HOCHSCHILD (CO)HOMOLOGY OF DIFFERENTIAL OPERATOR RINGS

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ABSTRACT. We show that the Hochschild homology of a differential operator k-algebra $E = A\#_f U(g)$, is the homology of a deformation of the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in $(M \otimes \overline{A}^*, b_*)$. Moreover, when A is smooth and k is a characteristic zero field, we obtain a type of Hochschild-Kostant-Rosenberg theorem for these algebras. When $A = k$ our complex reduce to the one obtained in [K] for the homology of filtrated algebras whose associated graded algebras are symmetric algebras. In the last section we give similar results for the cohomology.

INTRODUCTION

Let k be a field and A an associative k-algebra with 1. An extension $E \supseteq A$ of A is a differential operator ring on A if there exists a k-Lie algebra $\mathfrak{g}$ and a vector space embedding $x \mapsto \overline{x}$, of $\mathfrak{g}$ into E, such that for all $x, y \in \mathfrak{g}$, $a \in A$:

1) $\overline{x}a - a\overline{x} = a^x$, where $a \mapsto a^x$ is a derivation,
2) $\overline{x}\,\overline{y} - \overline{y}\,\overline{x} = \overline{[x,y]_{\mathfrak{g}}} + f(x,y)$, where $[-,-]_{\mathfrak{g}}$ is the bracket of $\mathfrak{g}$ and $f\colon \mathfrak{g} \times \mathfrak{g} \to A$ is a bilinear map,
3) for a given basis $(x_i)_{i \in I}$ of $\mathfrak{g}$, E is a free left A-module with the standard monomials in the x_i's as a basis.

This general construction was introduced in [Ch] and [Mc-R]. Several particular cases of this type of extensions have been considered previously in the literature. For instance:

- when $\mathfrak{g}$ is one dimensional, f is trivial and E is the Ore extension $A[x, \delta]$, where $\delta(a) = a^x$,
- when $A = k$, one obtain the algebras studied by Sridharan in [S], which are the quasi-commutative algebras E, whose associated graded algebras is a symmetric algebra,
- in [Mc,§2] this type of extensions was studied under the hypothesis that A is commutative and $(x, a) \mapsto a^x$ is an action and in [B-G-R, Theorem 4.2] the case in which the cocycle is trivial was considered.

In [B-C-M] and [D-T] the study of the crossed products $A\#_f H$ of an algebra A by a Hopf algebra H was begun and in [M] was proved that the differential operator rings on A are the crossed products of A by enveloping algebras of Lie algebras.

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In [G-G] we obtained a complex, simpler than the canonical one, giving the Hochschild homology of a general crossed product $E = A\#_f H$ with coefficients in an arbitrary E-bimodule M. In the present paper we show that, for differential operators rings, a complex simpler than the one obtained in [G-G] also works, and we give some applications of this result.

This paper is organized as follows: In Section 1 we recall the definition of differential operator rings following the Hopf algebra point of view of [B-C-M] and [D-T]. In Section 2 we recall a technical result, established in [G-G], that we need in order to carry out our computations. In Section 3 we get a resolution of a differential operator ring $E = A\#_f U(\mathfrak{g})$ as an E-bimodule. This resolution is a mixture of the canonical Hochschild normalized resolution of A and the Chevalley-Eilenberg resolution of $\mathfrak{g}$. In Section 4 we study the Hochschild homology of E with coefficients in an arbitrary E-bimodule M. The main result is Theorem 4.1, where the promised complex, which is a deformation of the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in $(M \otimes \overline{A}^*, b_*)$, is obtained. Then, we consider a natural filtration of this complex, and we derive from it the spectral sequence of [St] in a more explicit way than the original one. Then, we consider the case when A is a commutative smooth algebra. The result obtained by us under this condition is a common generalization of the Hochschild-Kostant-Rosenberg theorem and the computation given in [K] for the Hochschild homology of algebras whose associated graded algebras are symmetric algebras. Finally, in Section 5, we study the cohomology.

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1. Preliminaries

Let A be a k-algebra and H a Hopf algebra. A *weak action* of H on A is a bilinear map $(h, a) \mapsto a^h$ from $H \times A$ to A such that, for $h \in H$, $a, b \in A$

1) $(ab)^h = \sum_{(h)} a^{h^{(1)}} b^{h^{(2)}}$,
2) $1^h = \epsilon(h)1$,
3) $a^1 = a$.

By an *action* of H on A we mean a weak action such that

4) $(a^l)^h = a^{hl}$ for all $h, l \in H$, $a \in A$.

Let A be a k-algebra and H a Hopf algebra with a weak action on A. Given a k-linear map $f\colon H \otimes H \to A$ we let $A\#_f H$ denote the k-algebra (in general non associative and without 1) whose underlying vector space is $A \otimes H$ and whose multiplication is given by

$$(a \otimes h)(b \otimes l) = \sum_{(h)(l)} ab^{h^{(1)}} f(h^{(2)}, l^{(1)}) \otimes h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes h$ of $A\#_f H$ will usually be written $a\#h$ to remind us H is weakly acting on A. The algebra $A\#_f H$ is called a *crossed product* if it is associative with $1\#1$ as identity element. In [B-C-M] was proved that this happen if and only if f and the weak action satisfy the following conditions

1) (Normality of f) for all $h \in H$ we have $f(h, 1) = f(1, h) = \epsilon(h)1_A$,

2) (Cocycle condition) for all $h, l, m \in H$ we have

$$\sum_{(h)(l)(m)} f\big(l^{(1)}, m^{(1)}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)}m^{(2)}\big) = \sum_{(h)(l)} f\big(h^{(1)}, l^{(1)}\big) f\big(h^{(2)}l^{(2)}, m\big),$$

3) (Twisted module condition) for all $h, l \in H$, $a \in A$ we have

$$\sum_{(h)(l)} \big(a^{l^{(1)}}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)}\big) = \sum_{(h)(l)} f\big(h^{(1)}, l^{(1)}\big) a^{h^{(2)}l^{(2)}}.$$

From now on, we assume that H is the enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. In this case, item 1) of the definition of weak action implies that $(ab)^x = a^x b + ab^x$ for $x \in \mathfrak{g}$. So, a weak action determines a linear map $\delta\colon \mathfrak{g} \longrightarrow \mathrm{Der}_k(A)$ by $\delta(x)(a) = a^x$. Moreover if $(h, a) \mapsto a^h$ is an action, then δ is a homomorphism of Lie algebras. Reciprocally given a linear map $\delta\colon \mathfrak{g} \longrightarrow \mathrm{Der}_k(A)$, there exists a (generality non-unique) weak action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. When δ is a homomorphism of Lie algebras, there is a unique action of $U(\mathfrak{g})$ on A such that $\delta(x)(a) = a^x$. For a proof of these facts see [B-C-M].

Next we show that each normal cocycle $f\colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \longrightarrow A$ is convolution invertible, giving a formula for f^{-1}.

Remark 1.1. Each normal cocycle $f\colon U(\mathfrak{g}) \otimes U(\mathfrak{g}) \longrightarrow A$ is convolution invertible. Moreover, for each $h \in U(\mathfrak{g})$ and each family $x_1, \dots, x_r$ of elements of $\mathfrak{g}$, we have $f^{-1}(1, h) = f^{-1}(h, 1) = \epsilon(h)1_A$ and

$$f^{-1}(x_1 \cdots x_r, h) = \sum_{l=1}^{r} (-1)^l \sum_{\substack{1 \le p_1, \dots, p_l \\ p_1 + \cdots + p_l = r}} \sum_{\tau \in Sh_{p_1,\dots,p_l}} \sum_{(h)} f\big(x_{\tau(1)} \cdots x_{\tau(p_1)}, h^{(1)}\big) \times$$
$$\times f\big(x_{\tau(p_1+1)} \cdots x_{\tau(p_1+p_2)}, h^{(2)}\big) \cdots f\big(x_{\tau(p_1+\cdots+p_{l-1}+1)} \cdots x_{\tau(r)}, h^{(l)}\big),$$

where $Sh_{p_1,\dots,p_l}$ denotes the multishuffles associated to $p_1, \dots, p_l$. That is,

$$Sh_{p_1,\dots,p_l} = \left\{ \tau \in \mathfrak{S}_r : \tau\left(1 + \sum_{j=1}^{i} p_j\right) < \cdots < \tau\left(\sum_{j=1}^{i+1} p_j\right) \text{ for } 0 \le i < l \right\}.$$

This fact can be proved by a direct computation.

2. A Method for Constructing Resolutions

Let k be a commutative ring with 1 and E a k-algebra. In this section we recall a result that we will use in section 3. For the proof we remit to [G-G].

