$$\begin{cases} \gamma^{1/2}a\gamma^{1/2}\dot u u^{-1} + \dot u u^{-1}\gamma^{1/2}a\gamma^{1/2} = \left(\gamma^{1/2}\right)^{\cdot} a\gamma^{1/2} - \gamma^{1/2}a\left(\gamma^{1/2}\right)^{\cdot}, \\ u(0) = 1, \end{cases} \tag{9}$$

then $\omega = \gamma^{1/2}ua^{-1/2}$ is the horizontal lift of γ such that $\omega(0) = 1$.

Proof: A simple computation shows that if u is the solution of (9) and $\omega = \gamma^{1/2}ua^{-1/2}$, then $\dot\omega\omega^{-1}$ is a solution of (1) with $\omega(0) = 1$. □

Consider Y as a tangent field in a neighbourhood U_a of a and $X \in (T\mathbf{G}^+)_a = A_h$. Let $\gamma : [0,1] \to \mathbf{G}^+$ be a smooth curve with $\gamma(0) = a$, $\dot\gamma(0) = X$ and ω the horizontal lift of γ with $\omega(0) = 1$. For each $t \in [0,1]$ consider $L_{\omega(t)} : \mathbf{G}^+ \to \mathbf{G}^+$, $L_{\omega(t)}b = \omega(t)b\omega(t)^*$, $b \in \mathbf{G}^+$, and its tangent map at a,

$$(TL_{\omega(t)})_a : (T\mathbf{G}^+)_a \to (T\mathbf{G}^+)_{\gamma(t)}$$

which is invertible because $L_{\omega(t)}$ is a diffeomorphism. Then if

$$Y(t) = (TL_{\omega(t)})_a^{-1}Y_{\gamma(t)},$$

$Y(t) \in (T\mathbf{G}^+)_a$, for $t \in [0,1]$. Define the covariant derivative of the field Y in the direction X as

$$D_X(Y) = \frac{d}{dt}\left(TL_{\omega(t)}\right)_a^{-1} Y_{\gamma(t)}\Big|_{t=0}.$$

Observe that $D_X(Y) \in (T\mathbf{G}^+)_a$. In order to show that the definition of $D_X(Y)$ does not depend on the curve γ let us compute $K_a(D_X Y)$. Differentiating at $g = 1$ the equality $L_g \circ p \circ \mathrm{Aut}_{g^{-1}} = p_{gag^*}$, where $p = p_a$, $g \in \mathbf{G}$, we get

$$\left(TL_g\right)_a (Tp)_1 \,\mathrm{Aut}_{g^{-1}} = \left(Tp_{gag^*}\right)_1 .$$

Now observe that $A_h^a = \mathrm{Aut}_{g^{-1}}\left(A_h^{g^{*^{-1}}ag^{-1}}\right)$, and that

$$\left(Tp_{gag^*}\right)_1 \Big|_{A_h^{g^{*^{-1}}ag^{-1}}} : A_h^{g^{*^{-1}}ag^{-1}} \to A_h$$

is an isomorphism. In fact, for $Y \in A_h^{g^{*^{-1}}ag^{-1}}$,

$$Z = \left(Tp_{gag^*}\right)_1 (Y) = Ygag^* + ga^2g^{-1}Yga^{-1}g^*,$$

or equivalently

$$Zg^{*^{-1}}ag^{-1} = Yga^2g^{-1} + ga^2g^{-1}Y.$$

If $\left(Tp_{gag^*}\right)_1 (Y) = 0$ then Y is a solution of $Yga^2g^{-1} + ga^2g^{-1}Y = 0$, but this equation admits $Y = 0$ as the unique solution because $\sigma(ga^2g^{-1}) = \sigma(a^2) \subset \mathbb{R}^+$, so that $(Tp_{gag^*})_1$ is injective. Given $Z = Z^*$, consider the unique solution Y of $Yga^2g^{-1} + ga^2g^{-1}Y = Zg^{*^{-1}}ag^{-1}$, then $Y \in A_h^{g^{*^{-1}}ag^{-1}}$ and $\left(Tp_{gag^*}\right)_1 (Y) = Z$ because $Zg^{*^{-1}}ag^{-1}$ and $ga^2g^{-1} \in A_h^{g^{*^{-1}}ag^{-1}}$. Define

$$\widetilde{K}_g : A_h \to A_h^{g^{*^{-1}}ag^{-1}}, \quad \widetilde{K}_g = \left(\left(Tp_{gag^*}\right)_1 \Big|_{A_h^{g^{*^{-1}}ag^{-1}}}\right)^{-1},$$

