PROJECTIVE SPACE OF A C*-MODULE

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Dedicated to Norberto Fava, with affection and admiration

Abstract

Let X be a right Hilbert C*-module over A. We study the geometry and the topology of the projective space $\mathcal{P}(X)$ of X, consisting of the orthocomplemented submodules of X which are generated by a single element. We also study the geometry of the p-sphere $S_p(X)$ and the natural fibration $S_p(X) \to \mathcal{P}(X)$, where $S_p(X) = \{x \in X : \langle x, x\rangle = p\}$, for $p \in A$ a projection. The projective space and the p-sphere are shown to be homogeneous differentiable spaces of the unitary group of the algebra $\mathcal{L}_A(X)$ of adjointable operators of X. The homotopy theory of these spaces is examined.

1 Introduction

Let A be a C*-algebra, and X a right Hilbert C*-module over A. Denote by $\mathcal{L}_A(X)$ the C*-algebra of bounded adjointable operators on X. In this paper we examine the topology of the set $\mathcal{P}(X)$ of singly generated and orthocomplemented submodules of X, which we call the projective space of X. In the classical setting, when X is a finite dimensional vector space over the complex field ($A = \mathbb{C}$) carrying a positive definite inner product, the topology of the projective space is given by the quotient map

$$S^1_X \to \mathcal{P}(X),$$

where S^1_X is the unit sphere of X and $x \in S^1_X$ is mapped to the ray generated by x.

To follow this pattern in the C*-module case, one must first recognize which are the elements of X which generate orthocomplemented submodules. These turn out to be the $x \in X$ such that there exists $a \in A$ with xa generating the same module as x, and $\langle xa, xa\rangle = p$ a projection in A. So a sphere and a map come up, namely the p-sphere

$$S_p(X) = \{x \in X : \langle x, x\rangle = p\}$$

and the map

$$\rho : S_p(X) \to \mathcal{P}(X)\ ,\ \rho(x) = [x] := \{xa : a \in A\}.$$

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There are many spheres sitting over $\mathcal{P}(X)$. This fact is to be expected. Suppose that $[x(t)]$ is a continuous curve in a reasonable topology in $\mathcal{P}(X)$, with generators $x(t)$ chosen so that $\langle x(t), x(t)\rangle$ are projections, then the function $t \mapsto \langle x(t), x(t)\rangle$ should be continuous. It follows that elements $[x], [x']$ in $\mathcal{P}(X)$ with non equivalent projections $\langle x, x\rangle$ and $\langle x', x'\rangle$, can not be joined in $\mathcal{P}(X)$ by a continuous path. On the other hand, one does not need to take into acount all possible spheres: if $p \sim q$ (Murray-von Neumann equivalence) then $S_p(X)$ and $S_q(X)$ are isometrically isomorphic and they are mapped onto the same part of $\mathcal{P}(X)$. Indeed, if $v \in A$ satisfies $v^*v = p$ and $vv^* = q$, then $S_p(X)v^* = S_q(X)$, $S_p(X) = S_q(X)v$, and thus elements in both spheres generate the same submodules. In other words, any reasonable topology on $\mathcal{P}(X)$ would define in it at least as many connected components as there are classes of projections in A.

Another point of view would be to consider, instead of orthocomplemented singly generated submodules, the "rank one" projections in $\mathcal{L}_A(X)$, i.e. projections of the form $\theta_{x,x}$ (where, as is usual notation, $\theta_{x,y}(z) = \langle z, x\rangle y$ for $x, y, z \in X$), with the norm topology. It turns out that both topologies, the one induced by the p-spheres and this latter one, coincide. Therefore, in addition to the map ρ (which will be shown to be a fibre bundle) one has the action of the unitary group of $\mathcal{L}_A(X)$: if $\theta_{x,x}$ is a projection and U is a unitary,

$$U.\theta_{x,x} = U\theta_{x,x}U^* = \theta_{Ux,Ux},$$

or equivalently, if $[x] \in \mathcal{P}(X)$,

$$U.[x] = [Ux].$$

Once these basic facts are established, we proceed to describe the differentiable structure of the spheres and the projective space. This is done in section 2. In the next sections we consider certain particular examples, and focus on the homotopy theory of these spaces.

In section 3 we examine the case $X = H_A$. Using Mingo's theorem [16] one proves that the spheres $S_p(H_A)$ are contractible. This fact enables one to relate the homotopy groups of the connected components of $\mathcal{P}(H_A)$ with those of the unitary groups U_{pAp} of the algebras pAp, for p projections in A.

In section 4 we study the partial isometries of $\mathcal{L}_A(H_A) = M(A \otimes \mathcal{K})$. Here a clear distinction is drawn between the cases where the initial projection P is "compact" (i.e lies in $A \otimes \mathcal{K}$) or not. In the first case, the spheres $S_P(M(A \otimes \mathcal{K})) = S_P(A \otimes \mathcal{K})$ are contractible. In the latter case this is far from being the case. For example, as an easy consequence of Kasparov's stabilization theorem , π_0 of the unit sphere $S_1(M(A \otimes \mathcal{K}))$ is a semigroup which naturally parametrizes the equivalence classes of countably generated A-modules.

In section 5 we examine the von Neumann algebra case, i.e. A is von Neumann and X is selfdual. We prove first, using results of Handelman [11], that the unitary orbit of a projection in a von Neumann algebra is simply connected. As a consequence, the π_1-groups of the connected components of $\mathcal{P}(X)$ are trivial. Also several results concerning the homotopy groups of the p-spheres and the projective spaces are given, using again results of Handelman [11] and Araki, Smith and Smith [4]. For example, the Jones index of an inclusion of II_1 factors appears as a homotopic invariant of the projective space of the C^*-module induced by the inclusion.

For basic results and notations on C^*-modules, we shall follow E. C. Lance's book [14] and the papers [18] and [19] by W. L. Paschke.

2 Elementary properties of the p-spheres

Let A be a C*-algebra and X a right C*-module over A, which will be supposed to be full, i.e. $\langle X, X\rangle$ dense in A. As in the introduction, if $p \in A$ is a projection, $S_p(X) = \{x \in X : \langle x, x\rangle = p\}$, and the projective space $\mathcal{P}(X)$ is the set of submodules of X which are generated by a single element, and are also orthocomplemented, i.e. they are complemented by their orthogonal modules (with respect to the inner product in X).

If A is unital, $G = G_A$ will denote the group of invertibles of A and $U = U_A$ the unitary group of A. Let $p = p^2 = p^* \in A$ be a projection. Put

$$\mathcal{E}_p = \{q \in A : q^2 = q^* = q, \text{ such that there exists } v \in A \text{ with } v^*v = p, vv^* = q\}.$$

That is, $\mathcal{E}_p$ is the set of projections of A which are equivalent to p. There are many papers considering the geometric structure of the space of projections of a C*-algebra (see [21], [8], [24]). In [1] it was shown that $\mathcal{E}_p$ consists of a union of connected components of the set of all projections of A. Let us state some general properties of the spheres $S_p(X)$.