Let

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\downarrow \partial_3 & & & & & & \\
Y_2 & \xleftarrow{\mu_2} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & \cdots \\
\downarrow \partial_2 & & & & & & \\
Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \cdots \\
\downarrow \partial_1 & & & & & & \\
Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \cdots
\end{array}$$

be a diagram of E-bimodules and morphisms of E-bimodules verifying:

1) The column and the rows are chain complexes,
2) Each X_{rs} is isomorphic to a free E-bimodule $E \otimes \overline{X}_{rs} \otimes E$,
3) Each row is contractible as a complex of left E-modules, with a chain contracting homotopy $\sigma^0_{0s}\colon Y_s \to X_{0s}$ and $\sigma^0_{r+1,s}\colon X_{rs} \to X_{r+1,s}$ $(r \ge 0)$.

We define E-bimodule morphisms $d^l_{rs}\colon X_{rs} \to X_{r+l-1,s-l}$ ($r \ge 0$ and $1 \le l \le s$), recursively by

$$d^l_{rs}(\mathbf{x}) = \begin{cases} -\sigma^0_{0,s-1} \circ \partial_s \circ \mu_s(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1,\\ -\sum_{j=1}^{l-1} \sigma^0_{l-1,s-l} \circ d^{l-j}_{j-1,s-j} \circ d^j_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \le s,\\ -\sum_{j=0}^{l-1} \sigma^0_{r+l-1,s-l} \circ d^{l-j}_{r+j-1,s-j} \circ d^j_{rs}(\mathbf{x}) & \text{if } r > 0, \end{cases}$$

for $\mathbf{x} = 1 \otimes \overline{\mathbf{x}} \otimes 1$ with $\overline{\mathbf{x}} \in \overline{X}_{rs}$.

Theorem 2.1. *Let $\widetilde{\mu}\colon Y_0 \to E$ be a morphism of E-bimodules such that*

$$E \xleftarrow{\widetilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} Y_6 \xleftarrow{\partial_7} \dots,$$

is a complex that is contractible as a complex of left E-modules. Then

$$E \xleftarrow{\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} X_6 \xleftarrow{d_7} \dots,$$

where

$$\mu = \widetilde{\mu} \circ \mu_0, \qquad X_n = \bigoplus_{r+s=n} X_{rs} \qquad \text{and} \qquad d_n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=0}^{s} d^l_{rs},$$

is a relative projective resolution of E as an E-bimodule.

3. A resolution for a differential operator ring

Let $E = A\#_f U(\mathfrak{g})$ be a crossed product. In this section we obtain an E-bimodule resolution (X_*, d_*) of E, that is simpler than the canonical of Hochschild. Then an explicit expression of the boundary maps of this resolution is given. To begin, we fix some notations:

1) For each k-algebra B and each $r \in \mathbb{N}$, we write $\overline{B} = B/k$, $B^r = B \otimes \cdots \otimes B$ (r times) and $\overline{B}^r = \overline{B} \otimes \cdots \otimes \overline{B}$ (r times). Moreover, for $b \in B$, we also let b denote the class of b in $\overline{B}$.
2) Given $a_0 \otimes \cdots \otimes a_r \in A^{r+1}$ and $0 \le i < j \le r$, we write $\mathbf{a}_{ij} = a_i \otimes \cdots \otimes a_j$.
3) For each Lie k-algebra $\mathfrak{g}$ and each $s \in \mathbb{N}$, we write $\mathfrak{g}^{\wedge s} = \mathfrak{g} \wedge \cdots \wedge \mathfrak{g}$ (s times).
4) Given $\mathbf{x} = x_1 \wedge \cdots \wedge x_s \in \mathfrak{g}^{\wedge s}$ and $1 \le i \le s$, we write $\mathbf{x}_{\widehat{\imath}} = x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge x_s$.
5) Given $\mathbf{x} = x_1 \wedge \cdots \wedge x_s \in \mathfrak{g}^{\wedge s}$ and $1 \le i < j \le s$, we write $\mathbf{x}_{\widehat{\imath}\widehat{\jmath}} = x_1 \wedge \cdots \wedge \widehat{x}_i \wedge \cdots \wedge \widehat{x}_j \wedge \cdots \wedge x_s$.

3.1. The complex (Y'_*,∂'_X). Let $\widetilde{\mathfrak g}$ be the direct sum of two copies $\{y_x : x\in\mathfrak g\}$ and $\{z_x : x\in\mathfrak g\}$ of $\mathfrak g$, endowed with the bracket given by $[y_x,y_{x'}]_{\widetilde{\mathfrak g}} = y_{[x,x']_{\mathfrak g}}$ and $[y_x,z_{x'}]_{\widetilde{\mathfrak g}} = [z_x,z_{x'}]_{\widetilde{\mathfrak g}} = z_{[x,x']_{\mathfrak g}}$. Note that $\widetilde{\mathfrak g}$ is the semi-direct sum arising from the adjoint action of $\mathfrak g$ on itself. Let $\pi\colon U(\widetilde{\mathfrak g})\to U(\mathfrak g)$ be the algebra map defined by $\pi(y_x)=\pi(z_x)=x$. Let $\Lambda(\mathfrak g)$ be the exterior algebra generated by $\mathfrak g$. That is, the algebra generated by the elements e_x ($x\in\mathfrak g$) and the relations $e_{\lambda x+x'}=\lambda e_x+e_{x'}$ and $e_x^2=0$ ($\lambda\in k$, $x,x'\in\mathfrak g$). Let us consider the action of $U(\widetilde{\mathfrak g})$ on $\Lambda(\mathfrak g)$ determined by $e_{x'}^{y_x}=e_{[x,x']_{\mathfrak g}}$ and $e_{x'}^{z_x}=0$. The enveloping algebra $U(\widetilde{\mathfrak g})$ of $\widetilde{\mathfrak g}$ acts weakly on $A\otimes\Lambda(\mathfrak g)$ via $(a\otimes e)^u = a^{\pi(u)}\otimes e + a\otimes e^u$ ($a\in A$, $e\in\Lambda(\mathfrak g)$ and $u\in U(\widetilde{\mathfrak g})$). Moreover, the map $\widetilde f\colon U(\widetilde{\mathfrak g})\times U(\widetilde{\mathfrak g})\to A\otimes\Lambda(\mathfrak g)$, defined by $\widetilde f(u,v)=f(\pi(u),\pi(v))\otimes 1$, is a normal 2-cocycle which satisfies the twisted module condition.

Theorem 3.1.1. *Let Y'_* be the graded algebra generated by A, the degree zero elements y_x, z_x ($x\in\mathfrak g$), the degree one elements e_x ($x\in\mathfrak g$) and the relations*

$$
\begin{aligned}
&y_{\lambda x+x'}=\lambda y_x+y_{x'}, && y_xa=a^x+ay_x, && e_{x'}y_x=y_xe_{x'}+e_{[x',x]_{\mathfrak g}},\\
&z_{\lambda x+x'}=\lambda z_x+z_{x'}, && z_xa=a^x+az_x, && e_{x'}z_x=z_xe_{x'},\\
&e_{\lambda x+x'}=\lambda e_x+e_{x'}, && e_xa=ae_x, && e_x^2=0,\\
&y_{x'}y_x=y_xy_{x'}+y_{[x',x]_{\mathfrak g}}+f(x',x)-f(x,x'),\\
&z_{x'}y_x=y_xz_{x'}+z_{[x',x]_{\mathfrak g}}+f(x',x)-f(x,x'),\\
&z_{x'}z_x=z_xz_{x'}+z_{[x',x]_{\mathfrak g}}+f(x',x)-f(x,x').
\end{aligned}
$$

Let $(x_i)_{i\in I}$ be a basis of $\mathfrak g$ with indexes running on an ordered set I. For each $i\in I$ let us write $y_i=y_{x_i}$, $z_i=z_{x_i}$ and $e_i=e_{x_i}$. Then each Y'_s is a free A-module with basis

$$
y_{i_1}^{m_1}e_{i_1}^{\delta_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l}\qquad \begin{pmatrix} l\ge 0,\ i_1<\cdots<i_l\in I,\ m_j,n_j\ge 0,\ \delta_j\in\{0,1\}\\ m_j+\delta_j+n_j>0,\ \delta_1+\cdots+\delta_l=s\end{pmatrix}.
$$

Proof. Let $\vartheta\colon Y'_*\to (A\otimes\Lambda(\mathfrak g))\#_{\widetilde f}U(\widetilde{\mathfrak g})$ be the homomorphism of algebras defined by $\vartheta(a)=(a\otimes 1)\#1$ for all $a\in A$ and $\vartheta(y_x)=(1\otimes 1)\#y_x$, $\vartheta(z_x)=(1\otimes 1)\#z_x$ and $\vartheta(e_x)=(1\otimes e_x)\#1$ for all $x\in\mathfrak g$. Because of the Poincaré-Birkhoff-Witt theorem,

$$
\vartheta(y_{i_1}^{m_1}e_{i_1}^{\delta_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l})\quad (l\ge 0,\ i_1<\cdots<i_l\in I,\ m_j,n_j\ge 0 \text{ and } \delta_j\in\{0,1\}),
$$

is a basis of $(A\otimes\Lambda(\mathfrak g))\#_{\widetilde f}U(\widetilde{\mathfrak g})$ as an A-module. The theorem follows immediately from this fact. □

Remark 3.1.2. Note that E is a subalgebra of Y'_* by embedding $a\in A$ to a and $x\in\mathfrak g$ to y_x. This gives rise to an structure of left E-module on Y'_s. Similarly we consider Y'_* as a right E-module via the embedding of E in Y'_* that sends $a\in A$ to a and $x\in\mathfrak g$ to z_x.

