Acta Sci. Math. (Szeged)
67 (2001), 337–356

Oblique projections and Schur complements

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Dedicated to Horacio Porta, with affection and admiration

Communicated by L. Kérchy

Abstract. Let $\mathcal{H}$ be a Hilbert space, $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and $\langle , \rangle_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ the bounded sesquilinear form induced by a selfadjoint $A \in L(\mathcal{H})$, $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$, ξ, $\eta \in \mathcal{H}$. Given $T \in L(\mathcal{H})$, T is A-selfadjoint if $AT = T^*A$. If $\mathcal{S} \subseteq \mathcal{H}$ is a closed subspace, we study the set of A-selfadjoint projections onto $\mathcal{S}$,

$$\mathcal{P}(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q \ , \ \ R(Q) = \mathcal{S} \ , \ \ AQ = Q^*A\}$$

for different choices of A, mainly under the hypothesis that $A \geq 0$. There is a closed relationship between the A-selfadjoint projections onto $\mathcal{S}$ and the shorted operator (also called Schur complement) of A to $\mathcal{S}^\perp$. Using this relation we find several conditions which are equivalent to the fact that $\mathcal{P}(A, \mathcal{S}) \neq \emptyset$, in particular in the case of $A \geq 0$ with A injective or with $R(A)$ closed. If A is itself a projection, we relate the set $\mathcal{P}(A, \mathcal{S})$ with the existence of a projection with fixed kernel and range and we determine its norm.

Received May 2, 2000, and in revised form September 27, 2000.
AMS Subject Classification (1991): 47A64, 47A07, 46C99.
* Partially supported by Fundación Antorchas, CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TX92 and TW49) and ANPCYT (PICT97-2259).
° Partially supported CONICET (PIP 4463/96), Universidad de Buenos Aires (UBACYT TW49) and ANPCYT (PICT97-2259).

1. Introduction

If $\mathcal{H}$ is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle$ and $L(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$, consider the subset $\mathcal{Q}$ of $L(\mathcal{H})$ consisting of all projections onto (closed) subspaces of $\mathcal{H}$ and the subset $\mathcal{P}$ of $\mathcal{Q}$ of all orthogonal (i.e., selfadjoint) projections. Every $Q \in \mathcal{Q} \setminus \mathcal{P}$ is called an *oblique projection.* The structure of $\mathcal{Q}$ and $\mathcal{P}$ has been widely studied since the begining of the spectral theory. In recent times, applications of oblique projections to complex geometry [21], statistics [26], [27] and wavelet theory [1], [2], [24], [25] have renewed the interest on the subject. The reader is also referred to [6], [16].

In [21], [4] there is an analytic study of the map which assigns to any positive invertible operator $A \in L(\mathcal{H})$ and any subspace $\mathcal{S}$ of $\mathcal{H}$ the unique projection onto $\mathcal{S}$ which is selfadjoint for the scalar product $\langle \cdot, \cdot \rangle_A$ on $\mathcal{H}$ defined by $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$ $(\xi, \eta \in \mathcal{H})$. In this paper we study the existence of projections onto $\mathcal{S}$ which are selfadjoint for $\langle \cdot, \cdot \rangle_A$ if A is not necessarily invertible. More precisely, if $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $B: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is a Hermitian sesquilinear form, consider the subsets of $\mathcal{Q}$,

$$\mathcal{Q}_{\mathcal{S}} = \{Q \in \mathcal{Q} : Q(\mathcal{H}) = \mathcal{S}\} \quad \text{(projections with range S)}$$

and

$$\mathcal{Q}^B = \left\{Q \in \mathcal{Q} : B(\xi, Q\eta) = B(Q\xi, \eta), \quad \text{for all} \quad \xi, \eta \in \mathcal{H}\right\}$$
$$\text{(B-symmetric projections).}$$

The main theme of the paper is the characterization of the intersection of $\mathcal{Q}_{\mathcal{S}}$ and $\mathcal{Q}^B$. We shall limit our study to the case in which B is bounded, so that, by Riesz' theorem, there exists a unique selfadjoint operator $A \in L(\mathcal{H})$ such that $B(\xi, \eta) = B_A(\xi, \eta) = \langle A\xi, \eta \rangle$ $(\xi, \eta \in \mathcal{H})$; we search to characterize the set

$$\mathcal{P}(A, \mathcal{S}) = \mathcal{Q}_{\mathcal{S}} \cap \mathcal{Q}^{B_A}.$$

Observe that $\mathcal{P}(A, \mathcal{S})$ has a unique element if A is a positive invertible operator, but in general it can have 0, 1 or infinitely many elements. Even if we get a characterization of $\mathcal{P}(A, \mathcal{S})$ in general, much more satisfactory results can be obtained for a positive A ($A \geq 0$, i.e. $\langle A\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$). In this paper, a pair $(A, \mathcal{S})$ consisting of a bounded selfadjoint operator A and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$ is said to be *compatible* if $\mathcal{P}(A, \mathcal{S})$ is not empty.

The contents of the paper are the following:

In section 2 we recall the Douglas factorization and use it to prove that if $Q \in \mathcal{Q}$, $A \in L(H)$ and $R(QA) \subseteq R(A)$, then the unique operator $D \in L(\mathcal{H})$ with the properties

$$QA = AD\ , \quad \ker D = \ker QA \qquad \text{and} \qquad R(D) \subseteq \overline{R(A^*)},$$

(called the *reduced solution of* $AX = QA$) satisfies $D^2 = D$, i.e., $D \in \mathcal{Q}$.

In section 3, some characterizations of the compatibility of $(A, \mathcal{S})$ are given; some of them hold for general bounded selfadjoint operators A, and others hold only for positive operators A. Among other properties, it is shown that an oblique projection Q is A-selfadjoint (if $A \geq 0$) if and only if $0 \leq Q^*AQ \leq A$ (see Lemma 3.2). We establish that $\mathcal{P}(A, \mathcal{S})$ is an affine manifold and we give a parametrization for it. When $(A, \mathcal{S})$ is compatible, a distinguished element $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ can be defined. It is shown that the norm of $P_{A,\mathcal{S}}$ is minimal in $\mathcal{P}(A, \mathcal{S})$ (see Theorem 3.5).

In section 4 we consider the relationship between the compatibility of $(A, \mathcal{S})$ and some properties of the Schur complement. M. G. Krein [18] and W. N. Anderson and G. E. Trapp [3], extended the notion of Schur complement of matrices to Hilbert space operators, defining what is called the *shorted operator*. We recall the definition: if $A \in L(\mathcal{H})^+$, $\mathcal{S} \subseteq \mathcal{H}$ is a closed subspace, then the set

$$\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^\perp\}$$

has a maximum (for the natural order relation in $L(\mathcal{H})^+$), which is called the *shorted operator* of A to $\mathcal{S}^\perp$. We shall denote it by $\Sigma(P, A)$, where $P = P_\mathcal{S}$ stands for orthogonal projection onto $\mathcal{S}$. It is shown that, for any $Q \in \mathcal{P}(A, \mathcal{S})$, the Schur complement $\Sigma(P, A)$ is given by

$$\Sigma(P, A) = A(1 - Q)$$

(see Proposition 4.2). We also show that $(A, \mathcal{S})$ is compatible if and only if, in the characterization

$$\Sigma(P, A) = \inf\{R^*AR \ : \ R \in \mathcal{Q},\ \ker R = \mathcal{S}\ \},$$

due by Anderson and Trapp [3], the infimum is, indeed, a minimum (see Corollary 4.3).

