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A micropolar microplane theory

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Abstract

The classical micropolar constitutive theory is reformulated within the framework of the microplane concept in order to obtain constitutive equations and models including available and, hopefully, more precise information of the complex microstructure of engineering materials like concrete and other composites. The main objective is the macroscopic modeling and description of anisotropic material response behavior by means of the well-developed microplane concept applied within a micropolar continuum setting. A thermodynamically consistent approach is considered to derive the so called micropolar microplane theory. The main assumption of the present proposal is the integral relation between the macroscopic and the microscopic free energy as advocated by Carol et al. [Int. J. Solids Struct. 38 (2001) 2921] and Kuhl et al. [Int. J. Solids Struct. 38 (2001) 2933] whereby the microplane laws are chosen such that the macroscopic Clausius–Duhem inequality is fully satisfied. This theoretical framework is considered to derive both elastic and elastoplastic micropolar microplane models.

Keywords: Micropolar; Microplane; Elasticity; Plasticity

1. Introduction

The development of constitutive formulations for engineering materials has experienced a tremendous progress during the last decades. From this evolution we may extract two main conclusions: On the one hand the macroscopic response behavior of real solids and materials strongly depends on their microstructure and corresponding micromechanical features. Therefore, precise failure predictions require constitutive formulations which account for the relevant

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informations of the microstructure. On the other hand the computational simulation of the complex response and failure behavior of real structures does necessarily require efficient and robust macroscopic material models.

As a consequence we observe an increasing tendency to use macroscopic models based on fundamental aspects of the microstructure of the material. Without questions, one of the most successful attempts in this sense is the microplane theory pioneered by Bazant and Gambarova [3], Bazant [2] and Bazant and Oh [4,5] in the spirit of an idea originally proposed by Taylor [24].

Beside the positive aspect of incorporating microscopic information in the macroscopic material formulation by means of simple and consistent considerations, the most important contribution of the microplane theory comes from its superior capacity to model anisotropic material behaviors. The main assumption of the microplane theory is the relationship between the microscopic or rather local strain or stress components and the corresponding macroscopic or rather global tensorial quantities. Thereby, two different approaches may be considered, whereby the static or the kinematic constraint require that either the stresses or the strains on each microplane are the resolved components of their macroscopic counterparts. The static constraint was extensively used until the first application of the microplane theory to continuum damage mechanics and to cohesive-frictional materials by Bazant and Gambarova [3] and Bazant [2]. It was in those works were the terminology microplane theory appeared for the first time instead of the original terminology slip theory which was related to the plastic behavior assumed on slip planes by Taylor and other authors like, e.g., Batdorf and Budianski [1]. The potentials of the microplane theory for describing nonlinear response behaviors of engineering cohesive-frictional materials like concrete were extensively demonstrated in the early contributions by Bazant and coauthors related with the microplane theory and, more recently, in the works by Bazant and Prat [6], Carol et al. [7,8] and Carol and Bazant [9], among many others.

Despite the considerable progress of the microplane models and their excellent performance for brittle and quasi-brittle materials, the lack of a thermodynamically consistent approach for deriving microplane-based constitutive formulations was an important deficit. In this sense, the fact that the satisfaction of the second law of thermodynamics could generally not be guaranteed, as demonstrated by Carol et al. [10], was a crucial and fundamental shortcoming for further developments of the microplane models. To solve this deficiency these authors proposed a method for deriving microplane constitutive formulations within a thermodynamically consistent framework by means of the incorporation of a microscopic free Helmholtz energy on every microplane. This concept was successfully extended for inelastic material behavior such as damage and plasticity by Kuhl et al. [14,15]. However, both this work as well as the previous one by Carol et al. [10] were concerned with classical Boltzmann continua (elastic and inelastic).

In the present work the thermodynamically consistent approach to derive microplane models is further extended for micropolar continua in the spirit of the brothers Cosserat [11]. The main aim is to enrich the microscopic kinematic and strength features of the microplane formulation so as to reproduce particular and more complex behaviors of the internal structure of composite quasi-brittle materials like concrete whereby the presence of aggregates may contribute to the development of microrotations in characteristic planes during load histories beyond the elastic limit.

Another relevant motivation of the micropolar microplane theory in this work, related to the numerical solutions of BVP based, e.g., on the smeared crack concept, is the regularization of the post-peak response behavior of strain softening materials, see e.g., [12]. In this sense, the incorporation of the micropolar length scale leads to a quasi-non-local microplane model when the additional degrees of freedom of micropolar continua are activated. This characteristic length accounts for mesh objectivity during FE simulations of softening behaviors, see e.g., [21,22,26].

Section 2 of this paper refers to the most relevant equations of the micropolar continuum, including equilibrium, strain/curvature and stress/couple stress equations. In Section 3 the microplane theory is extended to the micropolar continuum. Thereby, both the static and the kinematic constraint are redefined to include the macroscopic couple stress and the macroscopic curvature projections at microplanes, respectively. Section 4 is concerned with the hemispherical integrations which are required for the closed form formulation of some micropolar microplane models. In Section 5 the attention focuses on the method for deriving micropolar microplane constitutive equations. Section 6 refers to the application of the proposed thermodynamically consistent method to the formulation of general 3D and, particularly, of 2D linear elastic constitutive equations. In Section 7, elastoplastic constitutive equations are derived both for the general case and for the von Mises type model. The results demonstrate the potentials of the proposed thermodynamically consistent approach to derive constitutive models based on enriched kinematic and strength properties at the microscopic level and thus allowing for computational simulations of complex anisotropic response behavior of engineering materials.

2. Micropolar continuum

In this section the relevant equations of the geometrically linear micropolar continuum in the spirit of the brothers Cosserat [11] are briefly reviewed. This theory was advocated by several authors during the last decades. One of the most prominent works in this regard was made by Eringen [13] who presented a detailed analysis of elastic micropolar continua and of their mechanical features. However, the first application of the micropolar continuum in non-linear computational solid mechanics took place at the end of the 1980's in the works by Mühlhaus [17] and de Borst [12] who analyzed the potentials of the elastoplastic micropolar constitutive theory to regularize the predictions of post-peak response behaviors of structural systems within the theoretical framework of the smeared-crack approach. In the same line, Steinmann and Willam [19], Willam and Dietsche [25], Sluys [18] and Willam et al. [26] analyzed the localization indicators and localization properties of nonlinear micropolar continua. Theoretical treatments of micropolar continua are also contained in [20,23].

2.1. Equilibrium at macro level

The quasi-static form of linear and angular momentum of a micropolar continuum in the three-dimensional configuration $\mathscr{B}$ (omitting body forces and body couples for simplicity) reads

$$\begin{aligned} \operatorname{div}\boldsymbol{\sigma}^{t} &= \mathbf{0} \\ \operatorname{div}\boldsymbol{\mu}^{t} + \boldsymbol{e} : \boldsymbol{\sigma} &= \mathbf{0} \end{aligned} \tag{1}$$

whereby $\boldsymbol{\mu}$ is a non-symmetric second-order tensor (with $\boldsymbol{\mu}^{t}$ its transpose) which represents the couple stresses of the micropolar continuum. In index notation the divergence operations read $\sigma_{ij,i}$

and $\mu_{ij,i}$. The local equilibrium equations of the classical continuum and the corresponding typical symmetric form of the stress tensor $\boldsymbol{\sigma}$ are restored when $\operatorname{div}\boldsymbol{\mu}^{\mathrm{t}} = \mathbf{0} \rightarrow \boldsymbol{e} : \boldsymbol{\sigma} = \mathbf{0}$. Here $\boldsymbol{e}$ denotes the third-order permutation tensor with the properties $e_{ijk} = 1$ and $e_{ijk} = -1$, respectively, for even and odd permutations of i, j, k, respectively, and $e_{ijk} = 0$ otherwise.

2.2. Strain and curvature at macro level

The deformation of a micropolar continuum is described in terms of two types of local motions: the classical translatory displacement field $\boldsymbol{u}$ and an independent micropolar rotation field characterized by the first-order tensor $\boldsymbol{\omega}$. This enriched configuration space leads to the following micropolar strain measures

$$\begin{aligned} \boldsymbol{\epsilon} &= \nabla_x \boldsymbol{u} - \boldsymbol{\Omega} \\ \boldsymbol{\kappa} &= \nabla_x \boldsymbol{\omega} \end{aligned} \tag{2}$$

with $\boldsymbol{\Omega} = -\boldsymbol{e} \cdot \boldsymbol{\omega}$ and the index notation $u_{i,j}$ for e.g. $\nabla_x \boldsymbol{u}$. Here $\boldsymbol{\epsilon}$ represents the non-symmetric micropolar strain tensor and $\boldsymbol{\kappa}$ is the micropolar curvature tensor which takes into account the differential changes of the micropolar rotations in the neighborhood of a point.