$\widetilde{K}_g(Z) = Y$ if and only if Y is the unique solution of $Yga^2g^{-1} + ga^2g^{-1}Y = Zg^{*^{-1}}ag^{-1}$.

Using these two facts in the last equality we get

$$\left(TL_g\right)_a (Tp)_1 \,\mathrm{Aut}_{g^{-1}} \Big|_{A_h^{g^{*^{-1}}ag^{-1}}} = \left(Tp_{gag^*}\right)_1 \Big|_{A_h^{g^{*^{-1}}ag^{-1}}}$$

and taking the inverses we get $\mathrm{Aut}_g K_a (TL_g)_a^{-1} = \widetilde{K}_g$.

If for each t, $g(t) = \omega(t)$ where ω is the horizontal lift of γ, then we get

$$\mathrm{Aut}_{\omega(t)} K_a \left(TL_{\omega(t)}\right)_a^{-1} (Y_{\gamma(t)}) = \widetilde{K}_{\omega(t)}(Y_{\gamma(t)})$$

or

$$K_a \left(TL_{\omega(t)}\right)_a^{-1} \left(Y_{\gamma(t)}\right) = \omega(t)^{-1}\widetilde{K}_{\omega(t)} \left(Y_{\gamma(t)}\right) \omega(t),$$

and differentiating at $t=0$,

$$K_a(D_XY) = \frac{d}{dt}\left(\omega^{-1}(t)\widetilde{K}_{\omega(t)}\left(Y_{\gamma(t)}\right)\omega(t)\right)\Big|_{t=0}.$$

Using $\omega(0)=1$ we get

$$K_a(D_XY) = -\dot{\omega}(0)\widetilde{K}_{\omega(0)}\left(Y_{\gamma(0)}\right) + \widetilde{K}_{\omega(0)}\left(Y_{\gamma(0)}\right)\dot{\omega}(0) + \frac{d}{dt}\widetilde{K}_{\omega(t)}\left(Y_{\gamma(t)}\right)\Big|_{t=0}.$$

Now, from $\gamma(0)=a$, $\dot{\gamma}(0)=X$ and ω the horizontal lift of γ with $\omega(0)=1$, $\dot{\omega}(0)$ is the unique solution of $Za+aZ^*=X$ and $\dot{\omega}(0)\omega^{-1}(0)=\dot{\omega}(0)\in A_h^a$. Then $\dot{\omega}(0)$ is the unique solution of $Za^2+a^2Z=Xa$ and depends only on X and a, in fact, $\dot{\omega}(0)=K_a(X)$, $\widetilde{K}_{\omega(0)}\left(Y_{\gamma(0)}\right)=\widetilde{K}_1(Y_a)$ but $\widetilde{K}_1(Y_a)$ is the unique solution of $Ya^2+a^2Y=Y_aa$, so that $\widetilde{K}_1(Y_a)=Y=K_a(Y_a)$.

We obtain

$$K_a(D_XY) = K_a(Y_a)K_a(X) - K_a(X)K_a(Y_a) + \frac{d}{dt}\widetilde{K}_{\omega(t)}\left(Y_{\gamma(t)}\right)\Big|_{t=0}$$

or

$$K_a(D_XY) = [K_a(Y_a), K_a(X)] + \frac{d}{dt}\widetilde{K}_{\omega(t)}\left(Y_{\gamma(t)}\right)\Big|_{t=0},$$

and for a fixed the last expression depends only on a and X.