Lemma 2.1 *Let $x \in X$. Then $\theta_{x,x}$ is a projection in $\mathcal{L}_A(X)$ if and only if $\langle x, x\rangle$ is a projection in A.*

Proof Straightforward. □

Note that, therefore, a singly generated module $\{xa : a \in A\} \subset X$ is orthocomplemented if and only if it has a generator xa_0 such that $\langle xa_0, xa_0\rangle$ is a projection in A. The result above also shows that $S_p(X) \subset Xp \subset X$. Therefore it can be regarded as the unit sphere of the module Xp over the algebra pAp. Unit spheres were studied in [1], where it was shown that they enjoy remarkable geometric properties.

Remark 2.2 Let $p \in A$ be a projection different from zero.

1. The sphere $S_p(X)$ is a C^∞ complemented submanifold of Xp, and therefore also of X (see [1]).
2. The unitary group $U_{\mathcal{L}_{pAp}(Xp)}$ of $\mathcal{L}_{pAp}(Xp)$ acts on $S_p(X)$ by means of

$$U \bullet x = U(x).$$

   The action is smooth and locally transitive. More precisely, two elements of $S_p(X)$ lying at distance less than 1/2 are conjugate by this action.

3. For any fixed $x_0 \in S_p(X)$, the map

$$\pi_{x_0} : U_{\mathcal{L}_{pAp}(Xp)} \to S_p(X) \quad \pi_{x_0}(U) = U(x_0)$$

is a homogeneous space, with isotropy group $I_{x_0} = \{V \in U_{\mathcal{L}_{pAp}(Xp)} : V(x_0) = x_0\}$. In particular, it is a principal bundle.

Example 2.3

1. If $X = A$ is a C*-algebra one obtains that the right ideal generated by b, $bA = \{ba : a \in A\}$, is a complemented submodule of A if and only if it contains a generator which is a partial isometry. In other words, bA is orthocomplemented in A if and only if there exists $bA = pA$ for some projection $p \in A$. That is, $\mathcal{P}(A)$ identifies with the space of projections of A. In this case the p sphere $S_p(A)$ consists of the partial isometries of A with initial space p. The elements of closed range are related to the differential geometry of the action of the group G_A of invertible elements on A by right multiplication. Namely, for $b \in A$ one has the following equivalent conditions:
   (a) The orbit $b\,G_A = \{bg : g \in G_A\}$ is a submanifold of A.
   (b) The mapping $\pi_b : G_A \to b\,G_A$, given by $\pi_b(g) = bg$ is a C$^\infty$ submersion.
   (c) bA is closed in A.
   (d) $bA = pA$ for some projection $p \in A$.
   (e) b is relatively regular in A.
2. Suppose that $X = B$ is a unital C*-algebra such that B contains A and there exists a finite index conditional expectation $E : B \to A$ defining the inner product in the usual way ([5], [14], [10]): $\langle b, b'\rangle = E(b^*b')$. In this case 2.1 reads: the submodule $\{ba : a \in A\}$ is complemented in B if and only if it contains a generator c such that $E(c^*c)$ is a projection in A. The unit sphere $S_1^E(B) = \{x \in B : E(x^*x) = 1\}$ (the superscript E stands to distinguish this sphere from the space of isometries $S_1(B)$, which in fact lies inside $S_1^E(B)$). The final projection $\theta_{1,1}$ corresponding to the element $1 \in S_1^E(B)$ is the Jones projection $e \in B_1$ of the conditional expectation E, where B_1 denotes the basic extension of $A \subset B$ using E.

The spheres $S_p(X)$ enable one to endow $\mathcal{P}(X)$ with a natural topology. Namely, the quotient topology given by the maps

$$\rho : S_p(X) \to \mathcal{P}(X), \quad \rho(x) = [x] = \{xa : a \in A\}.$$

Note that if $x, y \in S_p(X)$ are such that $\rho(x) = \rho(y)$, then $\langle x, y\rangle$ is a unitary element of pAp verifying $x\langle x, y\rangle = y$. Indeed, $y = xa$ for some a in pAp. Then $p = \langle y, y\rangle = a^*\langle x, y\rangle = \langle y, x\rangle a$. Therefore the connected components of $\mathcal{P}(X)$ are identified with the quotient spaces

$$S_p(X)/U_{pAp}$$

where U_{pAp} acts on $S_p(X)$ by the original right A-module action.

On the other hand, there is another natural mapping to consider,

$$\rho' : S_p(X) \to \mathcal{E} = \{\text{projections of } \mathcal{L}_A(X)\}, \quad \rho'(x) = \theta_{x,x}.$$

This map is C^∞ and its range is the set of projections which are (Murray-von Neumann) equivalent to θ_{x_0,x_0} for any fixed $x_0 \in S_p(X)$. Denote this set of projections by $\mathcal{E}_{x_0}$.

Proposition 2.4 *If $x_0 \in S_p(X)$, the mapping $\rho' : S_p(X) \to \mathcal{E}_{x_0}$ is a principal bundle with structure group equal to the unitary group of pAp.*

Proof. We omit the proof because it follows the lines of Proposition 2.5 and Corollary 2.6 of [1]. □

We may state the relation between ρ and ρ' in the following theorem. From now on we shall use the following notations: if $x_0 \in S_p(X)$, then $[x_0] \in \mathcal{P}(X)$ denotes the submodule generated by x_0, i.e., $[x_0] = \{x_0 a : a \in A\}$; $S_p(X)_{x_0}$ is the connected component of x_0 in $S_p(X)$ and $\mathcal{P}(X)_{[x_0]}$ is the connected component of $[x_0]$ in $\mathcal{P}(X)$.

Theorem 2.5 *Let X be a C^*-module over A and $x_0 \in S_p(X)$. Then $\mathcal{P}(X)_{[x_0]}$ is homeomorphic to the connected component of θ_{x_0,x_0} in the space of projections $\mathcal{E}_{x_0}$ of $\mathcal{L}_A(X)$.*

Proof. The proof follows by observing that both spaces are homeomorphic to the connected component of the class of x_0 in the quotient $S_p(X)/U_{pAp}$. □

One may summarize the situation in the following commutative diagram:

$$\begin{array}{ccc} S_p(X)_{x_0} & \xrightarrow{\rho'} & \mathcal{E}_{x_0} \\ & \searrow \rho & \big\downarrow r \\ & & \mathcal{P}(X)_{[x_0]} \end{array}$$

where r is the homeomorphism given by $r(\theta_{x,x}) =$ range of $\theta_{x,x}$.