Theorem 3.1.3. *Let $\widetilde\mu'\colon Y'_0\to E$ be the algebra map defined by $\widetilde\mu'(a)=a$ for $a\in A$ and $\widetilde\mu'(y_i)=\widetilde\mu'(z_i)=x_i$ for $i\in I$. There is a unique derivation $\partial'_*\colon Y'_*\to Y'_{*-1}$ such that $\partial'_1(e_i)=z_i-y_i$ for $i\in I$. Moreover, the chain complex of E-bimodules*

$$
E\xleftarrow{\widetilde\mu'}Y'_0\xleftarrow{\partial'_1}Y'_1\xleftarrow{\partial'_2}Y'_2\xleftarrow{\partial'_3}Y'_3\xleftarrow{\partial'_4}Y'_4\xleftarrow{\partial'_5}Y'_5\xleftarrow{\partial'_6}Y'_6\xleftarrow{\partial'_7}\dots
$$

is contractible as a complex of k-modules. A chain contracting homotopy is given by $\sigma_0(a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}) = az_{i_1}^{m_1}\cdots z_{i_l}^{m_l}$ *and*

$$\begin{aligned}&\sigma_{s+1}(ay_{i_1}^{m_1}e_{i_1}^{\delta_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l})\\&= -\sum_{\substack{j<\alpha\\0\le h<m_j}} az_{i_1}^{m_1+n_1}\cdots z_{i_{j-1}}^{m_{j-1}+n_{j-1}}y_{i_j}^h e_{i_j} z_{i_j}^{m_j+n_j-h-1} y_{i_{j+1}}^{m_{j+1}} e_{i_{j+1}}^{\delta_{j+1}} z_{i_{j+1}}^{n_{j+1}}\cdots y_{i_l}^{m_l}e_{i_l}^{\delta_l}z_{i_l}^{n_l},\end{aligned}$$

where $\alpha = \min\{k : \delta_k = 1\}$ *(in particular* $\delta_1 = \cdots = \delta_{\alpha-1} = 0$*).*

Proof. We must check that $\widetilde{\mu}'\circ\sigma_0 = id$, $\sigma_0\circ\widetilde{\mu}' + \partial_1'\circ\sigma_1 = id$ and $\partial_{s+1}'\circ\sigma_{s+1} + \sigma_s\circ\partial_s' = id$ for all $s>0$. It is immediate that

$$\begin{aligned}&\widetilde{\mu}'\circ\sigma_0(a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}) = \widetilde{\mu}'(az_{i_1}^{m_1}\cdots z_{i_l}^{m_l}) = a\#x_{i_1}^{m_1}\cdots x_{i_l}^{m_l}\quad\text{and}\\&\sigma_0\circ\widetilde{\mu}'(ay_{i_1}^{m_1}z_{i_1}^{n_1}\cdots y_{i_l}^{m_l}z_{i_l}^{n_l}) = \sigma_0(a\#x_{i_1}^{m_1+n_1}\cdots x_{i_l}^{m_l+n_l}) = az_{i_1}^{m_1+n_1}\cdots z_{i_l}^{m_l+n_l}.\end{aligned}$$

Let us compute $\partial_{s+1}'\circ\sigma_{s+1}$ for $s\ge 0$ and $\sigma_s\circ\partial_s'$ for $s>0$. To abbreviate we write

$$\begin{aligned}&\mathbf{M}_{\mathbf{i}_{uv}}^{\mathbf{m}\delta\mathbf{n}} = y_{i_u}^{m_u}e_{i_u}^{\delta_u}z_{i_u}^{n_u}\cdots y_{i_v}^{m_v}e_{i_v}^{\delta_v}z_{i_v}^{n_v} && \text{for } 1\le u<v\le l,\\&\mathbf{Z}_{\mathbf{i}_{uv}}^{\mathbf{m}+\mathbf{n}} = z_{i_u}^{m_u+n_u}\cdots z_{i_v}^{m_v+n_v} && \text{for } 1\le u<v\le l,\\&|\delta|_h = \delta_1+\cdots+\delta_h && \text{for } 1\le h\le l.\end{aligned}$$

We have

$$\begin{aligned}&\partial_{s+1}'\circ\sigma_{s+1}(a\mathbf{M}_{i_{1l}}^{\mathbf{m}\delta\mathbf{n}}) = \partial_s'\Bigg(-\sum_{\substack{u<\alpha\\0\le h<m_u}} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}}\Bigg)\\&= -\sum_{\substack{u<\alpha\\0\le h<m_u}} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}(y_{i_u}^h z_{i_u}^{m_u+n_u-h} - y_{i_u}^{h+1}z_{i_u}^{m_u+n_u-h-1})\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}}\\&-\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,v-1}}^{\mathbf{m}\delta\mathbf{n}} y_{i_v}^{m_v} z_{i_v}^{n_v+1}\mathbf{M}_{\mathbf{i}_{v+1,l}}^{\mathbf{m}\delta\mathbf{n}}\\&+\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,v-1}}^{\mathbf{m}\delta\mathbf{n}} y_{i_v}^{m_v+1} z_{i_v}^{n_v}\mathbf{M}_{\mathbf{i}_{v+1,l}}^{\mathbf{m}\delta\mathbf{n}},\end{aligned}$$

where $\alpha = \min\{k:\delta_k = 1\}$. Since

$$\begin{aligned}&\sum_{\substack{u<\alpha\\0\le h<m_u}} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}(y_{i_u}^h z_{i_u}^{m_u+n_u-h} - y_{i_u}^{h+1}z_{i_u}^{m_u+n_u-h-1})\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}}\\&=\sum_{u<\alpha} a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}}(z_{i_u}^{m_u+n_u} - y_{i_u}^{m_u}z_{i_u}^{n_u})\mathbf{M}_{\mathbf{i}_{u+1,l}}^{\mathbf{m}\delta\mathbf{n}} = a\mathbf{Z}_{\mathbf{i}_{1,\alpha-1}}^{\mathbf{m}+\mathbf{n}}\mathbf{M}_{\mathbf{i}_{\alpha l}}^{\mathbf{m}\delta\mathbf{n}} - a\mathbf{M}_{\mathbf{i}_{1l}}^{\mathbf{m}\delta\mathbf{n}},\end{aligned}$$

we obtain

$$\begin{aligned}&\partial_{s+1}'\circ\sigma_{s+1}(a\mathbf{M}_{i_{1l}}^{\mathbf{m}\delta\mathbf{n}}) = -a\mathbf{Z}_{\mathbf{i}_{1,\alpha-1}}^{\mathbf{m}+\mathbf{n}}\mathbf{M}_{\mathbf{i}_{\alpha l}}^{\mathbf{m}\delta\mathbf{n}} + a\mathbf{M}_{\mathbf{i}_{1l}}^{\mathbf{m}\delta\mathbf{n}}\\&-\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,v-1}}^{\mathbf{m}\delta\mathbf{n}} y_{i_v}^{m_v} z_{i_v}^{n_v+1}\mathbf{M}_{\mathbf{i}_{v+1,l}}^{\mathbf{m}\delta\mathbf{n}}\\&+\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}_{\mathbf{i}_{1,u-1}}^{\mathbf{m}+\mathbf{n}} y_{i_u}^h e_{i_u} z_{i_u}^{m_u+n_u-h-1}\mathbf{M}_{\mathbf{i}_{u+1,v-1}}^{\mathbf{m}\delta\mathbf{n}} y_{i_v}^{m_v+1} z_{i_v}^{n_v}\mathbf{M}_{\mathbf{i}_{v+1,l}}^{\mathbf{m}\delta\mathbf{n}}.\end{aligned}$$

