



# Index of Hadamard multiplication by positive matrices II

Gustavo Corach <sup>a,\*</sup>, Demetrio Stojanoff <sup>b,2</sup>

<sup>a</sup>*Instituto Argentino de Matemática, Saavedra 15 Piso 3 (1083), Buenos Aires, Argentina*

<sup>b</sup>*Depto. de Matemática, UNLP, 1 y 50 (1900), La Plata, Argentina*

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## Abstract

For each  $n \times n$  positive semidefinite matrix  $A$  we define the minimal index  $I(A) = \max\{\lambda \geq 0 : A \circ B \geq \lambda B \text{ for all } B \geq 0\}$  and, for each norm  $N$ , the  $N$ -index  $I_N(A) = \min\{N(A \circ B) : B \geq 0 \text{ and } N(B) = 1\}$ , where  $A \circ B = [a_{ij}b_{ij}]$  is the Hadamard or Schur product of  $A = [a_{ij}]$  and  $B = [b_{ij}]$  and  $B \geq 0$  means that  $B$  is a positive semidefinite matrix. A comparison between these indexes is done, for different choices of the norm  $N$ . As an application we find, for each bounded invertible selfadjoint operator  $S$  on a Hilbert space, the best constant  $M(S)$  such that  $\|STS + S^{-1}TS^{-1}\| \geq M(S)\|T\|$  for all  $T \geq 0$ . © 2001 Published by Elsevier Science Inc.

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## 1. Introduction

Given  $A = [a_{ij}]$ ,  $B = [b_{ij}] \in M_n = M_n(\mathbb{C})$ , the algebra of  $n \times n$  matrices over  $\mathbb{C}$ , denote by  $A \circ B$  the Hadamard product  $[a_{ij}b_{ij}]$ . In this paper  $A \geq 0$  means

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\* Corresponding author.

*E-mail addresses:* gcorach@mate.dm.uba.ar (G. Corach), demetrio@mate.dm.uba.ar (D. Stojanoff).

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that  $A$  is positive semidefinite;  $P_n = \{A \in M_n : A \geq 0\}$  denotes the set of positive semidefinite matrices.

Every  $A \in M_n$  defines a linear map  $\Phi_A : M_n \rightarrow M_n$  given by  $\Phi_A(B) = A \circ B$  for  $B \in M_n$ . By Schur’s product theorem [22] (see also [14, 7.5.3])  $A \circ B \in P_n$  if  $A, B \in P_n$  so that  $\Phi_A$  is a positive linear map. Actually it is completely positive, i.e., the inflation map  $\Phi_A^{(m)}$ , which acts entrywise as  $\Phi_A$  on  $M_m(M_n)$ , is positive for all  $m \in \mathbb{N}$ ; see [20, Proposition 1.2].

In [23], the second author studied conditions under which

$$\max \{ \lambda \geq 0 : \Phi_A(B) \geq \lambda B, \forall B \in P_n \} = \inf \{ \| \Phi_A(B) \| : B \in P_n, \| B \| = 1 \}.$$

The problem comes from the theory of conditional expectations. A *conditional expectation* on a  $C^*$ -algebra  $\mathcal{A}$  is a norm one projection  $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$  such that  $\mathcal{E}(\mathcal{A})$  is a sub- $C^*$ -algebra of  $\mathcal{A}$ . Every conditional expectation  $\mathcal{E}$  satisfies the condition

$$\begin{aligned} \sup \{ \lambda \geq 0 : \| \mathcal{E}(a) \| \geq \lambda \| a \| \forall a \in \mathcal{A}^+ \} \\ = \sup \{ \lambda \geq 0 : \mathcal{E}(a) \geq \lambda a \forall a \in \mathcal{A}^+ \}, \end{aligned} \tag{1}$$

where  $\mathcal{A}^+ = \{c \in \mathcal{A} : c \geq 0\}$ . The inverse of this number is called the *index* of  $\mathcal{E}$  and it is useful in the classification of inclusions of subalgebras of  $C^*$ -algebras. Note that a conditional expectation is completely positive. If  $\mathcal{E} : \mathcal{A} \rightarrow \mathcal{A}$  is a completely positive map that is not a conditional expectation, (1) fails in general and the problem arises of characterizing those  $\mathcal{E}$  such that (1) holds.

For  $A \in P_n$  define the *minimal index*  $I(A) = \max\{\lambda \geq 0 : A \circ B \geq \lambda B \forall B \in P_n\}$  and the *N-index*  $I_N(A) = \max\{\lambda \geq 0 : N(A \circ B) \geq \lambda N(B), \forall B \in P_n\}$  for any given norm  $N$  on  $M_n$ . We are mainly concerned with Schatten norms  $\| \cdot \|_p$  for  $p = 1, 2$ , and  $\infty$ ; we use the shorter notations  $I_1, I_2$ , and  $I_{sp}$  for  $I_{\| \cdot \|_1}, I_{\| \cdot \|_2}$ , and  $I_{\| \cdot \|_\infty}$ , respectively.  $I_{sp}$  is called the *spectral index*.

If  $\mathcal{A} = M_n$ , every conditional expectation  $\mathcal{E}$  has the form  $\mathcal{E}(C) = U \Phi_A(U^* C U) U^*$ , where  $U \in M_n$  is unitary and  $A \in P_n$  is a direct sum of matrices whose entries are all equal to one. In this case,  $\text{Ind}(\mathcal{E})^{-1} = 1/k = I_{sp}(A) = I(A)$ , where  $k$  is the number of diagonal blocks of  $A$ . We remove the inverse in our definition of minimal and  $N$ -index in order to avoid complications when the index is zero.

For references on the norm of  $\Phi_A$ , see [2,3,9–11,17,19,20] and references included therein. There is an extensive bibliography about the index of conditional expectations; see [21] and its references. For a deep study of the index theory of completely positive maps on operator algebras, see [5,12].

