Oblique Projections and Abstract Splines

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Given a closed subspace \( \mathcal{S} \) of a Hilbert space \( \mathcal{H} \) and a bounded linear operator \( A \in L(\mathcal{H}) \) which is positive, consider the set of all \( A \)-self-adjoint projections onto \( \mathcal{S} \):

\[
\mathcal{P}(A, \mathcal{S}) = \{ Q \in L(\mathcal{H}) : Q^2 = Q, \quad Q(\mathcal{H}) = \mathcal{S}, \quad AQ = Q^*A \}.
\]

In addition, if \( \mathcal{H}_1 \) is another Hilbert space, \( T : \mathcal{H} \rightarrow \mathcal{H}_1 \) is a bounded linear operator such that \( T^*T = A \) and \( \zeta \in \mathcal{H} \), consider the set of \( (T, \mathcal{S}) \) spline interpolants to \( \zeta \):

\[
s p(T, \mathcal{S}, \zeta) = \left\{ \eta \in \mathcal{S} + \mathcal{I} : ||T\eta|| = \min_{\sigma \in \mathcal{S}} ||T(\zeta + \sigma)|| \right\}.
\]

A strong relationship exists between \( \mathcal{P}(A, \mathcal{S}) \) and \( s p(T, \mathcal{S}, \zeta) \). In fact, \( \mathcal{P}(A, \mathcal{S}) \) is not empty if and only if \( s p(T, \mathcal{S}, \zeta) \) is not empty for every \( \zeta \in \mathcal{H} \). In this case, for any \( \zeta \in \mathcal{H} \) it holds

\[
s p(T, \mathcal{S}, \zeta) = \{(1 - Q)\zeta : Q \in \mathcal{P}(A, \mathcal{S})\}
\]

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and for any \( \xi \in \mathcal{H} \), the unique vector of \( sp(T, \mathcal{S}, \xi) \) with minimal norm is 
\((1 - P_{A, \mathcal{S}})\xi\), where \( P_{A, \mathcal{S}} \) is a distinguished element of \( \mathcal{P}(A, \mathcal{S}) \). These results offer a 
generalization to arbitrary operators of several theorems by de Boor, Atteia, Sard 
and others, which hold for closed range operators. © 2002 Elsevier Science (USA)

1. INTRODUCTION

Given two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}_1 \), \( T \in L(\mathcal{H}, \mathcal{H}_1) \), \( \mathcal{S} \subseteq \mathcal{H} \) a closed 
subspace and \( \xi \in \mathcal{H} \), an abstract spline or a \((T, \mathcal{S})\)-spline interpolant to \( \xi \) is 
every element of the set 
\[ sp(T, \mathcal{S}, \xi) = \left\{ \eta \in \xi + \mathcal{S} : \|T\eta\| = \min_{\sigma \in \mathcal{S}} \|T(\xi + \sigma)\| \right\}. \]

Observe that \( A = T^*T = |T|^2 \), as a positive bounded operator on \( \mathcal{H} \), defines 
a semiinner product \( \langle \cdot, \cdot \rangle_A : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C} \) by \( \langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle \), \( \xi, \eta \in \mathcal{H} \) 
and a corresponding seminorm \( \| \cdot \|_A : \mathcal{H} \rightarrow \mathbb{R}^+ \) given by \( \|\eta\|_A = \langle \eta, \eta \rangle_A^{1/2} = \langle A\eta, \eta \rangle_A^{1/2} = \|T\eta\| \). Thus, if for any \( \eta \in \mathcal{H} \) we consider \( d_A(\eta, \mathcal{S}) = \inf_{\sigma \in \mathcal{S}} \|\eta + \sigma\|_A \), then 
\[ sp(T, \mathcal{S}, \xi) = \{ \eta \in \xi + \mathcal{S} : \|\eta\|_A = d_A(\xi, \mathcal{S}) \}. \]

If \( A \) is an invertible operator, then \( \langle \cdot, \cdot \rangle_A \) is a scalar product, \((\mathcal{H}, \langle \cdot, \cdot \rangle_A)\) is a 
Hilbert space and, by the projection theorem, \( d_A(\xi, \mathcal{S}) = \|(1 - P_{A, \mathcal{S}})\xi\|_A \) and 
\( sp(T, \mathcal{S}, \xi) = \{(I - P_{A, \mathcal{S}})\xi\} \), where \( P_{A, \mathcal{S}} \) is unique orthogonal projection 
onto \( \mathcal{S} \) which is orthogonal to the inner product \( \langle \cdot, \cdot \rangle_A \). However, if \( A \) is not 
invertible then \( \| \cdot \|_A \) is or a seminorm or an incomplete norm and we cannot 
use the projection theorem unless we complete the quotient \( \mathcal{H} / \ker A \). One of 
the main goals of this paper is to get a simpler way of describing the set 
\( sp(T, \mathcal{S}, \xi) \).

We start with a positive bounded linear operator \( A \) on a Hilbert space \( \mathcal{H} \) 
and a closed subspace \( \mathcal{S} \) of \( \mathcal{H} \). The subspace \( \mathcal{S}^\perp = \{ \xi : \langle A\xi, \eta \rangle = 0 \forall \eta \in \mathcal{S} \} \) is called the 
A-orthogonal companion of \( \mathcal{S} \). Note the identities 
\[ \mathcal{S}^\perp = A^{-1}(\mathcal{S}^\perp) = A(\mathcal{S})^\perp = \ker(PA). \] (1)

Instead of defining adjoint operators with respect to \( \langle \cdot, \cdot \rangle_A \), we restrict our 
discussion to \( A \)-self-adjoint operators, i.e. \( W \in L(\mathcal{H}) \) such that \( AW = W^*A \).

Note that any such \( W \) satisfies \( \langle W\xi, \eta \rangle_A = \langle \xi, W^*\eta \rangle_A, \xi, \eta \in \mathcal{H} \).

The pair \((A, \mathcal{S})\) is said to be compatible if there exists a projection \( Q \in 
L(\mathcal{H}) \) such that \( Q(\mathcal{H}) = \mathcal{S} \) and \( AQ = Q^*A \). The main result in this paper is 
the description of the relationship between the set 
\[ \mathcal{P}(A, \mathcal{S}) = \{ Q \in \mathcal{B} : R(Q) = \mathcal{S}, \ AQ = Q^*A \} \]
and \( sp(T, \mathcal{I}, \xi) \), where \( T: \mathcal{H} \rightarrow \mathcal{H}_1 \) is any bounded linear operator such that \( T^*T = A \). A relevant point here is that this method allows to tackle the case of operators with non-closed range. Thus, several results by Atteia [3], Sard [18], Golomb [11], Shekhtman [19], de Boor [4], Izumino [13], Delvos [9], Deutsch [8] are generalized to any bounded linear operators \( T \).

If \((A, \mathcal{I})\) is compatible, there exists a distinguished element \( P_{A,\mathcal{I}} \in \mathcal{P}(A, \mathcal{I}) \). The study of the map \((A, \mathcal{I}) \rightarrow P_{A,\mathcal{I}}\) was initiated by Pasternak-Winiarski [15] at least for invertible \( A \). A geometrical description of that map can be found in [2]. In [7, 12] the inversibility hypothesis on \( A \) was removed, opening, in that way, the possibility that \( \mathcal{P}(A, \mathcal{I}) \) be empty or have many elements. This induces the notion of compatibility of a pair \((A, \mathcal{I})\). This paper is mainly devoted to explore the relationship of the compatibility of \((A, \mathcal{I})\) with the existence of spline interpolants for every \( \xi \in \mathcal{H} \). Section 2 contains a short study on compatibility of a pair \((A, \mathcal{I})\). If \((A, \mathcal{I})\) is compatible, the properties of the distinguished element \( P_{A,\mathcal{I}} \in \mathcal{P}(A, \mathcal{I}) \) are described. In Section 3, we show that \((A, \mathcal{I})\) is compatible if and only if \( sp(T, \mathcal{I}, \xi) \) is not empty for any \( \xi \in \mathcal{H} \) and that \( sp(T, \mathcal{I}, \xi) = \{(1 - Q)\xi : Q \in \mathcal{P}(A, \mathcal{I})\} \) for any \( \xi \in \mathcal{H} \setminus \mathcal{I} \). Moreover, the vector of \( sp(T, \mathcal{I}, \xi) \) with minimal norm is exactly \((1 - P_{A,\mathcal{I}})\xi\).

