

DIFFERENTIAL GEOMETRY ON THOMPSON'S COMPONENTS OF POSITIVE OPERATORS

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Consider the algebra $L(H)$ of bounded linear operators on a Hilbert space H , and let $L(H)^+$ be the set of positive elements of $L(H)$. For each $A \in L(H)^+$ we study differential geometry of the Thompson component of A , $C_A = \{B \in L(H)^+ : A \leq rB \text{ and } B \leq sA \text{ for some } s, r > 0\}$. The set of components is parametrized by means of all operator ranges of H . Each C_A is a differential manifold modelled in an appropriate Banach space and a homogeneous space with a natural connection. Moreover, given arbitrary $B, C \in C_A$, there exists a unique geodesic with endpoints B and C . Finally, we introduce a Finsler metric on C_A for which the geodesics are short and we show that it coincides with the so-called Thompson metric.

1. Introduction

It is the aim of this paper to study, from a differential geometric view point, the space $L(H)^+$ of all positive bounded operators on a Hilbert space H . More precisely, there is a natural partition of $L(H)^+$ in components C (the so-called Thompson's components of $L(H)^+$) and we prove that each C is (naturally) a differentiable submanifold of a suitable space of operators. One particular component is the set G^+ of all positive invertible operators on H . The geometry of G^+ is well known (see [3–5] and the references therein). Here we present a method which allows to study every component of $L(H)^+$.

The relevance of the search of geometrical structure of parts of $L(H)^+$ in mathematical physics has been emphasized by Uhlmann [19–21], Dąbrowski and Jadczyk [7], Dąbrowski and Grosse [6], Dittmann and Rudolph [8], Petz [15], Petz and Sudar [16] and by many others. Uhlmann provided a geometric method for studying

the so-called *geometric phase* (or *Berry phase* [2]). The reader will find in [20, 21] a nice exposition of the geometric ideas of the parallel transport for the Berry phase.

In Uhlmann's explanation of the geometrical phase, a central role is played by the set of generalized density operators, i.e. positive operators of the trace class. The main difficulty we find in studying these operators is that they have, in general, nonclosed range. The geometry of the space of positive operators with closed range has been studied in [4], where the first attempt of using Thompson's partition in components was made.

Also in Petz and Sudar's study of quantum systems [15, 16] there is evidence of the difficulty mentioned above, so they restricted the study to *finite* quantum systems, which in turn led them to consider only finite-dimensional Hilbert spaces and the so-called closed range operators.

The partitions of $\mathbf{L}(\mathbf{H})^+$ in components and the natural differentiable structure of each component is not a complete solution of the difficulties mentioned before. In fact, many natural curves are not contained in any component: if A has nonclosed range, each A^t belongs to a different component! However, in problems where the range of the considered operators remains constant, the results of the paper may be useful.

Let us describe the contents of the work. Section 2 contains description of the components of $\mathbf{L}(\mathbf{H})^+$. This result is essentially a consequence of a well-known theorem of R. G. Douglas [11]. One of the many characterizations of the components is the following: $A, B \in \mathbf{L}(\mathbf{H})^+$ belong to the same component if and only if $A^{1/2}$ and $B^{1/2}$ have the same range. For injective operators this is in turn equivalent to the boundedness of the operators $A^{1/2}B^{-1/2}$ and $B^{1/2}A^{-1/2}$ over their (common) domain. This description allows a parametrization of the set of all components of $\mathbf{L}(\mathbf{H})^+$ by means of the set of all *operator ranges* of \mathbf{H} : a subspace \mathbf{S} of \mathbf{H} is called an *operator range* if there is a bounded operator on \mathbf{H} with \mathbf{S} as its range. These subspaces are called "variétés de Julia" by J. Dixmier, who found many characteristics of them [9, 10]. Fillmore and Williams [12] give a very readable version of many of Dixmier's results. Using the results of this section we show that the means of operators, introduced by Pusz and Woronowicz [17], Anderson and Trapp [1], and Kubo and Ando [13] are consistent with components: if A, B belong to the component \mathbf{C} then the mean $A\sigma B$ belongs to \mathbf{C} , too. This is particularly useful for the geometrical mean $\#$ of [17], because $A\#B$ is the operator which rules the geodesic from A to B (see details in Section 4). In Section 3 we study different Hilbert spaces constructed from a single positive operator A . Typically, one defines an inner product $\langle \cdot, \cdot \rangle_A$ on \mathbf{H} by $\langle x, y \rangle_A = \langle Ax, y \rangle$ and considers the completion \mathbf{H}_A of $(\mathbf{H}, \langle \cdot, \cdot \rangle_A)$. The unique extension of $A^{1/2}$ to \mathbf{H}_A is an isometric isomorphism onto \mathbf{H} . On the other hand, $\mathbf{H}' = (\mathbf{R}(A^{1/2}), \langle \cdot, \cdot \rangle_{A^{-1}})$ is complete. It is shown that $\mathbf{L}(\mathbf{H}_A, \mathbf{H})$ is in a one-to-one correspondence to the left ideal $\mathbf{L}(\mathbf{H})A^{1/2}$, and $\mathbf{L}(\mathbf{H}, \mathbf{H}')$ to the right ideal $A^{1/2}\mathbf{L}(\mathbf{H})$. For noninjective operators the constructions are more involved. These operator spaces are used in Section 4 to study each component \mathbf{C}_A as a homogeneous space of the

group $\mathbf{GL}(\mathbf{H}')$ of all invertible operators on the space \mathbf{H}' . An isometric isomorphism $\mathbf{L}(\mathbf{H}) \rightarrow \mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ provides a way of defining a differentiable structure on \mathbf{C}_A , modelled in the Banach space of "Hermitian" elements of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$. The action of $\mathbf{GL}(\mathbf{H}')$ over \mathbf{C}_A provides a fibration $\mathbf{GL}(\mathbf{H}') \rightarrow \mathbf{C}_A$ with a principal connection which induces a linear connection on the tangent bundle \mathcal{TC}_A . We show how to lift curves in \mathbf{C}_A by means of the transport equation and characterize the geodesics of the connection. These results generalize the corresponding results on the component \mathbf{G}^+ . Finally, we introduce a Finsler metric on \mathbf{C}_A and show that it coincides with the so-called Thompson metric (or part metric), a complete study of which can be found in [14].

2. Thompson's components of $\mathbf{L}(\mathbf{H})^+$

Let K be a closed convex cone of a real Banach space which is normal in the sense that there exists a constant $r > 0$ such that $\|x\| \leq r\|y\|$ if $0 \leq x \leq y$ (here $x \leq y$ means that $y - x \in K$). Consider in K the following equivalence relation: $x \sim y$ if there exist constants $r, s > 0$ such that $x \leq ry$ and $y \leq sx$. A *component* is an equivalence class. Thompson [18] proved that

$$d_T(x, y) = \log \max\{\inf\{r > 0 : x \leq ry\}, \inf\{s > 0 : y \leq sx\}\}$$

defines a complete distance on each component of K . The reader will find in Nussbaum [14] an excellent exposition on this notion and its multiple applications.