In section 5 we consider the case of positive operators A which are *injective*. Using properties of the shorted operator $\Sigma(P, A)$, new conditions equivalent to the

fact that the pair $(A, \mathcal{S})$ is compatible are found. For example (see Proposition 5.4), it is shown that

$$(A, \mathcal{S}) \quad \text{is compatible} \quad \iff \mathcal{S}^\perp \subseteq R(A + \lambda(1-P)), \quad \text{for some} \quad \lambda > 0.$$

In section 6 we consider the case of positive operators A with *closed range*. Among other equivalences, it is shown that $(A, \mathcal{S})$ is compatible if and only if $\mathcal{S} + \ker A$ is closed (see Theorem 6.2). As a consequence it is shown that all manifolds $\mathcal{P}(B, \mathcal{S})$ for $R(B) = R(A)$ are "parallel" (see Corollary 6.4). So, in this sense, it suffices to study the case of the orthogonal projection $Q = P_{R(A)}$. This case is studied in section 7, where we give a formula for the norm of the projection $P_{Q,P} := P_{Q,\mathcal{S}}$ in $\mathcal{P}(Q, \mathcal{S})$. For example (see Proposition 7.2), if $\ker Q \cap R(P) = \{0\}$, then $PQP \in GL(\mathcal{S})$ and

$$\|P_{Q,P}\|^2 = \|(PQP)^{-1}\| = (1 - \|(1-Q)P\|^2)^{-1}.$$

In case that $R(P) \cap \ker Q = \{0\} = R(Q) \cap \ker P$ (e.g., if P and Q are in *position p* [12], [9] or *generic position* [15]), $P_{Q,P}$ is the oblique projection given by

$$\ker P_{Q,P} = \ker Q \quad \text{and} \quad R(P_{Q,P}) = R(P).$$

We thank several useful comments and suggestions of the referee. In particular, the present version of Theorem 3.5, in which the superfluous assumption $A \geq 0$ was removed, is due to him.

2. Preliminaries

In this paper $\mathcal{H}$ denotes a Hilbert space, $L(\mathcal{H})$ is the algebra of all linear bounded operators on $\mathcal{H}$, $L(\mathcal{H})^+$ is the subset of $L(\mathcal{H})$ of all (selfadjoint) positive operators, $GL(\mathcal{H})$ is the group of all invertible operators in $L(\mathcal{H})$ and $GL(\mathcal{H})^+ = GL(\mathcal{H}) \cap L(\mathcal{H})^+$ (positive invertible operators). For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$. We shall use the symbols $\dot{+}$ for direct sum and $\oplus$ for orthogonal sum of closed subspaces. Given $\mathcal{S}$ and $\mathcal{T}$ two closed subspaces of $\mathcal{H}$, we denote $\mathcal{S} \ominus \mathcal{T} = \mathcal{S} \cap \mathcal{T}^\perp$, the "relative orthogonal companion" of $\mathcal{T}$ in $\mathcal{S}$.

Denote by $\mathcal{Q}$ (resp. $\mathcal{P}$) the set of all projections (resp. selfadjoint projections) in $L(\mathcal{H})$:

$$\mathcal{Q} = \mathcal{Q}(L(\mathcal{H})) = \{Q \in L(\mathcal{H}) : Q^2 = Q\} \quad , \quad \mathcal{P} = \mathcal{P}(L(\mathcal{H})) = \{P \in \mathcal{Q} : P = P^*\}.$$

The nonselfadjoint elements of $\mathcal{Q}$ will be called *oblique projections*.

Recall that every $P \in \mathcal{P}$ induces a representation of elements of $L(\mathcal{H})$ as 2×2 operator matrices. Under this representation P can be identified with

$$\begin{pmatrix} I_{P(\mathcal{H})} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and all idempotents Q with the same range as P have the form

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some $x \in L(\ker P, R(P))$.

Definition 2.1. Let $A, B \in L(\mathcal{H})$. A bounded operator D which satisfies the conditions

$$AD = B\ , \quad \ker D = \ker B \quad \text{and} \quad R(D) \subseteq \overline{R(A^*)}$$

is called the *reduced solution* of the equation $AX = B$.

Now we state the well known criterium due to Douglas [13] (see also Fillmore–Williams [14]) about ranges and factorizations of operators:

Theorem 2.2. *Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent:*
1. $R(B) \subseteq R(A)$.
2. *There exists a positive number λ such that $BB^* \leq \lambda AA^*$.*
3. *There exists $D \in L(\mathcal{H})$ such that $B = AD$.*

Moreover, in this case there exists a unique reduced solution D of the equation $AX = B$ and $\|D\|^2 = \inf\{\lambda : BB^* \leq \lambda AA^*\}$.

Corollary 2.3. *Suppose that $Q \in \mathcal{Q}$, $A \in L(H)$ and $R(QA) \subseteq R(A)$. Then the reduced solution $D \in L(\mathcal{H})$ of $AX = QA$ satisfies that $D^2 = D$, i.e., $D \in \mathcal{Q}$.*

Proof. Note that $AD^2 = QAD = Q^2A = QA$. Also

$$\ker QA = \ker D \subseteq \ker D^2 \subseteq \ker AD^2 = \ker QA$$

and $R(D^2) \subseteq R(D) \subseteq \overline{R(A^*)}$. Thus, D^2 is a reduced solution of $AX = QA$ and, by uniqueness, it must be $D^2 = D$, i.e. $D \in \mathcal{Q}$.

■

3. A-selfadjoint projections, generic properties

Let $\mathcal{S}$ be a closed subspace of $\mathcal{H}$ and P be the orthogonal projection onto $\mathcal{S}$. Consider the bounded sesquilinear form $B = B_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ determined by a Hermitian operator $A \in L(\mathcal{H})$:

$$B_A(\xi, \eta) = \langle A\xi, \eta \rangle , \quad \xi,\ \eta \in \mathcal{H}.$$

This form induces the notion of A-orthogonality. In particular, the A-orthogonal companion of $\mathcal{S}$ is given by

$$\mathcal{S}^{\perp_A} := \{ \xi : \ \langle A\xi, \eta \rangle \ = 0 \ \ \forall \eta \in \mathcal{S} \ \} = A^{-1}(\mathcal{S}^{\perp}).$$

Given $T \in L(\mathcal{H})$, an operator $W \in L(\mathcal{H})$ is called an *A-adjoint* of T if

$$B_A(T\xi, \eta) = B_A(\xi, W\eta), \quad \xi,\ \eta \in \mathcal{H},$$

or, which is the same, if $T^*A = AW$. We shall study the existence and uniqueness of A-selfadjoint projections, i.e., $Q \in \mathcal{Q}$ such that $AQ = Q^*A$. Among them, we are interested in those whose range is exactly $\mathcal{S}$. Thus, the main goal of the paper is the study of the set

$$\mathcal{P}(A, \mathcal{S}) = \{ Q \in \mathcal{Q} : R(Q) = \mathcal{S}, AQ = Q^*A \}$$

for different choices of A.