In Section 4, we present some characterizations of \( P_{A,\mathcal{I}} \) which are useful for the study of the convergence of \( \{P_{A,\mathcal{I}}^{n}\xi\} \) if \((A, \mathcal{I})\) is compatible for every \( n \in \mathbb{N} \) and \( \mathcal{I}_n \) decreases to 0. This study is the goal of Section 5. Finally, Section 6 includes several examples of compatibility and spline projections.

In this paper, \( L(\mathcal{H}) \) is the algebra of all linear bounded operators on the Hilbert space \( \mathcal{H} \) and \( L(\mathcal{H})^+ \) is the subset of \( L(\mathcal{H}) \) of all self-adjoint positive (i.e., non-negative definite) operators. For every \( C \in L(\mathcal{H}) \) its range is denoted by \( R(C) \). If \( R(C) \) is closed, then \( C^\dagger \) denotes the Moore–Penrose pseudoinverse of \( C \). The orthogonal projections onto a closed subspace \( \mathcal{I} \) is denoted by \( P_{\mathcal{I}} \). The direct sum of subspaces \( \mathcal{I} \) and \( \mathcal{K} \) is denoted \( \mathcal{I} \oplus \mathcal{K} \). Finally, \( \mathcal{I} \cap \mathcal{K} \) denotes \( \mathcal{I} \cap \mathcal{K}^\perp \).

### 2. \( A \)-SELF-ADJOINT PROJECTIONS

Throughout this paper \( \mathcal{I} \) denotes a closed subspace of \( \mathcal{H} \) and \( A \) is a fixed operator in \( L(\mathcal{H})^+ \). Recall that \( \mathcal{I}^\perp = A^{-1}(\mathcal{I}^\perp) \). It is easy to see that a projection \( Q \) belongs to \( \mathcal{P}(A, \mathcal{I}) \) if and only if \( R(Q) = \mathcal{I} \) and \( \ker Q \subseteq A^{-1}(\mathcal{I}^\perp) \). Then

\[
\mathcal{I} \cap A^{-1}(\mathcal{I}^\perp) = \ker A \cap \mathcal{I}. \tag{2}
\]

In this case, \( \mathcal{P}(A, \mathcal{I}) \) has a single element if and only if \( \ker A \cap \mathcal{I} = \{0\} \) because

\[
\mathcal{I} \cap A^{-1}(\mathcal{I}^\perp) = \ker A \cap \mathcal{I}. \tag{3}
\]
If \( (A, \mathcal{S}) \) is compatible, then there is a distinguished element in \( \mathcal{P}(A, \mathcal{S}) \), namely the unique projection \( P_{A,\mathcal{S}} \) onto \( \mathcal{S} \) with kernel \( A^{-1}(\mathcal{S}^\perp) \oplus (\ker A \cap \mathcal{S}) \). The elements of \( \mathcal{P}(A, \mathcal{S}) \) can be parametrized by the set of relative supplements of \( \ker A \cap \mathcal{S} \) into \( A^{-1}(\mathcal{S}^\perp) \).

The set \( \mathcal{P}(A, \mathcal{S}) \) can also be characterized using the matrix operator decomposition induced by the orthogonal projection \( P = P_{\mathcal{S}} \). Under this representation, \( A \) has a matrix form

\[
A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},
\]

where \( a \in L(\mathcal{S}^\perp), \ b \in L(\mathcal{S}^\perp, \mathcal{S}) \) and \( c \in L(\mathcal{S}^\perp)^\perp \). Observe that \( P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \), \( PA = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \) and \( PAP = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \). Every projection \( Q \) with range \( \mathcal{S} \) has the matrix form \( Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \) for some \( x \in L(\mathcal{S}^\perp, \mathcal{S}) \). It is easy to see that \( Q \in \mathcal{P}(A, \mathcal{S}) \) if and only if \( x \) satisfies the equation \( ax = b \). Then

\[
\mathcal{P}(A, \mathcal{S}) = \left\{ Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} : x \in L(\mathcal{S}^\perp, \mathcal{S}) \text{ and } ax = b \right\}.
\]

Note that Eq. (5) implies that if \( (A, \mathcal{S}) \) is compatible, then \( R(b) \subseteq R(a) \). As a corollary of a well-known theorem of R.G. Douglas, it can be shown that these two conditions are, indeed, equivalent. First, we recall Douglas’ theorem [10]:

**Theorem 2.1.** Let \( B, C \in L(\mathcal{H}) \). Then the following conditions are equivalent:

1. \( R(B) \subseteq R(C) \).
2. There exists a positive number \( \lambda \) such that \( BB^* \leq \lambda CC^* \).
3. There exists \( D \in L(\mathcal{H}) \) such that \( B = CD \). Moreover, there exists a unique operator \( D \) which satisfies the conditions

\[
B = CD, \quad \ker D = \ker B \quad \text{and} \quad R(D) \subseteq R(C^*).
\]

In this case, \( \|D\|^2 = \inf \{ \lambda : BB^* \leq \lambda CC^* \} \); \( D \) is called the reduced solution of the equation \( CX = B \). If \( R(C) \) is closed, then \( D = C^*B \).

**Corollary 2.2.** Let \( A \in L(\mathcal{H})^+ \) and \( \mathcal{S} \subseteq \mathcal{H} \) a closed subspace. If \( A \) has matrix form as in (4), then \( (A, \mathcal{S}) \) is compatible if and only if \( R(b) \subseteq R(a) \).
The next theorem describes some properties of \( P(A, \mathcal{I}) \) and \( P_{A, \mathcal{I}} \). The norm of \( P_{A, \mathcal{I}} \) will be computed in Section 5.

**Theorem 2.3.** Let \( A \in L(\mathcal{H})^+ \) with matrix form (4), such that the pair \((A, \mathcal{I})\) is compatible.

1. The distinguished projection \( P_{A, \mathcal{I}} \in \mathcal{P}(A, \mathcal{I}) \) has the matrix form

\[
P_{A, \mathcal{I}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix},
\]

where \( d \in L(\mathcal{I}^\perp, \mathcal{I}) \) is the reduced solution of the equation \( ax = b \).

2. \( \mathcal{P}(A, \mathcal{I}) \) is an affine manifold which can be parametrized as

\[
\mathcal{P}(A, \mathcal{I}) = P_{A, \mathcal{I}} + L(\mathcal{I}^\perp, N),
\]

where \( N = A^{-1}(\mathcal{I}) \cap \mathcal{I} = \ker A \cap \mathcal{I} \) and \( L(\mathcal{I}^\perp, N) \) is viewed as a subspace of \( L(\mathcal{H}) \). A matrix representation of this parametrization is

\[
\mathcal{P}(A, \mathcal{I})Q = P_{A, \mathcal{I}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & z \\ 0 & 0 & 0 \end{pmatrix} \mathcal{I} \oplus N^\perp. \tag{6}
\]

3. \( P_{A, \mathcal{I}} \) has minimal norm in \( \mathcal{P}(A, \mathcal{I}) \), i.e. \( \|P_{A, \mathcal{I}}\| = \min \{\|Q\|: Q \in \mathcal{P}(A, \mathcal{I})\} \).