In this section we characterize the components of the normal closed convex cone $\mathbf{L}(\mathbf{H})^+$ of the real Banach space $\mathbf{L}(\mathbf{H})_h = \{X \in \mathbf{L}(\mathbf{H}) : X^* = X\}$. As a matter of fact, this section is a set of variations on the following result of [11]:

THEOREM 2.1. *If $A, B \in \mathbf{L}(\mathbf{H})$ the following conditions are equivalent:*

- (1) $\mathbf{R}(A) \subset \mathbf{R}(B)$,
- (2) *there exists $r > 0$ such that $AA^* \leq rBB^*$,*
- (3) *there exists $C \in \mathbf{L}(\mathbf{H})$ such that $A = BC$.*

If one of these conditions holds, then there exists a unique $C \in \mathbf{L}(\mathbf{H})$ such that

- (a) $\|C\|^2 = \inf\{r > 0 : AA^* \leq rBB^*\}$,
- (b) $\ker C = \ker A$,
- (c) $\mathbf{R}(C) \subset \overline{\mathbf{R}(B^*)}$.

COROLLARY 2.2. *If $A, B \in \mathbf{L}(\mathbf{H})$, then $\mathbf{R}(A) = \mathbf{R}(B)$ and $\dim \ker A = \dim \ker B$ if and only if there exists $C \in \mathbf{GL}(\mathbf{H})$ such that $A = BC$. In particular, positive operators A, B have the same range if and only if there exists $C \in \mathbf{GL}(\mathbf{H})$ such that $A = BC$.*

The reader is referred to [12] for a survey on these subjects and related matters.

We proceed now to characterize Thompson's components of $\mathbf{L}(\mathbf{H})^+$. In order to do this, we first consider the case of injective operators.

THEOREM 2.3. *Let \mathbf{C}_A be the component of an injective operator $A \in \mathbf{L}(\mathbf{H})^+$. For $B \in \mathbf{L}(\mathbf{H})^+$ the following conditions are equivalent:*

- (i) $B \in \mathbf{C}_A$,
- (ii) $\mathbf{R}(A^{1/2}) = \mathbf{R}(B^{1/2})$,
- (iii) *there exists a unique $V \in \mathbf{GL}(\mathbf{H})$ such that $B^{1/2} = A^{1/2}V$,*
- (iv) *there exists a unique $P \in \mathbf{GL}(\mathbf{H})^+$ such that $B = A^{1/2}PA^{1/2}$,*
- (v) $A^{-1/2}BA^{-1/2}$ *is a bounded operator on $\mathbf{R}(A^{1/2})$ and it has a unique extension to \mathbf{H} which is invertible.*

Proof: The equivalence between (i) and (ii) follows from Douglas' theorem. Its corollary shows the equivalence between (ii) and (iii). The uniqueness of V follows from the injectivity of A .

(iii) \Rightarrow (iv) If $B^{1/2} = A^{1/2}V$ for some $V \in \mathbf{GL}(\mathbf{H})$, then $B = A^{1/2}VV^*A^{1/2}$ and $P = VV^* \in \mathbf{GL}(\mathbf{H})^+$ is uniquely determined because A is injective.

(iv) \Rightarrow (v) If $B = A^{1/2}PA^{1/2}$ for some $P \in \mathbf{GL}(\mathbf{H})^+$, then $A^{-1/2}BA^{-1/2}$ is a bounded linear operator on $\mathbf{R}(A^{1/2})$, so that $A^{-1/2}BA^{-1/2}$ is a bounded linear operator on $\mathbf{R}(A^{1/2})$ which admits a unique extension to \mathbf{H} , namely P , which is invertible.

(v) \Rightarrow (i) The positive operator $A^{-1/2}BA^{-1/2}$ on $\mathbf{R}(A^{1/2})$ satisfies $\alpha I \leq A^{-1/2}BA^{-1/2} \leq \beta I$, where

$$\alpha = \inf\{\|A^{-1/2}BA^{-1/2}y\| : y \in \mathbf{R}(A^{1/2}), \|y\| = 1\}$$

and

$$\beta = \sup\{\|A^{-1/2}BA^{-1/2}y\| : y \in \mathbf{R}(A^{1/2}), \|y\| = 1\}.$$

This shows $\alpha A \leq B \leq \beta A$, and then $B \in \mathbf{C}_A$.

Observe that $B \in \mathbf{C}_A$ if and only if $B^{1/2}A^{-1/2}$ is a bounded operator on $\mathbf{R}(A^{1/2})$ (which can be extended to \mathbf{H}).

The general case is considered in the following result.

THEOREM 2.4. *For $A, B \in \mathbf{L}(\mathbf{H})^+$ the following conditions are equivalent:*

- (i) $B \in \mathbf{C}_A$,
- (ii) $\mathbf{R}(A^{1/2}) = \mathbf{R}(B^{1/2})$,
- (iii) *there exists $V \in \mathbf{GL}(\mathbf{H})$ such that $B^{1/2} = A^{1/2}V$,*
- (iv) *there exists $P \in \mathbf{GL}(\mathbf{H})^+$ such that $B = A^{1/2}PA^{1/2}$,*
- (v) $\mathbf{R}(B) \subset \mathbf{R}(A^{1/2}) \subset \overline{\mathbf{R}(B)}$ *and*

$$\left(A^{1/2}\Big|_M\right)^{-1} B \left(A^{1/2}\Big|_M\right)^{-1} : \mathbf{R}(A^{1/2}) \rightarrow \mathbf{R}(A^{1/2})$$

extends to a positive invertible operator in $\mathbf{L}(\mathbf{H})$, where $M = \overline{\mathbf{R}(A^{1/2})}$.

Proof: The equivalence (i) \Leftrightarrow (ii) \Leftrightarrow (iii) can be proved as in Theorem 2.3, by observing that the kernels of two positive operators with the same range must coincide.

(iii) \Rightarrow (iv) If $B^{1/2} = A^{1/2}V$ for some $V \in \mathbf{GL}(\mathbf{H})$, then $B = A^{1/2}VV^*A^{1/2}$ and $P = VV^* \in \mathbf{GL}(\mathbf{H})^+$.

(iv) \Rightarrow (v) If $B = A^{1/2}PA^{1/2}$ for some $P \in \mathbf{GL}(\mathbf{H})^+$, then $\mathbf{R}(B) \subset \mathbf{R}(A^{1/2})$ and $\ker B = \ker A^{1/2}$.