Definition 3.1. Let $A = A^* \in L(\mathcal{H})$ and let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. The pair $(A, \mathcal{S})$ is said to be *compatible* if there exists an A-selfadjoint projection with range $\mathcal{S}$, i.e. if $\mathcal{P}(A, \mathcal{S})$ is not empty.

For general results on A-selfadjoint operators the reader is referred to the papers by P. Lax [20] and J. Dieudonné [11]; a more recent paper by S. Hassi and K. Nordström [16] contains many interesting results on A-selfadjoint projections. Some of the results of this section overlapp with their work, but we include them because the methods used in our proofs are useful for the study of the case of a positive A, which is our main concern.

Lemma 3.2. *Let $A = A^* \in L(\mathcal{H})$ and $Q \in \mathcal{Q}$. Then the following conditions are equivalent:*

1. *Q satisfies that $AQ = Q^*A$, i.e. Q is A-selfadjoint.*
2. $\ker Q \subseteq A^{-1}(R(Q)^\perp) = R(Q)^{\perp_A}$.

If $A \in L(\mathcal{H})^+$, they are equivalent to

3. $Q^*AQ \leq A$.

Proof. $1 \leftrightarrow 2$. If $Q \in \mathcal{P}(A, \mathcal{S})$ and $\xi, \eta \in \mathcal{H}$, then

$$\langle A\eta, Q\xi\rangle = \langle Q^*A\eta, \xi\rangle = \langle AQ\eta, \xi\rangle = \langle Q\eta, A\xi\rangle, \tag{1}$$

so $\ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$. The converse can be proved in a similar way.

$1 \leftrightarrow 3$. Suppose that $0 \leq Q^*AQ \leq A$. Then, by Theorem 2.2, the reduced solution D of the equation $A^{1/2}X = Q^*A^{1/2}$ satisfies $\|D\| \leq 1$ and, by Corollary 2.3, $D^2 = D$. Thus, it must be $D^* = D$. Since $Q^*A = A^{1/2}DA^{1/2}$, we conclude that $Q^*A = AQ$. Conversely, note that $AQ = Q^*AQ \geq 0$ and, if $E = 1 - Q$, then also $AE = E^*AE$. Therefore, $A = A(Q + E) = Q^*AQ + E^*AE \geq Q^*AQ$ ∎

Throughout, we use the matrix representation determined by P.

Proposition 3.3. *Given $A = A^* \in L(\mathcal{H})$, the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible (i.e. $\mathcal{P}(A, \mathcal{S})$ is not empty).*
2. $R(PA) = R(PAP)$.
3. *If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ then $R(b) \subseteq R(a)$.*
4. $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$.

Proof. Note that

$$PA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad PAP = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

so $R(a) = R(PAP) \subseteq R(PA) = R(a) + R(b)$ and items 2 and 3 are equivalent. On the other hand, for any $Q \in \mathcal{Q}$ it holds $R(Q) = \mathcal{S}$ if and only if

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}.$$

Easy computations show that $Q^*A = AQ$ if and only if $ax = b$, so items 1 and 3 are equivalent by Theorem 2.2. Finally, if $Q \in \mathcal{P}(A, \mathcal{S})$ then, by Lemma 3.2, $\ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$, which implies 4. Conversely, if $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$, and if $\mathcal{N}$ is defined by $\mathcal{N} = \mathcal{S} \cap A^{-1}(\mathcal{S}^\perp)$, then $\mathcal{S} \dot{+} (A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}) = \mathcal{H}$. The projection Q defined by this decomposition of $\mathcal{H}$ satisfies, again by Lemma 3.2, the identity $Q^*A = AQ$ ∎

Definition 3.4. Let $A = A^* \in L(\mathcal{H})$ and suppose that the pair $(A, \mathcal{S})$ is compatible. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of the equation $ax = b$, we define the following oblique projection onto $\mathcal{S}$:

$$P_{A,\mathcal{S}} := \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}.$$

Theorem 3.5. *Let $A = A^* \in L(\mathcal{H})$ and suppose that $(A, \mathcal{S})$ is compatible. Denote $\mathcal{N} = A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S}$. Then the following properties hold:*

1. $\mathcal{N} = \ker a$ *and, if $A \geq 0$, then $\mathcal{N} = \ker A \cap \mathcal{S}$.*
2. $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$. *Moreover $P_{A,\mathcal{S}}$ is the projection onto $\mathcal{S}$ with kernel $A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$.*
3. *$\mathcal{P}(A, \mathcal{S})$ has a unique element (namely, $P_{A,\mathcal{S}}$) if and only if $\mathcal{S} \dot{+} A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$.*
4. *$\mathcal{P}(A, \mathcal{S})$ is an affine manifold and it can be parametrized as*

$$\mathcal{P}(A, \mathcal{S}) = P_{A,\mathcal{S}} \ + \ L(\mathcal{S}^\perp, \mathcal{N}),$$

where $L(\mathcal{S}^\perp, \mathcal{N})$ is viewed as a subspace of $L(\mathcal{H})$. A matrix representation of this parametrization is

$$\mathcal{P}(A, \mathcal{S}) \ni Q = P_{A,\mathcal{S}} + z = \begin{array}{c} \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathcal{S} \ominus \mathcal{N} \\ \mathcal{N} \\ \mathcal{S}^\perp \end{array} \tag{2}$$

with the notations of Definition 3.4.

5. *$P_{A,\mathcal{S}}$ has minimal norm in $\mathcal{P}(A, \mathcal{S})$:*

$$\|P_{A,\mathcal{S}}\| = \min\{ \ \|Q\| : Q \in \mathcal{P}(A, \mathcal{S})\}.$$

Nevertheless, $P_{A,\mathcal{S}}$ is not in general the unique $Q \in \mathcal{P}(A, \mathcal{S})$ that realizes the minimal norm.

Proof. 1. Let $\xi \in \mathcal{S}$. Then $A\xi = a\xi + b^*\xi$. Recall that $a\xi \in \mathcal{S}$ and $b^*\xi \in \mathcal{S}^\perp$. Therefore $A\xi \in \mathcal{S}^\perp$ if and only if $\xi \in \ker a$. If $A \geq 0$ and $\xi \in \mathcal{N}$, then $\|A^{1/2}\xi\| = \langle A\xi, \xi\rangle = 0$ so that $A\xi = 0$.