**Proof.**

(1) If \( Q = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix} \), then \( Q \in \mathcal{P}(A, \mathcal{I}) \) and \( \ker Q \subseteq A^{-1}(\mathcal{I}^\perp) \). Since \( P_{A, \mathcal{I}} \) is characterized by the properties \( R(P_{A, \mathcal{I}}) = \mathcal{I} \) and \( \ker P_{A, \mathcal{I}} = A^{-1}(\mathcal{I}^\perp) \oplus N^\perp \), then, in order to show that \( Q = P_{A, \mathcal{I}} \) it suffices to prove that \( \ker Q \subseteq N^\perp \). Let \( \xi \in \ker Q \) and write \( \xi = \xi_1 + \xi_2 \) with \( \xi_1 \in \mathcal{I} \) and \( \xi_2 \in \mathcal{I}^\perp \). Then \( 0 = Q\xi = \xi_1 + d\xi_2 \). If \( \eta \in N^\perp \), then \( \langle \xi, \eta \rangle = \langle \xi_1, \eta \rangle = -\langle d\xi_2, \eta \rangle = 0 \) because, by Theorem 2.1, \( R(d) \subseteq R(a) \) and, as an operator in \( L(\mathcal{I}) \), \( \ker a = \mathcal{I} \cap \ker PAP = \mathcal{I} \cap \ker A = N^\perp \).

(2) Let \( Q = \begin{pmatrix} 1 & y \\ 0 & 0 \end{pmatrix} \) with \( y \in L(\mathcal{I}^\perp, \mathcal{I}) \) and let \( d \in L(\mathcal{I}^\perp, \mathcal{I}) \) be the reduced solution of the equation \( ax = b \). Then \( Q \in \mathcal{P}(A, \mathcal{I}) \) if and only if \( ay = b \). Therefore, if \( z = y - d \), then \( Q \in \mathcal{P}(A, \mathcal{I}) \) if and only if \( Q = P_{A, \mathcal{I}} + z \) and \( R(z) \subseteq \ker a = N^\perp \). Concerning the matrix representation (6), recall that \( R(d) \subseteq R(a) = (\ker a)^\perp = \mathcal{I} \oplus N^\perp \). Therefore,

\[
Q = P_{A, \mathcal{I}} + z = \begin{pmatrix} 1 & 0 & d \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \mathcal{I} \oplus N^\perp.
\]
(3) If \( Q \in \mathcal{P}(A, \mathcal{S}) \) has the matrix form given in Eq. (6), then

\[
||Q||^2 = ||QQ^\ast|| = 1 + \left\| \begin{pmatrix} 0 & 0 & d \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \right\|^2 \geq 1 + ||d||^2 = ||P_{A,\mathcal{S}}||^2.
\]

Remark 2.4. Under additional hypothesis on \( A \), other characterizations of compatibility can be used. We mention a sample of these, taken from [6, 7]:

1. If \( A \) is injective then the following conditions are equivalent: (a) The pair \( (A, \mathcal{S}) \) is compatible. (b) \( \mathcal{S} \cap R(\lambda(1 - P)) \) for some (and then for any) \( \lambda > 0 \). (c) \( P(A(\mathcal{S})) = \mathcal{S} \) and \( A(\mathcal{S}) \cap \mathcal{S}^\perp = \{0\} \).

2. If \( A \) has closed range then the following conditions are equivalent: (a) The pair \( (A, \mathcal{S}) \) is compatible. (b) \( R(PAP) \) is closed. (c) \( \mathcal{S} + \ker A \) is closed.

3. If \( R(PAP) \) is closed (or, equivalently, if \( R(PA^{1/2}) \) or \( A^{1/2}(\mathcal{S}) \) are closed), then \( (A, \mathcal{S}) \) is compatible. Indeed, using the matrix form (4), the positivity of \( A \) implies that \( R(b) \subseteq R(a^{1/2}) \) (see, e.g., [1]). If \( R(PAP) = R(a) \) is closed, then \( R(b) \subseteq R(a^{1/2}) = R(a) \) so that \( (A, \mathcal{S}) \) is compatible by Corollary 2.2.

3. SPLINES AND \( A \)-SELF-ADJOINT PROJECTIONS

In this section, we characterize the existence of splines in terms of the existence of \( A \)-self-adjoint projections. The first result extends a theorem of Izumino [13] to operators whose ranges are not necessarily closed.

**Proposition 3.1.** Let \( T \in L(\mathcal{H}, \mathcal{H}_1) \), \( A = T^*T \in L(\mathcal{H}) \) and \( \mathcal{S} \subseteq \mathcal{H} \) a closed subspace. Then, for any \( \xi \in \mathcal{H}^\ast \),

\[
sp(T, \mathcal{S}, \xi) = (\xi + \mathcal{S}) \cap \mathcal{S}^\perp.
\]

In particular, \( sp(T, \mathcal{S}, \xi) \) is an affine manifold of \( L(\mathcal{H}) \) and, if \( \eta \in sp \), \( (T, \mathcal{S}, \xi) \), then \( sp(T, \mathcal{S}, \xi) = \eta + \ker T \cap \mathcal{S} \).

**Proof.** Suppose that \( \eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp) \) and \( \sigma \in \mathcal{S} \). Then \( \langle A\eta, \sigma \rangle = \langle A\sigma, \eta \rangle = 0 \) and

\[
||T(\eta + \sigma)||^2 = \langle A(\eta + \sigma), \eta + \sigma \rangle = \langle A\eta, \eta \rangle + \langle A\sigma, \sigma \rangle \geq \langle A\eta, \eta \rangle = ||T\eta||^2.
\]
Therefore, \( \eta \in s p (T, \mathcal{S}, \xi) \). Conversely, if \( \eta \in s p (T, \mathcal{S}, \xi) \) and \( \sigma \in \mathcal{S} \), then, for any \( t \in \mathbb{R} \),

\[
||T\eta||^2 \leq ||T(\eta + t\sigma)||^2 = \langle A(\eta + t\sigma), \eta + t\sigma \rangle \\
= \langle A\eta, \eta \rangle + t^2 \langle A\sigma, \sigma \rangle + 2t \text{Re} \langle A\eta, \sigma \rangle \\
= ||T\eta||^2 + t^2 \langle A\sigma, \sigma \rangle + 2t \text{Re} \langle A\eta, \sigma \rangle,
\]

therefore \( t^2 \langle A\sigma, \sigma \rangle + 2t \text{Re} \langle A\eta, \sigma \rangle \geq 0 \) for all \( t \in \mathbb{R} \) and a standard argument shows that \( \langle A\eta, \sigma \rangle = 0 \) and then \( \eta \in (\xi + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp) \).

**Theorem 3.2.** Let \( T \in L(\mathcal{H}, \mathcal{H}_1) \), \( A = T^*T \in L(\mathcal{H}) \) and \( \mathcal{S} \subseteq \mathcal{H} \) a closed subspace.

1. If \( \xi \in \mathcal{H} \), \( s p (T, \mathcal{S}, \xi) \) is not empty \( \xi \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp) \).
2. The following conditions are equivalent: (a) \( s p (T, \mathcal{S}, \xi) \) is not empty for every \( \xi \in \mathcal{H} \). (b) \( \mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H} \). (c) The pair \( (A, \mathcal{S}) \) is compatible.
3. If \( (A, \mathcal{S}) \) is compatible and \( \xi \in \mathcal{H} \setminus \mathcal{S} \), it holds \( s p (T, \mathcal{S}, \xi) = \{(I - Q)\xi : Q \in \mathcal{P}(A, \mathcal{S})\} \).
4. If \( (A, \mathcal{S}) \) is compatible, then for every \( \xi \in \mathcal{H} \), \( (I - P_{A,\mathcal{S}})\xi \) is the unique vector in \( s p (T, \mathcal{S}, \xi) \) with minimal norm.