This paper compares these notions of index and investigates how to compute them. The results obtained are useful in the study of certain operator inequalities. Recall that if  $L(\mathcal{H})$  is the algebra of bounded linear operators on a Hilbert space  $\mathcal{H}$  and  $S \in L(\mathcal{H})$  is a selfadjoint invertible operator, then

$$\| STS + S^{-1} T S^{-1} \| \geq 2 \| T \|$$

for all  $T \in L(\mathcal{H})$  [4]. It is natural to ask whether 2 is the best constant for each fixed  $S$ . Using a reduction to the finite dimensional case and a criterion for computing

$I_{\text{sp}}(B)$  for matrices  $B \in P_n$  such that  $B \geq 0$ , in terms of the principal submatrices of  $B$  (see Corollary 4.6), we are able to find for each  $S$ , the best constant  $M(S)$  such that  $\|STS + S^{-1}TS^{-1}\| \geq M(S)\|T\|$  for all  $T \geq 0$ .

In this paper we write  $A \geq 0$  for matrices (or vectors) with nonnegative entries. We write  $A \geq B$  or  $A \geq B$  if  $A - B \geq 0$  or  $A - B \geq 0$ , respectively.  $R(A)$  is the range of  $A$  and  $\ker A$  is the kernel of  $A$ , where  $A$  is thought of as acting on  $\mathbb{C}^n$ .  $A^T$  is the transpose matrix of  $A$ ,  $\bar{A} = [\bar{a}_{ij}]$  is the conjugate matrix of  $A$ , and  $A^* = \bar{A}^T$ .  $\rho(A)$  is the spectral radius of  $A$  and  $A^\dagger$  is the Moore–Penrose pseudoinverse of  $A$ . Throughout,  $p$  denotes the vector  $(1, \dots, 1)^T$  and  $E$  denotes the matrix  $pp^T$ , which has all its entries equal to 1.

Section 2 contains some elementary characterizations of the minimal index. We prove that, for a given  $A \in P_n$ ,  $I(A) > 0$  if and only if  $p \in R(A)$ ; and, in this case,  $I(A)^{-1}$  is the spectral radius of  $A^\dagger E$ .

Section 3 is devoted to a comparison of the minimal index with the spectral index. The main result in this section is the following: if  $A \in P_n$ ,  $A \geq 0$ , and there exists a vector  $u \in A^{-1}(\{p\})$  such that  $u \geq 0$ , then  $I(A) = I_{\text{sp}}(A)$ . The converse holds if  $I(A) \neq 0$ , without the hypothesis that  $A \geq 0$ .

In Section 4 we compare the indexes associated with the spectral and the Frobenius norms. The main result here is that  $I_2(A) = I_{\text{sp}}(\bar{A} \circ A)^{1/2}$  for every  $A \in P_n$ . As a consequence of the proof of this result we compute  $I_{\text{sp}}(B)$  for matrices  $B \in P_n$  such that  $B \geq 0$ , in terms of the principal submatrices of  $B$  (see Corollary 4.6). This criterion is the main tool used in Section 5, where we compute the minimal and spectral indexes of  $\Lambda = [\lambda_i \lambda_j + 1/\lambda_i \lambda_j]$  for any  $n$ -tuple of positive numbers  $\lambda_1, \dots, \lambda_n$  and use them to find, for each bounded Hermitian invertible operator  $S$  on a Hilbert space  $\mathcal{H}$ , the number

$$M(S) = \inf \left\{ \|STS + S^{-1}TS^{-1}\| : T \geq 0, \|T\| = 1 \right\}. \tag{2}$$

For example, if  $\|S\| \leq 1$ , then  $M(S) = \|S\|^2 + \|S\|^{-2}$ .

## 2. Elementary properties of the index

Let us give more detailed definitions:

**Definition 2.1.** The Hadamard *minimal* index of  $A \in P_n$  is

$$\begin{aligned} I(A) &= \max \left\{ \lambda \geq 0 : A \circ B \geq \lambda B \ \forall B \in P_n \right\} \\ &= \max \left\{ \lambda \geq 0 : (\Phi_A - \lambda Id)B \geq 0 \text{ for all } B \in P_n \right\} \\ &= \max \left\{ \lambda \geq 0 : A - \lambda E \geq 0 \right\}. \end{aligned}$$

The last equality follows from the fact that for  $C \in M_n$ , the map  $\Phi_C$  is positive if and only if  $C \geq 0$ .

**Definition 2.2.** Given a norm  $N$  in  $M_n$ , the Hadamard  $N$ -index for  $A \in P_n$  is

$$I_N(A) = \max \{ \lambda \geq 0 : N(A \circ B) \geq \lambda N(B) \forall B \in P_n \} \\ = \min \{ N(A \circ B) : B \in P_n \text{ and } N(B) = 1 \}.$$

The index associated with the spectral norm  $\| \cdot \|$  is denoted by  $I_{sp}(\cdot)$ ; we call it the *spectral index*. The index associated with the Frobenius norm  $\| \cdot \|_2$  is denoted by  $I_2(\cdot)$ .

**Example 2.3.** Let  $A = [a_{ij}]$  and  $B = [b_{ij}] \in P_n$ . Then, if  $\| \cdot \|_1$  denotes the trace norm,

$$\|B\|_1 = \text{tr}(B) = \sum_{i=1}^n b_{ii} \quad \text{and} \quad \|A \circ B\|_1 = \text{tr}(A \circ B) = \sum_{i=1}^n a_{ii} b_{ii}.$$

From these identities it is easy to see that, if  $I_1(\cdot)$  denotes the associated index, then  $I_1(A) = \min_{1 \leq i \leq n} a_{ii}$  for every  $A \in P_n$ .

**Remark 2.4.** Estimation of the  $N$ -index of a matrix  $A$  can be seen as an inequality, namely,  $N(A \circ B) \geq I_N(A)N(B)$  for every  $B \in P_n$ . It would also be interesting to get such inequalities without the restriction  $B \geq 0$  (of course, for matrices  $A$  without zero entries). But in this case, the constant involved is the inverse of the norm induced by  $N$  of the map  $\Phi_C$ , where  $c_{ij} = a_{ij}^{-1}$ . The computation of such norms is well known (see [9–11, 17, 19, 20]). For the index associated with the Frobenius norm, the computation of an infimum without the restriction  $B \geq 0$  becomes trivial, but with this restriction it is certainly not trivial, as shown in Theorem 4.3.

2.1. The minimal index  $I(A)$

The index  $I(\cdot)$  is called minimal because  $I(A) \leq I_N(A)$  for every unitary invariant norm  $N$ . Indeed, given  $B \in P_n$ , then  $A \circ B \geq I(A)B$  and, by Weyl’s monotonicity theorem,  $s_i(A \circ B) \geq I(A)s_i(B)$ ,  $1 \leq i \leq n$  (where  $s_i$  denote the  $i$ th singular value). Therefore  $N(A \circ B) \geq I(A)N(B)$  by Ky Fan’s dominance theorem; see [15, 3.5.9].