Therefore, $\overline{\mathbf{R}(B)} = (\ker B)^\perp = (\ker A^{1/2})^\perp = M$ and $\mathbf{R}(B) \subset \mathbf{R}(A^{1/2}) \subset M = \overline{\mathbf{R}(B)}$. On the other hand, $B|_M : M \rightarrow M$ is injective.

If Q denotes the orthogonal projection onto M , then $B = A^{1/2}QPQ A^{1/2}$. Let us abbreviate $P_1 = QPQ$. Then $P_1 \geq 0$ and $P_1|_M : M \rightarrow M$ is invertible: in fact, $\langle Px, x \rangle = \|P^{1/2}x\| \geq c\|x\|$, $\forall x \in \mathbf{H}$, for some $c > 0$; then

$$\langle P_1x, x \rangle = \langle PQx, Qx \rangle \geq c\|x\| \quad \forall x \in M,$$

and $P_1|_M$ is positive and bounded from below.

Then $B|_M = A^{1/2}P_1A^{1/2}|_M = A^{1/2}|_M P_1|_M A^{1/2}|_M$ which proves that

$$P_1 = \left(A^{1/2}|_M\right)^{-1} B \left(A^{1/2}|_M\right)^{-1} \quad \text{in } \mathbf{R}(A^{1/2}),$$

which admits an obvious bounded extension to M . This implies (v).

(v) \Rightarrow (i) If $P_1 : \mathbf{R}(A^{1/2}) \rightarrow \mathbf{R}(A^{1/2})$ extends to a positive invertible operator on \mathbf{H} , then in particular there exist $\alpha, \beta > 0$ such that

$$\alpha I \leq P_1 \leq \beta I \quad \text{in } \mathbf{R}(A^{1/2})$$

or, which is the same,

$$\alpha A|_M \leq B|_M \leq \beta A|_M.$$

But $\mathbf{R}(B) \subset \mathbf{R}(A^{1/2}) \subset \overline{\mathbf{R}(B)}$ and then $\ker A = \ker B = M^\perp$. This shows that

$$\alpha A \leq B \leq \beta A$$

and $B \in \mathbf{C}_A$.

REMARK 2.5. It should be noticed that the invertible operator V of the part (iii) is not unique, as it is in the injective case. However, $V(\ker A^{1/2}) = \ker A^{1/2}$ and $V(M) = M$. Then condition (iii) is equivalent to

$$(iii)' \quad \left(A^{1/2}|_M\right)^{-1} B^{1/2}|_M = V|_M \in \mathbf{GL}(M).$$

COROLLARY 2.6. *For every $A \in \mathbf{L}(\mathbf{H})^+$ the component \mathbf{C}_A coincides with the set $A^{1/2}\mathbf{GL}(\mathbf{H})^+A^{1/2}$. In particular, \mathbf{C}_A is contained in the trace class ideal if A is.*

As a consequence of Theorem 2.4 we obtain a parametrization of the set of components $\{\mathbf{C}_A : A \in \mathbf{L}(\mathbf{H})^+\}$ by means of the set of operator ranges of \mathbf{H} , i.e. subspaces \mathbf{S} of \mathbf{H} such that there exists a bounded linear operator $C \in \mathbf{L}(\mathbf{H})$ with $\mathbf{R}(C) = \mathbf{S}$. These subspaces have been studied by Dixmier [9, 10] under the name of “variétés de Julia”. The reader will find in [12] a modern treatment including simplified proofs of Dixmier’s results. Using the polar decomposition it can be proved that a subspace \mathbf{S} of \mathbf{H} is an operator range if and only if there exists $A \in \mathbf{L}(\mathbf{H})^+$ such that $\mathbf{R}(A) = \mathbf{S}$. Thus, there exist as many Thompson’s components of $\mathbf{L}(\mathbf{H})^+$ as the operator ranges of \mathbf{H} .

The next result shows that, for $A \in \mathbf{L}(\mathbf{H})^+$, the values of the curve $t \mapsto A^t$ lie in the same component if and only if $\mathbf{R}(A)$ is closed. Moreover, if $\mathbf{R}(A)$ is not closed, then each A^t lies in a different component.

THEOREM 2.7. *For a positive operator $A \in \mathbf{L}(\mathbf{H})^+$ the following alternative holds:*

- (1) *either $\mathbf{R}(A)$ is closed and then $\mathbf{R}(A^t) = \mathbf{R}(A)$ for every $t \in [0, 1]$*
- (2) *or $\mathbf{R}(A)$ is not closed and then $\mathbf{R}(A^t)$ is closed for $t = 0$, not closed for every $t \in (0, 1]$ and*

$$\mathbf{R}(A) \subset \mathbf{R}(A^t) \subsetneq \mathbf{R}(A^s) \subset \overline{\mathbf{R}(A)}$$

for $0 \leq s < t \leq 1$.

Proof: a) *First case: A is injective.* If $A \in \mathbf{L}(\mathbf{H})^+$ is injective, then $\mathbf{R}(A)^\perp = \ker A = 0$ so that $\mathbf{R}(A)$ is dense in \mathbf{H} . Then, the alternative reads:

- (1) *either $\mathbf{R}(A) = \mathbf{H}$ and then $\mathbf{R}(A^t) = \mathbf{H}$ for every $t \in [0, 1]$,*
- (2) *or $\mathbf{R}(A) \neq \mathbf{H}$ and then $\mathbf{R}(A^t) \neq \mathbf{H}$ for all $t \in [0, 1]$ and*

$$\mathbf{R}(A) \subsetneq \mathbf{R}(A^t) \subsetneq \mathbf{R}(A^s) \subsetneq \mathbf{H}$$

for $0 < s < t < 1$.

Since A is injective the following conditions are all equivalent:

- (i) $\mathbf{R}(A) = \mathbf{H}$,
- (ii) $A \in \mathbf{GL}(\mathbf{H})^+$,
- (iii) there exists $t \in (0, 1)$ such that $\mathbf{R}(A^t) = \mathbf{H}$,
- (iv) there exists $t \in (0, 1)$ such that $A^t \in \mathbf{GL}(\mathbf{H})$.

Observe also that, for $0 < t < 1$, we can factorize $A = A^t A^{1-t}$ and, more generally, if $0 < s < t < 1$,

$$A^t = A^s A^{t-s}.$$

In particular, $\mathbf{R}(A) \subset \mathbf{R}(A^t) \subset \mathbf{R}(A^s)$.

To complete the proof we must show that conditions (i) to (iv) are equivalent to

- (v) there exist t and h such that $0 \leq t < t+h \leq 1$ and $\mathbf{R}(A^t) = \mathbf{R}(A^{t+h})$.