The second-order strain tensor may finally be decomposed into a symmetric and skew-symmetric contribution $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}^{\mathrm{sym}} + \boldsymbol{\epsilon}^{\mathrm{skw}}$ with

$$\begin{aligned} \boldsymbol{\epsilon}^{\mathrm{sym}} &= \frac{1}{2}[\nabla_x \boldsymbol{u} + \nabla_x^{\mathrm{t}} \boldsymbol{u}] \\ \boldsymbol{\epsilon}^{\mathrm{skw}} &= \frac{1}{2}[\nabla_x \boldsymbol{u} - \nabla_x^{\mathrm{t}} \boldsymbol{u}] + \boldsymbol{e} \cdot \boldsymbol{\omega} \end{aligned} \tag{3}$$

2.3. Virtual work at macro level

To be complete and to highlight the work conjugacy of micropolar stress and couple stress with the micropolar strain and curvature we recall the corresponding expression for the virtual work

$$\int_{\mathscr{B}} [\delta\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{\mathrm{t}} + \delta\boldsymbol{\kappa} : \boldsymbol{\mu}^{\mathrm{t}}] \, \mathrm{d}V = \int_{\partial\mathscr{B}} [\delta\boldsymbol{u} \cdot \boldsymbol{t}_{\sigma}^{\mathrm{p}} + \delta\boldsymbol{\omega} \cdot \boldsymbol{t}_{\mu}^{\mathrm{p}}] \, \mathrm{d}A \tag{4}$$

Here $\boldsymbol{t}_{\sigma}^{\mathrm{p}}$ and $\boldsymbol{t}_{\mu}^{\mathrm{p}}$ denote the prescribed external surface stress and couple stress traction. Observe that expression like, e.g., $\delta\boldsymbol{\epsilon} : \boldsymbol{\sigma}^{\mathrm{t}}$ may be expanded into $\delta\boldsymbol{\epsilon}^{\mathrm{sym}} : \boldsymbol{\sigma}^{\mathrm{sym}} - \delta\boldsymbol{\epsilon}^{\mathrm{skw}} : \boldsymbol{\sigma}^{\mathrm{skw}}$. For a detailed account on different variational settings of micropolar continua we refer to Steinmann [23].

3. Microplane theory

In the microplane theory the macro-mechanical response behavior of materials is controlled by constitutive equations defined on characteristic planes, the so-called microplanes by means of either the static or the kinematic constraint, requiring that either the stresses or the strains on each microplane, respectively, can be derived by projections of their macroscopic counterparts.

3.1. Stresses and couple stresses at microplanes

Even though we shall not use the static constraint lateron, it is enlightning to distinguish this concept from the more useful kinematic constraint discussed in the sequel. For the case of the static constraint the stress and couple stress (traction) vectors on each microplane, see Fig. 1, are given by pre-multiplication with the microplane normal vector $\boldsymbol{n}$, i.e.

$$\bar{\boldsymbol{t}}_\sigma = \boldsymbol{n} \cdot \boldsymbol{\sigma} \qquad \bar{\boldsymbol{t}}_\mu = \boldsymbol{n} \cdot \boldsymbol{\mu} \tag{5}$$

Note that the first index of σ_{ij} and μ_{ij} denotes the surface normal in our definition. The microplane stresses and couple stresses follow as their normal and tangential components

$$\begin{aligned} \bar{\boldsymbol{\sigma}}_\mathrm{N} &= \bar{\sigma}_\mathrm{N}\boldsymbol{n} \qquad \bar{\boldsymbol{\sigma}}_\mathrm{T} = \bar{\boldsymbol{t}}_\sigma - \bar{\boldsymbol{\sigma}}_\mathrm{N} \\ \bar{\boldsymbol{\mu}}_\mathrm{N} &= \bar{\mu}_\mathrm{N}\boldsymbol{n} \qquad \bar{\boldsymbol{\mu}}_\mathrm{T} = \bar{\boldsymbol{t}}_\mu - \bar{\boldsymbol{\mu}}_\mathrm{N} \end{aligned} \tag{6}$$

and are obtained as projections of their macroscopic counterparts to the microplanes

$$\begin{aligned} \bar{\sigma}_\mathrm{N} &= \boldsymbol{N} : \boldsymbol{\sigma} \qquad \bar{\boldsymbol{\sigma}}_\mathrm{T} = \boldsymbol{T} : \boldsymbol{\sigma} \\ \bar{\mu}_\mathrm{N} &= \boldsymbol{N} : \boldsymbol{\mu} \qquad \bar{\boldsymbol{\mu}}_\mathrm{T} = \boldsymbol{T} : \boldsymbol{\mu} \end{aligned} \tag{7}$$

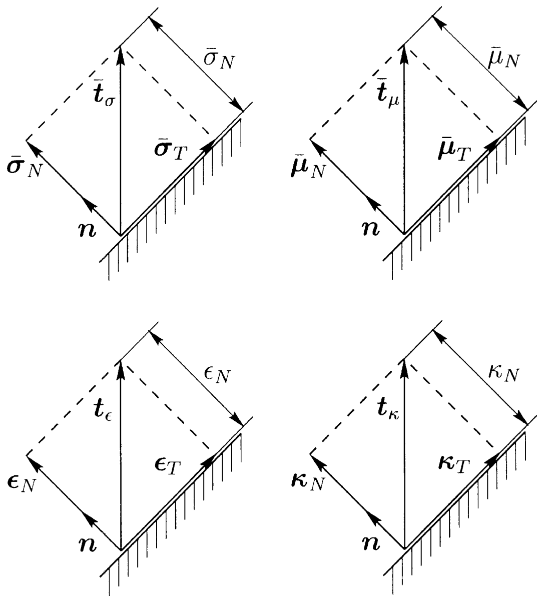

Fig. 1. Microplane normal and tangent components of the strain and curvature tensors.

Here, the second- and third-order projection tensors $\boldsymbol{N}$ and $\boldsymbol{T}$ are defined with $[\mathbf{I}]_{ijkl} = \delta_{ik}\delta_{jl}$ the fourth-order identity tensor and $\boldsymbol{n}$ the microplane normal vector as

$$\begin{aligned} \boldsymbol{N} &= \boldsymbol{n} \otimes \boldsymbol{n} \\ \boldsymbol{T} &= \boldsymbol{n} \cdot \mathbf{I} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \end{aligned} \tag{8}$$

Note again that the specific format of the third-order projection tensor $\boldsymbol{T}$ is due to our index convention as highlighted earlier in Eq. (5). In summary, $\bar{\sigma}_{\mathrm{N}}$ and $\bar{\mu}_{\mathrm{N}}$ represent the normal projected stress and normal projected couple stress, respectively, while $\bar{\boldsymbol{\sigma}}_{\mathrm{T}}$ and $\bar{\boldsymbol{\mu}}_{\mathrm{T}}$ denote the tangential projected stress and tangential projected couple stress vectors. Note, however, that these projected components of the macroscopic stress and couple stress tensors are in general different from those derived from constitutive equations at the microplanes, that we shall denote as σ_{N}, $\boldsymbol{\sigma}_{\mathrm{T}}$, μ_{N}, $\boldsymbol{\mu}_{\mathrm{T}}$ in the sequel.

The microplane normal and tangential stresses and couple stresses may be further decomposed into symmetric and skew-symmetric parts according to the usual decomposition strategy in a micropolar continuum. Nevertheless, since we shall not use explicitly the static constraint in the sequel, we refrain from doing so.

3.2. Strains and curvatures at microplanes

For the case of the kinematic constraint the strain and curvature vectors on each microplane, compare Fig. 1, are given by post-multiplication with the microplane normal vector $\boldsymbol{n}$, i.e.

$$\boldsymbol{t}_{\epsilon} = \boldsymbol{\epsilon} \cdot \boldsymbol{n} = \nabla_n \boldsymbol{u} - \boldsymbol{\omega} \times \boldsymbol{n} \qquad \boldsymbol{t}_{\kappa} = \boldsymbol{\kappa} \cdot \boldsymbol{n} = \nabla_n \boldsymbol{\omega} \tag{9}$$

The microplane strains and curvatures follow as their normal and tangential components

$$\begin{aligned} \boldsymbol{\epsilon}_{\mathrm{N}} &= \epsilon_{\mathrm{N}} \boldsymbol{n} & \boldsymbol{\epsilon}_{\mathrm{T}} &= \boldsymbol{t}_{\epsilon} - \boldsymbol{\epsilon}_{\mathrm{N}} \\ \boldsymbol{\kappa}_{\mathrm{N}} &= \kappa_{\mathrm{N}} \boldsymbol{n} & \boldsymbol{\kappa}_{\mathrm{T}} &= \boldsymbol{t}_{\kappa} - \boldsymbol{\kappa}_{\mathrm{N}} \end{aligned} \tag{10}$$