Given  $B, C \geq 0$  the following relation holds:

$$\max \{ \alpha \geq 0 : \alpha C \leq B \} = \|C^{1/2} B^\dagger C^{1/2}\|^{-1} = \rho(B^\dagger C)^{-1}. \tag{3}$$

In fact, if  $B$  is nonsingular, (3) follows from [14, 7.7.3] (see also [1, 6, 13, 16]). If  $B$  has rank  $r < n$ , there exist a unitary matrix  $U$  and

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix}$$

such that  $A_1$  is an  $r$ -by- $r$  invertible matrix and  $B = UAU^*$ . If  $\alpha \geq 0$  and  $B \geq \alpha C$  then, setting

$$D = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}^* & D_{22} \end{bmatrix} = U^* C U,$$

we get

$$\begin{bmatrix} A_1 & 0 \\ 0 & 0 \end{bmatrix} = A \succeq \alpha D$$

so that  $D_{22} = 0$  and, then,  $D_{12} = 0$ . Therefore  $A_1 \succeq \alpha D_{11}$  and, by the nonsingular case,  $\rho(A_1^{-1} D_{11}) \leq \alpha$ . The result follows by observing that  $\rho(A_1^{-1} D_{11}) = \rho(B^\dagger C)$ . Observe also that the block structure of  $D$  and the invertibility of  $A_1$  imply the inclusion  $R(C) \subset R(B)$ .

Taking  $B = A$  and  $C = E$  in (3) we get  $I(A) = \max\{\alpha \geq 0 : A \succeq \alpha E\} = \rho(A^\dagger E)^{-1}$  for every  $A \in P_n$  such that  $p \in R(A)$ . This proves part of the following result.

**Proposition 2.5.** *Let  $A \in P_n$ . Then  $I(A) \neq 0$  if and only if the vector  $p$  belongs to  $R(A)$ . In this case, for any vector  $y$  such that  $Ay = p$ , we have*

$$I(A) = \rho(A^\dagger E)^{-1} = \langle y, p \rangle^{-1} = \left( \sum_{i=1}^n y_i \right)^{-1}. \tag{4}$$

**Proof.** By definition,  $I(A) \neq 0$  if and only if there exists  $\alpha > 0$  such that  $A \succeq \alpha E$ . By the comments following (3), this means that  $R(E) \subset R(A)$  or, since  $p$  spans  $R(E)$ , that  $p \in R(A)$ . Finally,  $I(A)^{-1} = \rho(A^\dagger E) = \rho(A^\dagger p p^T) = \rho(p^T A^\dagger p) = p^T A^\dagger p = \langle A^\dagger p, p \rangle$ , and  $A(A^\dagger p) = p$ . If  $y$  is any vector such that  $Ay = p$ , then  $y - A^\dagger p \in \ker A = R(A)^\perp$ , so  $\langle y, p \rangle = \langle A^\dagger p, p \rangle$ .  $\square$

**Proposition 2.6.** *Let  $A \in P_n$ . Then  $I(A) = \min\{\langle z, Az \rangle : \sum_{i=1}^n z_i = 1\}$ .*

**Proof.** If  $\langle z, p \rangle = 1$ , then  $\langle z, Az \rangle \geq I(A) \langle z, Ez \rangle = I(A) z^* p p^* z = I(A) \langle z, p \rangle^2 = I(A)$ . If  $p \in R(A)$ , let  $x \in \mathbb{C}^n$  be such that  $Ax = p$ . Then  $z = I(A)x$  satisfies  $\langle z, p \rangle = I(A) \langle x, p \rangle = 1$  and  $\langle z, Az \rangle = I(A) \langle z, p \rangle = I(A)$  by Proposition 2.5. If  $p \notin R(A) = (\ker A)^\perp$ , then there exists  $z \in \ker A$  such that  $\langle z, p \rangle = 1$  and  $\langle z, Az \rangle = 0 = I(A)$ .  $\square$

**Remark 2.7.** Using Proposition 3.9 of [23] and the results of this section, it is easy to see that, for all  $A \in P_n$  and  $m \in \mathbb{N}$ , the inflation matrix  $A^{(m)} = E_m \otimes A$  (where  $E_m \in P_m$  has all its entries equal to 1) satisfies  $I_{\text{sp}}(A^{(m)}) = I_{\text{sp}}(A)$  and  $I(A^{(m)}) = I(A)$ . Note that the inflation map  $\Phi_A^{(m)} = \Phi_{A^{(m)}}$ . Therefore the indexes of  $\Phi_A$  are invariant under inflations and are invariants of  $\Phi_A$  as a completely positive map.

2.2.  $I_N(A)$  for general norms

Let  $A \in P_n$  and let  $N$  be a norm in  $M_n$ . If  $I_N(A) = 0$ , there is some positive semidefinite matrix  $C$  such that  $N(C) = 1$  (so  $C \neq 0$ ) and  $N(A \circ C) = 0$ . But then  $A \circ C = 0$ , so  $c_{ij} = 0$  whenever  $a_{ij} \neq 0$ . If all  $a_{ii} \neq 0$ , then all  $c_{ii} = 0$ , which forces  $C = 0$ . This contradiction shows that if all  $a_{ii} \neq 0$ , then  $I_N(A) > 0$ . Conversely, if some  $a_{ii} = 0$  just take  $C = E_{ii}/N(E_{ii})$ , so  $I(A) \leq N(A \circ C) = 0$ . Thus,

$$I_N(A) > 0 \quad \text{if and only if} \quad \text{all } a_{ii} > 0. \tag{5}$$

Let  $J \subseteq \{1, 2, \dots, n\}$  and let  $A_J$  denote the principal submatrix of  $A$  associated with  $J$ . Then, minimality ensures that

$$I_N(A) \leq I_N(A_J). \tag{6}$$

**Remark 2.8.** Let  $A \in P_n$ . Then, it can be shown that for every unitary invariant norm  $N$ , the following properties hold:

1. If  $A$  has rank 1, then  $I_N(A) = \min_{1 \leq i \leq n} A_{ii}$ .
2. If  $A$  is positive and diagonal, then  $I_N(D) = N'(D^{-1})^{-1} = \Phi'(a_{11}^{-1}, \dots, a_{nn}^{-1})^{-1}$ , where  $N'$  is the dual norm of  $N$  and  $\Phi'$  is the symmetric gauge function on  $\mathbb{R}^n$  associated with  $N'$ ; see [7, Chapter IV].