On the other hand

$$\sigma_s \circ \partial'_s(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1l}}) = \sigma_s\Biggl(\sum_{v\ge\alpha}(-1)^{|\delta|_v-1}\delta_v a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,v-1}}(y_{i_v}^{m_v}z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1}z_{i_v}^{n_v})\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{v+1,l}}\Biggr).$$

Since

$$\begin{aligned}\sigma_s\bigl(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,\alpha-1}}(y_{i_\alpha}^{m_\alpha}z_{i_\alpha}^{n_\alpha+1} - y_{i_\alpha}^{m_\alpha+1}z_{i_\alpha}^{n_\alpha})\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{\alpha+1,l}}\bigr) &= a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\alpha-1}}y_{i_\alpha}^{m_\alpha}e_{i_\alpha}z_{i_\alpha}^{n_\alpha}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha+1,l}}\\ -\sum_{\substack{u<\alpha\\0\le h<m_u}} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}&\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,\alpha-1}}(y_{i_\alpha}^{m_\alpha}z_{i_\alpha}^{n_\alpha+1} - y_{i_\alpha}^{m_\alpha+1}z_{i_\alpha}^{n_\alpha})\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha+1,l}}\end{aligned}$$

and, for $v>\alpha$,

$$\begin{aligned}&\sigma_s\bigl(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1,v-1}}(y_{i_v}^{m_v}z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1}z_{i_v}^{n_v})\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{v+1,l}}\bigr)\\ &= -\sum_{\substack{u<\alpha\\0\le h<m_u}} a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}}(y_{i_v}^{m_v}z_{i_v}^{n_v+1} - y_{i_v}^{m_v+1}z_{i_v}^{n_v})\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}},\end{aligned}$$

we obtain

$$\begin{aligned}\sigma_s\circ\partial'_s(a\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{i_{1l}}) &= a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,\alpha-1}}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{\alpha l}}\\ &+\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}}y_{i_v}^{m_v}z_{i_v}^{n_v+1}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}}\\ &-\sum_{\substack{u<\alpha\\0\le h<m_u}}\sum_{v\ge\alpha}(-1)^{|\delta|_v}\delta_v a\mathbf{Z}^{\mathbf{m}+\mathbf{n}}_{\mathbf{i}_{1,u-1}}y_{i_u}^{h}e_{i_u}z_{i_u}^{m_u+n_u-h-1}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{u+1,v-1}}y_{i_v}^{m_v+1}z_{i_v}^{n_v}\mathbf{M}^{\mathbf{m}\delta\mathbf{n}}_{\mathbf{i}_{v+1,l}}.\end{aligned}$$

The result follows immediately from these facts. □

3.2. The resolution (X_*, d_*)**.** Let $Y_s = E\otimes\mathfrak{g}^{\wedge s}\otimes U(\mathfrak{g})$ $(s\ge 0)$ and $X_{rs} = E\otimes\mathfrak{g}^{\wedge s}\otimes\overline{A}^r\otimes E$ $(r,s\ge 0)$. The groups X_{rs} are E-bimodules in an obvious way and the groups Y_s are E-bimodules via the left canonical action and the right action

$$\bigl(a_0\otimes(v\otimes\mathbf{x}\otimes w)\bigr)(a\#u) = \sum_{(u)(v)(w)} a_0(a^{w^{(1)}})^{v^{(1)}}f(w^{(2)},u^{(1)})^{v^{(2)}}\otimes\bigl(v^{(3)}\otimes\mathbf{x}\otimes w^{(3)}u^{(2)}\bigr),$$

where $\mathbf{x} = x_1\wedge\cdots\wedge x_s$. Let us consider the diagram

$$\begin{array}{ccccccc}\vdots & & & & & & \\ \downarrow{\scriptstyle\partial_3} & & & & & & \\ Y_2 & \xleftarrow{\mu_2} & X_{02} & \xleftarrow{d^0_{12}} & X_{12} & \xleftarrow{d^0_{22}} & \cdots\\ \downarrow{\scriptstyle\partial_2} & & & & & & \\ Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \cdots\\ \downarrow{\scriptstyle\partial_1} & & & & & & \\ Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \cdots\end{array},$$

where $\partial_*\colon Y_*\to Y_{*-1}$, $\mu_*\colon X_{0*}\to Y_*$ and $d^0_{**}\colon X_{**}\to X_{*-1,*}$, are defined by:

$$\begin{aligned}\partial_s\bigl(a\#v\otimes\mathbf{x}\otimes w\bigr) &= \sum_{i=1}^{s}(-1)^i\sum_{(v)(x_i)} af(v^{(1)},x_i^{(1)})\#v^{(2)}x_i^{(2)}\otimes\mathbf{x}_{\hat{\imath}}\otimes w\\ &\quad-\sum_{i=1}^{s}(-1)^i\sum_{(w)(x_i)} af(x_i^{(1)},w^{(1)})^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}_{\hat{\imath}}\otimes x_i^{(2)}w^{(2)}\\ &\quad-\sum_{1\le i<j\le s}(-1)^{i+j}a\#v\otimes[x_i,x_j]\wedge\mathbf{x}_{\widehat{\imath\jmath}}\otimes w,\end{aligned}$$

$$\mu_s\bigl(a_0\#v\otimes\mathbf{x}\otimes a_1\#w\bigr)=\sum_{(v)}a_0a_1^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}\otimes w,$$

$$\begin{aligned}d^0_{rs}\bigl(a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#w\bigr) &= \sum_{(v)}a_0a_1^{v^{(1)}}\#v^{(2)}\otimes\mathbf{x}\otimes\mathbf{a}_{2,r+1}\#w\\ &\quad+\sum_{i=1}^{r}(-1)^i a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,i-1}\otimes a_ia_{i+1}\otimes\mathbf{a}_{i+1,r+1}\#w,\end{aligned}$$

where $\mathbf{a}_{1,r+1}=a_1\otimes\cdots\otimes a_{r+1}$ and $\mathbf{x}=x_1\wedge\cdots\wedge x_s$. It is immediate that the μ_s's and the d^0_{rs}'s are E-bimodule maps. In the proof of Theorem 3.2.1 we will see that the ∂_s's also are. Each horizontal complex X_{*s} is the tensor product $(E\otimes U(\mathfrak{g}))\otimes_A(A\otimes\overline{A}^*,b'_*)\otimes_A E$, where $E\otimes U(\mathfrak{g})$ is a right A-module via the canonical inclusion of A in E. Hence, the family $\sigma^0_{0s}\colon Y_s\to X_{0s}$, $\sigma^0_{r+1,s}\colon X_{rs}\to X_{r+1,s}$ $(r\ge 0)$, of left E-module maps, defined by

$$\sigma^0_{r+1,s}\bigl(a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#w\bigr)=(-1)^{r+1}a_0\#v\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\otimes 1\#w\quad(r\ge -1),$$

is a contracting homotopy of

$$Y_s\xleftarrow{\mu_s}X_{0s}\xleftarrow{d^0_{1s}}X_{1s}\xleftarrow{d^0_{2s}}X_{2s}\xleftarrow{d^0_{2s}}X_{3s}\xleftarrow{d^0_{3s}}X_{4s}\xleftarrow{d^0_{4s}}\dots.$$

Moreover each X_{rs} is a projective relative E-bimodule. We define E-bimodule maps

$$d^l_{rs}\colon X_{rs}\to X_{r+l-1,s-l}\qquad(r\ge 0\text{ and }1\le l\le s),$$

recursively by:

$$d^l_{rs}(\mathbf{y})=\begin{cases}-\sigma^0_{0,s-1}\circ\partial_s\circ\mu_s(\mathbf{y}) & \text{if } r=0\text{ and } l=1,\\ -\sum_{j=1}^{l-1}\sigma^0_{l-1,s-l}\circ d^{l-j}_{j-1,s-j}\circ d^j_{0s}(\mathbf{y}) & \text{if } r=0\text{ and } 1<l\le s,\\ -\sum_{j=0}^{l-1}\sigma^0_{r+l-1,s-l}\circ d^{l-j}_{r+j-1,s-j}\circ d^j_{rs}(\mathbf{y}) & \text{if } r>0,\end{cases}$$

where $\mathbf{y}=1\#1\otimes x_1\wedge\cdots\wedge x_s\otimes\mathbf{a}_{1r}\otimes 1\#1\in X_{rs}$.