These equations are valid both for the symmetric as well as for the skew-symmetric parts of the strain and curvature measures. Taking into account the following properties

$$\begin{aligned} \boldsymbol{\epsilon}^{\mathrm{skw}} \cdot \boldsymbol{n} &= -\boldsymbol{n} \cdot \boldsymbol{\epsilon}^{\mathrm{skw}} & \boldsymbol{\kappa}^{\mathrm{skw}} \cdot \boldsymbol{n} &= -\boldsymbol{n} \cdot \boldsymbol{\kappa}^{\mathrm{skw}} \\ \boldsymbol{\epsilon}^{\mathrm{sym}} \cdot \boldsymbol{n} &= \quad \boldsymbol{n} \cdot \boldsymbol{\epsilon}^{\mathrm{sym}} & \boldsymbol{\kappa}^{\mathrm{sym}} \cdot \boldsymbol{n} &= \quad \boldsymbol{n} \cdot \boldsymbol{\kappa}^{\mathrm{sym}} \end{aligned} \tag{11}$$

and comparing to Eqs. (5) and (7) within the discussion of the static constraint the symmetric and skew-symmetric microplane strain components in the normal and tangential directions of the microplanes are then defined by

$$\begin{aligned} \epsilon_{\mathrm{N}} &= \boldsymbol{N} : \boldsymbol{\epsilon}^{\mathrm{sym}} = \boldsymbol{N} : \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{sym}} &= \boldsymbol{T} : \boldsymbol{\epsilon}^{\mathrm{sym}} = \boldsymbol{T}^{\mathrm{sym}} : \boldsymbol{\epsilon} \\ \boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{skw}} &= -\boldsymbol{T} : \boldsymbol{\epsilon}^{\mathrm{skw}} = -\boldsymbol{T}^{\mathrm{skw}} : \boldsymbol{\epsilon} \end{aligned} \tag{12}$$

while the corresponding microplane curvature components in the normal and tangential directions of the microplanes are given accordingly by

$$\begin{aligned} \kappa_{\mathrm{N}} &= \boldsymbol{N} : \boldsymbol{\kappa}^{\mathrm{sym}} = \boldsymbol{N} : \boldsymbol{\kappa} \\ \boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{sym}} &= \boldsymbol{T} : \boldsymbol{\kappa}^{\mathrm{sym}} = \boldsymbol{T}^{\mathrm{sym}} : \boldsymbol{\kappa} \\ \boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{skw}} &= -\boldsymbol{T} : \boldsymbol{\kappa}^{\mathrm{skw}} = -\boldsymbol{T}^{\mathrm{skw}} : \boldsymbol{\kappa} \end{aligned} \tag{13}$$

Here, in addition to the projection tensor $\boldsymbol{T}$ the symmetric and skew-symmetric projection tensors $\boldsymbol{T}^{\mathrm{sym}}$ and $\boldsymbol{T}^{\mathrm{skw}}$ with $\boldsymbol{T} = \boldsymbol{T}^{\mathrm{sym}} + \boldsymbol{T}^{\mathrm{skw}}$ are defined as

$$\begin{aligned} \boldsymbol{T}^{\mathrm{sym}} &= \boldsymbol{n} \cdot \mathsf{I}^{\mathrm{sym}} - \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \\ \boldsymbol{T}^{\mathrm{skw}} &= \boldsymbol{n} \cdot \mathsf{I}^{\mathrm{skw}} \end{aligned} \tag{14}$$

whereby $[\mathsf{I}^{\mathrm{sym}}]_{ijkl} = [\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]/2$ and $[\mathsf{I}^{\mathrm{skw}}]_{ijkl} = [\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}]/2$ are the symmetric and skew-symmetric parts of the fourth-order identity tensor $\mathsf{I} = \mathsf{I}^{\mathrm{sym}} + \mathsf{I}^{\mathrm{skw}}$. Note again that the apparently odd sign convention in Eqs. (12.3) and (13.3) is a direct consequence of the index definition as highlighted in Eq. (5) and the corresponding definition of the third-order projection tensor $\boldsymbol{T}$ in Eq. (8.2).

4. Hemispherical integrations

The integration properties of the microplane normal vector $\boldsymbol{n}$ are documented, e.g. in the works of by Bazant and Oh [5] and Lubarda and Krajcinovic [16] and are applied to perform analytical integrations over the hemisphere Ω

$$\begin{aligned} &\int_{\Omega} \mathrm{d}\Omega = 2\pi \\ &\int_{\Omega} \boldsymbol{n} \otimes \boldsymbol{n} \, \mathrm{d}\Omega = \frac{2\pi}{3} \boldsymbol{I} \\ &\int_{\Omega} \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \, \mathrm{d}\Omega = \frac{2\pi}{3} \left[\mathsf{I}_{\mathrm{vol}} + \frac{2}{5} \mathsf{I}_{\mathrm{dev}}^{\mathrm{sym}} \right] \end{aligned} \tag{15}$$

with $[\boldsymbol{I}]_{ij} = \delta_{ij}$ the second-order identity tensor and the volumetric and symmetric deviatoric fourth-order projection tensors defined as

$$\mathsf{I}_{\mathrm{vol}} = \frac{1}{3} \boldsymbol{I} \otimes \boldsymbol{I} \qquad \mathsf{I}_{\mathrm{dev}}^{\mathrm{sym}} = \mathsf{I}^{\mathrm{sym}} - \mathsf{I}_{\mathrm{vol}} \tag{16}$$

For later use the relevant products of the projection tensors $\boldsymbol{T}$ and $\boldsymbol{N}$ are given with $\boldsymbol{T}^{\mathrm{t}}$ denoting the appropriate transposition of the third-order projection tensor $\boldsymbol{T}$

$$[\boldsymbol{T}^{\mathrm{t}} \cdot \boldsymbol{T}]_{ijkl} := T_{aij} T_{akl} = n_i n_k \delta_{jl} - n_i n_j n_k n_l \qquad [\boldsymbol{N} \otimes \boldsymbol{N}]_{ijkl} = n_i n_j n_k n_l \tag{17}$$

and thus integrate over the hemisphere into

$$\frac{3}{2\pi}\int_{\Omega}\boldsymbol{T}^{\mathrm{t}}\cdot\boldsymbol{T}\,\mathrm{d}\Omega=\mathbf{I}^{\mathrm{skw}}+\frac{3}{5}\mathbf{I}^{\mathrm{sym}}_{\mathrm{dev}}\qquad\frac{3}{2\pi}\int_{\Omega}\boldsymbol{N}\otimes\boldsymbol{N}\,\mathrm{d}\Omega=\mathbf{I}_{\mathrm{vol}}+\frac{2}{5}\mathbf{I}^{\mathrm{sym}}_{\mathrm{dev}}\tag{18}$$

Accordingly, the products of $\boldsymbol{T}^{\mathrm{sym}}$ and $\boldsymbol{T}^{\mathrm{skw}}$ relevant in the sequel are given as

$$\begin{aligned}\left[[\boldsymbol{T}^{\mathrm{sym}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{sym}}\right]_{ijkl}&=\frac{1}{4}\left[n_in_k\delta_{jl}+n_in_l\delta_{jk}+\delta_{il}n_jn_k+\delta_{ik}n_jn_l\right]-n_in_jn_kn_l\\ \left[[\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{skw}}\right]_{ijkl}&=\frac{1}{4}\left[n_in_k\delta_{jl}-n_in_l\delta_{jk}-\delta_{il}n_jn_k+\delta_{ik}n_jn_l\right]\\ \left[[\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{sym}}\right]_{ijkl}&=\frac{1}{4}\left[n_in_k\delta_{jl}+n_in_l\delta_{kj}-\delta_{il}n_jn_k-\delta_{ik}n_jn_l\right]\end{aligned}\tag{19}$$

and thus integrate over the hemisphere into

$$\begin{aligned}&\frac{3}{2\pi}\int_{\Omega}[\boldsymbol{T}^{\mathrm{sym}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{sym}}\,\mathrm{d}\Omega=\frac{3}{5}\mathbf{I}^{\mathrm{sym}}_{\mathrm{dev}}\\ &\frac{3}{2\pi}\int_{\Omega}[\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{skw}}\,\mathrm{d}\Omega=\mathbf{I}^{\mathrm{skw}}\\ &\frac{3}{2\pi}\int_{\Omega}[\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}}\cdot\boldsymbol{T}^{\mathrm{sym}}\,\mathrm{d}\Omega=\mathbf{0}\end{aligned}\tag{20}$$

5. Thermodynamically consistent modeling

Based on the proposal by Carol et al. [10] and Kuhl et al. [15] we first develop here a general formulation for thermodynamically consistent micropolar microplane constitutive laws. The macroscopic Clausius–Duhem inequality for isothermal processes reads for a micropolar continuum

$$\mathscr{D}^{\mathrm{mac}}=\boldsymbol{\sigma}^{\mathrm{t}}:\dot{\boldsymbol{\epsilon}}+\boldsymbol{\mu}^{\mathrm{t}}:\dot{\boldsymbol{\kappa}}-\dot{\psi}^{\mathrm{mac}}\geqslant 0\tag{21}$$

Next, the main assumption is given by a relation between the microscopic and macroscopic free energy, compare Carol et al. [10] and Kuhl et al. [15]

$$\psi^{\mathrm{mac}}=\frac{3}{2\pi}\int_{\Omega}\psi^{\mathrm{mic}}\,\mathrm{d}\Omega\tag{22}$$

Moreover we consider the convenient uncoupled format of the microscopic free energy dependent on strain and curvatures components ϵ_{N}, $\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{sym}}$, $\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{skw}}$ and κ_{N}, $\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{sym}}$, $\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{skw}}$, respectively, as well as on the sets of internal variables $\boldsymbol{q}_u$, $\boldsymbol{q}_\omega$, related with the translatory motions and rotations, respectively

$$\psi^{\text{mic}} = \underbrace{\psi_u^{\text{mic}}(\epsilon_{\text{N}}, \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}}, \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}, \boldsymbol{q}_u)}_{\text{membrane energy}} + \underbrace{\psi_\omega^{\text{mic}}(\kappa_{\text{N}}, \boldsymbol{\kappa}_{\text{T}}^{\text{sym}}, \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}, \boldsymbol{q}_\omega)}_{\text{bending energy}} \tag{23}$$

Thus an additive decomposition of the total microscopic free energy into a membrane energy and a bending energy was assumed. This corresponds to the particular case of micropolar response behavior where the membrane-bending coupling diminishes to zero.