One does not need to take into consideration all the spheres $S_p(X)$ for all possible projections $p \in A$. If p is equivalent to q (for the Murray-von Neumann equivalence), then the spheres $S_p(X)$ and $S_q(X)$ are diffeomorphic and are carried by ρ onto the same part of $\mathcal{P}(X)$. Indeed, let $v \in A$ such that $v^*v = q$ and $vv^* = p$. Then, the C^∞ map

$$S_p(X) \ni x \mapsto xv \in S_q(X)$$

has inverse

$$S_q(X) \ni x \mapsto xv^* \in S_p(X),$$

and clearly, $\rho(x) = \rho(xv)$. Therefore, in order to cover $\mathcal{P}(X)$ one only needs to consider spheres $S_p(X)$ for projections p chosen one from each equivalence class. The next result states that such spheres and their corresponding parts of $\mathcal{P}(X)$ lie separated (considering in $\mathcal{P}(X)$ the norm metric of $\mathcal{L}_A(X)$).

Proposition 2.6 *Suppose that p and q are projections of A which are not equivalent. Then*

1. $d(S_p(X), S_q(X)) \geq \sqrt{2} - 1$.
2. $d(\rho'(S_p(X)), \rho'(S_q(X))) \geq 1$.

Proof. 1: Pick $x \in S_p(X)$ and $y \in S_q(X)$, and suppose that $\|x - y\| < \sqrt{2} - 1$. Then

$$p = \langle x, x\rangle = \langle (x - y) + y, (x - y) + y\rangle = \|x - y\|^2 + q + \langle x - y, y\rangle + \langle y, x - y\rangle.$$

Therefore

$$\|p - q\| \leq \|x - y\|^2 + 2\|x - y\|\|y\|.$$

Using that $\|y\| = 1$ and $\|x - y\| < \sqrt{2} - 1$, one gets that $\|p - q\| < 1$. This would imply that p and q are unitarily equivalent, a contradiction.
2: We claim that $\rho'(x) = \theta_{x,x}$ and $\rho'(y) = \theta_{y,y}$ are not unitarily equivalent if $x \in S_p(X)$ and $y \in S_q(X)$ with p and q non equivalent. Suppose that there exists a unitary U in $\mathcal{L}_A(X)$ such that $U\theta_{x,x}U^* = \theta_{U(x),U(x)} = \theta_{y,y}$. Then $v = \langle U(x), y\rangle$ is a partial isometry in A such that $vv^* = \langle x, x\rangle = p$ and $v^*v = \langle y, y\rangle = q$. Indeed,

$$vv^* = \langle U(x), y\rangle\langle y, U(x)\rangle = \langle U(x), \theta_{y,y}(U(x))\rangle = \langle U(x), \theta_{U(x),U(x)}(U(x))\rangle = p,$$

and analogously for $v^*v = q$. Note that our claim implies that $\|\theta_{x,x} - \theta_{y,y}\| \geq 1$. $\square$

If $p \in A$ is a projection, denote by $\Delta_p(X)$ the set

$$\Delta_p(X) = \{x \in Xp : \langle x, x\rangle \text{ is invertible in } pAp\}.$$

Clearly $S_p(X) \subset \Delta_p(X)$. Also it is clear that $\Delta_p(X)$ is an open subset of Xp, and therefore a complemented (analytic) submanifold of X.

Proposition 2.7 *$S_p(X)$ is a strong deformation retract of $\Delta_p(X)$.*

Proof. Put $F_t(x) = x\langle x, x\rangle^{-t/2}$, for $x \in \Delta_p(X)$ (the inverse of $\langle x, x\rangle^{t/2}$ taken in pAp). Clearly $F_0 = Id_{\Delta_p(X)}$, $F_t|_{S_p(X)} = Id_{S_p(X)}$ and $F_1 : \Delta_p(X) \to S_p(X)$ is a retraction. $\square$

Let $p \geq q$ be projections in A. Consider the map

$$m_{p,q} : S_p(X) \to S_q(X), \quad m_{p,q}(x) = xq.$$

Since $S_p(X)$ equals the unit sphere S_1 of the pAp-module Xp, we may restrict our attention to the unital case and the map

$$m_q = m_{1,q} : S_1(X) \to S_q(X).$$

Note also that the maps $m_{p,q}$ can also be defined on $\Delta_p(X)$ and $\Delta_q(X)$. Indeed, if $x = xp \in \Delta_p(X)$, then there exists a positive number d such that $\langle x, x \rangle \geq dp$. Therefore $\langle xq, xq \rangle = q\langle x, x \rangle q \geq dqpq = dq$, i.e. $xq \in \Delta_q(X)$.

The map m_p may not be surjective. Consider for instance $X = B(H)$ for H a separable Hilbert space. If s is the unilateral shift, put $p = ss^*$. Then $s^* \in S_p(B(H))$, but there exists no isometry $v \in S_1(B(H))$ such that $vp = s^*$.

Remark 2.8 It can be shown that the unitary group $U_{\mathcal{L}_A(X)}$ acts on $S_p(X)$ and the actions admits local smooth cross sections, defining in this way a principal bundle

$$\pi_{x_0} : U_{\mathcal{L}_A(X)} \to S_p(X)_{x_0} \quad , \quad \pi_{x_0}(U) = U(x_0)$$

with structure group $\{V \in U_{\mathcal{L}_A(X)} : V(x_0) = x_0\}$.

We may state now the main properties of m_p in the general case.

Proposition 2.9 *The map m_p fills connected components, i.e. if x belongs to the image of m_p, then every other element in the connected component of x in $S_p(X)$ also belongs to the image of m_p.*

Proof. Suppose that $y \in S_p(X)$ lies in the same component of x. Put $x_1 \in S_1(X)$ with $m_p(x_1) = x_1 p = x$. Since by 2.8 the action of the unitary group of $\mathcal{L}_A(X)$ on $S_p(X)$ is locally transitive, there exists a unitary operator U such that $U(x) = y$. Put $y_1 = U(x_1)$. Then clearly $y_1 \in S_1(X)$ and $m_p(y_1) = U(x_1)p = U(x_1 p) = y$. □

Proposition 2.10 *Suppose that A is unital and $m_p(x_1) = x$ for $x \in S_p(X)$ and $x_1 \in S_1(X)$. Denote by $S_p(X)_x$ the connected component of x in $S_p(X)$, and analogously for $S_1(X)_{x_1}$. Then the C^∞ map*

$$m_p : S_1(X)_{x_1} \to S_p(X)_x$$

is a fibre bundle.