Suppose that (v) holds. By Corollary 2.2 there exists $V \in \mathbf{GL}(\mathbf{H})$ such that $A^{t+h} = A^t V$. Since A^t is injective, we get $A^h = V$ which means $A^h \in \mathbf{GL}(\mathbf{H})^+$.

The factorization $A = A^{1/2} A^{1/2}$ shows that (i) \Rightarrow (v) obviously.

Let us show that the alternative is proven. If $\mathbf{R}(A)$ is not closed and $t \in (0, 1)$ then $\mathbf{R}(A^t)$ is not closed because (iii) \Rightarrow (i). That the inclusions are all proper follows from (v) \Rightarrow (i).

b) *General case: A is not necessarily injective.* Observe that $\ker A^t = \ker A$ for all t , so that $\overline{\mathbf{R}(A^t)} = (\ker A^t)^\perp = (\ker A)^\perp = \overline{\mathbf{R}(A)}$. In particular, $\mathbf{R}(A) \subset \mathbf{R}(A^t) \subset \overline{\mathbf{R}(A)}$. Thus, if $\mathbf{R}(A)$ is closed, then $\mathbf{R}(A^t) = \mathbf{R}(A)$ is closed for all t .

Suppose that $\mathbf{R}(A)$ is not closed and $\mathbf{R}(A^t) = \mathbf{R}(A^{t+h})$ for some $0 \leq t < t+h \leq 1$.

Again by Corollary 2.2 there exists an invertible V such that $A^{t+h} = A^t V$. It is easy to see that $V(\ker A) \subset \ker A$ and $V(\mathbf{M}) \subset \mathbf{M}$, where $\mathbf{M} = \overline{\mathbf{R}(A)}$. Then we can treat $V|_{\mathbf{M}}$ as an invertible operator on \mathbf{M} and write $A^{t+h}|_{\mathbf{M}} = (A^t|_{\mathbf{M}})(V|_{\mathbf{M}})$.

Then $A^h|_{\mathbf{M}} = V|_{\mathbf{M}}$, so $\mathbf{R}(A^h|_{\mathbf{M}})$ is closed. But $\mathbf{R}(A^h|_{\mathbf{M}}) = \mathbf{R}(A^h)$, so that $A^h|_{\overline{\mathbf{R}(A)}} \in \mathbf{GL}(\mathbf{M})^+$ and then $A|_{\overline{\mathbf{R}(A)}} \in \mathbf{GL}(\mathbf{M})^+$ by raising to the $\frac{1}{h}$ -th power. In particular, $\mathbf{R}(A) = \mathbf{R}(A|_{\mathbf{M}})$ is closed. \square

COROLLARY 2.8. *If $A \in \mathbf{L}(\mathbf{H})^+$ then $\mathbf{R}(A)$ is closed and then the curve $\gamma(t) = A^t$ lies in the Thompson component of A ; or $\mathbf{R}(A)$ is not closed, and then each A^t lies in a different component.*

3. Hilbert spaces associated to a positive operator

Each injective positive operator A defines a scalar product on \mathbf{H} by

$$\langle x, y \rangle_A = \langle Ax, y \rangle \quad (x, y \in \mathbf{H}),$$

and a norm $\| \cdot \|_A$ by

$$\|x\|_A = \langle x, x \rangle_A^{1/2} = \|A^{1/2}x\| \quad (x \in \mathbf{H}).$$

Thus $A^{1/2} : (\mathbf{H}, \| \cdot \|_A) \rightarrow (\mathbf{H}, \| \cdot \|)$ is an isometry onto $\mathbf{R}(A^{1/2})$. If \mathbf{H}_A denotes the completion of $(\mathbf{H}, \langle \cdot, \cdot \rangle_A)$, then $A^{1/2}$ admits an extension

$$\widetilde{A^{1/2}} : \mathbf{H}_A \rightarrow \mathbf{H}$$

which is an isometric isomorphism. Observe that the densely defined operator

$$A^{-1/2} : (\mathbf{R}(A^{1/2}), \| \cdot \|) \rightarrow (\mathbf{H}, \| \cdot \|_A)$$

is an isometry, so that it can be extended to an isometric isomorphism

$$\widetilde{A^{-1/2}} : \mathbf{H} \rightarrow \mathbf{H}_A$$

which is the inverse map of $\widetilde{A^{1/2}}$. (We shall use different symbols to denote extensions to \mathbf{H} and \mathbf{H}_A).

Every $\widetilde{B} \in \mathbf{L}(\mathbf{H}_A, \mathbf{H})$ induces, by restriction, an operator $B = \widetilde{B}|_{\mathbf{H}} : \mathbf{H} \rightarrow \mathbf{H}$ which is bounded because

$$\|Bx\| = \|\widetilde{B}x\| \leq \|\widetilde{B}\| \|x\|_A = \|\widetilde{B}\| \|A^{1/2}x\| \leq \|\widetilde{B}\| \|A^{1/2}\| \|x\|.$$

Moreover, by Douglas' theorem and by

$$B^*B \leq \|\widetilde{B}\|^2 A$$

it follows that $\mathbf{R}(B^*) \subset \mathbf{R}(A^{1/2})$.

Thus, the restriction defines a transformer

$$\begin{aligned} \mathbf{L}(\mathbf{H}_A, \mathbf{H}) &\rightarrow \mathbf{L}(\mathbf{H}), \\ \widetilde{B} &\mapsto B, \end{aligned}$$

with the image $\{B \in \mathbf{L}(\mathbf{H}) : \mathbf{R}(B^*) \subset \mathbf{R}(A^{1/2})\} = \mathbf{L}(\mathbf{H})A^{1/2}$ (the equality is, again, a consequence of Douglas' result). The same argument shows that $\tilde{B} \in \tilde{\mathbf{L}}(\mathbf{H}_A, \mathbf{H})$ is invertible (i.e., there exists $C \in \mathbf{L}(\mathbf{H}, \mathbf{H}_A)$ such that $\tilde{B} \circ C = 1_{\mathbf{H}}$, $C \circ \tilde{B} = 1_{\mathbf{H}_A}$) if and only if $B = VA^{1/2}$ for some $V \in \mathbf{GL}(\mathbf{H})$.