The rate of the microscopic free energy follows with the kinematic constraint relations in Eqs. (12)–(14) as

$$\begin{aligned}\dot{\psi}^{\text{mic}} = {} & \left[\sigma_{\text{N}}\boldsymbol{N} + \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} - \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}\right] : \dot{\boldsymbol{\epsilon}} \\ & + \left[\mu_{\text{N}}\boldsymbol{N} + \boldsymbol{\mu}_{\text{T}}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} - \boldsymbol{\mu}_{\text{T}}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}\right] : \dot{\boldsymbol{\kappa}} \\ & - \mathscr{D}_u^{\text{mic}} - \mathscr{D}_\omega^{\text{mic}}\end{aligned} \tag{24}$$

with σ_{N}, $\boldsymbol{\sigma}_{\text{T}}^{\text{sym}}$ and $\boldsymbol{\sigma}_{\text{T}}^{\text{skw}}$ the microplane constitutive stresses

$$\sigma_{\text{N}} := \frac{\partial\psi^{\text{mic}}}{\partial\epsilon_{\text{N}}} \qquad \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} := \frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{\epsilon}_{\text{T}}^{\text{sym}}} \qquad \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} := \frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}} \tag{25}$$

and μ_{N}, $\boldsymbol{\mu}_{\text{T}}^{\text{sym}}$ and $\boldsymbol{\mu}_{\text{T}}^{\text{skw}}$ the microplane constitutive couple stresses

$$\mu_{\text{N}} := \frac{\partial\psi^{\text{mic}}}{\partial\kappa_{\text{N}}} \qquad \boldsymbol{\mu}_{\text{T}}^{\text{sym}} := \frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{\kappa}_{\text{T}}^{\text{sym}}} \qquad \boldsymbol{\mu}_{\text{T}}^{\text{skw}} := \frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{\kappa}_{\text{T}}^{\text{skw}}} \tag{26}$$

and $\mathscr{D}_u^{\text{mic}}$, $\mathscr{D}_\omega^{\text{mic}}$ the microplane dissipation rate of membrane and bending type, respectively,

$$\mathscr{D}_u^{\text{mic}} := -\frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{q}_u} \star \dot{\boldsymbol{q}}_u \qquad \mathscr{D}_\omega^{\text{mic}} := -\frac{\partial\psi^{\text{mic}}}{\partial\boldsymbol{q}_\omega} \star \dot{\boldsymbol{q}}_\omega \tag{27}$$

whereby $\star$ indicates the appropriate contraction. Recall however, that the constitutive stresses and couple stresses on the microplanes are in general different from the projected microplane stress and couple stress components $\bar{\sigma}_{\text{N}}$, $\bar{\boldsymbol{\sigma}}_{\text{T}}^{\text{sym}}$, $\bar{\boldsymbol{\sigma}}_{\text{T}}^{\text{skw}}$ and $\bar{\mu}_{\text{N}}$, $\bar{\boldsymbol{\mu}}_{\text{T}}^{\text{sym}}$, $\bar{\boldsymbol{\mu}}_{\text{T}}^{\text{skw}}$ obtained by means of the static constraint.

Due to the membrane-bending decoupling assumption, the stress tensor components can be derived from that portion of the total microscopic free energy which is only related with the translatory motion ψ_u^{mic} while the components of the couple stress tensor follow from the other portion of the total microscopic free energy, related with micropolar rotations ψ_ω^{mic}.

The rate of the macroscopic free energy can then be expanded by applying the integral Eq. (22) to the rate of the microscopic free energy Eq. (24) as

$$\dot{\psi}^{\text{mac}} = \frac{3}{2\pi}\int_{\Omega}[\boldsymbol{N}\sigma_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{skw}}]\,\text{d}\Omega : \dot{\boldsymbol{\epsilon}} + \frac{3}{2\pi}\int_{\Omega}[\boldsymbol{N}\mu_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{\mu}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{\mu}_{\text{T}}^{\text{skw}}]\,\text{d}\Omega : \dot{\boldsymbol{\kappa}} - \frac{3}{2\pi}\int_{\Omega}[\mathscr{D}_u^{\text{mic}} + \mathscr{D}_\omega^{\text{mic}}]\,\text{d}\Omega \tag{28}$$

Comparing with Eq. (21) the macroscopic stress tensor and couple stress tensor are thus obtained from the microplane constitutive stress and couple stress components as follows

$$\begin{aligned}\boldsymbol{\sigma}^{\text{t}} &= \frac{3}{2\pi}\int_{\Omega}[\boldsymbol{N}\sigma_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{skw}}]\,\text{d}\Omega \\ \boldsymbol{\mu}^{\text{t}} &= \frac{3}{2\pi}\int_{\Omega}[\boldsymbol{N}\kappa_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}]\,\text{d}\Omega\end{aligned} \tag{29}$$

In order to satisfy the macroscopic dissipation inequality

$$\mathscr{D}^{\text{mac}} = \frac{3}{2\pi}\int_{\Omega}[\mathscr{D}_u^{\text{mic}} + \mathscr{D}_\omega^{\text{mic}}]\,\text{d}\Omega \geqslant 0 \tag{30}$$

we will require that the total microplane energy dissipation on every microplane is non-negative

$$\mathscr{D}^{\text{mic}} = \mathscr{D}_u^{\text{mic}} + \mathscr{D}_\omega^{\text{mic}} \geqslant 0 \tag{31}$$

which is a stronger requirement than that of Eq. (30) and therefore represents a sufficient condition to fulfill the second law of thermodynamics.

The rate of the microscopic free energy in Eq. (24) can be understood as the microplane format of the Clausius–Duhem inequality for the isothermal case, which can now be rewritten as

$$\mathscr{D}^{\text{mic}} = \mathscr{D}_u^{\text{mic}} + \mathscr{D}_\omega^{\text{mic}} = \mathscr{P}_u^{\text{mic}} - \dot{\psi}_u^{\text{mic}} + \mathscr{P}_\omega^{\text{mic}} - \dot{\psi}_\omega^{\text{mic}} \geqslant 0 \tag{32}$$

with the microplane stress and couple stress power

$$\begin{aligned}\mathscr{P}_u^{\text{mic}} &= \sigma_{\text{N}}\dot{\epsilon}_{\text{N}} + \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} \cdot \dot{\boldsymbol{\epsilon}}_{\text{T}}^{\text{sym}} + \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} \cdot \dot{\boldsymbol{\epsilon}}_{\text{T}}^{\text{skw}} \\ \mathscr{P}_\omega^{\text{mic}} &= \mu_{\text{N}}\dot{\kappa}_{\text{N}} + \boldsymbol{\mu}_{\text{T}}^{\text{sym}} \cdot \dot{\boldsymbol{\kappa}}_{\text{T}}^{\text{sym}} + \boldsymbol{\mu}_{\text{T}}^{\text{skw}} \cdot \dot{\boldsymbol{\kappa}}_{\text{T}}^{\text{skw}}\end{aligned} \tag{33}$$

6. Micropolar microplane elasticity

In the case of hyper-elastic behavior of both the membrane and bending stiffness components the internal variables are zero ($\boldsymbol{q}_u = \boldsymbol{q}_\omega \equiv \boldsymbol{0}$) and the microscopic free energy reduces to

$$\psi^{\text{mic}} = \psi_u^{\text{mic}}(\epsilon_{\text{N}}, \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}}, \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}) + \psi_\omega^{\text{mic}}(\kappa_{\text{N}}, \boldsymbol{\kappa}_{\text{T}}^{\text{sym}}, \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}) \tag{34}$$

For hyper-elasticity the free energy coincides with the stored energy which is here assumed to be composed of separate membrane and bending contributions of the following format

$$\begin{aligned}\psi_u^{\text{mic}} &= W_{\text{N u}}(\epsilon_{\text{N}}) + W_{\text{T u}}^{\text{sym}}(\boldsymbol{\epsilon}_{\text{T}}^{\text{sym}}) + W_{\text{T u}}^{\text{skw}}(\boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}) \\ \psi_\omega^{\text{mic}} &= W_{\text{N}\,\omega}(\kappa_{\text{N}}) + W_{\text{T}\,\omega}^{\text{sym}}(\boldsymbol{\kappa}_{\text{T}}^{\text{sym}}) + W_{\text{T}\,\omega}^{\text{skw}}(\boldsymbol{\kappa}_{\text{T}}^{\text{skw}})\end{aligned} \tag{35}$$

whereby especially for linear elasticity the elastic moduli $E_{\text{N u}}$, $\mathbf{E}_{\text{T u}}^{\text{sym}}$, $\mathbf{E}_{\text{T u}}^{\text{skw}}$, $E_{\text{N}\,\omega}$, $\mathbf{E}_{\text{T}\,\omega}^{\text{sym}}$ and $\mathbf{E}_{\text{T}\,\omega}^{\text{skw}}$ are introduced into the microscopic stored energy functions as

$$\begin{aligned}W_{\text{N u}} &= \frac{1}{2}\epsilon_{\text{N}} E_{\text{N u}} \epsilon_{\text{N}} \qquad W_{\text{T u}}^{\text{sym}} = \frac{1}{2}\boldsymbol{\epsilon}_{\text{T}}^{\text{sym}} \cdot \mathbf{E}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}} \qquad W_{\text{T u}}^{\text{skw}} = \frac{1}{2}\boldsymbol{\epsilon}_{\text{T}}^{\text{skw}} \cdot \mathbf{E}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}} \\ W_{\text{N}\,\omega} &= \frac{1}{2}\kappa_{\text{N}} E_{\text{N}\,\omega} \kappa_{\text{N}} \qquad W_{\text{T}\,\omega}^{\text{sym}} = \frac{1}{2}\boldsymbol{\kappa}_{\text{T}}^{\text{sym}} \cdot \mathbf{E}_{\text{T}\omega}^{\text{sym}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{sym}} \qquad W_{\text{T}\,\omega}^{\text{skw}} = \frac{1}{2}\boldsymbol{\kappa}_{\text{T}}^{\text{skw}} \cdot \mathbf{E}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}\end{aligned} \tag{36}$$