Theorem 3.2.1. *The complex*

$$E\xleftarrow{\mu}X_0\xleftarrow{d_1}X_1\xleftarrow{d_2}X_2\xleftarrow{d_3}X_3\xleftarrow{d_4}X_4\xleftarrow{d_5}X_5\xleftarrow{d_6}X_6\xleftarrow{d_7}\dots,$$

where $\mu(a_0\#v\otimes a_1\#w)=\sum_{(v)(w)} a_0a_1^{v^{(1)}}f(v^{(2)},w^{(1)})\#v^{(3)}w^{(2)}$,

$$X_n=\bigoplus_{r+s=n}X_{rs}\qquad \text{and}\qquad d_n=\sum_{\substack{r+s=n\\ r+l>0}}\sum_{l=0}^{s}d^l_{rs}$$

is a relative projective resolution of the E*-bimodule* E.

Proof. Let $\widetilde{\mu}'\colon Y_0'\to E$ and (Y'_*,∂'_*) be as in Theorem 3.1.1 and let $\widetilde{\mu}\colon Y_0\to E$ be the E-bimodule map defined by

$$\widetilde{\mu}(a\otimes(v\otimes w))=\sum_{(v)(w)}af(v^{(1)},w^{(1)})\#v^{(2)}w^{(2)}.$$

Let $\vartheta_*\colon(Y_*,\partial_*)\to(Y'_*,\partial'_*)$ be the isomorphism of E-bimodule complexes, determined by $\vartheta_s(x_1\wedge\cdots\wedge x_s)=e_{x_1}\wedge\cdots\wedge e_{x_s}$. Since $\widetilde{\mu}=\widetilde{\mu}'\circ\vartheta_0$, we obtain from Theorem 3.1.1, that the complex of E-bimodules

$$E\xleftarrow{\widetilde{\mu}}Y_0\xleftarrow{\partial_1}Y_1\xleftarrow{\partial_2}Y_2\xleftarrow{\partial_3}Y_3\xleftarrow{\partial_4}Y_4\xleftarrow{\partial_5}Y_5\xleftarrow{\partial_6}Y_6\xleftarrow{\partial_7}\dots,$$

is contractible as a complex of k-modules. Hence, the result follows immediately from Theorem 2.1. □

The boundary maps of the relative projective resolution of E that we just found are defined recursively. Next we compute these morphisms.

Theorem 3.3. *For* $x_i,x_j\in\mathfrak{g}$, *we put* $\widehat{f}_{ij}=f(x_i,x_j)-f(x_j,x_i)$. *We have:*

$$\begin{aligned}
d^1_{rs}\bigl(a_0\#1\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#1\bigr)&=\sum_{i=1}^{s}(-1)^{i+r+1}a_0\#x_i\otimes\mathbf{x}_{\hat{\imath}}\otimes\mathbf{a}_{1,r+1}\#1\\
&+\sum_{i=1}^{s}(-1)^{i+r}a_0\#1\otimes\mathbf{x}_{\hat{\imath}}\otimes\mathbf{a}_{1,r+1}\#x_i\\
&+\sum_{\substack{i=1\\1\le h\le r+1}}^{s}(-1)^{i+r}a_0\#1\otimes\mathbf{x}_{\hat{\imath}}\otimes\mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes\mathbf{a}_{h+1,r+1}\#1\\
&+\sum_{1\le i<j\le s}(-1)^{i+j+r}a_0\#1\otimes[x_i,x_j]\wedge\mathbf{x}_{\widehat{\imath\jmath}}\otimes\mathbf{a}_{1,r+1}\#1,\\
d^2_{rs}\bigl(a_0\#1\otimes\mathbf{x}\otimes\mathbf{a}_{1,r+1}\#1\bigr)&=\sum_{\substack{1\le i<j\le s\\0\le h\le r}}(-1)^{i+j+h}a_0\#1\otimes\mathbf{x}_{\widehat{\imath\jmath}}\otimes\mathbf{a}_{1h}\otimes\widehat{f}_{ij}\otimes\mathbf{a}_{h,r+1}\#1
\end{aligned}$$

and $d^l_{rs}=0$ *for all* $l\ge 3$, *where* $\mathbf{a}_{1,r+1}=a_1\otimes\cdots\otimes a_{r+1}$ *and* $\mathbf{x}=x_1\wedge\cdots\wedge x_s$.

Proof. To unify the expressions in the proof, we put $d^0_{0s}:=\mu_s$, $d^1_{-1,s}=\partial_s$ and $d^2_{-1,s}=0$. First, we compute the maps d^1_{rs}. For $r=-1$ the assertion is trivial. Suppose $r\ge 0$ and the result is valid for $d^1_{r's}$ with $-1\le r'<r$. Since, for all $r,s\ge 0$,

$$\sigma_{rs}\bigl(b_0\#v\otimes\mathbf{x}\otimes\mathbf{b}_{1,r-1}\otimes 1\#w)\bigr)=0,\tag{1}$$

then

$$\begin{aligned} d^1_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr) &= -\sigma^0_{r,s-1}\circ d^1_{r-1,s}\circ d^0_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)\\ &= (-1)^{r+1}\sigma^0_{r,s-1}\circ d^1_{r-1,s}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\#1\bigr). \end{aligned}$$

Hence, the formula for $d^1_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)$ follows immediately by induction on r. Now, let us compute d^2_{rs}. Suppose $r\ge 0$ and the result is valid for $d^2_{r's}$ with $0\le r'<r$. Using (1) twice, we get

$$\begin{aligned} & d^2_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)\\ &= -\sigma^0_{r+1,s-2}\circ \bigl(d^2_{r-1,s}\circ d^0_{rs}+d^1_{r,s-1}\circ d^1_{rs}\bigr)\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)\\ &= \sigma^0_{r+1,s-2}\left(\sum_{1\le i<j\le s}\sum_{h=0}^{r-1}(-1)^{i+j+h+r+1}a_0\#1\otimes \mathbf{x}_{\widehat{\imath\jmath}}\otimes \mathbf{a}_{1h}\otimes \widehat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\#1\right)\\ &\quad -\sigma^0_{r+1,s-2}\left(d^1_{r,s-1}\left(\sum_{j=1}^{s}(-1)^{j+r}a_0\#1\otimes \mathbf{x}_{\widehat{\jmath}}\otimes \mathbf{a}_{1r}\otimes 1\#1\right)(1\#x_j)\right)\\ &= \sum_{1\le i<j\le s}\sum_{h=0}^{r-1}(-1)^{i+j+h}a_0\#1\otimes \mathbf{x}_{\widehat{\imath\jmath}}\otimes \mathbf{a}_{1h}\otimes \widehat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes 1\#1\\ &\quad -\sigma^0_{r+1,s-2}\left(\sum_{1\le i<j\le s}(-1)^{i+j}a_0\#1\otimes \mathbf{x}_{\widehat{\imath\jmath}}\otimes \mathbf{a}_{1r}\otimes \widehat{f}_{ij}\#1\right)\\ &= \sum_{1\le i<j\le s}\sum_{l=0}^{r}(-1)^{i+j+h}a_0\#1\otimes \mathbf{x}_{\widehat{\imath\jmath}}\otimes \mathbf{a}_{1h}\otimes \widehat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes 1\#1. \end{aligned}$$

To prove that $d^l_{rs}=0$ for all $l>2$, it is sufficient to check that

$$\begin{aligned} &\sigma^0_{r+2,s-4}\circ d^2_{r+1,s-2}\circ d^2_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)=0,\\ &\sigma^0_{r+1,s-3}\circ d^1_{r+1,s-2}\circ d^2_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)=0,\\ &\sigma^0_{r+1,s-3}\circ d^2_{r,s-1}\circ d^1_{rs}\bigl(a_0\#1\otimes \mathbf{x}\otimes \mathbf{a}_{1r}\otimes 1\#1\bigr)=0. \quad\square \end{aligned}$$

Next, we give an explicit formula for the comparison map between (X_*,d_*) and the canonical normalized Hochschild resolution $(E\otimes \overline{E}^*\otimes E,b'_*)$. Using this map it is easy to obtain explicit quasi-isomorphisms from the complex obtained in the following section for the Hochschild homology into the canonical one, and similarly for the cohomology.