Regarding the norm of \tilde{B} , observe that

$$\|\tilde{B}\|_{\mathbf{L}(\mathbf{H}_A, \mathbf{H})} = \sup_{\tilde{x} \neq 0} \frac{\|\tilde{B}\tilde{x}\|}{\|\tilde{x}\|_A}.$$

Due to the fact that \mathbf{H} is dense in \mathbf{H}_A it follows that

$$\begin{aligned} \|\tilde{B}\|_{\mathbf{L}(\mathbf{H}_A, \mathbf{H})} &= \sup_{x \neq 0} \frac{\|Bx\|}{\|A^{1/2}x\|}, \quad x \in \mathbf{H}, \\ &= \sup \left\{ \frac{\|BA^{-1/2}z\|}{\|z\|} : z \in \mathbf{R}(A^{1/2}), z \neq 0 \right\}. \end{aligned}$$

Thus, \tilde{B} is a bounded operator if and only if $BA^{-1/2}$ is a bounded operator on $\mathbf{R}(A^{1/2})$ and, in this case,

$$\|\tilde{B}\|_{\mathbf{L}(\mathbf{H}_A, \mathbf{H})} = \|BA^{-1/2}\|.$$

If $\widehat{BA^{-1/2}}$ denotes the unique extension of $BA^{-1/2}$ to $(\mathbf{H}, \|\cdot\|)$ we obtain

$$\widehat{BA^{-1/2}} = \tilde{B} \circ \widehat{A^{-1/2}} = \tilde{B} \circ \widetilde{A^{1/2}}^{-1} = \tilde{B} \circ \widehat{A^{-1/2}},$$

because $\widetilde{A^{1/2}}^{-1} = \widetilde{A^{1/2}}^* \in \mathbf{L}(\mathbf{H}, \mathbf{H}_A)$.

Another useful remark is that $\mathbf{R}(A^{1/2})$ is complete with the norm $\|\cdot\|_{A^{-1}}$ induced by the inner product

$$\langle x, w \rangle_{A^{-1}} = \langle A^{-1/2}x, A^{-1/2}w \rangle, \quad x, w \in \mathbf{R}(A^{1/2}).$$

Moreover, $A^{1/2} : (\mathbf{H}, \|\cdot\|) \rightarrow (\mathbf{R}(A^{1/2}), \|\cdot\|_{A^{-1}})$ is an isometric isomorphism.

If \mathbf{H}' denotes the Hilbert space $(\mathbf{R}(A^{1/2}), \langle \cdot, \cdot \rangle_{A^{-1}})$, then $\mathbf{L}(\mathbf{H}, \mathbf{H}') = \{B \in \mathbf{L}(\mathbf{H}) : \text{there is } X \in \mathbf{L}(\mathbf{H}) \text{ such that } B = A^{1/2}X\} = A^{1/2}\mathbf{L}(\mathbf{H})$ and $\|B\|_{\mathbf{L}(\mathbf{H}, \mathbf{H}')} = \|A^{-1/2}B\|_{\mathbf{L}(\mathbf{H})}$.

It may be useful to characterize the elements of $\mathbf{L}(\mathbf{H}_A)$, i.e. the linear operators $\mathbf{H}_A \rightarrow \mathbf{H}_A$ which are bounded with respect to $\|\cdot\|_A$. It is easy to see that the map $\phi : B \mapsto \widehat{A^{-1/2}}B\widehat{A^{1/2}}$ is an isometric isomorphism from $\mathbf{L}(\mathbf{H})$ onto $\mathbf{L}(\mathbf{H}_A)$ which preserves the involution.

In a similar way, $\mathbf{L}(\mathbf{H}') = \{B : \mathbf{R}(A^{1/2}) \rightarrow \mathbf{R}(A^{1/2}) : \text{there exists } X \in \mathbf{L}(\mathbf{H}) \text{ such that } B = A^{1/2}XA^{-1/2}\} = A^{1/2}\mathbf{L}(\mathbf{H})A^{-1/2}$.

Finally, $\mathbf{L}(\mathbf{H}_A, \mathbf{H}') = A^{1/2}\mathbf{L}(\mathbf{H})\widetilde{A^{1/2}}$ and there is a natural isomorphism $\psi : \mathbf{L}(\mathbf{H}') \rightarrow \mathbf{L}(\mathbf{H}_A)$,

$$\psi(A^{1/2}XA^{-1/2}) = \widehat{A^{-1/2}}X^*\widetilde{A^{1/2}} = \widetilde{A^{1/2}}^{-1}X^*\widetilde{A^{1/2}}.$$

Using this notion we obtain another characterization of C_A which will be useful in the next sections.

PROPOSITION 3.1. *If $B \in L(H)^+$, then $B \in C_A$ if and only if there exists $\tilde{C} \in L(H_A, H)$, \tilde{C} invertible, such that $B = C^*C$.*

Proof: If $B \in C_A$, there exists $P \in GL(H)^+$ such that $B = A^{1/2}PA^{1/2}$. Then $C = P^{1/2}A^{1/2} : H \rightarrow H$ admits a unique extension $\tilde{C} : H_A \rightarrow H$ which is bijective and bicontinuous. Then $C^*C = A^{1/2}PA^{1/2} = B$. Conversely, if $B = C^*C$ for some invertible $\tilde{C} \in L(H_A, H)$ then $C = VA^{1/2}$ for some $V \in GL(H)$ and $B = C^*C = A^{1/2}V^*VA^{1/2} = A^{1/2}PA^{1/2}$, $P \in L(H)^+$. Thus $B \in C_A$.

Now consider an $A \in L(H)^+$ which is not necessarily injective. Denote $M = \overline{R(A)}$. Then

$$A|_M : M \rightarrow R(A)$$

is an injective operator in $L(M)^+$. In the same way as before $A|_M$ defines a scalar product in M by

$$\begin{aligned} \langle \cdot, \cdot \rangle_A : M \times M &\rightarrow \mathbb{C}, \\ \langle x, y \rangle_A &= \langle Ax, y \rangle, \quad x, y \in M. \end{aligned}$$

Denote by M_A the completion of $(M, \langle \cdot, \cdot \rangle_A)$.

From $\ker A = \ker A^{1/2}$, $\overline{R(A^{1/2})} = \overline{R(A)} = M$, and then

$$A^{1/2}|_M : M \rightarrow R(A^{1/2}) \subset M.$$

Again

$$A^{1/2}|_M : (M, \|\cdot\|_A) \rightarrow (M, \|\cdot\|)$$

is an isometry that can be extended to M_A . Denote this extension by $\widetilde{A^{1/2}}|_M : M_A \rightarrow M$. Then $\widetilde{A^{1/2}}|_M$ is an isometric isomorphism.

Again, $L(M_A, M)$ can be identified with the subset of $L(M)$,

$$\{B \in L(M) : R(B^*) \subset R(A^{1/2})\},$$

and if $GL(M_A, M)$ denotes the subset of $L(M_A, M)$ of the elements that admits an inverse in $L(M, M_A)$, then $GL(M_A, M)$ can be identified with the set

$$\{B \in L(M) : B = VA^{1/2}, \quad V \in GL(M)\}.$$

The sets $L(M, M_A)$, $L(M_A)$, $L(M_A, M')$ and $L(M')$ can be studied in the same way as in the injective case. (Here M' denotes the Hilbert space $(R(A^{1/2}), \langle \cdot, \cdot \rangle_{(A|_M)^{-1}})$).