The previous definition of the microplane Clausius–Duhem inequality in Eq. (32) then leads to the microplane constitutive stresses and couple stresses as thermodynamically conjugate variables to the microplane strain and curvature components, respectively

$$\begin{aligned}\sigma_{\text{N}} &= \frac{\partial \psi_u^{\text{mic}}}{\partial \epsilon_{\text{N}}} = E_{\text{N u}}\epsilon_{\text{N}} \qquad \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} = \frac{\partial \psi_u^{\text{mic}}}{\partial \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}}} = \boldsymbol{E}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}} \qquad \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} = \frac{\partial \psi_u^{\text{mic}}}{\partial \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}} = \boldsymbol{E}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}} \\ \mu_{\text{N}} &= \frac{\partial \psi_\omega^{\text{mic}}}{\partial \kappa_{\text{N}}} = E_{N\omega}\kappa_{\text{N}} \qquad \boldsymbol{\mu}_{\text{T}}^{\text{sym}} = \frac{\partial \psi_\omega^{\text{mic}}}{\partial \boldsymbol{\kappa}_{\text{T}}^{\text{sym}}} = \boldsymbol{E}_{\text{T}\,\omega}^{\text{sym}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{sym}} \qquad \boldsymbol{\mu}_{\text{T}}^{\text{skw}} = \frac{\partial \psi_\omega^{\text{mic}}}{\partial \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}} = \boldsymbol{E}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}\end{aligned} \tag{37}$$

From the macroscopic version of the Clausius–Duhem inequality the macroscopic stress and couple stress tensors follow as functions of the microplane components

$$\begin{aligned}\boldsymbol{\sigma}^{\text{t}} &= \frac{3}{2\pi}\int_\Omega [\boldsymbol{N}E_{\text{N u}}\epsilon_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{\epsilon}_{\text{T}}^{\text{skw}}]\,\mathrm{d}\Omega \\ \boldsymbol{\mu}^{\text{t}} &= \frac{3}{2\pi}\int_\Omega [\boldsymbol{N}E_{\text{N}\,\omega}\kappa_{\text{N}} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{sym}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{sym}} - [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{\kappa}_{\text{T}}^{\text{skw}}]\,\mathrm{d}\Omega\end{aligned} \tag{38}$$

The last equation can alternatively be rewritten as

$$\begin{aligned}\boldsymbol{\sigma}^{\text{t}} &= \mathsf{E}_u : \boldsymbol{\epsilon} \\ \boldsymbol{\mu}^{\text{t}} &= \mathsf{E}_\omega : \boldsymbol{\kappa}\end{aligned} \tag{39}$$

whereby the macroscopic membrane and bending constitutive moduli are defined as follows

$$\begin{aligned}\mathsf{E}_u &= \frac{3}{2\pi}\int_\Omega [E_{\text{N u}}\boldsymbol{N}\otimes\boldsymbol{N} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} + [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}]\,\mathrm{d}\Omega \\ \mathsf{E}_\omega &= \frac{3}{2\pi}\int_\Omega [E_{\text{N}\,\omega}\boldsymbol{N}\otimes\boldsymbol{N} + [\boldsymbol{T}^{\text{sym}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} + [\boldsymbol{T}^{\text{skw}}]^{\text{t}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}]\,\mathrm{d}\Omega\end{aligned} \tag{40}$$

Next, under the common assumption of microplane isotropy the tangential strain and curvature vectors and the tangential stress and couple stress vectors remain parallel during the entire load history. Consequently, we consider the following simplification

$$\begin{aligned}
\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{sym}} \| \boldsymbol{\sigma}_{\mathrm{T}}^{\mathrm{sym}} &\rightarrow \boldsymbol{E}_{\mathrm{T\,u}}^{\mathrm{sym}} = E_{\mathrm{T\,u}}^{\mathrm{sym}} \boldsymbol{I} \\
\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{skw}} \| \boldsymbol{\sigma}_{\mathrm{T}}^{\mathrm{skw}} &\rightarrow \boldsymbol{E}_{\mathrm{T\,u}}^{\mathrm{skw}} = E_{\mathrm{T\,u}}^{\mathrm{skw}} \boldsymbol{I} \\
\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{sym}} \| \boldsymbol{\mu}_{\mathrm{T}}^{\mathrm{sym}} &\rightarrow \boldsymbol{E}_{\mathrm{T}\omega}^{\mathrm{sym}} = E_{\mathrm{T}\omega}^{\mathrm{sym}} \boldsymbol{I} \\
\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{skw}} \| \boldsymbol{\mu}_{\mathrm{T}}^{\mathrm{skw}} &\rightarrow \boldsymbol{E}_{\mathrm{T}\omega}^{\mathrm{skw}} = E_{\mathrm{T}\omega}^{\mathrm{skw}} \boldsymbol{I}
\end{aligned} \tag{41}$$

Assuming further that the constitutive moduli are independent from the orientation of the microplanes we arrive at

$$\begin{aligned}
\mathbf{E}_u &= \frac{3}{2\pi}\left[E_{\mathrm{N\,u}} \int_\Omega \boldsymbol{N} \otimes \boldsymbol{N}\, \mathrm{d}\Omega + E_{\mathrm{T\,u}}^{\mathrm{sym}} \int_\Omega [\boldsymbol{T}^{\mathrm{sym}}]^{\mathrm{t}} \cdot \boldsymbol{T}^{\mathrm{sym}}\, \mathrm{d}\Omega + E_{\mathrm{T\,u}}^{\mathrm{skw}} \int_\Omega [\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}} \cdot \boldsymbol{T}^{\mathrm{skw}}\, \mathrm{d}\Omega\right] \\
\mathbf{E}_\omega &= \frac{3}{2\pi}\left[E_{\mathrm{N}\,\omega} \int_\Omega \boldsymbol{N} \otimes \boldsymbol{N}\, \mathrm{d}\Omega + E_{\mathrm{T}\,\omega}^{\mathrm{sym}} \int_\Omega [\boldsymbol{T}^{\mathrm{sym}}]^{\mathrm{t}} \cdot \boldsymbol{T}^{\mathrm{sym}}\, \mathrm{d}\Omega + E_{\mathrm{T}\,\omega}^{\mathrm{skw}} \int_\Omega [\boldsymbol{T}^{\mathrm{skw}}]^{\mathrm{t}} \cdot \boldsymbol{T}^{\mathrm{skw}}\, \mathrm{d}\Omega\right]
\end{aligned} \tag{42}$$

The integration formulae (15)–(20) allow an analytical evaluation of the integrals in Eq. (42) to render

$$\begin{aligned}
\mathbf{E}_u &= \left[\frac{3}{5}E_{\mathrm{N\,u}} - \frac{3}{5}E_{\mathrm{T\,u}}^{\mathrm{sym}}\right]\mathbf{I}_{\mathrm{vol}} + \left[\frac{2}{5}E_{\mathrm{N\,u}} + \frac{3}{5}E_{\mathrm{T\,u}}^{\mathrm{sym}}\right]\mathbf{I}^{\mathrm{sym}} + E_{\mathrm{T\,u}}^{\mathrm{skw}}\mathbf{I}^{\mathrm{skw}} \\
\mathbf{E}_\omega &= \left[\frac{3}{5}E_{\mathrm{N}\,\omega} - \frac{3}{5}E_{\mathrm{T}\,\omega}^{\mathrm{sym}}\right]\mathbf{I}_{\mathrm{vol}} + \left[\frac{2}{5}E_{\mathrm{N}\,\omega} + \frac{3}{5}E_{\mathrm{T}\,\omega}^{\mathrm{sym}}\right]\mathbf{I}^{\mathrm{sym}} + E_{\mathrm{T}\,\omega}^{\mathrm{skw}}\mathbf{I}^{\mathrm{skw}}
\end{aligned} \tag{43}$$

The comparison of Eq. (43) with the general isotropic non-symmetric elastic tensors for decoupled membrane-bending behavior

$$\begin{aligned}
\mathbf{E}_u &= \alpha_1 \mathbf{I}_{\mathrm{vol}} + [\alpha_2 + \alpha_3]\mathbf{I}^{\mathrm{sym}} + [\alpha_2 - \alpha_3]\mathbf{I}^{\mathrm{skw}} \\
\mathbf{E}_\omega &= \beta_1 \mathbf{I}_{\mathrm{vol}} + [\beta_2 + \beta_3]\mathbf{I}^{\mathrm{sym}} + [\beta_2 - \beta_3]\mathbf{I}^{\mathrm{skw}}
\end{aligned} \tag{44}$$

then leads finally to the identifications

$$\begin{aligned}
\alpha_1 &= \frac{3}{5}E_{\mathrm{N\,u}} - \frac{3}{5}E_{\mathrm{T\,u}}^{\mathrm{sym}}; & \beta_1 &= \frac{3}{5}E_{\mathrm{N}\,\omega} - \frac{3}{5}E_{\mathrm{T}\,\omega}^{\mathrm{sym}} \\
\alpha_2 + \alpha_3 &= \frac{2}{5}E_{\mathrm{N\,u}} + \frac{3}{5}E_{\mathrm{T\,u}}^{\mathrm{sym}}; & \beta_2 + \beta_3 &= \frac{2}{5}E_{\mathrm{N}\,\omega} + \frac{3}{5}E_{\mathrm{T}\,\omega}^{\mathrm{sym}} \\
\alpha_2 - \alpha_3 &= E_{\mathrm{T\,u}}^{\mathrm{skw}}; & \beta_2 - \beta_3 &= E_{\mathrm{T}\,\omega}^{\mathrm{skw}}
\end{aligned} \tag{45}$$

whereby $\alpha_1 := L$ and $\alpha_2 + \alpha_3 := 2G$ are recognized as the common Lamé parameters, while $\alpha_2 - \alpha_3 := 2G_{\mathrm{c}}$ is the micropolar shear modulus which couples the skew-symmetric stress–strain components.