2. In order to show that $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$, use the same argument as in the proof of Proposition 3.3. Then $\ker P_{A,\mathcal{S}} \subseteq A^{-1}(\mathcal{S}^\perp)$. Since $\mathcal{S} \dot{+} (A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}) = \mathcal{H}$, it suffices to show that $\ker P_{A,\mathcal{S}} \subseteq \mathcal{N}^\perp$. Let $\xi \in \ker P_{A,\mathcal{S}}$ and write $\xi = \xi_1 + \xi_2$ with $\xi_1 \in \mathcal{S}$ and $\xi_2 \in \mathcal{S}^\perp$. Then $0 = P_{A,\mathcal{S}}\xi = \xi_1 + d\xi_2$. If $\eta \in \mathcal{N}$, then $\langle \xi, \eta\rangle = \langle \xi_1, \eta\rangle = -\langle d\xi_2, \eta\rangle = 0$, because $R(d) \subseteq \overline{R(a)} = (\ker a)^\perp = \mathcal{N}^\perp$.

3. By Lemma 3.2, if $Q \in \mathcal{Q}$ and $R(Q) = \mathcal{S}$, then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $\ker Q \subseteq A^{-1}(\mathcal{S}^\perp)$. This clearly implies the assertion.

4. We have to show that every element $Q \in \mathcal{P}(A, \mathcal{S})$ can be written in an unique form as

$$Q = P_{A,\mathcal{S}} + z\ , \quad \text{with} \quad z \in L(\mathcal{S}^\perp, \mathcal{N}).$$

If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, $Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix}$ with $y \in L(\mathcal{S}^\perp, \mathcal{S})$ and if $d \in L(\mathcal{S}^\perp, \mathcal{S})$ is the reduced solution of the equation $ax = b$, then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $ay = b$ if and only if $a(y - d) = 0$. Therefore, if $z = y - d \in L(\mathcal{S}^\perp, \mathcal{S})$, then $Q \in \mathcal{P}(A, \mathcal{S})$ if and only if $Q = P_{A,\mathcal{S}} + z$ and $R(z) \subseteq \ker a = \mathcal{N}$.

Concerning the matrix representation, note that, by Theorem 2.2,

$$R(d) \subseteq \overline{R(a)} = (\ker a)^\perp \cap \mathcal{S} = \mathcal{S} \ominus \mathcal{N}.$$

5. If $Q \in \mathcal{P}(A, \mathcal{S})$ has the matrix form given in equation (2), then

$$\|Q\|^2 = 1 + \left\| \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\|^2 \geq 1 + \|d\|^2 = \|P_{A,\mathcal{S}}\|^2.$$

Choose $d \in L(\mathcal{S}^\perp, \mathcal{S})$ such that $\|d\| = 1$, $R(d) = \overline{R(d)} \neq \mathcal{S}$ and $\ker d \neq \{0\}$. Then the matrix

$$A = \begin{pmatrix} P_{R(d)} & d \\ d^* & 1 \end{pmatrix} \geq 0,$$

$\mathcal{N} = \ker A \cap \mathcal{S} = \mathcal{S} \ominus R(d)$ and d is the reduced solution of $P_{R(d)}x = d$. Let $z \in L(\ker d, \mathcal{N})$ with $0 < \|z\| \leq 1$; then the projection $Q = P_{A,\mathcal{S}} + z$ as in equation (2) satisfies $Q \in \mathcal{P}(A, \mathcal{S})$, $\|Q\| = \|P_{A,\mathcal{S}}\| = \sqrt{2}$ and $Q \neq P_{A,\mathcal{S}}$. ∎

4. Schur complements and A-selfadjoint projections

As before, let $P \in \mathcal{P}$ be the orthogonal projection onto the closed subspace $\mathcal{S} \subseteq \mathcal{H}$. Every $A \in GL(\mathcal{H})^+$ defines a scalar product on $\mathcal{H}$ which is equivalent to $\langle , \rangle$, namely

$$\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.$$

The unique projection $P_{A,\mathcal{S}}$ onto $\mathcal{S}$ which is A-orthogonal, i.e. A-selfadjoint, is uniquely determined by

$$P_{A,\mathcal{S}} = P(1 + P - A^{-1}PA)^{-1} = P(PAP + (1 - P)A(1 - P))^{-1}A.$$

Observe that $P_{A,\mathcal{S}} = A^{-1}P^*_{A,\mathcal{S}}A$, because A is invertible. In particular, in this case the set $\mathcal{P}(A,\mathcal{S})$ is a singleton. Analogously, there exists a unique projection $Q_{A,\mathcal{S}}$ which is A-orthogonal and has kernel $\mathcal{S}$: $Q_{A,\mathcal{S}} = 1 - P_{A,\mathcal{S}}$. Notice that $AQ_{A,\mathcal{S}} = Q^*_{A,\mathcal{S}}A$.

Consider the map $\Sigma\colon \mathcal{P} \times GL(\mathcal{H})^+ \to L(\mathcal{H})^+$ defined by $\Sigma(P,A) = AQ_{A,\mathcal{S}} = Q^*_{A,\mathcal{S}}A$. If $A \in GL(\mathcal{H})^+$ has matrix representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, then

$$P_{A,\mathcal{S}} = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}, \quad Q_{A,\mathcal{S}} = \begin{pmatrix} 0 & -a^{-1}b \\ 0 & 1 \end{pmatrix}, \quad \text{and } \Sigma(P,A) = \begin{pmatrix} 0 & 0 \\ 0 & c - b^*a^{-1}b \end{pmatrix}.$$

This reminds us the *Schur complement.* Recall that, given a square matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, with a and d square blocks, a Schur complement of a in M is $d - ca'b$, where a' is a generalized inverse of a. The reader is referred to [8] and [7] for concise surveys on the subject. This notion has been extended to positive Hilbert space operators by M. G. Krein [18] and, later and independently, by W. N. Anderson and G. E. Trapp [3] defining what is called the shorted operator: if $A \in L(\mathcal{H})^+$ then the set $\{X \in L(\mathcal{H})^+ : X \leq A \quad \text{and} \quad R(X) \subseteq \mathcal{S}^\perp\}$ has a maximum (for the natural order relation in $L(\mathcal{H})^+$), which is called the *shorted operator* of A to $\mathcal{S}^\perp$. If $A \in GL(\mathcal{H})^+$, it is shown in [3] that the shorted operator of A to $\mathcal{S}^\perp$ coincides with $\Sigma(P,A)$, where P is the orthogonal projection onto $\mathcal{S}$. Therefore we shall keep the notation $\Sigma(P,A)$ for the shorted operator of A to $R(P)^\perp$ for every pair $(P,A) \in \mathcal{P} \times L(\mathcal{H})^+$.

Next we collect some results of Anderson–Trapp and E. L. Pekarev [22] which are relevant in this paper. See also Krein [18] and Krein-Ovcharenko [19].