Proposition 3.2 of [23] tells us that

$$I_{\text{sp}}(A) = \inf \{ I_{\text{sp}}(D) : A \preceq D \text{ and } D \text{ is diagonal} \}, \tag{7}$$

and one could hope that a similar formula holds for any norm, but it does not. In fact, Corollary 4.4 says that, for every  $A \in P_n$  and the Frobenius norm,

$$\begin{aligned} I_2(A) &= \inf \left\{ \left( \sum_1^n D_{ii}^{-2} \right)^{-1/2} : D \text{ is diagonal and } A \circ \bar{A} \preceq D^2 \right\} \\ &= \inf \{ I_2(D) : D \text{ is diagonal and } A \circ \bar{A} \preceq D^2 \}. \end{aligned} \tag{8}$$

Note that the condition  $A \circ \bar{A} \preceq D^2$  is strictly less restrictive than  $A \preceq D$  (the reverse implication follows from Schur’s theorem). Nevertheless, Eq. (8) allows one to compute the 2-index for every positive semidefinite matrix using only diagonal matrices. We intend to study this type of characterizations of  $I_N(A)$  for general norms in a forthcoming paper.

3.  $I(A) = I_{\text{sp}}(A)$

In this section we characterize those matrices  $A \in P_n$  such that  $I(A) = I_{\text{sp}}(A)$ . In [23] it is shown that for

$$A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in P_2,$$

$$0 \neq I_{\text{sp}}(A) = I(A) \Leftrightarrow b \in \mathbb{R} \text{ and } 0 \leq b \leq \min\{a, c\} \neq 0. \tag{9}$$

This is easily seen to be equivalent to the conditions

1.  $A \geq 0$ .
2. There exists a vector  $z \geq 0$  such that  $Az = (1, 1)^T$  (if  $A$  is invertible, this means that  $A^{-1}(1, 1)^T \geq 0$ ).

We prove that, for positive semidefinite matrices of any size with nonnegative entries, condition 2 is equivalent to the identity  $I_{\text{sp}}(A) = I(A)$ . But first we need two lemmas:

**Lemma 3.1.** *Let  $A \in P_n$  and  $L = \{z \in \mathbb{R}^n : \sum_i z_i = 1\}$ . Consider the sets*

$$V_1 = \{z \in L : \langle Az, z \rangle = I(A)\} \text{ and } V_2 = \{z \in L : Az = I(A)p\}.$$

*Then  $V_1 = V_2 \neq \emptyset$ . Moreover, any local extreme point of the map  $G : L \rightarrow \mathbb{R}$  given by  $G(z) = \langle Az, z \rangle$ , belongs to  $V_2$ .*

**Proof.** It is clear that  $V_2 \subseteq V_1$ . By Proposition 2.6,  $I(A) \leq \min\{\langle Av, v \rangle : v \in L\}$ . Then the map  $G : L \rightarrow \mathbb{R}$  given by  $G(z) = \langle Az, z \rangle = \sum_{i,j} a_{ij}z_jz_i$  is differentiable and bounded from below. Thus  $G$  must have a minimum, which is also a critical point. Let the columns of  $X \in M_{n,n-1}$  be a basis for the orthogonal complement of  $p$ . Then we seek the unconstrained minimum of

$$\begin{aligned} \Phi(\xi) &= G(X\xi + p/n) \\ &= \langle A(X\xi + p/n), (X\xi + p/n) \rangle \\ &= (X + p/n)^T A(X\xi + p/n) \end{aligned}$$

over all  $\xi \in \mathbb{R}^{n-1}$ . But  $\nabla\Phi(\xi) = 2X^T A(X\xi + p/n) = 0$  says that at a critical point  $\xi_0$ ,  $Az_0 = Ap$  for some  $A$ , where  $z_0 \equiv X\xi_0 + p/n \in L$ . But, in that case,

$$I(A) \leq \langle Az_0, z_0 \rangle = \lambda \langle p, z_0 \rangle = \lambda.$$

If  $I(A) = 0$ , then  $\lambda = 0$ , because  $p \notin R(A)$ , by Proposition 2.5. If  $I(A) > 0$ , then also  $\lambda = I(A)$ , because  $y = \lambda^{-1}z_0$  satisfies  $Ay = p$  and

$$\lambda = \langle Az_0, z_0 \rangle = \lambda^2 \langle Ay, y \rangle = \lambda^2 I(A)^{-1}.$$

So  $\xi_0 \in \mathbb{R}^{n-1}$  is a critical point of  $\Phi$  if and only if  $z_0 = X\xi_0 + p/n \in V_2$ . Since each local extreme must be a critical point, this shows that  $\emptyset \neq V_1 \subseteq V_2$  and the final assertion is proved.  $\square$

**Lemma 3.2.** *Let  $A \in M_n$ , and suppose  $x \in \mathbb{C}^n$  with  $\|x\| = 1$ . Let  $y = x \circ \bar{x} = (|x_1|^2, \dots, |x_n|^2)^T$ .*

1. *If  $Ay = \lambda p$  for some  $\lambda \in \mathbb{C}$ , then  $(A \circ xx^*)x = \lambda x$ .*

2. Conversely, if all  $x_i \neq 0$  and  $(A \circ xx^*)x = \lambda x$  for some  $\lambda \in \mathbb{C}$ , then  $Ay = \lambda p$ . If  $A \in P_n$ , the eigenvalue  $\lambda$  of the matrix  $A \circ xx^*$  associated with the vector  $x$  must be  $I(A)$  and  $Ay = I(A) p$ .