2D Elasticity: In this case the number of elastic material parameters reduces from six to four with e.g. $\beta_1 = \beta_3 = 0$ and $\beta_2 := 2Gl_{\mathrm{c}}^2$, where l_{c} denotes the elastic characteristic length of the micropolar continuum, see e.g. Steinmann and Willam [19]. In this case we thus obtain $\beta_2 = E_{\mathrm{N}\,\omega} = E_{\mathrm{T}\omega}^{\mathrm{sym}} = E_{\mathrm{T}\omega}^{\mathrm{skw}}$ which leads to

$$l_c^2 = \frac{E_{N\omega}}{2G} = \frac{E_{T\omega}^{sym}}{2G} = \frac{E_{T\omega}^{skw}}{2G} \tag{46}$$

Note that the constants $E_{N\omega}$, $E_{T\omega}^{sym}$ and $E_{T\omega}^{skw}$ contained in the micropolar curvature (dimension [length^{-1}]) versus couple stress (dimension [force × length^{-1}]) constitutive law have the dimension [force] whereas G has dimension [force × length^{-2}]; thus l_c has dimension [length]. Eq. (46) together with the three relations on the left hand side of Eq. (45) define the four independent parameters of the two-dimensional elastic micropolar microplane formulation, whereby the bending macroscopic elastic constitutive modulus results in

$$\mathbf{E}_\omega = 2Gl_c^2\mathbf{I} \tag{47}$$

The index notation of the constitutive equation related with $\mathbf{E}_\omega$ takes the form

$$\mu_{\alpha 3} = 2Gl_c^2\omega_{3,\alpha} \quad \alpha = 1, 2 \tag{48}$$

whereby the subindex 3 indicates the out of plane direction. The last expression does fully coincide with the elastic constitutive equation of the classical 2D micropolar continuum.

Elastic Boltzmann continuum: An interesting particular case of microplane hyper-elasticity is obtained for

$$E_{Tu}^{skw} = 0 \quad \text{and} \quad \mathbf{E}_\omega = \mathbf{O} \tag{49}$$

which leads to the well-known elastic operator of the elastic Boltzmann continuum and to the classical format of the elastic moduli

$$\mathbf{E}_u = \left[\frac{3}{5}E_{Nu} - \frac{3}{5}E_{Tu}^{sym}\right]\mathbf{I}_{vol} + \left[\frac{2}{5}E_{Nu} + \frac{3}{5}E_{Tu}^{sym}\right]\mathbf{I}_{sym} = L\mathbf{I}_{vol} + 2G\mathbf{I}_{sym} \tag{50}$$

Recall that the limitations for the values of L and $2G$, see e.g. Bazant and Prat [6] who argued that a macroscopic Poisson ration larger than 0.25 is not admissible in the standard microplane model (since it requires a negative tangential stiffness at the microplanes), may be eliminated by considering a volumetric deviatoric decomposition as advocated e.g. by Carol et al. [7] on each microplane. Thereby $\boldsymbol{\epsilon}$ is decomposed in its volumetric and deviatoric contribution $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_{vol} + \boldsymbol{\epsilon}_{dev}$ with $\boldsymbol{\epsilon}_{vol} = \mathbf{I}_{vol} : \boldsymbol{\epsilon}$ and $\boldsymbol{\epsilon}_{dev} = \mathbf{I}_{dev}^{sym} : \boldsymbol{\epsilon}$ with subsequent projection of these contributions on the microplanes, i.e. $\epsilon_N^{vol} = \boldsymbol{N} : \boldsymbol{\epsilon}_{vol}$, $\epsilon_N^{dev} = \boldsymbol{N} : \boldsymbol{\epsilon}_{dev}$, $\boldsymbol{\epsilon}_T^{vol} = \boldsymbol{T}^{sym} : \boldsymbol{\epsilon}_{vol} \equiv \mathbf{0}$ and $\boldsymbol{\epsilon}_T^{dev} = \boldsymbol{T}^{sym} : \boldsymbol{\epsilon}_{dev} \equiv \boldsymbol{\epsilon}_T$. A stored energy based on the arguments ϵ_N^{vol}, ϵ_N^{dev} and $\boldsymbol{\epsilon}_T$ then allows for arbitrary Poisson ratios in the range −1 to 0.5.

7. Micropolar microplane elastoplasticity

In this section the thermodynamically consistent formulation of the microplane-based micropolar elastoplastic model is presented both for the general case and for the von Mises type model.

7.1. General case

The elastoplastic micropolar microplane response behavior is motivated by the usual additive decomposition of the macroscopic total strain and curvature tensors into elastic and plastic contributions

$$\begin{aligned} \boldsymbol{\epsilon} &= \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p \\ \boldsymbol{\kappa} &= \boldsymbol{\kappa}_e + \boldsymbol{\kappa}_p \end{aligned} \tag{51}$$

Thus the kinematic constraint assumption extends the applicability of the additive decomposition to the microplane level. As a consequence, the total strain and curvature components at microplanes can be expressed as

$$\begin{aligned} \epsilon_{\mathrm{N}} &= \epsilon_{\mathrm{N\,e}} + \epsilon_{\mathrm{N\,p}} & \kappa_{\mathrm{N}} &= \kappa_{\mathrm{N\,e}} + \kappa_{\mathrm{N\,p}} \\ \boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{sym}} &= \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{sym}} + \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{sym}} & \boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{sym}} &= \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{sym}} + \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{sym}} \\ \boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{skw}} &= \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{skw}} + \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{skw}} & \boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{skw}} &= \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{skw}} + \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{skw}} \end{aligned} \tag{52}$$

In the most general case the tensor of internal variables includes the plastic contributions of all the strain and curvature components at the microplanes

$$\boldsymbol{q} = \boldsymbol{q}(\epsilon_{\mathrm{N\,p}}, \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{sym}}, \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{skw}}, \kappa_{\mathrm{N\,p}}, \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{sym}}, \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{skw}}, \xi^{\mathrm{mic}}) \tag{53}$$

whereby the scalar internal variable ξ^{mic} accounts for the simplest isotropic hardening/softening response.

Next the microscopic free energy follows the definition of the elastic free energy as proposed in Eqs. (34)–(36) as

$$\begin{aligned} \psi^{\mathrm{mic}} = {} & W_{\mathrm{N\,u}}(\epsilon_{\mathrm{N}} - \epsilon_{\mathrm{N\,p}}) + W_{\mathrm{T\,u}}^{\mathrm{sym}}(\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{sym}} - \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{sym}}) + W_{\mathrm{T\,u}}^{\mathrm{skw}}(\boldsymbol{\epsilon}_{\mathrm{T}}^{\mathrm{skw}} - \boldsymbol{\epsilon}_{\mathrm{T\,p}}^{\mathrm{skw}}) + W_{N\omega}(\kappa_{\mathrm{N}} - \kappa_{\mathrm{N\,p}}) \\ & + W_{\mathrm{T}\,\omega}^{\mathrm{sym}}(\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{sym}} - \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{sym}}) + W_{\mathrm{T}\,\omega}^{\mathrm{skw}}(\boldsymbol{\kappa}_{\mathrm{T}}^{\mathrm{skw}} - \boldsymbol{\kappa}_{\mathrm{T\,p}}^{\mathrm{skw}}) + \int_0^{\xi^{\mathrm{mic}}} \phi^{\mathrm{mic}}(\tilde{\xi}^{\mathrm{mic}})\,\mathrm{d}\tilde{\xi}^{\mathrm{mic}} \end{aligned} \tag{54}$$

whereby the restricted format of isotropic hardening/softening behavior is taken into account by the term $\int_0^{\xi^{\mathrm{mic}}} \phi^{\mathrm{mic}}(\tilde{\xi}^{\mathrm{mic}})\,\mathrm{d}\tilde{\xi}^{\mathrm{mic}}$.