Theorem 4.1. *Let $A \in L(\mathcal{H})^+$ have the operator matrix representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$.*

1. *$R(b) \subseteq R(a^{1/2})$ and if $d \in L(\mathcal{H})$ is the reduced solution of the equation $a^{1/2}\, x = b$ then*
$$\Sigma(P,A) = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.$$
2. *If $\mathcal{M} = A^{-1/2}(\mathcal{S}^\perp)$ and $P_{\mathcal{M}}$ is the orthogonal projection onto $\mathcal{M}$ then*
$$\Sigma(P,A) = A^{1/2}P_{\mathcal{M}}A^{1/2}.$$
3. *$\Sigma(P,A)$ is the infimum of the set $\{R^*AR \;:\; R \in \mathcal{Q},\ \ker R = \mathcal{S}\ \}$; in general, the infimum is not attained.*

4. $R(A) \cap \mathcal{S}^\perp \subseteq R(\Sigma(P, A)) \subseteq R(\Sigma(P, A)^{1/2}) = R(A^{1/2}) \cap \mathcal{S}^\perp$; *in general, the inclusions are strict.*

The reader is referred to [3] and [22] for proofs of these facts. We prove now that the infimum of item 3 is attained if and only if $(A, \mathcal{S})$ is compatible by relating the notions of shorted operators and A-selfadjoint projections (when there is one). As a consequence, we complete item 4 of the previous theorem in case that $(A, \mathcal{S})$ is compatible.

Proposition 4.2. *Let $A \in L(\mathcal{H})^+$ such that the pair $(A, \mathcal{S})$ is compatible. Let $E \in \mathcal{P}(A, \mathcal{S})$ and $Q = 1 - E$. Then*

1. $\Sigma(P, A) = AQ = Q^*AQ$.
2. $\Sigma(P, A) = \min\{R^*AR : R \in \mathcal{Q}, \ker R = \mathcal{S}\}$ *and the minimum is attained at Q.*
3. $R(\Sigma(P, A)) = R(A) \cap \mathcal{S}^\perp$.

Proof. 1. Note that $0 \le AQ = Q^*AQ \le A$, by Lemma 3.2. Also $R(AQ) = R(Q^*A) \subseteq R(Q^*) = \mathcal{S}^\perp$. Given $X \le A$ with $R(X) \subseteq \mathcal{S}^\perp$, then, since $\ker Q = \mathcal{S}$, we have that

$$X = Q^*XQ \le Q^*AQ = AQ,$$

where the first equality can be easily checked because X has the form $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$.

2. By item 1, $Q^*AQ = \Sigma(P, A)$ and $\ker Q = \mathcal{S}$. So the minimum is attained at Q by Theorem 4.1.

3. Clearly the equation $\Sigma(P, A) = AQ$ implies that $R(\Sigma(P, A)) \subseteq R(A) \cap \mathcal{S}^\perp$. The other inclusion always holds by Theorem 4.1. ■

Corollary 4.3. *If $A \in L(\mathcal{H})^+$ the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *The set $\{S^*AS : S \in \mathcal{Q}, \ \ker S = \mathcal{S}\}$ attains its minimum at some projection R.*
3. *There exists $R \in \mathcal{Q}$ such that $\ker R = \mathcal{S}$ and $R^*AR \le A$.*

Proof. $1 \to 2$. Follows from Proposition 4.2.

$2 \to 3$. Follows from Theorem 4.1.

$3 \to 1$. By Lemma 3.2, any projection R such that $R^*AR \leq A$ satisfies that $AR = R^*A$. If also $\ker R = \mathcal{S}$, then $1 - R \in \mathcal{P}(A, \mathcal{S})$. ■

In the next sections we shall study the existence of A-selfadjoint projections onto a closed subspace $\mathcal{S}$, under particular hypothesis on the positive operator A.

5. A-selfadjoint projections: the injective case

As before, let $P \in \mathcal{P}$ be the orthogonal projection onto $\mathcal{S}$. In this section we study the case of injective operators $A \in L(\mathcal{H})^+$. We define the notion of A-admissibility for $\mathcal{S}$, in terms of the shorted operator $\Sigma(P, A)$. In Example 5.5 below, we show that A-admissibility for $\mathcal{S}$ is in general a strictly weaker property than compatibility for the pair $(A, \mathcal{S})$. But under the assumption of A-admissibitity for $\mathcal{S}$, the fact that $(A, \mathcal{S})$ is compatible becomes equivalent to the equality $R(\Sigma(P, A)) = \mathcal{S}^\perp \cap R(A)$ (see item 4 of Theorem 4.1 and item 3 of Proposition 4.2).

Definition 5.1. We shall say that $\mathcal{S}$ is *A-admissible* if $\ker \Sigma(P, A) = \mathcal{S}$.

Lemma 5.2. *If $A \in L(\mathcal{H})^+$ is injective and $(A, \mathcal{S})$ is compatible, then $\mathcal{S}$ is A-admissible.*

Proof. Let $E \in \mathcal{P}(A, \mathcal{S})$ and $Q = 1 - E$. Then, by Proposition 4.2, $\Sigma(P, A) = AQ$ and $\ker \Sigma(P, A) = \ker Q = \mathcal{S}$. ■

Remark 5.3. Let $A \in L(\mathcal{H})^+$. It is easy to see that a closed subspace $\mathcal{S}$ is A-admissible if and only if $\mathcal{S}^\perp \cap R(A^{1/2})$ is dense in $\mathcal{S}^\perp$. If $(A, \mathcal{S})$ is compatible then a condition which is stronger than A-admissibility holds. Indeed, $\Sigma(P, A) = A(1 - P_{A,\mathcal{S}})$ implies that $\ker \Sigma(P, A) = \mathcal{S}$. But in this case, $R(\Sigma(P, A)) \subseteq R(A) \cap \mathcal{S}^\perp$ which must be dense in $\mathcal{S}^\perp$. Note that $R(A^{1/2})$ strictly contains $R(A)$ if $R(A)$ is not closed.

Nevertheless, we restrict ourselves to the weaker notion of A-admissibility because under the hypothesis that $\mathcal{S}$ is A-admissible, the pair $(A, \mathcal{S})$ is compatible if and only if $R(\Sigma(P, A)) \subseteq R(A)$. Observe that $R(\Sigma(P, A)) \subseteq R(A)$ is false in general (recall item 4 of Theorem 4.1 and item 3 of Proposition 4.2)

Proposition 5.4. *If $A \in L(\mathcal{H})^+$ is injective then the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. i) $\ker \Sigma(P, A) = \mathcal{S}$ *(i.e. $\mathcal{S}$ is A-admissible) and* ii) $R(\Sigma(P, A)) \subseteq R(A)$.
3. *$\mathcal{S}$ is A-admissible and, if $\mathcal{M} = A^{-1/2}(\mathcal{S}^\perp)$, then $P_\mathcal{M} A P_\mathcal{M} \leq \mu A$ for some $\mu > 0$.*
4. *$\mathcal{S}^\perp \subseteq R(A + \lambda(1 - P))$ for some (and then for any) $\lambda > 0$.*

Proof. $1 \to 2$. By Lemma 5.2, $\mathcal{S}$ must be A-admissible. If $Q_{A,\mathcal{S}} = 1 - P_{A,\mathcal{S}}$, then $\Sigma(P, A) = AQ_{A,\mathcal{S}}$ and item 2 follows.