Remark 3.4. There is a map of complexes $\theta_*\colon (X_*,d_*)\to(E\otimes \overline{E}^*\otimes E,b'_*)$, given by

$$\begin{aligned} &\theta_{r+s}\bigl(1_E\otimes x_1\wedge\cdots\wedge x_s\otimes a_1\otimes\cdots\otimes a_r\otimes 1_E\bigr)\\ &= \sum_{\tau\in\mathfrak{S}_s}\operatorname{sg}(\tau)1_E\otimes\bigl((1\#x_{\tau(1)})\otimes\cdots\otimes(1\#x_{\tau(s)})\bigr)*\bigl((a_1\#1)\otimes\cdots\otimes(a_r\#1)\bigr)\otimes 1_E, \end{aligned}$$

where $*$ denotes the shuffle product defined by

$$(e_1\otimes\cdots\otimes e_s)*(e_{s+1}\otimes\cdots\otimes e_n)=\sum_{\sigma\in\{(s,n-s)-\text{shuffles}\}}\operatorname{sg}(\sigma)e_{\sigma(1)}\otimes\cdots\otimes e_{\sigma(n)}.$$

4. The Hochschild homology

Let $E = A\#_f U(\mathfrak{g})$ and M an E-bimodule. We use Theorem 3.2.1 in order to construct a complex $\overline{X}_*(E,M)$, simpler than the canonical one, giving the Hochschild homology of E with coefficients in M.

The complex $\overline{X}_*(E,M)$. Let $r,s,l \ge 0$ with $l \le \min(2,s)$ and $r+l>0$. We define the morphism $\overline{d}^l_{rs}\colon M\otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s} \to M\otimes \overline{A}^{r+l-1}\otimes \mathfrak{g}^{\wedge s-l}$, by:

$$\begin{aligned}
\overline{d}^0_{rs}(m\otimes \mathbf{a}_{1r}\otimes \mathbf{x}) &= ma_1\otimes \mathbf{a}_{2r}\otimes \mathbf{x} + \sum_{i=1}^{r-1}(-1)^i m\otimes \mathbf{a}_{1,i-1}\otimes a_i a_{i+1}\otimes \mathbf{a}_{i+2,r}\otimes \mathbf{x}\\
&\quad + (-1)^r a_r m\otimes \mathbf{a}_{1,r-1}\otimes \mathbf{x},\\
\overline{d}^1_{rs}(m\otimes \mathbf{a}_{1r}\otimes \mathbf{x}) &= \sum_{i=1}^{s}(-1)^{i+r}\big((1\#x_i)m - m(1\#x_i)\big)\otimes \mathbf{a}_{1r}\otimes \mathbf{x}_{\widehat{\imath}}\\
&\quad + \sum_{\substack{i=1\\ 1\le h\le r}}^{s}(-1)^{i+r} m\otimes \mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\widehat{\imath}}\\
&\quad + \sum_{1\le i<j\le s}(-1)^{i+j+r} m\otimes \mathbf{a}_{1r}\otimes [x_i,x_j]\wedge \mathbf{x}_{\widehat{\imath\jmath}},\\
\overline{d}^2_{rs}(m\otimes \mathbf{a}_{1r}\otimes \mathbf{x}) &= \sum_{\substack{1\le i<j\le s\\ 0\le h\le r}}(-1)^{i+j+h} m\otimes \mathbf{a}_{1h}\otimes \widehat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\widehat{\imath\jmath}},
\end{aligned}$$

where $\mathbf{a}_{1r} = a_1\otimes\cdots\otimes a_r$, $\mathbf{x} = x_1\wedge\cdots\wedge x_s$ and $\widehat{f}_{ij} = f(x_i,x_j) - f(x_j,x_i)$.

Theorem 4.1. *The Hochschild homology* $\mathrm{H}_*(A,M)$, *of E with coefficients in M, is the homology of*

$$\overline{X}_*(E,M) = \quad \overline{X}_0 \xleftarrow{\overline{d}_1} \overline{X}_1 \xleftarrow{\overline{d}_2} \overline{X}_2 \xleftarrow{\overline{d}_3} \overline{X}_3 \xleftarrow{\overline{d}_4} \overline{X}_4 \xleftarrow{\overline{d}_5} \overline{X}_5 \xleftarrow{\overline{d}_6} \overline{X}_6 \xleftarrow{\overline{d}_7} \dots,$$

where

$$\overline{X}_n = \bigoplus_{r+s=n} M\otimes \overline{A}^r\otimes \mathfrak{g}^{\wedge s} \qquad \textit{and} \qquad \overline{d}_n = \sum_{\substack{r+s=n\\ r+l>0}}\ \sum_{l=0}^{\min(s,2)} \overline{d}^l_{rs}.$$

Proof. It follows from the fact that $\overline{X}_*(E,M)$ is the complex obtained by taking the tensor product $M\otimes_{E^e}(X_*,d_*)$, where (X_*,d_*) is the complex of Theorem 3.2.1, and using the identifications $\vartheta_{rs}\colon M\otimes\overline{A}^r\otimes\mathfrak{g}^{\wedge s} \longrightarrow M\otimes_{E^e} E\otimes \mathfrak{g}^{\wedge s}\otimes \overline{A}^r\otimes E$, given by $\vartheta_{rs}(m\otimes \mathbf{a}_{1r}\otimes\mathbf{x}) = m\otimes(1\#1\otimes \mathbf{x}\otimes a_{1r}\otimes 1\#1)$. □

Note that when f takes its values in k, then $\overline{X}_*(E,M)$ is the total complex of the double complex $\big(M\otimes\overline{A}^*\otimes\mathfrak{g}^{\wedge *}, \overline{d}^0_{**}, \overline{d}^1_{**}\big)$.

4.2. Stefan's Spectral sequence. Next we show that the complex $\overline{X}_*(E,M)$ has a natural filtration, which gives a more explicit version of the homology spectral sequence obtained in [St].

For each $x\in\mathfrak{g}$, we have the morphism $\Theta^x_*\colon (M\otimes\overline{A}^*, b_*) \longrightarrow (M\otimes\overline{A}^*, b_*)$, defined by $\Theta^x_r(m\otimes\mathbf{a}_{1r}) = \big((1\#x)m - m(1\#x)\big)\otimes\mathbf{a}_{1r} + \sum_{1\le h\le r} m\otimes \mathbf{a}_{1,h-1}\otimes a_h^x\otimes\mathbf{a}_{h+1,r}$.

Proposition 4.2.1. *For each $x, x' \in \mathfrak{g}$ the endomorphisms of $\mathrm{H}_*(A,M)$ induced by $\Theta_*^{x'} \circ \Theta_*^{x} - \Theta_*^{x} \circ \Theta_*^{x'}$ and by $\Theta_*^{[x,x']}$ coincide. Consequently $\mathrm{H}_*(A,M)$ is a right $U(\mathfrak{g})$-module.*

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}_0(A,M)$. In this case, the assertion can be easily checked, using that $(1\#x')(1\#x) = f(x',x)\#1 + 1\#x'x$ for all $x, x' \in \mathfrak{g}$. □

The chain complex $\overline{X}_*(E,M)$ has the filtration $F_0 \subseteq F_1 \subseteq F_2 \subseteq \dots$, where

$$F_i(\overline{X}_n) = \bigoplus_{\substack{r+s=n \\ r\ge 0,\, 0\le s\le i}} M \otimes \overline{A}^r \otimes \mathfrak{g}^{\wedge s}.$$

Using this fact and Proposition 4.2.1, we obtain the following:

Corollary 4.2.2. *There is a converging spectral sequence*

$$E^2_{rs} = \mathrm{H}_s(\mathfrak{g}, \mathrm{H}_r(A,M)) \Rightarrow \mathrm{H}_{r+s}(E,M).$$

Given an A-bimodule M we let $[A,M]$ denote the k-submodule of M generated by the commutators $am - ma$ ($a \in A$ and $m \in M$).

Corollary 4.2.3. *If A is separable, then $\mathrm{H}_*(E,M) = \mathrm{H}_*\big(\mathfrak{g}, \frac{M}{[A,M]}\big)$.*

4.3. Smooth algebras. Let A be a commutative ring and let M be a symmetric A-bimodule. In the famous paper [H-K-R] was proved that if A is a commutative smooth k-algebra, then $\mathrm{H}_n(A,M) = M \otimes_A \Omega^n_{A/k}$, where $\Omega^n_{A/k}$ denotes the A-module of differential n-forms of A. Next, we generalize this result by computing the Hochschild homology of a differential operator ring $E = A\#_f U(\mathfrak{g})$ with coefficients in an E-bimodule M which is symmetric as an A-bimodule, under the hypothesis that $\mathbb{Q} \subseteq k$ and A is a commutative smooth k-algebra.