The constitutive stresses and couple stresses at microplanes are then obtained from the evaluation of the microplane Clausius–Duhem inequality

$$\begin{aligned} \sigma_{\mathrm{N}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \epsilon_{\mathrm{N\,e}}} = E_{\mathrm{N\,u}} \epsilon_{\mathrm{N\,e}} & \mu_{\mathrm{N}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \kappa_{\mathrm{N\,e}}} = E_{\mathrm{N}\,\omega} \kappa_{\mathrm{N\,e}} \\ \boldsymbol{\sigma}_{\mathrm{T}}^{\mathrm{sym}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{sym}}} = \boldsymbol{E}_{\mathrm{T\,u}}^{\mathrm{sym}} \cdot \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{sym}} & \boldsymbol{\mu}_{\mathrm{T}}^{\mathrm{sym}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{sym}}} = \boldsymbol{E}_{\mathrm{T}\,\omega}^{\mathrm{sym}} \cdot \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{sym}} \\ \boldsymbol{\sigma}_{\mathrm{T}}^{\mathrm{skw}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{skw}}} = \boldsymbol{E}_{\mathrm{T\,u}}^{\mathrm{skw}} \cdot \boldsymbol{\epsilon}_{\mathrm{T\,e}}^{\mathrm{skw}} & \boldsymbol{\mu}_{\mathrm{T}}^{\mathrm{skw}} &= \frac{\partial \psi^{\mathrm{mic}}}{\partial \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{skw}}} = \boldsymbol{E}_{\mathrm{T}\,\omega}^{\mathrm{skw}} \cdot \boldsymbol{\kappa}_{\mathrm{T\,e}}^{\mathrm{skw}} \end{aligned} \tag{55}$$

The evolution of the internal variables is restricted by the microplane dissipation inequality

$$\mathscr{D}^{\text{mic}} = \sigma_{\text{N}}\dot{\epsilon}_{\text{N p}} + \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} \cdot \dot{\boldsymbol{\epsilon}}_{\text{T p}}^{\text{sym}} + \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} \cdot \dot{\boldsymbol{\epsilon}}_{\text{T p}}^{\text{skw}} + \mu_{\text{N}}\dot{\kappa}_{\text{N p}} + \boldsymbol{\mu}_{\text{T}}^{\text{sym}} \cdot \dot{\boldsymbol{\kappa}}_{\text{T p}}^{\text{sym}} + \boldsymbol{\mu}_{\text{T}}^{\text{skw}} \cdot \dot{\boldsymbol{\kappa}}_{\text{T p}}^{\text{skw}} - \phi^{\text{mic}}\dot{\xi}^{\text{mic}} \geqslant 0 \tag{56}$$

Thus, the yield function Φ^{mic} on each microplane can be defined in terms of the microplane constitutive stresses and couple stresses

$$\Phi^{\text{mic}} = \varphi(\sigma_{\text{N}}, \boldsymbol{\sigma}_{\text{T}}^{\text{sym}}, \boldsymbol{\sigma}_{\text{T}}^{\text{skw}}, \mu_{\text{N}}, \boldsymbol{\mu}_{\text{T}}^{\text{sym}}, \boldsymbol{\mu}_{\text{T}}^{\text{skw}}) - \phi^{\text{mic}}(\xi^{\text{mic}}) \leqslant 0 \tag{57}$$

For later use the corresponding gradients of Φ^{mic} or rather φ are abbreviated by

$$\begin{aligned} &v_{\text{N u}} \doteq \partial\varphi/\partial\sigma_{\text{N}} \qquad &&\boldsymbol{v}_{\text{T u}}^{\text{sym}} \doteq \partial\varphi/\partial\boldsymbol{\sigma}_{\text{T}}^{\text{sym}} \qquad &&\boldsymbol{v}_{\text{T u}}^{\text{skw}} \doteq \partial\varphi/\partial\boldsymbol{\sigma}_{\text{T}}^{\text{skw}} \\ &v_{\text{N}\,\omega} \doteq \partial\varphi/\partial\mu_{\text{N}} \qquad &&\boldsymbol{v}_{\text{T}\,\omega}^{\text{sym}} \doteq \partial\varphi/\partial\boldsymbol{\mu}_{\text{T}}^{\text{sym}} \qquad &&\boldsymbol{v}_{\text{T}\,\omega}^{\text{skw}} \doteq \partial\varphi/\partial\boldsymbol{\mu}_{\text{T}}^{\text{skw}} \end{aligned} \tag{58}$$

For the associated case the plastic strain and curvature evolution laws are obtained from the variational problem defined by the dissipation inequality (56) under consideration of the convexity condition and of the constraint (57). For the general non-associated case we postulate instead

$$\begin{aligned} &\dot{\epsilon}_{\text{N p}} = \dot{\gamma}^{\text{mic}}\vartheta_{\text{N u}} \qquad &&\dot{\boldsymbol{\epsilon}}_{\text{T p}}^{\text{sym}} = \dot{\gamma}^{\text{mic}}\boldsymbol{\vartheta}_{\text{T u}}^{\text{sym}} \qquad &&\dot{\boldsymbol{\epsilon}}_{\text{T p}}^{\text{skw}} = \dot{\gamma}^{\text{mic}}\boldsymbol{\vartheta}_{\text{T u}}^{\text{skw}} \\ &\dot{\kappa}_{\text{N p}} = \dot{\gamma}^{\text{mic}}\vartheta_{\text{N}\,\omega} \qquad &&\dot{\boldsymbol{\kappa}}_{\text{T p}}^{\text{sym}} = \dot{\gamma}^{\text{mic}}\boldsymbol{\vartheta}_{\text{T}\,\omega}^{\text{sym}} \qquad &&\dot{\boldsymbol{\kappa}}_{\text{T p}}^{\text{skw}} = \dot{\gamma}^{\text{mic}}\boldsymbol{\vartheta}_{\text{T}\,\omega}^{\text{skw}} \qquad \dot{\xi}^{\text{mic}} = \dot{\gamma}^{\text{mic}} \end{aligned} \tag{59}$$

with the flow directions at each microplane

$$\begin{aligned} &\vartheta_{\text{N u}} = \partial\breve{\Phi}/\partial\sigma_{\text{N}} \qquad &&\boldsymbol{\vartheta}_{\text{T u}}^{\text{sym}} = \partial\breve{\Phi}/\partial\boldsymbol{\sigma}_{T}^{\text{sym}} \qquad &&\boldsymbol{\vartheta}_{\text{T u}}^{\text{skw}} = \partial\breve{\Phi}/\partial\boldsymbol{\sigma}_{T}^{\text{skw}} \\ &\vartheta_{\text{N}\,\omega} = \partial\breve{\Phi}/\partial\mu_{\text{N}} \qquad &&\boldsymbol{\vartheta}_{\text{T}\,\omega}^{\text{sym}} = \partial\breve{\Phi}/\partial\boldsymbol{\mu}_{T}^{\text{sym}} \qquad &&\boldsymbol{\vartheta}_{\text{T}\,\omega}^{\text{skw}} = \partial\breve{\Phi}/\partial\boldsymbol{\mu}_{T}^{\text{skw}} \end{aligned} \tag{60}$$

in terms of the plastic multiplier $\dot{\gamma}^{\text{mic}}$ and of the gradients to the microplane plastic potentials $\breve{\Phi}$.

The Karush–Kuhn–Tucker loading–unloading conditions as well as the consistency condition can be defined on each microplane as

$$\Phi^{\text{mic}} \leqslant 0 \qquad \dot{\gamma}^{\text{mic}} \geqslant 0 \qquad \Phi^{\text{mic}}\dot{\gamma}^{\text{mic}} = 0 \qquad \dot{\Phi}^{\text{mic}}\dot{\gamma}^{\text{mic}} = 0 \tag{61}$$

An explicit solution for the plastic multiplier can be obtained from the consistency condition

$$\begin{aligned} \dot{\gamma}^{\text{mic}} = &\frac{1}{h}\left[v_{\text{N u}}E_{\text{N u}}\boldsymbol{N} + \boldsymbol{v}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} - \boldsymbol{v}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{E}_{\text{T u}}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}\right] : \dot{\boldsymbol{\epsilon}} \\ &+ \frac{1}{h}\left[v_{\text{N}\,\omega}E_{\text{N}\,\omega}\boldsymbol{N} + \boldsymbol{v}_{\text{T}\,\omega}^{\text{sym}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{sym}} \cdot \boldsymbol{T}^{\text{sym}} - \boldsymbol{v}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{E}_{\text{T}\,\omega}^{\text{skw}} \cdot \boldsymbol{T}^{\text{skw}}\right] : \dot{\boldsymbol{\kappa}} \end{aligned} \tag{62}$$

whereby

$$h = v_{\mathrm{N\,u}} E_{\mathrm{N\,u}} \vartheta_{\mathrm{N\,u}} + \boldsymbol{v}^{\mathrm{sym}}_{\mathrm{T\,u}} \cdot \boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T\,u}} \cdot \boldsymbol{\vartheta}^{\mathrm{sym}}_{\mathrm{T\,u}} - \boldsymbol{v}^{\mathrm{skw}}_{\mathrm{T\,u}} \cdot \boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T\,u}} \cdot \boldsymbol{\vartheta}^{\mathrm{skw}}_{\mathrm{T\,u}}$$
$$+ v_{\mathrm{N}\,\omega} E_{\mathrm{N}\,\omega} \vartheta_{\mathrm{N}\,\omega} + \boldsymbol{v}^{\mathrm{sym}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{\vartheta}^{\mathrm{sym}}_{\mathrm{T}\,\omega} - \boldsymbol{v}^{\mathrm{skw}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{\vartheta}^{\mathrm{skw}}_{\mathrm{T}\,\omega} + H^{\mathrm{mic}} \tag{63}$$

and

$$H^{\mathrm{mic}} = \frac{\partial \phi^{\mathrm{mic}}(\xi^{\mathrm{mic}})}{\partial \xi^{\mathrm{mic}}} \tag{64}$$