Proof. Since the connected component of the identity in the unitary group of $\mathcal{L}_A(X)$ acts transitively on both $S_1(X)_{x_1}$ and $S_p(X)_x$, it suffices to show that m_p has local cross sections around x. If $x' \in S_p(X)$ satisfies $\|x - x'\| < 1/2$, then by 2.8 there exists a unitary operator U depending continuously (in fact, smoothly) on x' such that $U(x) = x'$. Put $s(x') = U(x_1)$. Then it is apparent that s is a C^∞ local cross section for m_p. □

3 Partial isometries of $\mathcal{L}_A(H_A)$

Let H_A be the module introduced by Kasparov

$$H_A = \{(a_n) : a_n \in A \text{ and } \sum_{n \in I\!N} a_n^* a_n \text{ converges in norm } \}.$$

H_A becomes a right A-module with the action $(a_n)b = (a_n b)$, and a Hilbert C*-module with the A-valued inner product $\langle (a_n), (b_n) \rangle = \sum_{n \in I\!N} a_n^* b_n$ (see [13]). The C*-algebra $\mathcal{L}_A(H_A)$ is isomorphic to $M(A \otimes \mathcal{K})$, where $\mathcal{K}$ denotes the ideal of compact operators of an infinite dimensional separable Hilbert space ([13]). We will show that the p-spheres of H_A are contractible, using the Mingo-Cuntz-Higson theorem([16],[9]), which states that if A has a countable approximate unit, then both the invertible and unitary groups of $\mathcal{L}_A(H_A)$ are contractible. Our result will be valid though for arbitrary C*-algebras, and will use only the unital version [16] of the theorem. We give the statement without proof, because it will be a consequence of Corollary 3.10.

Theorem 3.1 *For any C*-algebraA and any projection $p \in A$, $S_p(H_A)$ is contractible.*

Corollary 3.2 *For any projection $p \in A$, the mapping $m_p : S_1(H_A) \to S_p(H_A)$ is a C^∞ fibre bundle.*

Proof. It suffices to combine 2.9 and 2.10. □

Corollary 3.3 *For any projection $p \in A$, the space $\Delta_p(H_A)$ is contractible.*

As noted before, $S_p(H_A)$ can be regarded as the unit sphere of the module $H_A p$. But since clearly $H_A p$ is different from H_{pAp}, the analysis does not reduce directly to the unital case.

Let us now consider the map m_p on the spaces Δ.

Proposition 3.4 *Suppose that A is unital. Then the C^∞ map*

$$m_p : \Delta_1(H_A) \to \Delta_p(H_A) \ , \quad m_p(x) = xp$$

is a fibre bundle.

Proof. First let us show that m_p is surjective. Let $x = xp \in \Delta_p(H_A)$. Note that if $x = (a_n)$ then $a_n p = a_n$ for all n. Given $\epsilon > 0$, there exists N such that $\|a_N\| < \epsilon$. Pick $x_0 = x + e_N(1-p)$ (e_N denotes the sequence with 1 in the N-th entry and zero elsewhere). Then clearly $x_0 p = x$ and

$$\langle x_0, x_0 \rangle = 1 - p + \langle x, x \rangle + (1-p)x_N + x_N^*(1-p)$$

is invertible in A, if we choose ϵ small such that $\langle x,x\rangle \geq 2\epsilon p$.
In order to show that m_p is a fibre bundle, it will suffice to show that it has C^∞ local cross sections around any point in $\Delta_p(H_A)$. Note that the connected (in fact, contractible [16]) group $G_{\mathcal{L}_A(H_A)}$ acts transitively on Δ_q for any projection q. Indeed, first note that $G_{\mathcal{L}_A(H_A)}$ acts on Δ_q: if $x \in \Delta_q$ and $G \in G_{\mathcal{L}_A(H_A)}$,

$$\langle G(x), G(x)\rangle = \langle G^*G(x), x\rangle \geq r\langle x,x\rangle,$$

for some $r > 0$, because G^*G is positive and invertible (see [14]). Next we show that $G_{\mathcal{L}_A(H_A)}$ acts transitively on Δ_1. To do this, observe that if $x \in S_1$ and $b \in A$ is positive and invertible, then there exists an invertible operator $R \in \mathcal{L}_A(H_A)$ such that $R(x) = xb$: take $R = \theta_{xb,x} + 1 - \theta_{x,x}$. It is straightforward to verify that R is invertible and $R(x) = xb$. Now take $x, y \in \Delta_1$. Then $x' = x\langle x,x\rangle^{-1/2}$ and $y' = y\langle y,y\rangle^{-1/2}$ lie in S_1. Since the unitary group of $\mathcal{L}_A(H_A)$ acts transitively on S_1, there exists a unitary U such that $U(x') = y'$. Let R_1, R_2 be invertibles in $\mathcal{L}_A(H_A)$ such that $R_1(x') = x'\langle x,x\rangle^{1/2} = x$ and $R_2(y') = y$. Then $R_2UR_1^{-1}(x) = y$. Finally, we can see that $G_{\mathcal{L}_A(H_A)}$ acts transitively on any Δ_q: if $x, y \in \Delta_q$, pick $x_0, y_0 \in S_1$ such that $x_0q = x$ and $y_0q = y$ (recall that m_q is surjective). If R is an invertible operator such that $R(x_0) = y_0$, then also $R(x) = y$.
The transitivity of the action of $G_{\mathcal{L}_A(H_A)}$ on both Δ_1 and Δ_p makes it sufficient to find local cross sections for m_p around a fixed element of Δ_p, say e_1p. Using the action the cross section can be carried over any point in Δ_p. Define the C^∞ (affine) map

$$\sigma : \Delta_p(H_A) \to H_A \ , \quad \sigma(x) = x + e_2(1-p),$$

for $x = (a_n) \in \Delta_p(H_A)$. Note that $m_p(\sigma(x)) = x$ and

$$\langle \sigma(x), \sigma(x)\rangle = \langle x,x\rangle + 1 - p + (1-p)a_2 + a_2^*(1-p).$$

Note also, that

$$\|(1-p)a_2\|^2 \leq \|a_2\|^2 = \|pa_2^*a_2p\| \leq \|p(1-a_1)^*(1-a_1)p + \sum_{n\geq 2} pa_n^*a_np\|$$
$$= \|x - e_1p\|.$$

Fix any scalar $0 < \delta < 1$ and let $\mathcal{V}_\delta$ be the open neighbourhood of e_1p

$$\mathcal{V} = \{x \in \Delta_p(H_A) : \langle x,x\rangle > \delta p \text{ and } \|x - e_1p\| < \delta/2\}.$$

Then it is clear (using the inequality above) that if $x \in \mathcal{V}_\delta$ then $\sigma(x) \in \Delta_1(H_A)$. In other words, σ is a local cross section for m_p as claimed. □

Remark 3.5 Suppose that A is unital. The fibre F of $m_p : \Delta_1(H_A) \to \Delta_p(H_A)$ (over e_1p) is the space

$$F = \{x = (a_n) \in \Delta_1(H_A) : a_1p = p \text{ and } a_np = 0 \text{ for all } n \geq 2\}.$$

Note that therefore F splits as

$$F = \{a = a_1 \in A : ap = p\} \times \{x' = (a_n)_{n\geq 2} : a_n p = 0 \text{ for all } n \geq 2\}.$$

The first factor is clearly a convex set, and the second one is diffeomorphic to $\Delta_{1-p}(H_A)$. Indeed, if $x = (a_n) \in F$,

$$\langle x, x\rangle = \sum a_n^* a_n = p + \sum_{n\geq 2}(1-p)a_n^* a_n(1-p).$$

Since $\langle x, x\rangle$ is invertible in A, it follows that $\sum_{n\geq 2}(1-p)a_n^* a_n(1-p)$ must be invertible in $(1-p)A(1-p)$. In particular, it follows that F is homotopically equivalent to $\Delta_{1-p}(H_A)$.