Means in C_A

A binary operation σ on the class of positive operators of $L(H)$, $(A, B) \rightarrow (A\sigma B)$, is called a mean if

- (i) $A \leq B$ and $B \leq D$ imply $A\sigma B \leq C\sigma D$,

- (ii) $C(A\sigma B) \leq (CAC)\sigma(CBC)$,
- (iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $(A_n\sigma B_n) \downarrow A\sigma B$ ($A_n \downarrow A$ denotes $A_1 \geq A_1 \geq \dots$ and A_n converges strongly to A),
- (iv) $1\sigma 1 = 1$.

For each mean σ , the map $\sigma \rightarrow f$, defined by $f(x) = 1\sigma x$ for $x > 0$ establishes an isomorphism from the class of means onto the class of normalized operator-monotone functions f .

In fact, for $A, B \in \mathbf{GL}(\mathbf{H})^+$ the following relation yields

$$A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2},$$

with f the monotone function associated to σ . This formula shows that σ can be recovered from the function f , see [13].

Consider $B, C \in \mathbf{L}(\mathbf{H})^+$ such that $B \leq rC$, $r > 0$, or equivalently such that $\mathbf{R}(B^{1/2}) \subset \mathbf{R}(C^{1/2})$. Then

$$\mathbf{R}(B^{1/2}) \subset \mathbf{R}[(B\sigma C)^{1/2}] \subset \mathbf{R}(C^{1/2}).$$

In particular, for B and C in the same component \mathbf{C}_A , $B\sigma C$ lies in \mathbf{C}_A for any mean σ .

To see this suppose that if there exists $r > 0$ such that $B \leq rC$, take $\alpha = \max\{1, r\}$. Then from $B \leq \alpha B$ and $B \leq \alpha C$ we get $B \leq \alpha(B\sigma C)$, or $\mathbf{R}(B^{1/2}) \subset \mathbf{R}[(B\sigma C)^{1/2}]$.

In the same way, if $\beta = \min\{1, \frac{1}{r}\}$, from $\beta B \leq C$ and $\beta C \leq C$ we get $\beta(B\sigma C) \leq C$, or $\mathbf{R}[(B\sigma C)^{1/2}] \leq \mathbf{R}(C^{1/2})$.

Observe that if $A \in \mathbf{L}(\mathbf{H})^+$ has closed range $\mathbf{R}(A)$ then \mathbf{C}_A identifies with $\mathbf{GL}(\mathbf{R}(A))^+$ and then the formula

$$A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$$

which is valid for invertible elements A and B , is still true if we consider

$$(A|_{\mathbf{R}(A)})^{-1/2}B(A|_{\mathbf{R}(A)})^{-1/2}.$$

4. \mathbf{C}_A as a homogeneous space

In this section we define an action on \mathbf{C}_A and study the induced homogeneous structure.

Consider an injective $A \in \mathbf{L}(\mathbf{H})^+$, and let \mathbf{C}_A be the component of A . If $B \in \mathbf{C}_A$ then $B = A^{1/2}PA^{1/2}$, with $P \in \mathbf{GL}(\mathbf{H})^+$, uniquely determined, as seen in Section 2. If $\tilde{B} = A^{1/2}P\tilde{A}^{1/2}$, with $\tilde{A}^{1/2}$ the extension of $A^{1/2}$ to $\langle \mathbf{H}_A, \| \cdot \|_A \rangle$, then \mathbf{C}_A identifies with a subset of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$.

As in the preceding section

$$\begin{aligned} \mathbf{GL}(\mathbf{H}') &= \{W \in \mathbf{L}(\mathbf{H}') : W = A^{1/2}VA^{-1/2}, V \in \mathbf{GL}(\mathbf{H})\} \\ &= A^{1/2}\mathbf{GL}(\mathbf{H})A^{-1/2} \end{aligned}$$

and

$$\mathbf{GL}(\mathbf{H}_A) = \psi(\mathbf{GL}(\mathbf{H}'))$$

with the map $\psi : \mathbf{L}(\mathbf{H}') \rightarrow \mathbf{L}(\mathbf{H}_A)$ defined in Section 3 by

$$\psi(A^{1/2}ZA^{-1/2}) = \left(\widetilde{A^{1/2}}\right)^{-1} Z^* \widetilde{A^{1/2}}.$$

Consider the following action on \mathbf{C}_A ,

$$\begin{aligned} \mathbf{L} : \mathbf{GL}(\mathbf{H}') \times \mathbf{C}_A &\rightarrow \mathbf{C}_A, \\ (W, \widetilde{B}) &\rightarrow \mathbf{L}_W \widetilde{B} = W \widetilde{B} \psi(W). \end{aligned}$$

Then

$$\mathbf{L}_W \widetilde{B} = A^{1/2} V P V^* \widetilde{A^{1/2}},$$

where $W = A^{1/2} V A^{-1/2}$ and $\widetilde{B} = A^{1/2} P \widetilde{A^{1/2}}$, $V \in \mathbf{GL}(\mathbf{H})$, $P \in \mathbf{GL}(\mathbf{H})^+$, are uniquely determined.

Also, as $P = (A^{-1/2} B A^{-1/2})^\wedge$ (see Section 3),

$$\mathbf{L}_W \widetilde{B} = A^{1/2} V (A^{-1/2} B A^{-1/2})^\wedge V^* \widetilde{A^{1/2}}.$$

It is easy to see that \mathbf{L} satisfies:

- (i) $\mathbf{L}_W \mathbf{L}_T = \mathbf{L}_{WT}$ because $\psi(WT) = \psi(T)\psi(W)$, $W, T \in \mathbf{GL}(\mathbf{H}')$.
- (ii) $\mathbf{L}_1 = \text{id}$ because $\psi(1_{\mathbf{H}'}) = 1_{\mathbf{H}_A}$.
- (iii) \mathbf{L} is transitive: if $W = A^{1/2} V A^{-1/2}$, $V \in \mathbf{GL}(\mathbf{H})$, then $\mathbf{L}_W \widetilde{A} = A^{1/2} V V^* \widetilde{A^{1/2}}$.

\mathbf{C}_A can be considered as a differential submanifold of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ because $\mathbf{C}_A = g(\mathbf{GL}(\mathbf{H})^+)$, where

$$\begin{aligned} g : \mathbf{L}(\mathbf{H}) &\rightarrow \mathbf{L}(\mathbf{H}_A, \mathbf{H}'), \\ X &\rightarrow A^{1/2} X \widetilde{A^{1/2}} \end{aligned}$$

is an isometric isomorphism.