**Proof.** The first assertion follows directly from Proposition 3.1. Indeed, if \( \eta \in s p (T, \mathcal{S}, \xi) \) and \( \eta = \xi + \sigma \) with \( \sigma \in \mathcal{S} \), then \( \xi = -\sigma + \eta \in \mathcal{S} + A^{-1}(\mathcal{S}^\perp) \); the converse implication is similar. The second assertion follows from the first one and Eq. (2). In order to prove the third item, let \( \xi \in \mathcal{H} \) and \( Q \in \mathcal{P}(A, \mathcal{S}) \). Then, by Proposition 3.1 and Eq. (2),

\[
(I - Q)\xi = \xi - Q\xi \in (\mathcal{S} + \mathcal{S}) \cap \ker Q \subseteq (\mathcal{S} + \mathcal{S}) \cap A^{-1}(\mathcal{S}^\perp) = s p (T, \mathcal{S}, \xi).
\]

Conversely, let \( \eta \in s p (T, \mathcal{S}, \xi) \) and \( \sigma \in \mathcal{S} \) such that \( \xi = \sigma + \eta \). We are looking for some \( Q \in \mathcal{P}(A, \mathcal{S}) \) such that \( Q\xi = \sigma \). Let \( \eta_1 = (I - P_{A,\mathcal{S}})\xi \) and \( \sigma_1 = \xi - \eta_1 = P_{A,\mathcal{S}}\xi \in \mathcal{S} \). Then, by Proposition 3.1,

\[
\sigma - \sigma_1 = \eta_1 - \eta \in \mathcal{S} \cap A^{-1}(\mathcal{S}^\perp) = \ker A \cap \mathcal{S}.
\]

If \( \xi = \sigma_2 + \rho \) with \( \sigma_2 \in \mathcal{S} \) and \( 0 \neq \rho \in \mathcal{S}^\perp \), choose \( z \in L(\mathcal{S}^\perp, \ker A \cap \mathcal{S}) \) \((\subseteq L(\mathcal{H}))\) such that \( z(\rho) = \sigma - \sigma_1 \). By Theorem 2.3, \( Q = P_{A,\mathcal{S}} + z \in \mathcal{P}(A, \mathcal{S}) \) and clearly \( Q\xi = \sigma \).

The minimality of \( ||(I - P_{A,\mathcal{S}})\xi|| \) is proved as follows. If \( \xi \in \mathcal{S} \), then \( (I - P_{A,\mathcal{S}})\xi = 0 \), which must be minimal. If \( \xi \notin \mathcal{S} \), let \( \xi = \sigma_2 + \rho \) with \( \sigma_2 \in \mathcal{S} \) and \( 0 \neq \rho \in \mathcal{S}^\perp \). By Theorem 2.3, any \( Q \in \mathcal{P}(A, \mathcal{S}) \) has the form \( Q = P_{A,\mathcal{S}} + z \), with \( z \in L(\mathcal{S}^\perp, \ker A \cap \mathcal{S}) \) \((\subseteq L(\mathcal{H}))\). Recall that \( R(P_{A,\mathcal{S}}) = \mathcal{S} \cap (\ker A \cap \mathcal{S}) \)
\[ \|I - Q\|z\|^2 = \|I - Q\|p\|^2 = \|p - PA,\xi(p)\|^2 = \|p\|^2 + \|PA,\xi(p)\|^2 + \|\xi(p)\|^2 \]

\[ \geq |p|^2 + \|PA,\xi(p)\|^2 = \|p - PA,\xi(p)\|^2 = \|(I - PA,\xi)\|^2. \]

**Corollary 3.3.** Let \( T \in L(H, H_1) \), \( A = T^*T \in L(H) \) and \( \mathcal{J} \subseteq H \) a closed subspace. Then the following are equivalent:

1. \( sp (T, \mathcal{J}, \xi) \) has a unique element for every \( \xi \in H \).
2. The pair \((A, \mathcal{J})\) is compatible and \( \ker T \cap \mathcal{J} = \{0\} \).

**Remark 3.4.** Let \( T \in L(H, H_1) \), \( A = T^*T \in L(H) \) and \( \mathcal{J} \subseteq H \) a closed subspace.

1. If \((A, \mathcal{J})\) is compatible then, by item 4 of Theorem 3.2, the projection \( 1 - PA,\mathcal{J} \) coincides with the so-called spline projection for \( T \) and \( \mathcal{J} \) when \( T \) has a closed range.
2. If \( R(T) \) is closed, then, by Remark 2.4 and Theorem 3.2, \( sp (T, \mathcal{J}, \xi) \neq \emptyset \) for every \( \xi \in H \) if and only if \( \ker T + \mathcal{J} \) is closed. In case that \( \ker T \cap \mathcal{J} = \{0\} \), then it is equivalent to the condition that the inclination between \( \ker T \) and \( \mathcal{J} \) is less than one (see [4, 8]).
3. If \( \xi \in \mathcal{J} \), then \( sp (T, \mathcal{J}, \xi) = \ker T \cap \mathcal{J} \). On the other hand, \( (I - Q)\xi = 0 \) for every \( Q \in P(A, \mathcal{J}) \). So the equality of item 3 of Theorem 3.2 may be false in this case.

### 4. CHARACTERIZATIONS OF THE SPLINE PROJECTION \( PA,\mathcal{J} \)

Fix \( A \in L(H)^+ \) and a closed subspace \( \mathcal{J} \subseteq H \). As before, we denote \( P = P_{\mathcal{J}} \). In this section, two different descriptions of the spline projection \( PA,\mathcal{J} \) are given and, as a consequence, we relate \( PA,\mathcal{J} \) with the shorted operator (see [1] and Remark 4.4 below).

By Corollary 2.2, it holds that the pair \((A, \mathcal{J})\) is compatible if and only if \( R(PA) \subseteq R(PAP) \). In case that \( A \) is invertible, it is known (see [2]) that, in the matrix form (4), \( a \) is invertible in \( L(\mathcal{J}) \) and

\[
PA = \begin{pmatrix}
a^{-1} & 0 \\
0 & 0
\end{pmatrix}
\]

because \( a^{-1}b \) is the reduced solution of \( ax = b \) (see Theorem 2.3). Rewriting (7), we get \( (PAP)PA,\mathcal{J} = PA \). Thus, if \( A \) is invertible, \( PA,\mathcal{J} \) is the reduced solution of the equation \( (PAP)X = PA \). Let us consider the general case, in other words, if the pair \((A, \mathcal{J})\) is compatible, let us relate \( PA,\mathcal{J} \) with the
reduced solution $Q$ of the equation

$$(PAP)X = PA.$$ 

(8)

Observe that, in general, $R(PAP)$ is strictly contained in $\mathcal{S}$. Therefore, $R(Q)$ may be smaller than $\mathcal{S} = R(P_{A,\mathcal{S}})$.

**Proposition 4.1.** If the pair $(A, \mathcal{S})$ is compatible, $Q$ is the reduced solution of Eq. (8) and $\mathcal{N} = \ker A \cap \mathcal{S}$, then

$$P_{A,\mathcal{S}} = P_{A\mathcal{N}} + Q.$$ 

Moreover, $Q$ verifies the following properties:

1. $Q^2 = Q$, $\ker Q = A^{-1}(\mathcal{S}^\perp)$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$.
2. $Q$ is $A$-self-adjoint.
3. $Q = P_{A,\mathcal{S} \ominus \mathcal{N}}$.