Finally, the macroscopic elastoplastic constitutive equations can be expressed as

$$\begin{bmatrix} \dot{\boldsymbol{\sigma}}^{\mathrm{t}} \\ \dot{\boldsymbol{\mu}}^{\mathrm{t}} \end{bmatrix} = \begin{bmatrix} \mathbf{E}^{u,u}_{\mathrm{ep}} & \mathbf{E}^{u,\omega}_{\mathrm{ep}} \\ \mathbf{E}^{\omega,u}_{\mathrm{ep}} & \mathbf{E}^{\omega,\omega}_{\mathrm{ep}} \end{bmatrix} : \begin{bmatrix} \dot{\boldsymbol{\epsilon}} \\ \dot{\boldsymbol{\kappa}} \end{bmatrix} \tag{65}$$

with the elastoplastic operators (the integrals extend only over those microplanes that are yielding)

$$\begin{aligned}
\mathbf{E}^{u,u}_{\mathrm{ep}} &= \mathbf{E}_u - \frac{3}{2\pi} \int_\Omega \frac{1}{h} \tilde{\boldsymbol{n}}_u \otimes \tilde{\boldsymbol{m}}_u \, \mathrm{d}\Omega \\
\mathbf{E}^{\omega,\omega}_{\mathrm{ep}} &= \mathbf{E}_\omega - \frac{3}{2\pi} \int_\Omega \frac{1}{h} \tilde{\boldsymbol{n}}_\omega \otimes \tilde{\boldsymbol{m}}_\omega \, \mathrm{d}\Omega \\
\mathbf{E}^{u,\omega}_{\mathrm{ep}} &= \quad - \frac{3}{2\pi} \int_\Omega \frac{1}{h} \tilde{\boldsymbol{n}}_u \otimes \tilde{\boldsymbol{m}}_\omega \, \mathrm{d}\Omega \\
\mathbf{E}^{\omega,u}_{\mathrm{ep}} &= \quad - \frac{3}{2\pi} \int_\Omega \frac{1}{h} \tilde{\boldsymbol{n}}_\omega \otimes \tilde{\boldsymbol{m}}_u \, \mathrm{d}\Omega
\end{aligned} \tag{66}$$

whereby the modified gradients are defined as

$$\begin{aligned}
\tilde{\boldsymbol{n}}_u &= E_{\mathrm{N\,u}} v_{\mathrm{N\,u}} \boldsymbol{N} + \boldsymbol{T}^{\mathrm{sym}} \cdot [\boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T\,u}} \cdot \boldsymbol{v}^{\mathrm{sym}}_{\mathrm{T\,u}}] - \boldsymbol{T}^{\mathrm{skw}} \cdot [\boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T\,u}} \cdot \boldsymbol{v}^{\mathrm{skw}}_{\mathrm{T\,u}}] \\
\tilde{\boldsymbol{m}}_u &= E_{\mathrm{N\,u}} \vartheta_{\mathrm{N\,u}} \boldsymbol{N} + \boldsymbol{T}^{\mathrm{sym}} \cdot [\boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T\,u}} \cdot \boldsymbol{\vartheta}^{\mathrm{sym}}_{\mathrm{T\,u}}] - \boldsymbol{T}^{\mathrm{skw}} \cdot [\boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T\,u}} \cdot \boldsymbol{\vartheta}^{\mathrm{skw}}_{\mathrm{T\,u}}] \\
\tilde{\boldsymbol{n}}_\omega &= E_{\mathrm{N}\,\omega} v_{\mathrm{N}\,\omega} \boldsymbol{N} + \boldsymbol{T}^{\mathrm{sym}} \cdot [\boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{v}^{\mathrm{sym}}_{\mathrm{T}\,\omega}] - \boldsymbol{T}^{\mathrm{skw}} \cdot [\boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{v}^{\mathrm{skw}}_{\mathrm{T}\,\omega}] \\
\tilde{\boldsymbol{m}}_\omega &= E_{\mathrm{N}\,\omega} \vartheta_{\mathrm{N}\,\omega} \boldsymbol{N} + \boldsymbol{T}^{\mathrm{sym}} \cdot [\boldsymbol{E}^{\mathrm{sym}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{\vartheta}^{\mathrm{sym}}_{\mathrm{T}\,\omega}] - \boldsymbol{T}^{\mathrm{skw}} \cdot [\boldsymbol{E}^{\mathrm{skw}}_{\mathrm{T}\,\omega} \cdot \boldsymbol{\vartheta}^{\mathrm{skw}}_{\mathrm{T}\,\omega}]
\end{aligned} \tag{67}$$

Please note the resulting format of the micropolar microplane elastoplastic tangent operator is quite similar to that of the classical micropolar model (compare Willam et al. [26]) with exception of the integrals which account for the microplane contribution to the macroscopic operator in case of the micropolar microplane formulation.

7.2. von Mises type model

The classical micropolar elastoplastic von Mises type model, see e.g. de Borst [12], is characterized by the yield condition

$$\Phi^{\mathrm{mac}} = \sqrt{3J_2} - \phi^{\mathrm{mac}} = 0 \qquad J_2 = \frac{1}{4}\boldsymbol{s} : \boldsymbol{s} + \frac{1}{4}\boldsymbol{s} : \boldsymbol{s}^{\mathrm{t}} + \frac{1}{2l_c^2}\boldsymbol{\mu} : \boldsymbol{\mu} \tag{68}$$

with $\boldsymbol{s}$ the deviator of $\boldsymbol{\sigma}$ and with linear hardening yield stress

$$\phi^{\text{mac}} = \phi_0^{\text{mac}} + H^{\text{mac}} \xi^{\text{mac}} \tag{69}$$

Here the evolution of the hardening/softening parameter is given by

$$\dot{\xi}^{\text{mac}} = \sqrt{\frac{1}{3}\dot{\boldsymbol{\epsilon}}_p : \dot{\boldsymbol{\epsilon}}_p + \frac{1}{3}\dot{\boldsymbol{\epsilon}}_p : \dot{\boldsymbol{\epsilon}}_p^{\text{t}} + \frac{2}{3} l_{\text{c}}^2 \dot{\boldsymbol{\kappa}}_p : \dot{\boldsymbol{\kappa}}_p} = \dot{\gamma}^{\text{mac}} \tag{70}$$

Mimicking the above formulation for the micropolar microplane case, the corresponding von Mises yield condition at the microplane level is proposed in the format

$$\Phi^{\text{mic}} = \varphi - \phi^{\text{mic}} \leqslant 0 \qquad \varphi^2 = \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{sym}} + \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} \cdot \boldsymbol{\sigma}_{\text{T}}^{\text{skw}} + \frac{1}{l_{\text{c}}^2}[\boldsymbol{\mu}_{\text{T}}^{\text{sym}} \cdot \boldsymbol{\mu}_{\text{T}}^{\text{sym}} + \boldsymbol{\mu}_{\text{T}}^{\text{skw}} \cdot \boldsymbol{\mu}_{\text{T}}^{\text{skw}}] \tag{71}$$

with the linear hardening yield stress

$$\phi^{\text{mic}} = \phi_0^{\text{mic}} + H^{\text{mic}} \xi^{\text{mic}} \tag{72}$$

Here the evolution of the hardening/softening parameter is given by

$$\dot{\xi}^{\text{mic}} = \sqrt{\dot{\boldsymbol{\epsilon}}_{\text{Tp}}^{\text{sym}} \cdot \dot{\boldsymbol{\epsilon}}_{\text{Tp}}^{\text{sym}} + \dot{\boldsymbol{\epsilon}}_{\text{Tp}}^{\text{skw}} \cdot \dot{\boldsymbol{\epsilon}}_{Tp}^{\text{skw}} + l_{\text{c}}^2[\dot{\boldsymbol{\kappa}}_{\text{Tp}}^{\text{sym}} \cdot \dot{\boldsymbol{\kappa}}_{\text{Tp}}^{\text{sym}} + \dot{\boldsymbol{\kappa}}_{\text{Tp}}^{\text{skw}} \cdot \dot{\boldsymbol{\kappa}}_{\text{Tp}}^{\text{skw}}]} = \dot{\gamma}^{\text{mic}} \tag{73}$$

which, similarly to the macroscopic description, coincides with the plastic multiplier.

8. Conclusions

In this work the thermodynamically consistent approach for deriving microplane constitutive formulations by Carol et al. [10] and Kuhl et al. [15] was reformulated for elastic and inelastic micropolar continua. As in the previous work, the main assumption is the incorporation of a microscopic free Helmholtz energy on every microplane, which in the present case includes the contributions of the additional degree of freedom and stiffness of the micropolar continuum, represented by the micropolar rotations and the couple stresses. Moreover, an uncoupled format of the free energy in terms of the membrane and bending contributions is considered.

The resulting constitutive equations for the micropolar microplane elastic model typically contain a characteristic length as the ratio between the bending elastic moduli and the micropolar shear modulus. The solutions for the micropolar microplane elastoplastic model include the macroscopic explicit formulation of the constitutive tangential moduli in terms of the microplane contributions. The general elastoplastic formulation for the micropolar microplane model was particularized for von Mises type elastoplasticity.

The proposed constitutive theory allows the formulation of material models based on relevant aspects of the microstructure of engineering materials which exceeds the capacity of the theoretical framework developed so far for the macroscopic modeling of their anisotropic response behaviors.

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