Let us assume that $\mathbb{Q} \subseteq k$, A is a commutative ring and M is symmetric as an A-bimodule. For each $r, s, l \ge 0$ with $1 \le l \le \min(2,s)$, we define the morphism $\widetilde{d}^l_{rs} \colon M \otimes_A \Omega^r_{A/k} \otimes \mathfrak{g}^{\wedge s} \to M \otimes_A \Omega^{r+l-1}_{A/k} \otimes \mathfrak{g}^{\wedge s-l}$, by:

$$\begin{aligned}
\widetilde{d}^1_{rs}(m \otimes_A da_1 \cdots da_r \otimes \mathbf{x}) &= \sum_{i=1}^{s} (-1)^{i+r} \big((1\#x_i)m - m(1\#x_i)\big) \otimes_A da_1 \cdots da_r \otimes \mathbf{x}_{\hat{\imath}} \\
&\quad + \sum_{\substack{i=1 \\ 1\le h\le r}}^{s} (-1)^{i+r} m \otimes da_1 \cdots da_{h-1} da_h^{x_i} da_{h+1} \cdots da_r \otimes \mathbf{x}_{\hat{\imath}} \\
&\quad + \sum_{1\le i<j\le s} (-1)^{i+j+r} m \otimes_A da_1 \cdots da_r \otimes [x_i, x_j] \wedge \mathbf{x}_{\widehat{\imath\jmath}},
\end{aligned}$$

$$\widetilde{d}^2_{rs}(m \otimes_A da_1 \cdots da_r \otimes \mathbf{x}) = \sum_{1\le i<j\le s} (-1)^{i+j} m \otimes_A d\widehat{f}_{ij} da_1 \cdots da_r \otimes \mathbf{x}_{\widehat{\imath\jmath}},$$

where $\mathbf{x} = x_1 \wedge \cdots \wedge x_s$ and $\widehat{f}_{ij} = f(x_i,x_j) - f(x_j,x_i)$. Consider the complex

$$\widetilde{X}_*(E,M) = \quad \widetilde{X}_0 \xleftarrow{\widetilde{d}_1} \widetilde{X}_1 \xleftarrow{\widetilde{d}_2} \widetilde{X}_2 \xleftarrow{\widetilde{d}_3} \widetilde{X}_3 \xleftarrow{\widetilde{d}_4} \widetilde{X}_4 \xleftarrow{\widetilde{d}_5} \widetilde{X}_5 \xleftarrow{\widetilde{d}_6} \widetilde{X}_6 \xleftarrow{\widetilde{d}_7} \dots,$$

where

$$\widetilde{X}_n = \bigoplus_{r+s=n} M \otimes_A \Omega^r_{A/k} \otimes \mathfrak{g}^{\wedge s} \qquad \text{and} \qquad \widetilde{d}_n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=1}^{\min(s,2)} \widetilde{d}^l_{rs}.$$

Let $\vartheta_n \colon \overline{X}_n \to \widetilde{X}_n$ be the map $\vartheta_n(m\otimes \mathbf{a}_{1r}\otimes \mathbf{x}) = \frac{1}{r!} m\otimes_A da_1\cdots da_r\otimes \mathbf{x}$. It is easy to check that $\vartheta_* \colon \overline{X}_*(E,M) \to \widetilde{X}_*(E,M)$ is a morphism of complexes, which is a quasi-isomorphism when A is smooth. Hence, in this case, the Hochschild homology of E with coefficients in M is the homology of $\widetilde{X}_*(E,M)$.

A filtrated algebra E is called quasi-commutative if its associated graded algebra is commutative. In [S] was proved that if E a quasi-commutative algebra whose associated graded algebra is a polynomial ring, then E is isomorphic to a differential polynomial ring $k\#_f U(\mathfrak{g})$, with $\mathfrak{g}$ a finite dimensional Lie algebra. The Hochschild homology of this type of algebras was computed in [K, Theorem 3]. Next, we generalize this result.

Remark 4.3.1. Now, we assume that $\mathfrak{g}$ acts trivially on A. Consider the symmetric algebra $S = S_A(\mathfrak{g})$, endowed with the Poisson bracket defined by $\{a,x\} = \{a,b\} = 0$ and $\{x,y\} = f(x,y) - f(y,x) + [x,y]_{\mathfrak{g}}$ ($a\in A$, $x,y\in \mathfrak{g}$). It is easy to check that $\widetilde{X}_*(E,E)$ is isomorphic to the canonical complex $(\Omega^*_{S/k},\delta_*)$ introduced by Brylinski and Koszul in [B] and [Ko] respectively. In fact an isomorphism $\Theta_* \colon (\Omega^*_{S/k},\delta_*) \to \widetilde{X}_*(E,E)$ is given by $\Theta_{r+s}(P da_1\cdots da_r dx_1\cdots dx_s) = (-1)^s \eta(P) da_1\cdots da_r x_1\wedge\cdots\wedge x_s$, where $P\in S$, $a_1,\dots,a_r\in A$, $x_1,\dots,x_s\in\mathfrak{g}$ and $\eta\colon S\to E$ is the symmetrization $\eta(ay_1\cdots y_n) = \frac{a}{n!}\sum_{\sigma\in\mathfrak{S}_n} y_{\sigma(1)}\cdots y_{\sigma(n)}$.

4.4. Compatibility with the canonical decomposition. Let us assume that $k\supseteq \mathbb{Q}$, A is a commutative ring, M is symmetric as an A-bimodule and the cocycle f takes its values in k. In [G-S] was obtained a decomposition of the canonical Hochschild complex $(M\otimes \overline{A}^*, b_*)$. It is easy to check that the maps $\overline{d}_0$ and $\overline{d}_1$ are compatible with this decomposition. Since $\overline{d}_2$ is the zero map, we obtain a decomposition of $\overline{X}_*(E,M)$, and then a decomposition of $\mathrm{H}_*(E,M)$.

5. The Hochschild cohomology

Let $E = A\#_f U(\mathfrak{g})$ and M an E-bimodule. Using again Theorem 3.5 we construct a complex $\overline{X}^*(E,M)$, simpler than the canonical one, giving the Hochschild cohomology of E with coefficients in M.

The complex $\overline{X}^*(E,M)$. Let $r,s,l\ge 0$ with $l\le \min(2,s)$ and $r+l>0$. We define the morphism $\overline{d}_l^{rs}\colon \operatorname{Hom}_k(\overline{A}^{r+l-1}\otimes \mathfrak{g}^{\wedge s-l}, M) \to \operatorname{Hom}_k(\overline{A}^{r}\otimes \mathfrak{g}^{\wedge s}, M)$, by:

$$\begin{aligned}\overline{d}_0^{rs}(\varphi)(\mathbf{a}_{1r}\otimes\mathbf{x}) &= a_1\varphi(\mathbf{a}_{2r}\otimes\mathbf{x}) + \sum_{i=1}^{r-1}(-1)^i\varphi(\mathbf{a}_{1,i-1}\otimes a_ia_{i+1}\otimes\mathbf{a}_{i+2,r}\otimes\mathbf{x})\\ &\quad + (-1)^r\varphi(\mathbf{a}_{1,r-1}\otimes\mathbf{x})a_r,\end{aligned}$$

$$\overline{d}_1^{rs}(\varphi)(\mathbf{a}_{1r}\otimes\mathbf{x}) = \sum_{i=1}^{s}(-1)^{i+r}\big[\varphi(\mathbf{a}_{1r}\otimes\mathbf{x}_{\widehat{\imath}})(1\#x_i) - (1\#x_i)\varphi(\mathbf{a}_{1r}\otimes\mathbf{x}_{\widehat{\imath}})\big]$$

$$
\begin{aligned}
&+ \sum_{\substack{i=1\\1\le h\le r}}^{s} (-1)^{i+r}\varphi(\mathbf{a}_{1,h-1}\otimes a_h^{x_i}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\hat{\imath}})\\
&+ \sum_{1\le i<j\le s} (-1)^{i+j+r}\varphi(\mathbf{a}_{1r}\otimes [x_i,x_j]\wedge \mathbf{x}_{\widehat{\imath\jmath}}),\\
\overline{d}_2^{rs}(\varphi)(\mathbf{a}_{1r}\otimes \mathbf{x}) &= \sum_{\substack{1\le i<j\le s\\0\le h\le r}} (-1)^{i+j+h}\varphi(\mathbf{a}_{1h}\otimes \widehat{f}_{ij}\otimes \mathbf{a}_{h+1,r}\otimes \mathbf{x}_{\widehat{\imath\jmath}}),
\end{aligned}
$$

where $\mathbf{a}_{1r} = a_1\otimes\cdots\otimes a_r$, $\mathbf{x} = x_1\wedge\cdots\wedge x_s$ and $\widehat{f}_{ij} = f(x_i,x_j)-f(x_j,x_i)$. Applying the functor $\operatorname{Hom}_{E^e}(-,M)$ to the complex (X_*,d_*) of Theorem 3.2.1, and using the identifications