**Proof.** Using the matrix form (4) of $A$, observe that, in $L(\mathcal{S})$, $\ker a = \mathcal{N}$ and $\overline{R(a)} = \overline{R(a^{1/2})} = \mathcal{S} \ominus \mathcal{N}$. Note that $R(Q) \subseteq \overline{R(a)}$. Also $\ker Q = \ker (PA) = A^{-1}(\mathcal{S}^\perp)$. If $\xi \in \mathcal{S} \ominus \mathcal{N}$, then

$$a(Q\xi) = (PAP)Q\xi = PA\xi = PAP\xi = a(\xi).$$

Since $a$ is injective in $\mathcal{S} \ominus \mathcal{N}$, we can deduce that $Q\xi = \xi$ for all $\xi \in \mathcal{S} \ominus \mathcal{N}$. Now, the compatibility of $(A, \mathcal{S})$ implies that $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{H}$. Also $A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \ker A \cap \mathcal{S} = \mathcal{N}$. Therefore $A^{-1}(\mathcal{S}^\perp) \oplus (\mathcal{S} \ominus \mathcal{N}) = \mathcal{H}$. Then $Q^2 = Q$ and $R(Q) = \mathcal{S} \ominus \mathcal{N}$. Note that

$$\ker Q = A^{-1}(\mathcal{S}^\perp) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^\perp) = R(Q)^{1/2},$$

so that $Q$ is $A$-self-adjoint by Eq. (2). On the other hand, $(\mathcal{S} \ominus \mathcal{N}) \cap \ker A = \{0\}$, so that $Q$ is the unique element of $P(A, \mathcal{S} \ominus \mathcal{N})$, by Theorem 2.3. Observe that $R(Q) \subseteq \mathcal{N}^{1/2}$ and $\mathcal{N} \subseteq \ker A \subseteq A^{-1}(\mathcal{S}^\perp) = \ker Q$. Therefore, $(P_{\mathcal{N}} + Q)^2 = P_{\mathcal{N}} + Q$, $R(P_{\mathcal{N}} + Q) = \mathcal{S}$ and $\ker (P_{\mathcal{N}} + Q) = (A^{-1}(\mathcal{S}^\perp)) \ominus \mathcal{N}$. These formulae clearly imply that $P_{\mathcal{N}} + Q = P_{A,\mathcal{S}}$ (see Theorem 2.3).

**Proposition 4.2.** If $(A, \mathcal{S})$ is compatible and $\mathcal{M} = A^{1/2}(\mathcal{S})$, then $R(P_{A^{1/2}}A^{1/2}) \subseteq R(A^{1/2}P)$. Moreover, Eq. (8) and

$$(A^{1/2}P)X = P_{A^{1/2}}A^{1/2}$$

(9)
have the same reduced solution. In particular, if \( A^{1/2}(\mathcal{S}) \) is closed and \( \ker A \cap \mathcal{S} = \{0\} \), then

\[
P_{A,\mathcal{S}} = (A^{1/2}P)^\dagger P_{\mathcal{H}} A^{1/2} = (A^{1/2}P)^\dagger A^{1/2} = (TP)^\dagger T
\]

for every \( T \in L(\mathcal{H}, \mathcal{H}_1) \) such that \( T^*T = A \).

**Proof.** Denote \( B = A^{1/2} \). Recall that \( \mathcal{M} = \overline{B(\mathcal{S})} = B^{-1}(\mathcal{S}^\perp)^\perp \). Observe that

\[
BP_{\mathcal{H}} B = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}} \quad \text{(11)}
\]
in fact, for \( \zeta \in \mathcal{H} \), let \( \eta = P_{A,\mathcal{S}} \zeta \) and \( \rho = \zeta - \eta \in A^{-1}(\mathcal{S}^\perp) \); then \( B\eta \in \mathcal{M} \) and \( B\rho \in B^{-1}(\mathcal{S}^\perp)^\perp = \mathcal{M}^\perp \). Hence, \( BP_{\mathcal{H}} B = A\eta = AP_{A,\mathcal{S}} \zeta \). By Proposition 4.1, the projection \( Q = P_{A,\mathcal{S}} - P_{\mathcal{S}} \) is the reduced solution of the equation \( PAP X = PA \). We shall see that \( Q \) is the reduced solution of Eq. (9). First note that, by Eq. (11), \( BP_{\mathcal{H}} B = (AP)P_{A,\mathcal{S}} = (AP)Q \), so \( B(P_{\mathcal{H}} B - BPQ) = 0 \). But \( R(P_{\mathcal{H}} B - BPQ) \subseteq \overline{R(B)} = (\ker B)^\perp \). Hence, \( Q \) is a solution of (9). Note that \( \ker P_{\mathcal{H}} B = B^{-1}(\ker B) \). Finally,

\[
\overline{R((BP)^*)} = \overline{R(PB)} = \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N} = R(Q).
\]

The first equality of Eq. (10) follows directly. The second, from the fact that \( (A^{1/2}P)^\dagger P_{\mathcal{H}} = (A^{1/2}P)^\dagger \). The last equality follows easily using the polar decomposition of \( T \) because \( A^{1/2} = |T| \).

Formula (10), for operators with closed range, is due to Golomb [11].

**Corollary 4.3.** Under the notations of Proposition 4.2, the pair \((A, \mathcal{S})\) is compatible if and only if \( R(P_{\mathcal{H}} A^{1/2}) \subseteq R(A^{1/2}P) \).

**Proof.** Suppose that \( R(P_{\mathcal{H}} A^{1/2}) \subseteq R(A^{1/2}P) \). Then, given \( \zeta \in \mathcal{H} \), there must exist \( \sigma \in \mathcal{S} \) such that \( P_{\mathcal{H}} A^{1/2} \zeta = A^{1/2} \sigma \). Therefore, \( A^{1/2}(\zeta - \sigma) = (1 - P_{\mathcal{H}})A^{1/2} \zeta \) and

\[
||A^{1/2}(\zeta - \sigma)|| = ||(1 - P_{\mathcal{H}})A^{1/2} \zeta|| = d(A^{1/2} \zeta, A^{1/2}(\mathcal{S}))
\]

\[
= \inf \{||A^{1/2}(\zeta + \tau)||: \tau \in \mathcal{S} \}.
\]

Hence, \( \zeta - \sigma \in \text{sp} \ (T, \mathcal{S}, \zeta) \) and \( \text{sp} \ (T, \mathcal{S}, \zeta) \neq \emptyset \) for every \( \zeta \in \mathcal{H} \). This implies compatibility by Theorem 3.2. The converse implication was shown in Proposition 4.2.
Remark 4.4. If $A \in L(\mathcal{H})^+$ and $\mathcal{S} \in \mathcal{H}$ is a closed subspace, then the set

$$\{ X \in L(\mathcal{H})^+ : X \leq A \text{ and } R(X) \subseteq \mathcal{S}^\perp \}$$

has a maximum (for the natural order relation in $L(\mathcal{H})^+$), which is called the \emph{shorted operator} of $A$ to $\mathcal{S}^\perp$. We denote it by $\Sigma(P, A)$. This notion, due to Krein [14] and Anderson–Trapp [1], has many applications to electrical engineering. It is well known (see [16]) that

$$\Sigma(P, A) = A^{1/2} P_{\mathcal{S}} A^{1/2},$$

where $\mathcal{F} = A^{-1/2}(\mathcal{S}^\perp) = A^{1/2}(\mathcal{S})^\perp$. From the proof of Proposition 4.2, it follows that, if $(A, \mathcal{S})$ is compatible, then $A^{1/2}(1 - P_{\mathcal{S}})A^{1/2} = AP_{A, \mathcal{S}}$. Therefore, in this case, $\Sigma(P, A) = A(1 - P_{\mathcal{S}})$. More generally, it can be shown that $\Sigma(P, A) = A(1 - Q)$ for every $Q \in \mathcal{P}(A, \mathcal{S})$ (see [7]).

5. CONVERGENCE OF SPLINE PROJECTIONS

This section is devoted to the study of the convergence of abstract splines in the general (i.e. not necessarily closed range) case. Given $A \in L(\mathcal{H})^+$, let us consider a sequence of closed subspaces $\mathcal{S}_n$ such that all pairs $(A, \mathcal{S}_n)$ are compatible. Following de Boor [4] and Izumino [13], it is natural to look for conditions which are equivalent to the fact that $P_{A, \mathcal{S}_n} \rightharpoonup^{\text{SOT}} 0$ (i.e. the spline projections converge to $I$), where $\rightharpoonup^{\text{SOT}}$ means convergence in the strong operator topology. This problem has a well-known solution under the assumption that $R(A)$ is closed (see [4] or [13]). However, in our more general setting, it is possible that the sequence $\{\mathcal{S}_n\}$ decreases to $\{0\}$, while $\|P_{A, \mathcal{S}_n}\|$ tends to infinity (see Example 5.7). This induces us to consider the following weaker convergence:

**Definition 5.1.** Let $A \in L(\mathcal{H})^+$ and $T_n, T \in L(\mathcal{H}), n \in \mathbb{N}$. We shall say that the sequence $T_n$ converges $A$-SOT to $T$: $T_n \rightharpoonup^{A\text{-SOT}} T$ if

$$\|(T_n - T)\xi\|_A \to 0 \quad \text{for every } \xi \in \mathcal{H}.$$ 

Note that $T_n \rightharpoonup^{A\text{-SOT}} T$ if and only if $A^{1/2}T_n \rightharpoonup^{\text{SOT}} A^{1/2}T$.