One can relate the homotopy groups of $\mathcal{P}(H_A)$ to the homotopy groups of U_{pAp}, and eventually compute them.

Corollary 3.6 *Fix $x_0 \in S_p(H_A)$ and let $[x_0]$ be the submodule of H_A generated by x_0. Then for all $n \geq 1$,*

$$\pi_n(\mathcal{P}(H_A), [x_0]) \simeq \pi_{n-1}(U_{pAp}, p).$$

In particular, if A is a von Neumann algebra and p is a properly infinite projection of A, then the connected component of $[x_0]$ in $\mathcal{P}(H_A)$ is contractible.

Proof. The result follows easily from the homotopy exact sequence of the fibre bundle $\rho : S_p(H_A) \to \mathcal{P}(H_A)_{[x_0]}$, with fibre U_{pAp}.

If A is a von Neumann algebra and p is a properly infinite projection, then U_{pAp} is contractible (see [7]). Therefore the connected component of $[x_0]$ in $\mathcal{P}(H_A)$ has trivial homotopy groups. Since it is homeomorphic to $\mathcal{E}_{x_0}$, which is a differentiable manifold modeled on a Banach space (namely, $\mathcal{L}_A(H_A)$), it follows [17] that it is contractible. $\square$

Remark 3.7 In [25] Zhang proved that if A is a non elementary C*-algebra ($\neq \mathcal{K}$ or $M_n(\mathbb{C})$) with real rank zero and stable rank one, and p is any projection in A, then for $k \geq 1$

$$\pi_{2k}(U_{pAp}) \simeq K_1(A) \simeq \pi_{2k+1}(\mathcal{P}(A))$$

and

$$\pi_{2k+1}(U_{pAp}) \simeq K_0(A) \simeq \pi_{2k+2}(\mathcal{P}(A)).$$

Here $\mathcal{P}(A)$ denotes the space of selfadjoint projections of A, which agrees with the projective space of the module $X = A$. These results, applied for both A and $A \otimes \mathcal{K}$ imply 3.6 for this class of algebras.

If $A = B(H)$ with H an infinite dimensional Hilbert space, then the connected components of $\mathcal{P}(H_{B(H)})$ corresponding to infinite rank projections are contractible. Note that because infinite rank projections in $B(H)$ are equivalent, a single sphere, $S_1(H_A)$, suffices to cover the whole connected component. The other connected components of $\mathcal{P}(H_{B(H)})$, corresponding to finite rank projections p, can be parametrized by the ranks of the projections. If $\mathrm{rk}(p) = n$, $1 \leq n < \infty$, then $\mathcal{P}(H_{B(H)})_n$ has trivial π_0 and π_1 and π_2 equal to $\mathbb{Z}_n$.

More generally, if A is a von Neumann algebra, $\pi_0(\mathcal{P}(H_A))$ is parametrized by the equivalence classes of projections in A.

If p is finite in A, then again $\pi_1(\mathcal{P}(H_A), [x_0])$ ($x_0 \in S_p(H_A)$) is trivial and π_2 depends on the type decomposition of the finite algebra pAp. Suppose that $pAp = q_c(pAp) \oplus_{n \in \mathbb{N}} q_n(pAp)$, with q_c and q_n the central projections of pAp decomposing it in its type II_1 and I_n parts, one has

$$\pi_2(\mathcal{P}(H_A), e_1 p) \simeq C(\Omega_c, \mathbb{R}) \oplus_{n \in \mathbb{N}} C(\Omega_n, \mathbb{Z}),$$

where Ω_c and Ω_n denote the Stone spaces of the centres of $q_c(pAp)$ and $q_n(pAp)$, respectively.

If $x_0 \in S_p(X)$, then $S_p(X)$ is (isometrically) isomorphic to $S_{\theta_{x_0,x_0}}(\mathcal{L}_A(X))$. Indeed, any partial isometry V in $\mathcal{L}_A(X)$ with initial space θ_{x_0,x_0} is of the form $V = \theta_{V(x_0),x_0}$. Then

$$S_{\theta_{x_0,x_0}}(\mathcal{L}_A(X)) \ni V \mapsto V(x_0) \in S_p(X)$$

has inverse $y \mapsto \theta_{y,x_0}$. These maps are clearly isometric since $\|\theta_{y,x_0} - \theta_{y',x_0}\| = \|\theta_{y-y',x_0}\| = \|y - y'\|$, where the last equality holds because $y - y' = (y - y')p = (y - y')\langle x_0, x_0\rangle$.

Let us consider from now on the case $X = H_A$. The space $S_1(\mathcal{L}_A(H_A))$ is of particular interest because it consists of all isometries of H_A. Recall the bundle

$$\rho : S_1(\mathcal{L}_A(H_A)) \to \mathcal{E}_1$$

from the space of isometries onto the space of projections of $\mathcal{L}_A(H_A)$ which are equivalent to the identity, namely $\rho(V) = VV^*$. The fibre of this map is the unitary group of $\mathcal{L}_A(H_A)$. It follows that $\pi_0(S_1(\mathcal{L}_A(H_A))) = \pi_0(\mathcal{E}_1)$. Now $\mathcal{E}_1$ can be thought of as the set of (orthogonally) complemented submodules of H_A which are isomorphic to H_A. Recall Kasparov's stability theorem, which states that any countably generated Hilbert module is isomorphic to a submodule of H_A with complement isomorphic to the full H_A [13]. Since the unitary group of $\mathcal{L}_A(H_A)$ is connected, unitary equivalence classes of such submodules correspond to connected components of $\mathcal{E}_1$. Therefore one has a natural bijection

$$\{\text{ classes of countably generated Hilbert modules over } A\} \leftrightarrow \pi_0(S_1(\mathcal{L}_A(H_A))).$$

Note that $\pi_0(S_1(\mathcal{L}_A(H_A))$ has a natural semigroup structure, which can be induced on the set of classes of countably generated A-modules.

Proposition 3.8 *Suppose that $P \in \mathcal{K}_A(H_A)$ is a (compact) projection. Then the map*

$$m_P : S_1(\mathcal{L}_A(H_A)) \to S_P(\mathcal{L}_A(H_A))$$

is surjective, and therefore a C^∞ fibre bundle.