Consider $p : \mathbf{GL}(\mathbf{H}') \rightarrow \mathbf{C}_A$ as the map defined by $p(W) = \mathbf{L}_W \widetilde{A} = W \widetilde{A} \psi(W) = A^{1/2} V V^* \widetilde{A^{1/2}}$, with $W = A^{1/2} V A^{-1/2}$.

The isotropy group \mathcal{I}_A of A is

$$\mathcal{I}_A = \{W \in \mathbf{GL}(\mathbf{H}') : p(W) = \widetilde{A}\} = A^{1/2} \mathcal{U}(\mathbf{H}) A^{-1/2} = \mathcal{U}(\mathbf{H}'),$$

where $\mathcal{U}(\mathbf{H})$ and $\mathcal{U}(\mathbf{H}')$ are the unitary groups of \mathbf{H} and \mathbf{H}' .

In general, for $B \in \mathbf{C}_A$, the isotropy group \mathcal{I}_B of B is

$$\mathcal{I}_B = A^{1/2} \mathcal{I}_{(A^{-1/2} B A^{-1/2})^\wedge} A^{-1/2}$$

with $\mathcal{I}_{(A^{-1/2} B A^{-1/2})^\wedge}$ being the isotropy group of (the positive invertible element of $\mathbf{L}(\mathbf{H})$) $(A^{-1/2} B A^{-1/2})^\wedge$ corresponding to the action $\mathbf{L} : \mathbf{GL}(\mathbf{H}) \times \mathbf{GL}(\mathbf{H})^+ \rightarrow \mathbf{GL}(\mathbf{H})^+$, $\mathbf{L}_V P = V P V^*$, $V \in \mathbf{GL}(\mathbf{H})^*$, $P \in \mathbf{GL}(\mathbf{H})^+$ and $P(V) = \mathbf{L}_V 1$, see [5].

The tangent space $(\mathcal{TI}_A)_1$ coincides with $A^{1/2}\mathbf{L}(\mathbf{H})_{\text{ah}}A^{-1/2}$, where $\mathbf{L}(\mathbf{H})_{\text{ah}} = \{X \in \mathbf{L}(\mathbf{H}) : X^* = -X\}$, or also

$$(\mathcal{TI}_A)_1 = \{Y \in \mathbf{L}(\mathbf{H}') : Y^\# = -Y\} = \mathbf{L}(\mathbf{H}')_{\text{ah}},$$

where $Y^\#$ is the adjoint of Y in $\mathbf{L}(\mathbf{H}')$,

$$Y^\# = A^{1/2}X^*A^{-1/2}, \quad \text{if } Y = A^{1/2}XA^{-1/2}, \quad X \in \mathbf{L}(\mathbf{H}).$$

For $\tilde{B} \in \mathbf{C}_A$, the tangent space to \mathbf{C}_A at \tilde{B} , there is

$$(\mathcal{TC}_A)_{\tilde{B}} = \{Y \in \mathbf{L}(\mathbf{H}_A, \mathbf{H}') : Y = A^{1/2}X\tilde{A}^{1/2}, \quad X = X^* \in \mathbf{L}(\mathbf{H})\}$$

with $\mathbf{L}(\mathbf{H})_{\text{h}}$ the subspace of hermitian elements of $\mathbf{L}(\mathbf{H})$, $\mathbf{L}(\mathbf{H})_{\text{h}} = \{X \in \mathbf{L}(\mathbf{H}) : X^* = X\}$.

Observe that $(\mathcal{TC}_A)_{\tilde{B}}$ is a (real) closed subspace of $\mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ because

$$(\mathcal{TC}_A)_{\tilde{B}} = g(\mathbf{L}(\mathbf{H})_{\text{h}}),$$

$g : \mathbf{L}(\mathbf{H}) \rightarrow \mathbf{L}(\mathbf{H}_A, \mathbf{H}')$ is the isometry defined before.

The tangent map of p at 1 is

$$\begin{aligned} (\mathcal{T}_p)_1 : \mathbf{L}(\mathbf{H}') &\rightarrow (\mathcal{TC}_A)_{\tilde{A}}, \\ (\mathcal{T}_p)_1 Y &= Y\tilde{A} + \tilde{A}\psi(Y). \end{aligned}$$

A linear connection is defined on \mathbf{C}_A by giving the following distribution of subspaces of $\mathbf{L}(\mathbf{H}')$ at each $W \in \mathbf{GL}(\mathbf{H}')$:

$$\begin{aligned} \mathcal{H}_1 &= \mathbf{GL}(\mathbf{H}')_{\text{h}} = A^{1/2}\mathbf{L}(\mathbf{H})_{\text{h}}A^{-1/2}, \\ \mathcal{V}_1 &= (\mathcal{TI}_A)_1 = \mathbf{GL}(\mathbf{H}')_{\text{ah}}, \end{aligned}$$

and

$$\mathcal{H}_W = W\mathcal{H}_1, \quad \mathcal{V}_W = W\mathcal{V}_1.$$

We have that

$$(\mathcal{T}_p)_1|_{\mathcal{H}_1} : \mathcal{H}_1 \rightarrow (\mathcal{TC}_A)_{\tilde{A}}$$

is an isomorphism, $(\mathcal{T}_p)_1(Y) = 2Y\tilde{A}$, $Y \in \mathcal{H}$, because $\tilde{H}\psi(Y) = Y\tilde{A}$ if $Y^\# = Y$.

Define

$$K_{\tilde{A}} = \left((\mathcal{T}_p)_1|_{\mathcal{H}_1} \right)^{-1} : (\mathcal{TC}_A)_{\tilde{A}} \rightarrow \mathcal{H}_1,$$

then

$$K_{\tilde{A}}(Z) = \frac{1}{2}Z\tilde{A}^{-1}.$$

Observe that if $Z \in (\mathcal{TC}_A)_{\tilde{A}}$ then $Z = A^{1/2}X\tilde{A}^{1/2}$ for some hermitian X in $\mathbf{L}(\mathbf{H})$, and $K_{\tilde{A}}(Z) = \frac{1}{2}A^{1/2}XA^{-1/2} \in \mathbf{L}(\mathbf{H}')_{\text{h}}$.

Given $\gamma : [0, 1] \rightarrow \mathbf{C}_A$, a smooth curve in \mathbf{C}_A , consider a lift $W : [0, 1] \rightarrow \mathbf{GL}(\mathbf{H}')$ of γ , i.e., W is a curve in $\mathbf{GL}(\mathbf{H}')$ such that $p(W(t)) = \gamma(t)$. W is called horizontal if $\dot{W}(t) \in \mathcal{H}_{W(t)}$ for all t or, equivalently, if $W^{-1}\dot{W} \in \mathcal{H}_1$.