We start with the computation of the norm of $P_{A, \mathcal{S}}$ for any compatible pair $(A, \mathcal{S})$. Before that, recall the following formula, due to Ptak [17] (see also [5, 7]): if $Q_1$ and $Q_2$ are orthogonal projections such that $R(Q_1)^\perp + R(Q_2) = \mathcal{H}$, then the norm of the unique projection $Q_3$ with $ker Q_3 = R(Q_1)$
and $R(Q_3) = R(Q_2)$ is
\[ \|Q_3\| = (1 - \|Q_1Q_2\|^2)^{-1/2}. \] (13)

**Proposition 5.2.** Let $A \in L(H)^+$ such that the pair $(A, \mathcal{F})$ is compatible. Then,
\[ \|P_{A,\mathcal{F}}\|^2 = \inf \{ \lambda > 0 : PA^2P \leq \lambda(PAP)^2 \}. \] (14)

If, in addition, $\ker A \cap \mathcal{F} = \{0\}$, then
\[ \|P_{A,\mathcal{F}}\| = (1 - \|QP\|^2)^{-1/2}, \] (15)
where $Q$ denotes the orthogonal projection onto $A^{-1}(\mathcal{F}^\perp)$.

**Proof.** Let $Q$ be the reduced solution of the equation $(PAP)X = PA$. Then $\|Q\|^2$ equals the infimum of Eq. (14) by Douglas Theorem. On the other hand, by Proposition 4.1, $\|Q\| = \|P_{A,\mathcal{F}}\|$, showing formula (14). If $\ker A \cap \mathcal{F} = \{0\}$, then Theorem 2.3 assures that $R(P_{A,\mathcal{F}}) = \mathcal{F}$ and $\ker P_{A,\mathcal{F}} = A^{-1}(\mathcal{F}^\perp)$. Therefore, (15) follows from Ptak formula (13). \[ \blacksquare \]

**Remark 5.3.** Let $A \in L(H)^+$ such that the pair $(A, \mathcal{F})$ is compatible and $\ker A \cap \mathcal{F} = \{0\}$. Then, if $P_{\ker A}$ is the orthogonal projection onto $\ker A$, then
\[ \|P_{A,\mathcal{F}}\| \geq (1 - \|P_{\ker A}P\|^2)^{-1/2}. \]
Indeed, if $Q$ is the projection of Eq. (15), then $P_{\ker A}Q \leq Q$ because $\ker A \subseteq A^{-1}(\mathcal{F}^\perp)$. Then $\|P_{\ker A}P\|^2 = \|PP_{\ker A}P\| \leq \|PQP\| = \|QP\|^2$. This inequality, shown by de Boor [4] in the closed range case, relates the norm of $P_{A,\mathcal{F}}$ with the angle between $\ker A$ and $\mathcal{F}$.

**Proposition 5.4.** Let $A \in L(H)^+$ and let $\mathcal{F}_n (n \in \mathbb{N})$ be closed subspaces such that all pairs $(A, \mathcal{F}_n)$ are compatible. Denote $\mathcal{H}_n = A^{1/2}(\mathcal{F}_n)$, $n \in \mathbb{N}$.

1. The following conditions are equivalent: (a) $P_{A,\mathcal{F}_n} \rightarrow A$-$\text{SOT}$ 0, (b) $\langle AP_{A,\mathcal{F}_n}x, x \rangle \rightarrow 0$, for every $x \in H$ (i.e. $AP_{A,\mathcal{F}_n} \rightarrow WOT$ 0 by polarization),
(c) $AP_{A,\mathcal{F}_n} \rightarrow \text{SOT}$ 0. (d) $\Sigma(\mathcal{F}_n, A) \rightarrow \text{SOT}$ $A$. (e) $P_{\mathcal{H}_n}A^{1/2} \rightarrow \text{SOT}$ 0.
2. If there exists $C \geq 0$ such that $\|P_{A,\mathcal{F}_n}\| \leq C$ for all $n \in \mathbb{N}$ and $P_{\mathcal{F}_n}A \rightarrow \text{SOT}$ 0, then $P_{A,\mathcal{F}_n} \rightarrow A$-$\text{SOT}$ 0.
3. If $P_{A,\mathcal{F}_n} \rightarrow A$-$\text{SOT}$ 0, then $P_{\mathcal{F}_n}A \rightarrow \text{SOT}$ 0.

**Proof.**

1. Because $P_{A,\mathcal{F}_n}^* = AP_{A,\mathcal{F}_n}$, it is clear that conditions (a)–(c) are equivalent. By Remark 4.4, $\Sigma(\mathcal{F}_n, A) = A(1 - P_{A,\mathcal{F}_n})$ so that (c) is equivalent
to (d). Finally, by Proposition 4.2, we know that $A^{1/2}P_{A,\mathcal{S}} = P_{A,\mathcal{S}}A^{1/2}$ and this shows that (a) is equivalent to (e).

2. Suppose that there exists $C \geq 0$ such that $\|P_{A,\mathcal{S}}\| \leq C$ for all $n \in \mathbb{N}$ and that $P_{\mathcal{S},A} \to \text{SOT } 0$. Denote $P_n = P_{\mathcal{S}}$. The fact that $R(P_{A,\mathcal{S}}) = R(P_n)$ implies that $P_nP_{A,\mathcal{S}} = P_{A,\mathcal{S}}$. Therefore, for every $\xi \in \mathcal{H}$,

$$\|P_{\mathcal{S}}^*A\|\xi\| = \|P_{A,\mathcal{S}}^*P_nA\|\xi\| \to 0,$$

since $\|P_{A,\mathcal{S}}\|$ is bounded. Hence $P_{A,\mathcal{S}}^*A = AP_{A,\mathcal{S}} \to \text{SOT } 0$ so that $P_{A,\mathcal{S}} \to \text{SOT } 0$ by item 1.

3. Suppose that $P_{A,\mathcal{S}} \to \text{A-SOT } 0$. Then, by item 1, $AP_{A,\mathcal{S}} \to \text{SOT } 0$. Note that $P_{A,\mathcal{S}}P_n = P_n$, so that $P_nP_{A,\mathcal{S}} = P_n$. Given $\xi \in \mathcal{H}$, we have that

$$\|P_nA\|\xi\| = \|P_nP_{A,\mathcal{S}}^*A\|\xi\| = \|P_nAP_{A,\mathcal{S}}\|\xi\| \leq \|AP_{A,\mathcal{S}}\|\xi\| \to 0. \quad \blacksquare$$

**Remark 5.5.** With the notations of Proposition 5.4, it follows that $P_{A,\mathcal{S}} \to \text{A-SOT } 0$ if and only if $A^{1/2}(1 - P_{A,\mathcal{S}})\|\xi\| = A^{1/2}\|\xi\|$ for every $\xi \in \mathcal{H}$ or, equivalently, the spline interpolants $\xi_n = (1 - P_{A,\mathcal{S}})\|\xi\|$ satisfy that $T\|\xi_n\| \to A\|\xi\|$ in $\mathcal{H}$, if $T \in L(\mathcal{H}, \mathcal{H})$ and $T^*T = A$. In particular, if $P_{A,\mathcal{S}} \to \text{A-SOT } 0$, then

$$\min \{\|T(\|\xi + \tau\|): \tau \in \mathcal{S}_n\} = \|T(1 - P_{A,\mathcal{S}})\|\xi\| \to \|A\|\xi\||. $$

**Proposition 5.6.** Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}_2 \subseteq \mathcal{S}_1 \subseteq \mathcal{H}$ be closed subspaces. Suppose that $(A, \mathcal{S}_1)$ is compatible. Denote by $P_i = P_{\mathcal{S}_i}$, $i = 1, 2$ and $a_1 = P_1AP_1 \in L(\mathcal{S}_1)^+$. Then

$(A, \mathcal{S}_2)$ is compatible if and only if $(a_1, \mathcal{S}_2)$ is compatible in $L(\mathcal{S}_1)$.