Proof. Recall that $\mathcal{K}_A(H_A) = \mathcal{K} \otimes A = \lim_n M_n(A)$. Since P is compact, it is unitarily equivalent to a matrix projection P_0 in $M_n(A)$ for some n. It clearly suffices to show that m_{P_0} is a fibre bundle. Therefore we may suppose P a matrix. Let us verify first that m_P is onto. Pick $V \in S_P(\mathcal{L}_A(H_A))$. Let $Q = VV^*$ be the final projection of V. Again, since Q is compact it is unitarily equivalent to a matrix projection, $Q_0 = UQU^* \in M_n(A)$ (without loss of generality we may suppose the same size for P and Q_0). Then the partial isometry UV with initial projection P and final projection Q_0 is also an $n \times n$ matrix. Let W be the operator with

$$\begin{pmatrix} UV & 1 - Q_0 \\ 1 - P & V^*U^* \end{pmatrix}$$

on the first $2n \times 2n$ corner, and the identity on the rest. Then W is a unitary operator in $\mathcal{L}_A(H_A)$, satisfying $WP = UVP = UV$. Therefore $V = U^*WP = m_P(U^*W)$, with $U^*W \in U_{\mathcal{L}_A(H_A)} \subset S_1(\mathcal{L}_A(H_A))$. $\square$

Remark 3.9 In 4.1 it is shown in fact that $m_P(U_{\mathcal{L}_A(H_A)}) = S_P(\mathcal{L}_A(H_A))$, and therefore that the restriction of m_P to the unitary group of $\mathcal{L}_A(H_A)$ (= the connected component of the identity in $S_1(\mathcal{L}_A(H_A)))$ is a fibre bundle. In particular, this implies that $S_P(\mathcal{L}_\mathcal{A}(\mathcal{H}_\mathcal{A}))$ is connected. Moreover

Corollary 3.10 *Suppose that $P \in \mathcal{K} \otimes A$ is a projection, then the space $S_P(\mathcal{K} \otimes A)$ of partial isometries in $\mathcal{K} \otimes A$ with initial space P is contractible.*

Proof. Note that $S_P(\mathcal{K} \otimes A)$ coincides with $S_P(\mathcal{L}_A(H_A))$. Consider the restriction of m_p to the unitary group of $\mathcal{L}_A(H_A) = M(\mathcal{K} \otimes \mathcal{A})$,

$$m_P : U_{M(\mathcal{K} \otimes A)} \to S_P(\mathcal{K} \otimes A).$$

This map is a bundle with fibre equal to $\{V : VP = P\}$. Clearly this set identifies with the unitary group of the submodule $R(P)^\perp \subset H_A$. Since P is compact, [16] there exists an isometry W in $M(A \otimes \mathcal{K})$ such that $1 - P = WW^*$. Then $R(P)^\perp \simeq H_A$, and therefore the fibre is contractible. Then $S_P(\mathcal{K} \otimes A)$ is a differentiable manifold, and the base space of a contractible fibre bundle with contractible fibre. It follows from the already cited result of [17] that $S_P(\mathcal{K} \otimes A)$ is contractible. $\square$

As a consequence, putting $P = \theta_{x,x}$ for some $x \in H_A$ with $\langle x, x\rangle = p$, one obtains the proof of the contractibility of $S_p(H_A)$, because, as noted before, $S_{\theta_{x,x}}(A \otimes \mathcal{K}) \simeq S_p(H_A)$. In [16] Mingo defines an index map (generalizing the index for Fredholm operators in Hilbert spaces) which induces an isomorphism

$$\{\text{ Fredholm partial isometries of } H_A\} \to K_0(A)$$

where Fredholm operators of H_A are elements of $\mathcal{L}_A(H_A)$ which have closed complemented range, with finitely generated kernel and cokernel. If one restricts this map to the semigroup of classes of (Fredholm) isometries, one obtains $-K_0^+(A)$. In other words, (classes of) Fredholm isometries correspond to finitely generated A-modules. The remaining part of S_1, which could be called the set of semi-Fredholm isometries, corresponds to the infinite (countably) generated A-modules.

4 Projective space of a selfdual module

In this section we consider the case when A is a von Neumann algebra and X is selfdual [18]. Then $\mathcal{L}_A(X)$ is a von Neumann algebra with the same centre as A.

Let us first state the following result, which can be proved analogously as in 6.2 of [1].

Lemma 4.1 *Let M be a von Neumann algebra of type II_1 (resp. I_n, $n < \infty$), Tr its center valued trace and $e \in M$ a projection. Let $i : U_{eMe} \hookrightarrow U_M$ be the inclusion map $i(eue) = eue + 1 - e$. Then the image $i^*(\pi_1(U_{eMe})) \subset \pi_1(U_M)$ is the group of multiples of $Tr(e)\hat{}$, where, as in [11], $\pi_1(U_M)$ is identified with the additive group $C(\Omega, \mathbb{R})$ (resp. $C(\Omega, \mathbb{Z})$), Ω is the Stone space of the centre of M and $Tr(e)\hat{}$ is the Gelfand transform of $Tr(e)$.*

Denote by $\mathcal{Z}(A)$ the centre of A, and $\mathcal{Z}(A)_{sa}$ the subspace of selfadjoint elements of $\mathcal{Z}(A)$

Remark 4.2 In [1] we computed the fundamental group of the sphere $S_1(X)$. The procedure to compute the fundamental group of $S_p(X)$ for any projection p is similar. Fix an element $x_0 \in S_p(X)$ and consider the bundle

$$\pi_{x_0} : U_{\mathcal{L}_A(X)} \to S_p(X),\ \pi_{x_0}(U) = U(x_0)$$

with fibre

$$F = \{V \in U_{\mathcal{L}_A(X)} : V x_0 = x_0\}$$

introduced in 2.8. $\mathcal{L}_A(X)$ and A have the same central projections decomposing them in their type I,II and III parts (see [19]), although it clearly can happen A finite with $\mathcal{L}_A(X)$ infinite. Using these projections the spheres $S_p(X)$ split and one is set in the case when $\mathcal{L}_A(X)$ and A are of one and the same definite type.