We define the covariant derivative of a tangent field Y_t along a curve $\gamma\subset \mathbf{G}^+$, $\frac{D}{dt}Y$, as

$$K_\gamma\left(\frac{D}{dt}Y\right) = \left[K_\gamma(Y_t), K_\gamma(\dot{\gamma})\right] + \frac{d}{dt}\widetilde{K}_\omega(Y_t),$$

where ω is the horizontal lift of γ, with $\omega(0)=1$. γ is called a *geodesic* if $\dot{\gamma}$ is parallel along γ, i.e., if $K_\gamma\left(\frac{D}{dt}\dot{\gamma}\right)=0$.

In this case, if ω is the horizontal lift of γ with $\omega(0)=1$ and $\gamma(0)=a$, $\dot{\gamma}(0)=X$ then $0=\frac{d}{dt}\widetilde{K}_{\omega(t)}(\dot{\gamma}(t))$ so that $\widetilde{K}_{\omega(t)}(\dot{\gamma}(t))=\widetilde{K}_{\omega(0)}(\dot{\gamma}(0))=\widetilde{K}_1(X)=K_a(X)=Z$, $\forall t$, or equivalently, ω is a solution of the differential equation

$$\begin{cases}\omega^{-1}\dot{\omega}a^2 + a\dot{\omega}^*{\omega^*}^{-1}a = \omega^{-1}Z\omega a^2 + a^2\omega^{-1}Z\omega,\\ \omega(0)=1.\end{cases} \tag{10}$$

Also, as $Z=K_a(X)=\widetilde{K}_{\omega(t)}(\dot{\gamma}(t))$, $\forall t$, we observe that $Z\in A_h^{\omega^*a\omega^{-1}}$ $\forall t$, so

$$Z\omega a\omega^* + \omega a^2\omega^{-1}Z\omega a^{-1}\omega^* = \dot{\gamma}$$

or

$$Z\gamma + \gamma{\omega^*}^{-1}a\omega^{-1}Z\omega a^{-1}\omega^* = \dot{\gamma}$$

and then, as $Z\in A_h^{{\omega^*}^{-1}a\omega^{-1}}$, $Z\gamma+\gamma Z^*=\dot{\gamma}$, and using the fact that $Z\in A_h^a$ we obtain

$$Z\gamma a + \gamma a Z = \dot{\gamma}a. \tag{11}$$

Then γ is a geodesic if γ is a solution of (11), $\gamma(0) = a$ and $Z \in A_h^a$ is the solution of $Xa = Za^2 + a^2 Z$, for a given $X \in A_h$. (Observe that $\dot{\gamma}(0) = X$.)

THEOREM 5.6. *Given $X \in A_h$, if Z is the solution of $Za^2 + a^2 Z = Xa$ then $\gamma(t) = e^{Zt} a e^{Z^* t}$ is the unique geodesic such that $\gamma(0) = a$ and $\dot{\gamma}(0) = X$.*

Proof: A simple computation shows that γ is a solution of (11) with $\gamma(0) = a$ and $\dot{\gamma}(0) = X$. Observe that if ω is the horizontal lift of a solution β of (11) with $\omega(0) = 1$ then $\dot{\omega}\omega^{-1}$ is a solution of $\dot{\beta} a = \dot{\omega}\omega^{-1}\beta a + \beta a \dot{\omega}\omega^{-1}$ and β is a solution of (4.13). Thus $\dot{\beta} a = Z\beta a + \beta a Z$ so that $0 = \left(\dot{\omega}\omega^{-1} - Z\right)\beta a + \beta a \left(\dot{\omega}\omega^{-1} - Z\right)$ and

$$\begin{cases} \dot{\omega}\omega^{-1} = Z, \\ \omega(0) = 1, \end{cases}$$

hence $\omega = e^{Zt}$ and $\beta = \omega a \omega^* = e^{Zt} a e^{Z^* t} = \gamma$. □

REMARK 5.7. i) For every $a \in \mathbf{G}^+$ and $X \in A_h$ there exists a unique geodesic γ, with $\gamma(0) = a$ and $\dot{\gamma}(0) = X$.