$2 \to 3$. If $R(A^{1/2}P_M A^{1/2}) \subseteq R(A)$ then $R(P_M A^{1/2}) \subseteq R(A^{1/2})$, because $\ker A^{1/2} = \ker A = \{0\}$. Then, by Douglas' theorem, item 3 follows.

$3 \to 1$. Note that $P_M A P_M \leq \mu A$ if and only if $R(P_M A^{1/2}) \subseteq R(A^{1/2})$ if and only if there exists a unique $F \in L(\mathcal{H})$ such that $A^{1/2}F = P_M A^{1/2}$, $\ker(P_M A^{1/2}) \subseteq \ker F$ and $R(F) \subseteq R(A^{1/2})$. We shall see that $1 - F \in \mathcal{P}(A, \mathcal{S})$. Indeed, $F^2 = F$ by Corollary 2.3. F is A-selfadjoint because $AF = A^{1/2}P_M A^{1/2} = \Sigma(P, A)$ which is selfadjoint. Finally, $\ker F = \mathcal{S}$. Indeed, $AF = \Sigma(P, A)$, so $\ker F = \ker \Sigma(P, A) = \mathcal{S}$ because $\mathcal{S}$ is A-admissible.

$4 \leftrightarrow 1$. Using Proposition 3.3, we know that the fact that $(A, \mathcal{S})$ is compatible only depends on the first row PA of A. Therefore we can freely change A by $A + \lambda(1 - P)$, for $\lambda > 0$. In this case conditions 2 can be rewritten as condition 4, because $\Sigma(P, A + \lambda(1 - P)) = \Sigma(P, A) + \lambda(1 - P)$. ∎

Example 5.5. Given a positive injective operator $A \in L(\mathcal{H})$ with non-closed range, it is easy to show that there exists $\xi \in R(A^{1/2}) \setminus R(A)$. Let P_ξ be the orthogonal projection onto the subspace generated by ξ. Then $R(P_\xi) \subseteq R(A^{1/2})$, so that, by Douglas' theorem, $P_\xi \leq \lambda A$ for some positive number λ which we can suppose equal to 1, by changing A by λA. It is well known that this implies that the operator $B \in L(\mathcal{H} \oplus \mathcal{H})$ defined by

$$B = \begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix}$$

is positive. Let $\mathcal{S} = \mathcal{H}_1 = \mathcal{H} \oplus 0$. Then $\mathcal{S}^\perp = \mathcal{H}_2 = 0 \oplus \mathcal{H}$. We shall see that B is injective, $\mathcal{H}_1$ is B-admissible and $\mathcal{H}_2 \cap R(B)$ is dense in $\mathcal{H}_2$; but $\mathcal{P}(B, \mathcal{S})$ is empty.

Indeed, it is clear that B does not verify condition 3 of Proposition 3.3, so $\mathcal{P}(B, \mathcal{S})$ is empty. Let D be the reduced solution of $P_\xi = A^{1/2}X$. Then $\Sigma(P, B) = A - D^*D$. Note that $\ker D = \ker P_\xi$ implies $DP_\xi = D$. So $D^*D = P_\xi D^*D$. Then, if $0 \oplus \eta \in \ker \Sigma(P, B)$,

$$A\eta = D^*D\eta = P_\xi D^*D\eta = \lambda\xi \ \text{ for some } \ \lambda \in \mathbb{C} \quad \Rightarrow \quad \eta = 0$$

because $\xi \notin R(A)$ and A is injective. So $\ker \Sigma(P, B) = \mathcal{S}$ and $\mathcal{H}_1$ is B-admissible. Also

$$(3) \qquad B(\omega \oplus \eta) \in 0 \oplus \mathcal{H} \iff A\omega + P_\xi \eta = 0 \iff \omega = 0 \text{ and } \eta \in \{\xi\}^\perp.$$

Then $R(B) \cap \mathcal{H}_2 = \{B(0 \oplus \eta) : \eta \in \{\xi\}^\perp\} = 0 \oplus A(\{\xi\}^\perp)$. We shall see that $A(\{\xi\}^\perp)$ is dense in $\mathcal{H}$. Indeed, if $\zeta \in [A(\{\xi\}^\perp)]^\perp$, then $\langle \eta, A\zeta \rangle = \langle A\eta, \zeta \rangle = 0$ for all $\eta \in \{\xi\}^\perp$. So $A\zeta = \mu\xi$ for some $\mu \in \mathbb{C}$. As before this implies that $\zeta = 0$. Finally, the injectivity of B can be deduced by a similar argument as was used in (3).

6. A-selfadjoint projections: the closed range case

As before we fix $P \in \mathcal{P}$ with $R(P) = \mathcal{S}$. In this section A denotes a positive operator with closed range. We shall see that, in this case, the fact that $(A, \mathcal{S})$ is compatible depends only on the positivity of angle between $\ker A$ and $\mathcal{S}$. Namely, $(A, \mathcal{S})$ is compatible if and only if $\ker A + \mathcal{S}$ is closed. To establish the link between compatibility and the angle condition, we need to determine when $R(PAP)$ is closed. This is done in the following Lemma:

Lemma 6.1. *Let $A \in L(\mathcal{H})^+$.*

1. *The following equality holds: $\overline{R(PAP)} = \mathcal{S} \cap (\mathcal{S} \cap \ker A)^\perp$.*
2. *Moreover, if $R(A)$ is closed, then $R(PAP)$ is closed if and only if the subspace $\ker A + \mathcal{S}$ is closed.*

Proof. Observe that $\ker PAP = \ker AP = \mathcal{S}^\perp \oplus (\mathcal{S} \cap \ker A)$. Therefore

$$\overline{R(PAP)} = (\ker PAP)^\perp = \mathcal{S} \ominus (\mathcal{S} \cap \ker A) = \mathcal{S} \cap (\mathcal{S} \cap \ker A)^\perp := \mathcal{M}.$$

Clearly $\mathcal{M} \cap \ker A = \{0\}$. Suppose that $\mathcal{N} = \ker A + \mathcal{S} = \ker A \dot{+} \mathcal{M}$ is closed. Let Q be the projection from $\mathcal{N}$ onto $\mathcal{M}$ with $\ker Q = \ker A$; observe that Q is bounded. If $Q = 0$ then $\mathcal{M} = \{0\}$, $\mathcal{S} \subseteq \ker A$ and $PAP = 0$. If $\mathcal{M} \neq \{0\}$, given $\xi \in \mathcal{M}$, let $\eta \in R(A)$ such that $A\eta = A\xi$ (A is invertible in $R(A)$). Clearly $\eta = \xi + \zeta$ with $\zeta \in \ker A$. Then $\eta \in \mathcal{N}$, $Q\eta = \xi$ and $\|\xi\| \leq \|Q\| \, \|\eta\|$. Therefore

$$\langle PAP\xi, \xi \rangle = \langle A\xi, \xi \rangle = \langle A\eta, \eta \rangle \geq \lambda \|\eta\|^2 \geq \lambda \|Q\|^{-2} \|\xi\|^2$$

for some $\lambda > 0$, because $R(A)$ is closed and so $A|_{R(A)}$ is bounded from below. Conversely, if $R(PAP)$ is closed then $R(PAP) = \mathcal{M}$. Then there exists $\mu > 0$ such that $\langle A\xi, \xi \rangle = \|A^{1/2}\xi\|^2 \geq \mu \|\xi\|^2$ for $\xi \in \mathcal{M}$ and $A^{1/2}(\mathcal{M})$ is closed. So $\mathcal{N} = A^{-1/2}(A^{1/2}(\mathcal{M}))$ must be also closed. ∎