If A is properly infinite, one has the following:

Theorem 4.3 *If A is properly infinite, then the connected components of $S_p(X)$ are contractible.*

Proof. If A is properly infinite, then $\mathcal{L}_A(X)$ is properly infinite. In [1] it was proven that A infinite implies $\mathcal{L}_A(X)$ infinite. Suppose that $\mathcal{L}_A(X)$ is not properly infinite, then there would exist a finite central projection in $\mathcal{L}_A(X)$, which would imply the existence of a finite central projection in A. Therefore in the bundle π_p one has that $U_{\mathcal{L}_A(X)}$ is contractible ([7]). The fibre F identifies with the unitary group of the submodule $Y = [x_0]^\perp$, which is also a selfdual module over A. If Y is non trivial, then again F is contractible, and the connected component of $[x_0]$ in $S_p(X)$ is contractible, being a manifold with trivial homotopy groups. If Y is trivial, then $X = [x_0]$, which implies that X is isomorphic to pAp (as pAp modules). Then F reduces to the trivial group, and the result follows (in this case, $S_p(X)$ is just the isometries of pAp, whose connected components are homeomorphic to the unitary group of pAp). $\square$

If A is finite, then $S_p(X)$ is connected. Indeed, the projections $\theta_{x,x}$ for $x \in S_p(X)$ are equivalent and, in this case, finite. Then they are unitarily equivalent, which implies that the action of $U_{\mathcal{L}_A(X)}$ is transitive in $S_p(X)$ (see [1]). In this case $\mathcal{L}_A(X)$ can be either finite or infinite, and there exists a central projection p_f in A such that $p_f\mathcal{L}_{\mathcal{X}}(\mathcal{X})$ is finite and $(1-p_f)\mathcal{L}_A(X)$ is properly infinite. Note that the first algebra identifies with $\mathcal{L}_{Ap_f}(Xp_f)$, and the second with $\mathcal{L}_{A(1-p_f)}(Xp_f)$. The spheres and the projective space split, $S_p(X) \cong S_{pp_f}(Xp_f) \times S_{p(1-p_f)}(X(1-p_f))$ and $\mathcal{P}(X) \cong \mathcal{P}(Xp_f) \times \mathcal{P}(X(1-p_f))$. So one may consider separately the cases when $\mathcal{L}_A(X)$ is finite or properly infinite.

If $\mathcal{L}_A(X)$ is properly infinite, the unitary group of $\mathcal{L}_A(X)$ is contractible. Moreover, since $\theta_{x,x}$ is finite, $(1-\theta_{x,x})\mathcal{L}_A(X)(1-\theta_{x,x})$ is also properly infinite. The unitary group of this algebra identifies with the fibre of the bundle π_x over the sphere $S_p(X)$, $\{V \in U_{\mathcal{L}_A(X)} : V(x) = x\}$. Therefore one has the following:

Proposition 4.4 *If A is finite, and p_f is defined as above, then the sphere $S_p(X)$ is homotopically equivalent to its finite part $S_{pp_f}(Xp_f)$.*

Proof. The proof proceeds as in the above result, observing that the infinite parts corresponding to the central projection $1 - p_f$ are contractible. $\square$

The fundamental groups of the spheres in the finite case were computed in [1]. If $\mathcal{L}_A(X)$ is of type II_1, $S_p(X)$ is connected and one has the tail of the homotopy exact sequence

$$\pi_1(F) \stackrel{i^*}{\hookrightarrow} \pi_1(U_{\mathcal{L}_A(X)}) \to \pi_1(S_p(X)) \to 0,$$

where $F = \{V \in U_{\mathcal{L}_A(X)} : Vx_0 = x_0\}$. Applying 5.1 above and the results of [11] characterizing the fundamental groups of the unitary groups of von Neumann algebras, one obtains that the image of i^* equals the (additive) group $\{z(1 - Tr(\theta_{x_0,x_0})) : z \in \mathcal{Z}(A)_{sa}\}$, and therefore

$$\pi_1(S_p(X)) = \mathcal{Z}(A)_{sa}/\{z(1 - Tr(\theta_{x_0,x_0})) : z \in \mathcal{Z}(A)_{sa}\},$$

for any $x_0 \in S_p(X)$.

Proceeding analogously, if $\mathcal{L}_A(X)$ is of type I_n $(n < \infty)$, one has

$$\pi_1(S_p(X)) = C(\Omega, \mathbb{Z})/\{f(1 - Tr(\theta_{x_0,x_0})\hat{}) : f \in C(\Omega, \mathbb{Z})\},$$

where Ω is the Stone space of $\mathcal{Z}(A)$. Note that also in this case $S_p(X)$ is connected.

We will show that the connected components of the projective space $\mathcal{P}(X)$ have trivial fundamental group. This will be done again using the results of Handelman [11], and the principal bundle

$$U_{\mathcal{L}_A(X)} \to \mathcal{P}(X)_{[x]} \ , \ U \mapsto [U(x)]$$

for a given $[x] \in \mathcal{P}(X)$. In fact, this result will follow from considering the general case, of a von Neumann algebra M and an arbitrary projection $p \in M$.

Theorem 4.5 *The unitary orbit $U_M(p) = \{upu^* : u \in U_M\}$ is simply connected.*

Proof. Clearly $U_M(p)$ is connected. Consider the principal bundle π_p

$$\pi_p : U_M \to U_M(p) \ , \ \pi_p(u) = upu^*,$$

with fibre $\{v \in U_M : vpv^* = p\} = U_M \cap \{p\}'$. Note that the fibre is the unitary group of the commutant $M \cap \{p\}'$, and is therefore connected.

If q is a projection in the center of M, then the unitary orbit factors as the unitary orbit of qp under the action of the unitary group of qM, times the unitary orbit of $(1-q)p$ under the action of the unitary group of $(1-q)M$. Therefore, using the type decomposition central projections of M, one may consider separately the cases M properly infinite, type II_1 and type I_n, for $n < \infty$.

If M is properly infinite, U_M has trivial π_1-group. It follows that in this case also the unitary orbit has trivial π_1-group.

Suppose now that M is either of type II_1 or type I_n with $n < \infty$. Note that the fibre $U_M \cap \{p\}'$ factors as $U_{pMp} \times U_{(1-p)M(1-p)}$. Let us show that the homomorphism $i^* : \pi_1(U_M \cap \{p\}') \to \pi_1(U_M)$ induced by the inclusion map $i : U_M \cap \{p\}' \hookrightarrow U_M$ is surjective. The image of i^* contains both $i^*(\pi_1(U_{pMp}))$ and $i^*(\pi_1(U_{(1-p)M(1-p)}))$, which, by the above lemma, are generated by the multiples of $Tr(p)\hat{}$ and $1 - Tr(p)\hat{}$ respectively. Therefore i^* is surjective. Using the homotopy exact sequence of the bundle π_p,

$$\ldots \pi_1(U_M \cap \{p\}') \to^{i^*} \pi_1(U_M) \to \pi_1(U_M(p)) \to \pi_0(U_M \cap \{p\}') = 0,$$

since i^* is surjective, it follows that $\pi_1(U_M(p))$ is trivial. □

Corollary 4.6 *If M is a properly infinite von Neumann algebra and $p \in M$ is a properly infinite projection with $1-p$ also properly infinite, then $U_M(p)$ is contractible.*

Proof. In this case, one has that in the proof of the preceeding result, the structure group $U_{pMp} \times U_{(1-p)M(1-p)}$ and the space of the bundle, U_M, are both contractible. The proof follows because $U_M(p)$ is a differentiable manifold. $\square$

Corollary 4.7 *Let X be a selfdual right C^*-module over the von Neumann algebra A. Pick $[x] \in \mathcal{P}(X)$, with $x \in S_p(X)$, then the connected component $\mathcal{P}(X)_{[x]}$ of $[x]$ has trivial π_1-group.*

If A is properly infinite, then

$$\pi_n(\mathcal{P}(X), [x]) \simeq \pi_{n-1}(U_{pAp}, p).$$

If moreover p is properly infinite, then $\mathcal{P}(X)_{[x]}$ is contractible.