ii) For every $a \in \mathbf{G}^+$ and every $b \in \mathbf{G}^+$ of the form $b = e^Z a e^{Z^*}$, with $Z \in A_h^a$, there exists a geodesic γ joining a and b, namely

$$\gamma(t) = \left(b^{1/2} u a^{-1/2}\right)^t a \left(a^{-1/2} u^{-1} b^{1/2}\right)^t,$$

where $u \in \mathcal{U}$ is such that if $b = gag^*$, $g \in \mathbf{G}_h^a$, then $g = b^{1/2} u a^{-1/2}$.

iii) If $a = 1$, there is a unique geodesic joining 1 and b, for every $b \in \mathbf{G}^+$, namely $\gamma(t) = b^t$.

6. Comparison between both approaches

i) In Sections 3 and 5 the fibre bundles $\left(\mathbf{G}, \mathbf{G}^+, \mathcal{U}^{a^{-1}}\right)$ have been studied with different connections in each case. In Section 3, we have defined the connection by $V_1 = A_{ah}^{a^{-1}}$, the set of A of a^{-1}-antihermitian elements of a, and $H_1 = A_h^{a^{-1}} = \ker(Tp_a)_1$, the set of a^{-1}-hermitian elements of A, and the corresponding horizontal and vertical spaces at each $g \in \mathbf{G}$ as

$$H_g = gH_1, \quad V_g = gV_1 = \ker (Tp_a)_g .$$

In Section 5 we have considered, for the same vertical space at 1, $V_1 = \ker(Tp_a)_1 = A_{ah}^{a^{-1}}$, a different complement in A, namely $H_1 = A_h^a$, the set of a-hermitian elements of A with the corresponding vertical and horizontal spaces at each g,

$$H_g = H_1 g, \quad V_g = gV_1 = \ker(Tp_a)_g.$$

These different connections give rise to different horizontal lifts of a given curve γ in $\mathbf{G}^+$, which have been compared in Section 4 (see Remarks), and also different covariant derivatives.

The geodesic γ with $\gamma(0) = a$ and $\dot\gamma(0) = X$ is, in the first case, the curve $\gamma(t) = a^{1/2}e^{a^{-1/2}Xa^{-1/2}t}a^{1/2}$, and, in the second case, $\gamma(t) = e^{Zt}ae^{Z^*t}$, where Z is the solution to $Xa = Za^2 + a^2Z$.

ii) Observe that, taking $a = 1$ in Section 4, we have obtained Uhlmann's equations, described also in Section 4. More precisely, for $a = 1$ we have $p_1(g) = gg^*$. In this case $(Tp_1)_1(X) = X+X^*$ and $\ker(Tp_1)_1 = A_{ah} = V_1$; $H_1 = A_h$. The isotropy group is the unitary group. The connection is given by $H_g = A_h g$ and $V_g = gA_{ah}$. Also the transport equation for a given curve γ in $\mathbf{G}^+$ is

$$\begin{cases} \gamma\dot\omega\omega^{-1} + \dot\omega\omega^{-1}\gamma = \dot\gamma, \\ \omega(0) = g, \end{cases} \tag{12}$$

which is equivalent to Uhlmann's equation (5).

Observe that if ω is a lift of γ which is a solution of (12) then ω satisfies the parallel condition, because in this case ω is horizontal and $\dot\omega\omega^{-1} \in A_h$ or equivalently $(\dot\omega\omega^{-1})^* = \dot\omega\omega^{-1}$, or $\dot\omega^*\omega = \omega^*\dot\omega$.

The equation for the unitary part of a horizontal lift ω of γ is, in this case,

$$\gamma\dot u u^{-1} + \dot u u^{-1}\gamma = (\gamma^{1/2})^{\cdot}\gamma^{1/2} - \gamma^{1/2}(\gamma^{1/2})^{\cdot}.$$

Finally, the unique geodesic joining 1 with $b \in \mathbf{G}^+$ is $\gamma(t) = b^t$ so that geodesics with origin 1 coincide in both cases.

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