**Proof.** Suppose that  $Ay = \lambda p$ . Then

$$\begin{aligned} (A \circ xx^*)x &= (a_{ij}x_i\bar{x}_j) \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} (\sum_j a_{1j}|x_j|^2) x_1 \\ \vdots \\ (\sum_j a_{nj}|x_j|^2) x_n \end{pmatrix} \\ &= \begin{pmatrix} (Ay)_1 x_1 \\ \vdots \\ (Ay)_n x_n \end{pmatrix} = \lambda x. \end{aligned} \tag{10}$$

Eq. (10) shows that if all  $x_i \neq 0$  and  $(A \circ xx^*)x = \lambda x$ , then  $Ay = \lambda p$ . If  $A \in P_n$  and  $I(A) = 0$ , then  $\lambda = 0$  because  $p \notin R(A)$ . If  $I(A) \neq 0$ , then  $p \in R(A) = (\ker A)^\perp$ . So  $Ay \neq 0$  because  $1 = \|x\|^2 = \langle p, y \rangle \neq 0$ . Then  $\lambda \neq 0$ . If  $z = \lambda^{-1}y$ , then  $Az = p$  and  $1 = \langle p, y \rangle = \lambda \langle Az, z \rangle = \lambda I(A)^{-1}$ , by Proposition 2.5,  $\lambda = I(A)$ .  $\square$

**Theorem 3.3.** Let  $A \in P_n$ .

1. If  $I_{\text{sp}}(A) = I(A) \neq 0$ , then there exists a vector  $u \geq 0$  such that  $Au = p$ .
2. If  $A \geq 0$  and there exists a vector  $u \geq 0$  such that  $Au = p$ , then  $I_{\text{sp}}(A) = I(A)$ .

**Proof.**

1. Observe that  $I(A)B \preceq A \circ B \preceq \|A \circ B\| I$ . By Lemma 2.1 of [23], there exists  $x \in \mathbb{R}^n$  such that  $\|x\| = 1$  and  $I_{\text{sp}}(A) = \|A \circ xx^*\|$ . So, if  $y = x \circ x$ , then  $\langle y, p \rangle = 1$  and

$$I(A)xx^T \preceq D_x A D_x \preceq I(A)I,$$

which implies that

$$I(A) = I(A)(x^T x)^2 \leq x^T D_x A D_x x = y^T A y \leq I(A)x^T x = I(A).$$

We have  $\langle Ay, y \rangle = I(A)$ ,  $y \geq 0$  and  $\langle y, p \rangle = 1$ . Then, by Lemma 3.1,  $Ay = I(A)p$ . Take  $u = I(A)^{-1}y$ .

2. Let  $u$  be a nonnegative vector such that  $Au = p$ . Let  $y = I(A)u$  and  $x = (y_1^{1/2}, \dots, y_n^{1/2})^T$ . Note that  $\|x\|^2 = \langle y, p \rangle = 1$ . By Lemma 3.2 we know that  $x$  is an eigenvector of  $A \circ xx^*$  with eigenvalue  $I(A)$ . Recall that always  $I(A) \leq I_{\text{sp}}(A)$ .

**Case 1.** Suppose that  $x$  has strictly positive entries. Since  $A \circ xx^* \geq 0$ , it is well known (see Corollary 8.1.30 of [14]) that the eigenvalue  $I(A)$  of  $x$  must be the



spectral radius of  $A \circ xx^*$ . Since  $A \circ xx^* \in P_n$  we deduce that  $I(A) = \|A \circ xx^*\| \geq I_{\text{sp}}(A)$ .

**Case 2.** Let  $J = \{i : x_i \neq 0\}$ ,  $A_J$  the principal submatrix of  $A$  determined by the indexes of  $J$  and similarly define  $x_J$ . Then  $x_J$  is an eigenvector of  $A_J \circ x_J x_J^*$  with eigenvalue  $I(A)$ . Note also that  $A_J \circ x_J x_J^* \geq I(A_J)x_J x_J^*$  and  $x_J x_J^*(x_J) = \|x_J\|^2 x_J = x_J$ . Then

$$0 \leq \langle (A_J \circ x_J x_J^* - I(A_J)x_J x_J^*)x_J, x_J \rangle = I(A) - I(A_J)$$

and, by the definition of  $I$ ,  $I(A_J) = I(A)$ . Now, as in Case 1, we can deduce that

$$I(A) = I(A_J) = \|A_J \circ x_J x_J^*\| \geq I_{\text{sp}}(A_J) \geq I_{\text{sp}}(A),$$

where the last inequality holds by (6).  $\square$

**Corollary 3.4.** Let  $A \in P_n$  such that  $A \geq 0$  and  $I_{\text{sp}}(A) = I(A)$ . Let  $u$  be a nonnegative vector such that  $Au = p$  and  $y = I(A)u$ .

1. Let  $x = (y_1^{1/2}, \dots, y_n^{1/2})^T$ . Then  $\|x\| = 1$  and  $\|A \circ xx^*\| = I_{\text{sp}}(A)$ .
2. Let  $J = \{i : u_i \neq 0\}$  and denote by  $A_J$  the principal submatrix of  $A$  determined by  $J$ . Then  $I(A) = I(A_J) = I_{\text{sp}}(A_J) = I_{\text{sp}}(A)$ .

**Proof.** This follows from the proof of Theorem 3.3.  $\square$

**Remark 3.5.** In Theorem 3.3(2), the hypothesis that  $A \geq 0$  is essential. Indeed, consider

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \quad \text{and} \quad u = (2, 3)^T.$$

Then  $Au = (1, 1)^T$  but  $1/5 = I(A) \neq I_{\text{sp}}(A) = 1$ . For  $A \in P_n$ , we conjecture that  $I(A) = I(sp, A) \neq 0$  implies that  $A \geq 0$ , as in the  $2 \times 2$  case.

#### 4. $I_{\text{sp}}(A)$ and $I_2(A)$

In this section we study the relation between the indexes associated with the spectral and Frobenius norms. In Lemma 2.1 of [23] it is shown that the index  $I_{\text{sp}}(\cdot)$  is always attained at rank-1 projections. The index  $I_1(\cdot)$  has the same property (see Example 2.3). It is natural to conjecture that the same result holds for any unitary invariant norm  $N$ . We show that the conjecture is true for the Frobenius norm:

**Proposition 4.1.** Let  $A \in P_n$ . Then there exists an  $x \in \mathbb{C}^n$  such that  $\|x\| = 1$  and  $I_2(A) = \|A \circ xx^*\|_2$ . That is,  $I_2(A)$  is attained at a rank-1 projection.