**Proof.** We know that, if $A = \begin{pmatrix} a_1 & b_1 \\ b_1^* & c_1 \end{pmatrix}$, in the matrix decomposition induced by $P_1$, then $R(b_1) \subseteq R(a_1)$. Hence also $R(P_2b_1) \subseteq R(P_2a_1)$. If $a_1 = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$, using now the matrix decomposition induced by $P_2$, then $P_2a_1 = a_2 + b_2$ and $P_2A(1 - P_2) = b_2 + P_2b_1$. Hence,

$$R(P_2b_1) \subseteq R(P_2a_1) = R(a_2) + R(b_2) \text{ and } R(P_2A(1 - P_2)) = R(b_2) + R(P_2b_1).$$

Therefore, the pair $(A, \mathcal{S}_2)$ is compatible if and only if $R(P_2A(1 - P_2)) \subseteq R(P_2P_2) = R(a_2)$ if and only if $R(b_2) \subseteq R(a_2)$ if and only if the pair $(a_1, \mathcal{S}_2)$ is compatible. \blacksquare

**Example 5.7.** Let $A \in L(\mathcal{H})^+$ injective but not invertible. With the notations of Proposition 5.6 it is easy to see that $P_1P_{A,\mathcal{S}_2}P_1 = P_{A,\mathcal{S}_2}P_1 \in \mathcal{S}(a_1, \mathcal{S}_2)$. Note that $a_1$ is injective, so that $\mathcal{S}(a_1, \mathcal{S}_2)$ has a unique
element and

\[ P_{n_1, \mathcal{F}} = P_{A, \mathcal{F}} P_1 \Rightarrow \|P_{A, \mathcal{F}}\| \geq \|P_{n_1, \mathcal{F}}\|. \]  \hspace{1cm} (16)

We shall see that there exists a sequence \( \mathcal{F}_n, n \in \mathbb{N} \), of closed subspaces of \( \mathcal{H} \) such that

1. the pair \((A, \mathcal{F})\) is compatible for every \( n \in \mathbb{N} \),
2. \( \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \) for every \( n \in \mathbb{N} \),
3. \( \bigcap_{n \geq 1} \mathcal{F}_n = \{0\} \), so that \( P_{\mathcal{F}_n} \to^{SOT} 0 \),
4. \( \|P_{A, \mathcal{F}_n}\| \to \infty \).

In order to prove this fact, we need the following lemma:

**Lemma 5.8.** Let \( B \in L(\mathcal{H})^+ \) be injective non-invertible. Then, for every \( \varepsilon > 0 \), there exists a closed subspace \( \mathcal{F} \subseteq \mathcal{H} \) such that the pair \((B, \mathcal{F})\) is compatible, \( P_{\mathcal{F}} B P_{\mathcal{F}} \) is not invertible in \( L(\mathcal{F}) \) and \( \|P_{\mathcal{F}}\| \geq \varepsilon^{-1} \).

**Proof.** Let \( \eta \in \mathcal{H} \) be a unit vector. Denote by \( \zeta = B \eta \) and consider the subspace \( \mathcal{F} = \{\zeta\}^\perp \) and \( P = P_{\mathcal{F}} \). It is clear that \( \eta \in B^{-1}(\mathcal{F}^\perp) \). First note that \( \langle \zeta, \eta \rangle = \langle B \eta, \eta \rangle > 0 \), so that \( \eta \not\in \mathcal{F} \). Since \( \mathcal{F} \) is an hyperplane, this implies that \( \mathcal{F} + B^{-1}(\mathcal{F}^\perp) = \mathcal{H} \) and the pair \((B, \mathcal{F})\) is compatible. Also \( P_{\mathcal{F}} B P_{\mathcal{F}} \) is not invertible because \( \dim \mathcal{F}^\perp = 1 < \infty \). Note that \( B^{-1}(\mathcal{F}^\perp) \) is the subspace generated by \( \eta \). Hence, if \( Q = P_{B^{-1}(\mathcal{F}^\perp)} \), it is easy to see that \( \|PQ\| = \|P\eta\| \). Then, by Eq. (15),

\[ \|P_{\mathcal{F}}\| = (1 - \|PQ\|^2)^{-1/2} = (1 - \|P\eta\|^2)^{-1/2} = \|(1 - P)\eta\|^{-1} \]

and

\[ \|(1 - P)\eta\| = \left| \langle \eta, \zeta \rangle \right| / \|\zeta\| = \frac{\langle \eta, B \eta \rangle}{\|B \eta\|}. \]

So, it suffices to show that there exists a unit vector \( \eta \) such that \( \langle \eta, B \eta \rangle \leq \varepsilon \|B \eta\| \). Consider \( \rho \in \mathcal{H} \setminus \mathcal{R}(B^{1/2}) \) a unit vector. Let \( \rho_n \) be a sequence of unit vectors in \( \mathcal{R}(B^{1/2}) \) such that \( \rho_n \to \rho \). Let \( \mu_n \in \mathcal{H} \) such that \( B^{1/2} \mu_n = \rho_n, n \in \mathbb{N} \), and denote by \( \xi_n = B^{1/2} \rho_n = B \mu_n \), and \( \xi = B^{1/2} \rho \). It is easy to see, using that \( B(\mu_n) = \xi_n \to \zeta \not\in \mathcal{R}(B) \), that \( \|\mu_n\| \to \infty \). Denote by \( \eta_n = \mu_n / \|\mu_n\|^{-1} \). Then

\[ \frac{\langle \eta_n, B \eta_n \rangle}{\|B \eta_n\|} = \frac{\|\mu_n\|^2}{\|\mu_n\|^2 \|B \eta_n\|} = \frac{\|B^{1/2} \mu_n\|^2}{\|\mu_n\| \|B \eta_n\|} = \frac{1}{\|\mu_n\| \|\xi\|} \to 0 \]

because \( \xi_n \to \zeta \neq 0 \).
By an inductive argument, using Lemma 5.8, Proposition 5.6 and Eq. (16), we can construct a sequence of compatible subspaces \( S_n, n \in \mathbb{N} \), such that 
\[
S_n \perp S_{n+1}/C_{18} S_n \quad \text{and} \quad \left\| P_{A,S_n} \right\| \to \infty.
\]
We can also get that 
\[
P_{A,S_n} \to \text{SOT} 0.
\]
This implies that 
\[
\left\| P_{A,S_n} \right\| \leq \left\| A \right\|, \quad \text{while} \quad \left\| P_{A,S_n} \right\| \text{can be arbitrarily large.}
\]

6. SOME EXAMPLES

In this section, we present several examples of pairs \((A, \mathcal{S})\) which are not compatible and pairs \((A, \mathcal{S})\) which are compatible and such that the spline projector \(P_{A,S_n}\) can be explicitly computed. Observe that Example 6.4 cannot be studied under the closed range hypothesis, considered by Atteia, de Boor and Izumino.

**Example 6.1.** Let \( A \in L(\mathcal{H})^+ \) and 
\[
M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathcal{H} \oplus \mathcal{H})^+.
\]
Denote by \( \mathcal{S} = \mathcal{H} \oplus \{0\} \) and by \( N = \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \). Since \( M = N^*N \), then \( \ker M = \ker N = \{ \xi \oplus -A^{1/2} \xi : \xi \in \mathcal{H} \} \) which is the graph of \(-A^{1/2}\). Note that \( R(N) = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathcal{S} \), so that \( R(M) \) is also closed. If \( A \) is injective with non-closed range, then \((M, \mathcal{S})\) is not compatible (because \( R(A) \) is properly included in \( R(A^{1/2}) \)). Observe that this implies that the inclination between \( \mathcal{S} \) and \( \ker M \) is one, cf. [4].