$$\vartheta^{rs}\colon \operatorname{Hom}_k(\overline{A}^r\otimes \mathfrak{g}^{\wedge s},M)\longrightarrow \operatorname{Hom}_{E^e}\big(E\otimes\mathfrak{g}^{\wedge s}\otimes\overline{A}^r\otimes E,M\big),$$

given by $\vartheta^{rs}(\varphi)(1\#1\otimes\mathbf{x}\otimes\mathbf{a}_{1r}\otimes 1\#1) = \varphi(\mathbf{a}_{1r}\otimes\mathbf{x})$, we obtain the complex

$$\overline{X}^*(E,M) = \quad \overline{X}^0 \xrightarrow{\overline{d}^1} \overline{X}^1 \xrightarrow{\overline{d}^2} \overline{X}^2 \xrightarrow{\overline{d}^3} \overline{X}^3 \xrightarrow{\overline{d}^4} \overline{X}^4 \xrightarrow{\overline{d}_5} \overline{X}^5 \xrightarrow{\overline{d}^6} \overline{X}^6 \xrightarrow{\overline{d}^7} \dots,$$

where

$$\overline{X}^n = \bigoplus_{r+s=n}\operatorname{Hom}_k(\overline{A}^r\otimes\mathfrak{g}^{\wedge s},M)\qquad\text{and}\qquad \overline{d}^n = \sum_{\substack{r+s=n\\r+l>0}}\sum_{l=0}^{\min(s,2)}\overline{d}_l^{rs}.$$

Note that when f takes its values in k, then $\overline{X}^*(E,M)$ is the total complex of the double complex $\big(\operatorname{Hom}_k(\overline{A}^*\otimes\mathfrak{g}^{\wedge *},M),\overline{d}_0^{**},\overline{d}_1^{**}\big)$.

Theorem 5.1. *The Hochschild cohomology* $\mathrm{H}^*(E,M)$*, of* E *with coefficients in* M*, is the homology of* $\overline{X}^*(E,M)$*.*

Proof. It is an immediate consequence of the above discussion. □

5.2. Stefan's Spectral sequence. Next we show that the complex $\overline{X}^*(E,M)$ has a natural filtration, which gives a more explicit version of the cohomology spectral sequence obtained in [St].

For each $x\in\mathfrak{g}$, we have the map $\Theta_x^*\colon\big(\operatorname{Hom}_k(\overline{A}^*,M),b^*\big)\longrightarrow\big(\operatorname{Hom}_k(\overline{A}^*,M),b^*\big)$, defined by $\Theta_x^r(\varphi)(\mathbf{a}_{1r}) = (1\#x)\varphi(\mathbf{a}_{1r})-\varphi(\mathbf{a}_{1r})(1\#x)-\sum_{h=1}^r\varphi(\mathbf{a}_{1,h-1}\otimes a_h^x\otimes\mathbf{a}_{h+1,r})$.

Proposition 5.2.1. *For each* $x,x'\in\mathfrak{g}$ *the endomorphisms of* $\mathrm{H}^*(A,M)$ *induced by* $\Theta_{x'}^*\circ\Theta_x^*-\Theta_x^*\circ\Theta_{x'}^*$ *and by* $\Theta_{[x',x]}^*$ *coincide. Consequently* $\mathrm{H}^*(A,M)$ *is a left* $U(\mathfrak{g})$*-module.*

Proof. It is similar to the proof of Proposition 4.2.1. □

The cochain complex $\overline{X}^*(E,M)$ has the filtration $F^0\supseteq F^1\supseteq\dots$, where

$$F^i(\overline{X}^n) = \bigoplus_{\substack{r+s=n\\r\ge 0,\ s\ge i}}\operatorname{Hom}_k\big(\overline{A}^r\otimes\mathfrak{g}^{\wedge s},M\big).$$

From this fact and Proposition 5.2.1, we obtain the following:

Corollary 5.2.2. *There is a converging spectral sequence*

$$E_2^{rs} = \mathrm{H}^s(\mathfrak{g}, \mathrm{H}^r(A,M)) \Rightarrow \mathrm{H}^{r+s}(E,M).$$

Given an A-bimodule M we let M^A denote the k-submodule of M consisting of the elements m verifying $am = ma$ for all $a \in A$.

Corollary 5.2.3. *If A is separable, then* $\mathrm{H}^*(E,M) = \mathrm{H}^*\big(\mathfrak{g}, M^A\big)$.

5.3. Smooth algebras. Let us assume that $k \supseteq \mathbb{Q}$, A is a commutative ring and M is symmetric as an A-bimodule. For each $r,s,l \ge 0$ with $1 \le l \le \min(2,s)$, we define the morphism $\widetilde{d}_l^{rs}\colon \mathrm{Hom}_A(\Omega_{A/k}^{r+l-1} \otimes \mathfrak{g}^{\wedge s-l}, M) \to \mathrm{Hom}_A(\Omega_{A/k}^{r} \otimes \mathfrak{g}^{\wedge s}, M)$, by:

$$\begin{aligned}
\widetilde{d}_1^{rs}(\varphi)(da_1\cdots da_r \otimes \mathbf{x}) &= \sum_{i=1}^{s} (-1)^{i+r}\big[\varphi(da_1\cdots da_r \otimes \mathbf{x}_{\hat{\imath}}), (1\#x_i)\big] \\
&+ \sum_{\substack{i=1\\ 1\le h\le r}}^{s} (-1)^{i+r}\varphi(da_1\cdots d_{h-1} da_h^{x_i} d_{h+1}\cdots d_r \otimes \mathbf{x}_{\hat{\imath}}) \\
&+ \sum_{1\le i<j\le s} (-1)^{i+j+r}\varphi(da_1\cdots da_r \otimes [x_i,x_j]\wedge \mathbf{x}_{\hat{\imath}\hat{\jmath}}), \\
\widetilde{d}_2^{rs}(\varphi)(da_1\cdots da_r \otimes \mathbf{x}) &= \sum_{1\le i<j\le s} (-1)^{i+j}\varphi\big(d\widehat{f}_{ij} da_1\cdots da_r \otimes \mathbf{x}_{\hat{\imath}\hat{\jmath}}\big),
\end{aligned}$$

where $\mathbf{x} = x_1\wedge\cdots\wedge x_s$, $\widehat{f}_{ij} = f(x_i,x_j) - f(x_j,x_i)$ and $\big[\varphi(da_1\cdots da_r \otimes \mathbf{x}_{\hat{\imath}}), (1\#x_i)\big] = \varphi(da_1\cdots da_r\otimes \mathbf{x}_{\hat{\imath}})(1\#x_i) - (1\#x_i)\varphi(da_1\cdots da_r\otimes \mathbf{x}_{\hat{\imath}})$. Consider the complex

$$\widetilde{X}^*(E,M) = \quad \widetilde{X}^0 \xrightarrow{\widetilde{d}^1} \widetilde{X}^1 \xrightarrow{\widetilde{d}^2} \widetilde{X}^2 \xrightarrow{\widetilde{d}^3} \widetilde{X}^3 \xrightarrow{\widetilde{d}^4} \widetilde{X}^4 \xrightarrow{\widetilde{d}_5} \widetilde{X}^5 \xrightarrow{\widetilde{d}^6} \widetilde{X}^6 \xrightarrow{\widetilde{d}^7} \dots,$$

where

$$\widetilde{X}^n = \bigoplus_{r+s=n} \mathrm{Hom}_k(\Omega_{A/k}^r \otimes \mathfrak{g}^{\wedge s}, M) \qquad \text{and} \qquad \widetilde{d}^n = \sum_{\substack{r+s=n\\ r+l>0}} \sum_{l=1}^{\min(s,2)} \widetilde{d}_l^{rs}.$$

Let $\vartheta^n\colon \widetilde{X}^n \to \overline{X}^n$ be the map $\vartheta^n(\varphi)(\mathbf{a}_{1r}\otimes \mathbf{x}) = \frac{1}{r!}\varphi(da_1\cdots da_r\otimes \mathbf{x})$. It is easy to check that $\vartheta^*\colon \widetilde{X}^*(E,M) \to \overline{X}^*(E,M)$ is a morphism of complexes, which is a quasi-isomorphism when A is smooth. Hence, in this case, the Hochschild cohomology of E with coefficients in M is the cohomology of $\widetilde{X}^*(E,M)$.

5.4. Compatibility with the canonical decomposition. Let us assume that $k \supseteq \mathbb{Q}$, A is a commutative ring, M is symmetric as an A-bimodule and the cocycle f takes its values in k. Then, the Hochschild cohomology $\mathrm{H}^*(E,M)$ has a decomposition similar to the one obtained in 4.4 for the Hochschild homology.

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