**Proof.** Let  $\lambda = \max\{\mu \geq 0 : \|A \circ B\|_2 \geq \mu \|B\|_2 \text{ for all } B \in P_n \text{ with rank } 1\}$ . By its definition  $\lambda \geq I_2(A)$ . Let us prove that  $\|A \circ B\|_2 \geq \lambda \|B\|_2$  for all  $B \in P_n$ . Indeed, for  $B \geq 0$ , write  $B = \sum_{i=1}^k B_i$ , where each  $B_i$  has rank 1,  $B_i \in P_n$ , and  $B_i B_j = 0$  if  $i \neq j$ . Then

$$\lambda^2 \|B\|_2^2 = \lambda^2 \sum_{i=1}^k \|B_i\|_2^2 \leq \sum_{i=1}^k \|A \circ B_i\|_2^2.$$

On the other hand, using  $\text{tr}(XY) \geq 0$  for positive semidefinite matrices  $X$  and  $Y$ ,

$$\begin{aligned} \|A \circ B\|_2^2 &= \text{tr}((A \circ B)^*(A \circ B)) = \sum_{ij} \text{tr}(A \circ B_i)^*(A \circ B_j) \\ &\geq \sum_i \text{tr}(A \circ B_i)^*(A \circ B_i) = \sum_i \|A \circ B_i\|_2^2. \quad \square \end{aligned}$$

**Proposition 4.2.** Let  $A \in P_n$ .

1. There exists a nonnegative vector  $x$  such that  $\|x\| = 1$  and  $\|A \circ xx^*\|_2 = I_2(A)$ .
2. Any such vector  $x$  satisfies  $(A \circ \bar{A} \circ xx^*)x = I(B_J)x$ , where  $B = A \circ \bar{A}$  and  $J = \{i : x_i \neq 0\}$ .

**Proof.** Let  $y$  be a unit vector such that  $\|A \circ yy^*\|_2 = I_2(A)$ . Let  $x_i = |y_i|$ . It is easily checked that  $\|x\| = 1$  and  $\|A \circ xx^*\|_2 = \|A \circ yy^*\|_2 = I_2(A)$ , which proves 4.2(1). Let  $B = A \circ \bar{A} \in P_n$ . Let  $y$  be a nonnegative unit vector and let  $z = (y_1^2, \dots, y_n^2)^T$ . Then

$$\|A \circ yy^*\|_2^2 = \sum_{i,j} |a_{ij}|^2 y_i^2 y_j^2 = \sum_{i,j} b_{ij} z_j z_i = \langle Bz, z \rangle$$

and  $\sum_1^n z_i = 1$ . Moreover,  $\|A \circ yy^*\|_2 = I_2(A)$  if and only if  $\langle Bz, z \rangle$  is the minimum of the map  $G(v) = \langle Bv, v \rangle$  restricted to the simplex  $\Delta = \{v \in \mathbb{R}^n : v \geq 0 \text{ and } \sum_1^n v_i = 1\}$ . Using Lemma 3.1, we know that if  $z$  belongs to the interior  $\Delta^\circ$  of  $\Delta$ , then  $z$  is a local extremum of  $G$  in the plane  $L = \{z \in \mathbb{R}^n : \sum_i z_i = 1\}$ , so  $Bz = I(B)p$ .

If the vector  $x$  of item 1 satisfies  $x_i > 0$  for all  $i$ , then  $z = x \circ x \in \Delta^\circ$  and  $Bz = I(B)p$ . By Lemma 3.2,  $(A \circ \bar{A} \circ xx^*)x = I(B)x$ , showing 4.2(2) in this case. If some  $x_i = 0$ , let  $J = \{i : x_i \neq 0\}$ , let  $B_J$  be the principal submatrix of  $B$  determined by the indexes of  $J$ , and similarly define  $x_J$ . Then  $I_2(A) = \|A \circ xx^*\|_2 = \|A_J \circ x_J x_J^*\|_2 \geq I_2(A_J)$  and

$$I_2(A) = I_2(A_J) = \|A_J \circ x_J x_J^*\|_2,$$

because the converse inequality always holds by (6). Note that, by its definition,  $x_J$  has no zero entries. By the previous case,  $x_J$  is an eigenvector of  $B_J \circ x_J x_J^*$  with eigenvalue  $I(B_J)$ . But clearly  $B \circ xx^*$  has zeroes outside  $J \times J$ , so  $x$  is an eigenvector of  $B \circ xx^*$  if and only if  $x_J$  is an eigenvector of  $B_J \circ x_J x_J^*$ . Note that the eigenvalue of  $x$  is always  $I(B_J)$ .  $\square$

**Theorem 4.3.** Let  $A \in P_n$ . Then  $I_2(A) = I_{\text{sp}}(\bar{A} \circ A)^{1/2}$ .

**Proof.** If  $B = \bar{A} \circ A$  and  $y \in \mathbb{C}^n$  with  $\|y\| = 1$ , then

$$\|A \circ yy^*\|_2^2 = \sum_{i,j} |a_{ij}|^2 |y_i|^2 |y_j|^2 = \langle (B \circ yy^*)y, y \rangle \leq \|B \circ yy^*\|.$$

Therefore  $I_2(A)^2 \leq I_{\text{sp}}(B)$ . On the other hand, let  $x$  be a nonnegative unit vector such that  $I_2(A)^2 = \|A \circ xx^*\|_2^2$  and  $J = \{i : x_i \neq 0\}$ . Then, by Proposition 4.2,  $(B \circ xx^*)x = I(B_J)x$  and

$$I_2(A)^2 = \|A \circ xx^*\|_2^2 = \langle (B \circ xx^*)x, x \rangle = I(B_J).$$

But  $x_J$  is a unit eigenvector of  $B_J \circ x_J x_J^*$  with strictly positive entries. So, by Lemma 3.2,  $B_J(x_J \circ x_J) = I(B_J)(1, \dots, 1)^T$ . Suppose that  $I_2(A) \neq 0$ . Then  $I(B_J) \neq 0$ ,  $B_J \geq 0$ , the vector  $u = I(B_J)^{-1}(x_J \circ x_J)$  has strictly positive entries, and  $B_J u = (1, \dots, 1)^T$ . Hence we can apply Theorem 3.3 to  $B_J$  and, by (6),

$$I(B_J) = I_{\text{sp}}(B_J) \geq I_{\text{sp}}(B) \geq I_2(A)^2 = I(B_J).$$

If  $I_2(A) = 0$ , then (5) ensures that some  $a_{ii} = 0$ , so  $I_{\text{sp}}(B) = 0$  by (5).  $\square$