**Remark 6.2.** Let \( P \in \mathcal{P}, R(P) = \mathcal{S} \) and \( A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})^+ \). It is well known that the positivity of \( A \) implies that \( R(b) \subseteq R(a^{1/2}) \). Therefore, if \( \dim \mathcal{S} < \infty \) then the pair \((A, \mathcal{S})\) is compatible: in fact in this case \( R(a) = R(PAP) \) must be closed, so \( R(b) \subseteq R(a^{1/2}) = R(a) \) and Corollary 2.2, can be
applied. On the other hand, if \( \dim \mathcal{S}^\perp < \infty \) and \( R(A) \) is closed then, by Remark 2.4, \((A, \mathcal{S})\) is compatible. However, if \( R(A) \) is not closed, then the pair \((A, \mathcal{S})\) can be non-compatible:

**Proposition 6.3.** Let \( P \in \mathcal{P} \), \( R(P) = \mathcal{S} \) and \( A \in L(\mathcal{H})^+ \). Suppose that \( A \) is injective non-invertible and \( \dim \mathcal{S}^\perp < \infty \). Then \((A, \mathcal{S})\) is compatible if and only if \( \mathcal{S}^\perp \subseteq R(A) \).

**Proof.** By Eq. (2), \((A, \mathcal{S})\) is compatible if and only if \( A^{-1}(\mathcal{S}^\perp) + \mathcal{S} = \mathcal{H} \). Since \( A \) is injective, Eq. (3) says that \( A^{-1}(\mathcal{S}^\perp) \cap \mathcal{S} = \{0\} \). Now the result becomes clear because \( \dim A^{-1}(\mathcal{S}^\perp) = \dim (\mathcal{S}^\perp \cap R(A)) \).

**Example 6.4.** Let \( T \in L(\mathcal{H}, L^2) \) given by \( T e_m = e_{m+1} \), where \( e_m \) \((m \in \mathbb{N})\) is an orthonormal basis of \( \mathcal{H} \). Then \( A = T^* T \) is given by \( A e_m = \frac{e_m}{m} \), which is injective non-invertible. Let \( \xi_1, \ldots, \xi_n \in R(A) \), denote by \( \mathcal{S} = \{\xi_1, \ldots, \xi_n\}^\perp \) and \( P = P_{\mathcal{S}} \). If \( \xi_i = (\xi_i^{(1)}, \xi_i^{(2)}, \ldots, \xi_i^{(m)}), 1 \leq i \leq n \), denote by

\[
\eta_i = (\xi_i^{(1)}, \xi_i^{(2)}, \ldots, m^2 \xi_i^{(m)}, \ldots) \in \mathcal{H}, \quad 1 \leq i \leq n,
\]

and \( Q \) the orthogonal projection onto the subspace \( \mathcal{T} \) generated by \( \eta_1, \ldots, \eta_n \). It is clear that \( \mathcal{T} = A^{-1}(\mathcal{S}^\perp) \). Then \((A, \mathcal{S})\) is compatible and \( P_{A, \mathcal{S}} \) is the projection onto \( \mathcal{S} \) with kernel \( \mathcal{T} \). Therefore (cf. [5] or [17]), \( \|P Q\| < 1 \),

\[
P_{A, \mathcal{S}} = (1 - Q P)^{-1}(1 - Q) = \sum_{k=0}^{\infty} (Q P)^k (1 - Q)
\]

and \( \|P_{A, \mathcal{S}}\| = \|1 - P_{A, \mathcal{S}}\| = (1 - \|P Q\|^2)^{-1/2} \).

**Remark 6.5.** Let \( B \in L(\mathcal{H})^+ \) be injective and non-invertible. Let \( \xi \in \mathcal{H} \) be a unit vector, \( \mathcal{S} = \{\xi\}^\perp \), \( P = P_\mathcal{S} \) and \( P_\xi = 1 - P \). Let \( B = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \) in terms of \( P \). By Proposition 6.3, \((B, \mathcal{S})\) is compatible if and only if \( \xi \in R(B) \). Note that the sequence \( \xi_n \) \((in \( R(B) \)) of Lemma 5.8 converges to \( \xi \notin R(B) \). This is, precisely, the fact which implies that \( \|P_{B, \{\xi_n\}^\perp}\| \) converges to infinity.

**Example 6.6.** Fix \( \mathcal{S} \) a closed subspace of \( \mathcal{H} \) and consider the set

\[
\mathcal{A}_{\mathcal{S}} = \{A \in L(\mathcal{H})^+ : \text{the pair } (A, \mathcal{S}) \text{ is compatible}\}
\]

and the map \( \mathcal{A} : \mathcal{A}_{\mathcal{S}} \to \mathcal{P} \) given by \( \mathcal{A}(A) = P_{A, \mathcal{S}} \). We shall see that \( \mathcal{A} \) is not continuous. Indeed, let \( A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \), and suppose that \( R(b) = R(a) \) is a closed subspace \( \mathcal{M} \) properly included in \( \mathcal{S} \). Denote by \( \mathcal{N} = \mathcal{S} \ominus \mathcal{M} \) and consider the projection \( P_{\mathcal{S}} \) and some element \( u \in L(\mathcal{S}^\perp, \mathcal{N}) \subseteq L(\mathcal{H}) \), \( u \neq 0 \). Consider,
for every \( n \in \mathbb{N} \),

\[
A_n = A + \frac{1}{n} (P_{N'} + u) (P_{N'} + u) = A + \frac{1}{n} = \begin{pmatrix}
1 & 0 & u \\
0 & 0 & 0 \\
u^* & 0 & u^* u
\end{pmatrix}_{N'}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
u^* & 0 & u^* u
\end{pmatrix}_{M'_{\perp}}
\]

\[
= \begin{pmatrix}
\frac{1}{n} & 0 & \frac{1}{n} u \\
0 & a & b \\
\frac{1}{n} u^* & b^* & c + \frac{1}{n} u^* u
\end{pmatrix} \geq A \geq 0.
\]

It is clear that \( A_n \to A \). Note that \( a \) is invertible in \( L(M) \). Then, by Theorem 2.3,

\[
P_{A, \mathcal{P}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & a^{-1} b \\
0 & 0 & 0
\end{pmatrix}_{N'}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
u^* & 0 & u^* u
\end{pmatrix}_{N'}
\]

Also \( a + \frac{1}{n} P_{N'} \) is invertible in \( L(\mathcal{S}) \) for every \( n \in \mathbb{N} \). Then,

\[
P_{A_n, \mathcal{P}} = \begin{pmatrix}
n & 0 & 0 \\
0 & a^{-1} & 0 \\
0 & 0 & 0
\end{pmatrix}_{N'} \begin{pmatrix}
\frac{1}{n} & 0 & \frac{1}{n} u \\
0 & a & b \\
\frac{1}{n} u^* & b^* & c + \frac{1}{n} u^* u
\end{pmatrix}_{M'_{\perp}}
\]

\[
= \begin{pmatrix}
1 & 0 & u \\
0 & 1 & a^{-1} b \\
0 & 0 & 0
\end{pmatrix}_{N'}
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
u^* & 0 & u^* u
\end{pmatrix}_{N'}
\]

for all \( n \in \mathbb{N} \). Therefore, \( \pi(A_n) = P_{A_n, \mathcal{P}} P_{A, \mathcal{P}} = \pi(A) \). Note that the sequence \( \pi(A_n) \) converges (actually, it is constant) to \( P_{A, \mathcal{P}} + u \), which belongs to \( \mathcal{P}(A, \mathcal{S}) \) by Theorem 2.3.

**REFERENCES**