If

$$W(t) = A^{1/2}V(t)A^{-1/2}, \quad V(t) \in \mathbf{GL}(\mathbf{H}),$$

and

$$\gamma(t) = A^{1/2}P(t)\widetilde{A^{1/2}}, \quad P(t) \in \mathbf{GL}(\mathbf{H})^+,$$

then W is horizontal if and only if

$$V^{-1}\dot{V} \in \mathbf{L}(\mathbf{H})_{\mathbf{h}}.$$

But $V^{-1}\dot{V} \in \mathbf{L}(\mathbf{H})_{\mathbf{h}}$ if and only if V is a horizontal lift of $P(t)$. Then W is a horizontal lift of γ if and only if V is a horizontal lift of P , or, equivalently, if V is a solution to the associated transport equation

$$\dot{V} = \frac{1}{2}\dot{P}P^{-1}V.$$

But then W is a solution of the transport equation

$$\dot{W} = \frac{1}{2}\dot{\gamma}\gamma^{-1}W.$$

The transport equation induces a *covariant derivative* of a tangent field X along a curve γ , $\frac{DX}{dt}$, and the field X is *parallel* along γ if $\frac{DX}{dt} = 0$. A curve γ is a *geodesic* if $\dot{\gamma}$ is parallel along γ .

For \widetilde{B} and $\widetilde{C} \in \mathbf{C}_A$ there exists a geodesic joining them, $\gamma_{\widetilde{B}, \widetilde{C}}$, namely

$$\gamma_{\widetilde{B}, \widetilde{C}}(t) = B^{1/2}((B^{-1/2}CB^{-1/2})^\wedge)^t \widetilde{B}^{1/2},$$

where again $(B^{-1/2}CB^{-1/2})^\wedge$ is the extension of $B^{-1/2}CB^{-1/2}$ to an operator in $\mathbf{L}(\mathbf{H})^+$.

Observe that the geodesic $\gamma_{\widetilde{B}, \widetilde{C}}$, joining \widetilde{B} and \widetilde{C} , only depends on \widetilde{B} and \widetilde{C} , and not on A .

Also, if $\widetilde{B} = A^{1/2}P_1\widetilde{A^{1/2}}$ and $\widetilde{C} = A^{1/2}P_2\widetilde{A^{1/2}}$ with P_1 and P_2 in $\mathbf{GL}(\mathbf{H})^+$, then $A^{-1/2}\gamma_{\widetilde{B}, \widetilde{C}}(\widetilde{A^{1/2}})^{-1}$ is the geodesic in $\mathbf{GL}(\mathbf{H})^+$ joining P_1 and P_2 : if $A^{-1/2}B^{1/2} = P_1^{1/2}U$ and $A^{-1/2}C^{1/2} = P_2^{1/2}V$, with U and V in $\mathcal{U}(\mathbf{H})$, then

$$A^{-1/2}\gamma_{\widetilde{B}, \widetilde{C}}(t)(\widetilde{A^{1/2}})^{-1} = P_1^{1/2}U((B^{-1/2}CB^{-1/2})^\wedge)^t U^{-1}P_1^{1/2},$$

and

$$(B^{-1/2}CB^{-1/2})^\wedge = U^{-1}P_1^{-1/2}P_2P_1^{-1/2}U,$$

therefore

$$((B^{-1/2}CB^{-1/2})^\wedge)^t = U^{-1}(P_1^{-1/2}P_2P_1^{-1/2})^t U,$$

and then

$$\begin{aligned} A^{-1/2}\gamma_{\widetilde{B}, \widetilde{C}}(\widetilde{A^{1/2}})^{-1} &= P_1^{1/2}(P_1^{-1/2}P_2P_1^{-1/2})^t P_1^{1/2} \\ &= \gamma_{P_1, P_2}(t). \end{aligned}$$

Finally, we define a Finsler structure by setting a norm $\|\cdot\|_{\tilde{B}}$ on the tangent space $(\mathcal{TC}_A)_{\tilde{B}}$: if $Y \in (\mathcal{TC}_A)_{\tilde{B}}$,

$$\|Y\|_{\tilde{B}} = \|Y\|_{L(\mathbf{H}_B, \mathbf{H}'_B)} = \|B^{-1/2}Y(\widetilde{B^{1/2}})^{-1}\|_{L(\mathbf{H})},$$

where \mathbf{H}'_B denotes the Hilbert space $(\mathbf{R}(B^{1/2}), \langle \cdot, \cdot \rangle_{B^{-1}})$.

Observe that this metric coincides with the metric defined for positive invertible operators (see [5]), because in this case, if $X \in \mathbf{L}(\mathbf{H})_h$

$$\|X\|_A = \|A^{-1/2}XA^{-1/2}\|, \quad A \in \mathbf{GL}(\mathbf{H})^+.$$

The length of a curve $\gamma : [0, 1] \rightarrow \mathbf{C}_A$ is

$$L(\gamma) = \int_0^1 \|\dot{\gamma}\|_{\gamma} dt,$$

and if $\gamma(t) = A^{1/2}P(t)\widetilde{A^{1/2}}$, with $P(t) \in \mathbf{GL}(\mathbf{H})^+$, then $L(\gamma(t)) = L(P(t))$.

Then as a consequence of the results for invertible positive operators, if $\gamma : [0, 1] \rightarrow \mathbf{C}_A$ is a curve with $\gamma(0) = \tilde{B}$ and $\gamma(1) = \tilde{C}$,

$$L(\gamma) \geq L(\gamma_{\tilde{B}, \tilde{C}}) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|,$$

where $\gamma_{\tilde{B}, \tilde{C}}$ is the geodesic joining B and C (see [5]).

If

$$d(\tilde{B}, \tilde{C}) = \inf\{L(\gamma) : \gamma[0, 1] \rightarrow \mathbf{C}_A, \quad \gamma(0) = \tilde{B}, \gamma(1) = \tilde{C}\}$$

is the geodesic metric, then

$$d(\tilde{B}, \tilde{C}) = L(\gamma_{\tilde{B}, \tilde{C}}) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|.$$

But $\|\log(B^{-1/2}CB^{-1/2})^\wedge\|$ coincides with the Thompson metric $d_T(B, C)$, see [4].

Therefore the geodesic metric coincides with the Thompson metric defined in each component,

$$d(\tilde{B}, \tilde{C}) = d_T(B, C) = \|\log(B^{-1/2}CB^{-1/2})^\wedge\|.$$

In the general case, if $A \in \mathbf{L}(\mathbf{H})^+$ is not necessarily injective, \mathbf{C}_A can be considered as a differential submanifold of $\mathbf{L}(\mathbf{M}_A, \mathbf{M}')$ (see Section 3) and as in the injective case, we obtain a homogeneous space with a Finsler structure.

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