**Corollary 4.4.** Let  $A \in P_n$ . Then

$$I_2(A) = \inf \left\{ \left( \sum_1^n d_{ii}^{-2} \right)^{-1/2} : D \text{ is positive diagonal and } A \circ \bar{A} \leq D^2 \right\} \\ = \inf \left\{ I_2(D) : D \text{ is positive diagonal and } A \circ \bar{A} \leq D^2 \right\}.$$

**Proof.** This is a direct consequence of Theorem 4.3 and Proposition 3.2 of [23].  $\square$

**Remark 4.5.** In Theorem 4.3 we get information about  $A \in P_n$  using  $B = \bar{A} \circ A$ . But it can also be used to get information about any  $B \in P_n$  with  $B \geq 0$ , using  $A = (b_{ij}^{1/2})$ . Unfortunately it may certainly happen that  $A \notin P_n$ . Nevertheless this obstruction can be removed in the following way: given a selfadjoint (but not necessarily positive semidefinite) matrix  $A \in M_n$ , consider the index

$$I_2(A) = \min \{ \|A \circ xx^*\|_2 : \|x\| = 1 \},$$

which, by Proposition 4.1, is consistent with Definition 2.2 when  $A \geq 0$ . A careful inspection of the proofs of Proposition 4.2 and Theorem 4.3 shows that they remain true using this new index if the condition “ $A \in P_n$ ” is replaced by “ $A = A^*$  and  $B = \bar{A} \circ A \in P_n$ ”. Note that Lemmas 3.1 and 3.2, and Theorem 3.3 are applied only to the positive semidefinite matrix  $B$  or its principal submatrices. The inequality  $I_2(A) \leq I_2(A_J)$  in (6) (which is also used in the proofs) remains valid for this new index. This observation is useful in order to avoid the unpleasant condition “ $A = (b_{ij}^{1/2}) \in P_n$ ” in the following result.

**Corollary 4.6.** *Suppose  $B \in P_n$  and  $B \geq 0$ . Then there exists a subset  $J_0$  of  $\{1, 2, \dots, n\}$  such that  $I_{\text{sp}}(B) = I_{\text{sp}}(B_{J_0}) = I(B_{J_0})$ . Therefore*

$$I_{\text{sp}}(B) = \min \{ I_{\text{sp}}(B_J) : I_{\text{sp}}(B_J) = I(B_J) \}.$$

If  $A = (b_{ij}^{1/2})$  (which may be not positive semidefinite), then  $J_0$  can be characterized as  $J_0 = \{i : x_i \neq 0\}$  for any unit vector  $x$  such that  $I_2(A) = \|A \circ xx^*\|_2$ . Also  $I_{\text{sp}}(B) = \|B \circ xx^*\| = \langle By, y \rangle$ , where  $y = (|x_1|^2, \dots, |x_n|^2)^T$ .

**Proof.** Use Remark 4.5 and the proof of Theorem 4.3.  $\square$

### 5. An operator inequality

In this section we compute the indexes of a particular class of matrices and, as an application, we get a new operator inequality, closely related to the inequality proved in [8]; see also [4,18].

Let  $x = (\lambda_1, \dots, \lambda_n)^T \in \mathbb{R}_+^n$ ,  $S = \{\lambda_1, \dots, \lambda_n\}$ , and

$$A = A_x = \left( \lambda_i \lambda_j + \frac{1}{\lambda_i \lambda_j} \right)_{ij} \in P_n.$$

Observe that  $A$  has rank 1 or 2.

#### 5.1. Computation of $I(A)$

1. If all  $\lambda_i$  are equal, then  $A = (\lambda_1^2 + \lambda_1^{-2}) E$  and  $I(A) = \lambda_1^2 + \lambda_1^{-2}$ .
2. If  $\#S > 1$ , then the range of  $A$  is spanned by the independent vectors  $x = (\lambda_1, \dots, \lambda_n)^T$  and  $y = (\lambda_1^{-1}, \dots, \lambda_n^{-1})^T$ , because  $A = xx^* + yy^* = [xy][xy]^*$  has rank 2.
3. If  $\#S = 2$ , say  $S = \{\lambda, \mu\}$ , then  $p = ax + by$ , with  $a = (\lambda + \mu)^{-1}$  and  $b = \lambda\mu(\lambda + \mu)^{-1}$ . If a vector  $z$  satisfies  $Az = p$ , then

$$p = Az = (xx^* + yy^*)z = \langle z, x \rangle x + \langle z, y \rangle y.$$

Therefore

$$I(A) = \langle z, p \rangle^{-1} = (\langle z, x \rangle^2 + \langle z, y \rangle^2)^{-1} = \frac{(\lambda + \mu)^2}{1 + \lambda^2 \mu^2} = I(A_0),$$

where the last equality is shown in Remark 4.3 of [23].

4. If  $\#S > 2$ , it is easy to see that  $p$  is not in the subspace spanned by  $x$  and  $y$ . Then  $I(A)$  must be zero by Proposition 2.5.

Note that  $I(A) \neq 0$  if and only if  $\#S \leq 2$ .

5.2. Computation of  $I_{\text{sp}}(A)$

We shall compute  $I_{\text{sp}}(A)$  using Corollary 4.6 and therefore use the principal submatrices of  $A$ , which are matrices of the same type. Let  $J \subset \{1, 2, \dots, n\}$ , let  $S_J = \{\lambda_j : j \in J\}$ , and let  $x_J$  be the induced vector. Then  $A_J = A_{x_J}$  and so  $I_{\text{sp}}(A_J) \neq 0$ . Suppose that  $I_{\text{sp}}(A_J) = I(A_J)$ . Then  $\#S_J \leq 2$  by Section 5.1. If  $\#S_J = 2$ , let  $i_1, i_2 \in J$  be such that  $\lambda_{i_1} \neq \lambda_{i_2}$ . By Theorem 3.3 there exists a vector  $y \in \mathbb{R}^J$  such that  $y \geq 0$  and  $A_J y = p_J$ . Let  $z_1 = \sum\{y_k : k \in J \text{ and } \lambda_k = \lambda_{i_1}\} \geq 0$  and  $z_2 = \sum\{y_j : j \in J \text{ and } \lambda_j = \lambda_{i_2}\} \geq 0$ . Easy computations show that  $A_{\{i_1, i_2\}}(z_1, z_2)^T = (1, 1)^T$ . Then, by Theorem 3.3 and Section 5.1,

$$I_{\text{sp}}(A_J) = I(A_J) = \frac{(\lambda_{i_1} + \lambda_{i_2})^2}{1 + \lambda_{i_1}^2 \lambda_{i_2}^2} = I(A_{\{i_1, i_2\}}) = I_{\text{sp}}(A_{\{i_1, i_2\}}).$$