Theorem 6.2. *If $A \in L(\mathcal{H})^+$ has closed range then the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *$R(PAP)$ is closed.*
3. *$\mathcal{S} + \ker A$ is closed.*
4. *$R(PA)$ is closed.*
5. *$\mathcal{S}^\perp + R(A)$ is closed.*
6. *$R(AP) = A(\mathcal{S})$ is closed.*

Proof. $2 \leftrightarrow 3$. Use Lemma 6.1.

$2 \rightarrow 1$. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be decomposed by means of P. Note that $a = PAP$, so $R(a)$ is closed. Therefore, since $A \geq 0$, $R(b) \subseteq R(a^{1/2}) = R(a)$. Then $(A, \mathcal{S})$ is compatible by Proposition 3.3.

$1 \rightarrow 3$. Suppose that $(A, \mathcal{S})$ is compatible. Let $P_{A,\mathcal{S}} \in \mathcal{P}(A, \mathcal{S})$ and let $Q_{A,\mathcal{S}} = 1 - P_{A,\mathcal{S}}$. Then

$$\ker A \subseteq \ker(Q^*_{A,\mathcal{S}}A) = \ker(AQ_{A,\mathcal{S}}) = \mathcal{S} \dot{+} (\ker A \cap R(Q_{A,\mathcal{S}})) \subseteq \ker A + \mathcal{S}.$$

Therefore $\ker A + \mathcal{S} = (\ker AQ_{A,\mathcal{S}})$ which is closed.

$4 \leftrightarrow 5$. This is an easy consequence of the identity

$$R(A) + \mathcal{S}^\perp = P^{-1}[P(R(A))] = P^{-1}[R(PA)].$$

$3 \leftrightarrow 5$. In fact, it holds in general that the sum of two closed subspaces is closed if and only if the sum of their orthogonal complements is closed (see [10]).

$4 \leftrightarrow 6$. It is a general fact that $R(C)$ is closed if and only if $R(C^*)$ is closed. ■

Remark 6.3. Conditions 3, 4 and 5, 6 are known to be equivalent, because $R(P) = \mathcal{S}$ and $\ker P = \mathcal{S}^\perp$ (see Thm. 22 of [10]). They are also equivalent to, for example, the angle condition

$$c(\mathcal{S}, \ker A) < 1,$$

where $c(\mathcal{S}, \mathcal{T})$ is the cosine of the Friedrichs angle between the two subspaces $\mathcal{S}$, $\mathcal{T}$, defined by:

$$(4)\quad c(\mathcal{S}, \mathcal{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathcal{S} \cap (\mathcal{S} \cap \mathcal{T})^\perp,\ \|\xi\| \leq 1,\ \eta \in \mathcal{T} \cap (\mathcal{S} \cap \mathcal{T})^\perp,\ \|\eta\| \leq 1\}.$$

Also Lemma 6.1 can be deduced from the results of [10].

Corollary 6.4. *For every $A \in L(\mathcal{H})^+$ with closed range, the following conditions are equivalent:*

1. *The pair $(A, \mathcal{S})$ is compatible.*
2. *For all $B \in L(\mathcal{H})^+$ with $R(B) = R(A)$, the pair $(B, \mathcal{S})$ is compatible.*
3. *The pair $(P_{R(A)}, \mathcal{S})$ is compatible, if $P_{R(A)}$ denotes the orthogonal projection onto the closed subspace $R(A)$.*

Moreover, if $B \in L(\mathcal{H})^+$ and $R(B) = R(A)$, then the affine manifolds $\mathcal{P}(A, \mathcal{S})$ and $\mathcal{P}(B, \mathcal{S})$ are "parallel", i.e.

$$\mathcal{P}(B, \mathcal{S}) = (P_{B,\mathcal{S}} - P_{A,\mathcal{S}}) + \mathcal{P}(A, \mathcal{S}). \tag{5}$$

Proof. If $R(B) = R(A)$ then $\ker B = \ker A = \ker P_{R(A)}$ and, by Theorem 6.2, the three conditions are equivalent. Equality (5) follows from the parametrization given in Theorem 3.5, since

$$A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \ker B \cap \mathcal{S} = B^{-1}(\mathcal{S}^\perp) \cap \mathcal{S}$$

∎

Condition 3 is an invitation to consider the sets $\mathcal{P}(Q, \mathcal{S})$ for $Q \in \mathcal{P}$, which we study in the next section.

7. The case of two projections

In this section we shall study the case in which A is an orthogonal projection, i.e., $A = Q \in \mathcal{P}$. Then, by Theorem 6.2 (items 3 and 6), $\ker Q + R(P)$ is closed if and only $\mathcal{P}(Q, R(P))$ is not empty. In this case we shall denote by $P_{Q,P}$ the projection $P_{Q,R(P)}$ of Definition 3.4. In the following theorem we collect several conditions which are equivalent to the existence of $P_{Q,P}$. Notice, however, that the equivalence of items 3 to 10 can be deduced from results by R. Bouldin [5] and S. Izumino [17]; a nice survey on this and related subjects can be found in [10]. Observe that Theorem 6.2 provides alternative proofs of some of the equivalences.

Theorem 7.1. *Let $P, Q \in \mathcal{P}$ with $R(P) = \mathcal{S}$ and $R(Q) = \mathcal{T}$. The following are equivalent:*

1. *$(Q, \mathcal{S})$ is compatible.*
2. *$(P, \mathcal{T})$ is compatible.*
3. *$\ker Q + R(P)$ is closed.*
4. *$\ker P + R(Q)$ is closed.*

5. $R(PQ)$ *is closed.*
6. $R(QP)$ *is closed.*
7. $R(1 - P + Q)$ *is closed.*
8. $R(1 - Q + P)$ *is closed.*
9. $c(\mathcal{S}, \mathcal{T}^\perp) = c(\mathcal{T}, \mathcal{S}^\perp) < 1$.

If $\ker Q \cap \mathcal{S} = \{0\}$, *they are equivalent to*

10. $\|(1 - Q)P\| < 1$.

Proof. $1 \leftrightarrow 2 \leftrightarrow 3$. Follows from Theorem 6.2.

$3 \leftrightarrow 4 \leftrightarrow 9$. Follows from Theorem 13 of [10].

$3 \leftrightarrow 6$ and $4 \leftrightarrow 5$. Follows from Theorem 22 of [10].