Therefore, in order to compute  $I_{\text{sp}}(A)$  using Corollary 4.6, we need to consider only the diagonal entries of  $A$  and some of the principal submatrices of size  $2 \times 2$ . If  $\lambda_i \neq \lambda_j$ , (9) ensures that

$$I_{\text{sp}}(A_{\{i, j\}}) = I(A_{\{i, j\}}) \Leftrightarrow \lambda_i \lambda_j + \frac{1}{\lambda_i \lambda_j} \leq \min \left\{ \lambda_i^2 + \frac{1}{\lambda_i^2}, \lambda_j^2 + \frac{1}{\lambda_j^2} \right\}.$$

If  $\lambda_i < \lambda_j$ , this condition is equivalent to

$$\lambda_i^2 \leq \frac{1}{\lambda_i \lambda_j} \leq \lambda_j^2. \tag{11}$$

In particular, this implies that  $\lambda_i < 1 < \lambda_j$ . Then, by Corollary 4.6,

$$I_{\text{sp}}(A) = \min\{M_1, M_2\} \tag{12}$$

where  $M_1 = \min_i \lambda_i^2 + \lambda_i^{-2} = \min_i A_{ii}$  and

$$M_2 = \inf \left\{ \frac{(\lambda_i + \lambda_j)^2}{1 + \lambda_i^2 \lambda_j^2} : \lambda_i < 1 < \lambda_j \text{ and } \lambda_i^2 \leq \frac{1}{\lambda_i \lambda_j} \leq \lambda_j^2 \right\}.$$

For example, if all  $\lambda_i \geq 1$  (or all  $\lambda_i \leq 1$ ), then by (11),  $I_{\text{sp}}(A) = M_1 = \min_i \lambda_i^2 + \lambda_i^{-2}$ . On the other hand, if  $\lambda \neq 1$  and  $x = (\lambda, \lambda^{-1})^T$ , then

$$I_{\text{sp}}(A_x) = M_2 = \frac{\lambda^2 + \lambda^{-2}}{2} + 1 < M_1 = \lambda^2 + \lambda^{-2}.$$

**Proposition 5.1.** *Let  $\mathcal{H}$  be a Hilbert space and let  $S$  be a bounded selfadjoint invertible operator on  $\mathcal{H}$ . Let  $M(S)$  be the best constant such that*

$$\|STS + S^{-1}TS^{-1}\| \geq M(S)\|T\| \quad \text{for all } 0 \leq T \in L(\mathcal{H}).$$

Then  $M(S) = \min\{M_1(S), M_2(S)\}$ , where

$$M_1(S) = \min_{\lambda \in \sigma(S)} \lambda^2 + \lambda^{-2}$$

and

$$M_2(S) = \inf \left\{ \frac{(|\lambda| + |\mu|)^2}{1 + \lambda^2 \mu^2} : \lambda, \mu \in \sigma(S), |\lambda| < |\mu| \text{ and } \lambda^2 \leq \frac{1}{|\lambda \mu|} \leq \mu^2 \right\}.$$

In particular, if  $\|S\| \leq 1$  (or  $\|S^{-1}\| \leq 1$ ), then

$$M(S) = \|S\|^2 + \|S\|^{-2} \quad (\text{resp. } \|S^{-1}\|^2 + \|S^{-1}\|^{-2}).$$

**Proof.** We follow the same steps as in [8]. By taking the polar decomposition of  $S$ , we can assume that  $S > 0$ , because the unitary part of  $S$  is also the unitary part of  $S^{-1}$ ; it commutes with  $S$  and  $S^{-1}$  and it preserves norms. Note that we must change  $\sigma(S)$  by  $\sigma(|S|) = \{|\lambda| : \lambda \in \sigma(S)\}$ .

By the spectral theorem, we can assume that  $\sigma(S)$  is finite, because  $S$  can be approximated in norm by operators  $S_n$  such that each  $\sigma(S_n)$  is a finite subset of  $\sigma(S)$ ,  $\sigma(S_n) \subset \sigma(S_{n+1})$  for all  $n \in \mathbb{N}$  and  $\bigcup_n \sigma(S_n)$  is dense in  $\sigma(S)$ . Then  $M(S_n)$  (and  $M_i(S_n)$ ,  $i = 1, 2$ ) converges to  $M(S)$  (resp.  $M_i(S)$ ,  $i = 1, 2$ ).

We can suppose also that  $\dim \mathcal{H} < \infty$ , by choosing an appropriate net of finite rank projections  $\{P_F\}_{F \in \mathcal{F}}$  that converges strongly to the identity and replacing  $S$ ,  $T$  by  $P_F S P_F$ ,  $P_F T P_F$ . Indeed, the net may be chosen in such a way that  $S P_F = P_F S$  and  $\sigma(P_F S P_F) = \sigma(S)$  for all  $F \in \mathcal{F}$ . Note that for every  $A \in L(H)$ ,  $\|P_F A P_F\|$  converges to  $\|A\|$ .

Finally, we can suppose that  $S$  is diagonal by a unitary change of basis in  $\mathbb{C}^n$ . In this case, if  $\lambda_1, \dots, \lambda_n$  are the eigenvalues of  $S$  (with multiplicity) and  $x = (\lambda_1, \dots, \lambda_n)^T$ , then

$$S T S + S^{-1} T S^{-1} = A_x \circ T.$$

None of our reductions (unitary equivalences and compressions) change the fact that  $0 \leq T$ . Now the statement follows from formula (12). If  $\|S\| \leq 1$ , then  $M(S) = M_1(S)$ , because  $M_2(S)$  is the infimum of the empty set. Clearly  $M_1(S)$  is attained at the element  $\lambda \in \sigma(S)$  such that  $|\lambda| = \|S\|$ .  $\square$

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