$5 \leftrightarrow 7$ and $6 \leftrightarrow 8$. Follows from 2.5 of [17]

$3 \leftrightarrow 10$. Follows from Theorem 13 of [10].

■

Suppose that any of the conditions of Theorem 7.1 is satisfied by $P, Q \in \mathcal{P}$. As a final result, we shall compute $\|P_{Q,P}\|$. First, we assume that $\ker Q \cap R(P) = \{0\}$.

Proposition 7.2. *Let* $P, Q \in \mathcal{P}$. *Denote* $R(P) = \mathcal{S}$. *Suppose that* $\ker Q \cap \mathcal{S} = \{0\}$ *and* $\ker Q + \mathcal{S}$ *is closed. Then* $Q|_\mathcal{S}$ *is invertible in* $L(\mathcal{S}, Q(\mathcal{S}))$, PQP *is invertible in* $L(\mathcal{S})$ *and*

$$\|P_{Q,P}\| = \|(Q|_\mathcal{S})^{-1}\| = \|(PQP)^{-1}\|^{1/2} = (1 - \|(1 - Q)P\|^2)^{-1/2}.$$

Proof. Using Theorem 7.1, we know that $\|(1 - Q)P\| < 1$. Then

$$\|P - PQP\| = \|P(1 - Q)P\| = \|(1 - Q)P\|^2 < 1,$$

showing that PQP is invertible in $L(\mathcal{S})$. On the other hand consider $Q|_\mathcal{S}\colon \mathcal{S} \to Q(\mathcal{S})$. By Theorem 6.2, $Q(\mathcal{S})$ is closed, so $Q|_\mathcal{S}$ is invertible in $L(\mathcal{S}, Q(\mathcal{S}))$.

If $P_{Q,P} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$, then $\|P_{Q,P}\|^2 = 1 + \|d\|^2$. Recall that d is the reduced solution of the equation $PQPX = PQ(1 - P)$. So, by Theorem 2.2,

$$\begin{aligned}
\|d\|^2 &= \inf\{\lambda > 0 : PQ(1 - P)QP \le \lambda PQPQP\} \\
&= \inf\{\lambda > 0 : PQP \le (1 + \lambda)(PQP)^2\} \\
&= \inf\{\lambda > 0 : P \le (1 + \lambda)PQP\} = \inf\{\lambda > 0 : (PQP)^{-1} \le (1 + \lambda)P\} \\
&= \|(PQP)^{-1}\| - 1.
\end{aligned}$$

So $\|P_{Q,P}\|^2 = \|(PQP)^{-1}\|$. Note also that

$$P \le (1 + \lambda)PQP \iff \|\xi\|^2 \le (1 + \lambda)\langle PQP\xi, \xi\rangle = (1 + \lambda)\|Q\xi\|^2 \quad \text{for all } \xi \in \mathcal{S}.$$

Taking infimum over λ, we get $\|P_{Q,P}\| = (1+\|d\|^2)^{1/2} = \|(Q|_{\mathcal{S}})^{-1}\|$.

It is easy to see that, if $0 < A \le I$ in $L(\mathcal{H})$, then $\|I-A\| = 1 - \|A^{-1}\|^{-1}$. Applying this identity to PQP in $L(\mathcal{S})$ we get

$$\|(PQP)^{-1}\| = (1-\|P-PQP\|)^{-1} = (1-\|P(1-Q)P\|)^{-1} = (1-\|(1-Q)P\|^2)^{-1} \quad \blacksquare$$

Remark 7.3. Let $P, Q \in \mathcal{P}$ with $R(P) = \mathcal{S}$ and $R(Q) = \mathcal{T}$ and suppose that any of the conditions of Theorem 7.1 hold. By Proposition 3.5,

$$\begin{aligned} \ker P_{Q,P} &= Q^{-1}(\ker P) \ominus (\ker Q \cap R(P)) \\ &= (\ker Q + R(Q) \cap \ker P) \ominus (\ker Q \cap R(P)). \end{aligned}$$

Therefore, in the case that

$$R(Q) \cap \ker P = \{0\} = \ker Q \cap R(P) \tag{6}$$

(e.g., if P and Q are in *position p* [12], [9] or *generic position* [15]) we can conclude that $P_{Q,P}$ is the projection given by

$$\ker P_{Q,P} = \ker Q \quad \text{and} \quad R(P_{Q,P}) = R(P).$$

Then $\mathcal{S} \dot{+} \ker Q = \mathcal{H}$ and $P_{Q,P}$ is the oblique projection given by this decomposition of $\mathcal{H}$. In this case, formula $\|P_{Q,P}\| = (1-\|(1-Q)P\|^2)^{-1/2}$ has been proved by Ptak in [23] (see also [6]).

Theorem 7.4. *Let $P, Q \in \mathcal{P}$ and assume that $\ker Q + R(P)$ is closed. Denote by $\mathcal{N} = \ker Q \cap R(P)$, $\mathcal{M} = R(P) \ominus \mathcal{N}$ and $P_0 = P_{\mathcal{M}}$. Then*

1. *$\mathcal{P}(Q, \mathcal{M})$ has only one element, namely P_{Q,P_0}.*
2. *$P_{Q,P} = P_{\mathcal{N}} + P_{Q,P_0}$.*
3. *$\|P_{Q,P}\| = \|P_{Q,P_0}\| = (1-\|(1-Q)P_0\|^2)^{-1/2}$.*

Proof. If $\mathcal{N} = \{0\}$, we can use Proposition 7.2. Assume now that $\mathcal{N}$ is not trivial. Then, by the results of section 3, we get the matrix form

$$P_{Q,P} = \begin{array}{c} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathcal{N} \\ \mathcal{M} \\ \ker P \end{array} .$$

Denote

$$(7) \qquad T = P_{Q,P} - P_{\mathcal{N}} = \begin{array}{c} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & d \\ 0 & 0 & 0 \end{pmatrix} \end{array} \begin{array}{l} \mathcal{N} \\ \mathcal{M} \\ \ker P \end{array} .$$

We must show that $T = P_{Q,P_0}$. Note that $\ker Q \cap \mathcal{M} = \{0\}$, so $\mathcal{P}(Q, \mathcal{M})$ has, at most, one element. On the other hand, $T^2 = T$ and $R(T) = \mathcal{M}$ by equation (7). Also

$$T^*Q = (T^* + P_{\mathcal{N}})Q = P_{Q,P}^*Q = QP_{Q,P} = Q(P_{\mathcal{N}} + T) = QT,$$

because $QP_{\mathcal{N}} = 0$. So, $T = P_{Q,P_0}$ as claimed. By equation (7) and Proposition 7.2,

$$P_{Q,P} = P_{\mathcal{N}} + P_{Q,P_0} \quad \text{and} \quad \|P_{Q,P}\| = \|P_{Q,P_0}\| = (1 - \|(1-Q)P_0\|^2)^{-1/2},$$

because $\ker Q \cap R(P_0) = \{0\}$. ∎

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