Proof. The proof follows by observing that $\mathcal{P}(X)_{[x]}$ is homeomorphic to the unitary orbit $U_{\mathcal{L}_A(X)}(\theta_{x,x})$. If A is properly infinite, then $S_p(X)$ is contractible, and the second statement follows considering the bundle ρ. Finally, if p is properly infinite, U_{pAp} is contractible, and therefore $\mathcal{P}(X)_{[x]}$ is contractible. $\square$

If A is of type II_1, then the second homotopy group of the projective space is non trivial. We shall consider this fact next. Recall that one can consider separately the cases when $\mathcal{L}_A(X)$ is finite and properly infinite, using the central projection p_f defined before. First, we need the following result:

Lemma 4.8 *Suppose that B is a von Neumann algebra of type II_1 and $p \in B$ a projection. Then the inclusion $i : U_{pBp} \to U_B$, $i(pup) = pup + 1 - p$ induces the (additive) group homomorphism between the π_1 groups*

$$i^* : \pi_1(U_{pBp}, p) \to \pi_1(U_B, 1), \ i^*(xp) = Tr(xp)$$

where Tr is the centre valued trace of B, and $\pi_1(U_B)$ (resp. $\pi_1(U_{pBp})$) is identified with $\mathcal{Z}(B)_{sa}$ (resp. $\mathcal{Z}(pBp)_{sa} = \mathcal{Z}(B)_{sa}p$).

Proof. In [11] and [4] it was shown that the classes of the loops e^{itq}, with q a projection, generate the fundamental group of the unitary group. Moreover, the isomorphism identifying $\pi_1(U_B)$ with $\mathcal{Z}(B)_{sa}$ takes the class of the loop e^{itq} to the element $Tr(q)$. The analogous fact holds for pBp, except that one considers projections $q \le p$, and one uses Tr_{pBp} the centre valued trace of pBp. Now, it is clear that $Tr_{pBp}(pxp) = Tr(pxp)p$. Therefore, if $q \le p$, $Tr_{pBp}(q) = Tr(q)p \in \mathcal{Z}(B)p$ which corresponds to the class of the loop $e^{itq}p$ in U_{pBp}, is mapped to the class of the loop $e^{itq}p + 1 - p = e^{itq}$ in U_B. This class corresponds to the element $Tr(q)$ in $\mathcal{Z}(B)_{sa}$. That is, $i^*(xp) = Tr(xp)$ for all $x \in \mathcal{Z}(B)_{sa}p$. $\square$

Proposition 4.9 *Suppose that A is of type II_1. Fix $[x] \in \mathcal{P}(X)$ and denote by e the projection $\theta_{x,x}$.*

a) If $\mathcal{L}_A(X)$ is properly infinite, then

$$\pi_2(\mathcal{P}(X), [x]) \simeq \mathcal{Z}(A)_{sa}e.$$

b) If $\mathcal{L}_A(X)$ is finite, denote by Tr the centre valued trace of $\mathcal{L}_A(X)$. Then

$$\pi_2(\mathcal{P}(X),[x]) \simeq \{(a,b) \in \mathcal{Z}(A)_{sa}e \times \mathcal{Z}(A)_{sa}(1-e) : Tr(a+b) = 0\}.$$

Both sets on the right hand are considered as additive groups.

Proof. Recall the homotopy exact sequence of the bundle $U_{\mathcal{L}_\mathcal{A}(\mathcal{X})} \to \mathcal{P}(X)_{[x]}$, with fibre equal to the unitary group of $\mathcal{L}_A(X) \cap \{e\}'$. The fibre is homeomorphic to the product of the unitary groups of $e\mathcal{L}_A(X)e$ and $(1-e)\mathcal{L}_\mathcal{A}(\mathcal{X})(\infty - \rceil)$. In case a), since e is finite and $\mathcal{L}_A(X)$ is properly infinite, the unitary group of $(1-e)\mathcal{L}_A(X)(1-e)$ is contractible. Therefore in

$$\pi_2(\mathcal{L}_A(X),1) \to \pi_2(\mathcal{P}(X)_{[x]},[x]) \to \pi_1(U_A,1) \times \pi_1(U_{(1-e)\mathcal{L}_A(X)(1-e)}, 1-e) \to \pi_1(\mathcal{L}_A(X),1)$$

one has that $\pi_i(U_{\mathcal{L}_A(X)})$, $i \geq 0$ and $\pi_1(U_{(1-e)\mathcal{L}_A(X)(1-e)})$ are trivial.

Then $\pi_2(\mathcal{P}(X)_{[x]},[x]) \simeq \pi_1(U_A,1) \simeq \mathcal{Z}(A)_{sa}$.

In case b), i.e. $\mathcal{L}_A(X)$ finite, Schröder [23] proved that $\pi_2(\mathcal{L}_A(X)) = 0$. In this case $\pi_1(U_{(1-e)\mathcal{L}_A(X)(1-e)}) \simeq \mathcal{Z}(A)_{sa}(1-e)$, i.e. the selfadjoint elements of the centre of $(1-e)\mathcal{L}_A(X)(1-e)$. Therefore one has

$$0 \to \pi_2(\mathcal{P}(X)_{[x]},[x]) \xrightarrow{\partial} \mathcal{Z}(A)_{sa}e \times \mathcal{Z}(A)_{sa}(1-e) \xrightarrow{i^*} \mathcal{Z}(A)_{sa} \to 0.$$

By 5.8, the morphism i^* is given by $i^*(a,b) = Tr(ae + b(1-e))$. On the other hand, the sequence above shows that the map $\partial : \pi_2(\mathcal{P}(X)_{[x]},[x]) \to \mathcal{Z}(A)_{sa} \times \mathcal{Z}(A)_{sa}(1-e)$ is injective. Therefore

$$\pi_2(\mathcal{P}(X)_{[x]},[x]) \simeq \partial(\pi_2(\mathcal{P}(X)_{[x]},[x])) = \ker i^*,$$

which ends the proof. $\square$

If A is a factor, then $\mathcal{L}_A(X)$ is either finite or properly infinite. In [1] it was noted that $\mathcal{L}_A(X)$ is finite if and only if X is finitely generated. In both situations, it follows from the preceeding result that the second homotopy group of $\mathcal{P}(X)$ is isomorphic to $(I\!R,+)$.

Remark 4.10 Let us recall the example where X is a von Neumann algebra B containing A, and there exists a finite index conditional expectation $E : B \to A$, which will be automatically normal. Then ([5]) B is a selfdual A-module and the results above apply. The algebra $\mathcal{L}_A(B)$ is isomorphic to the Jones extension B_1, namely the von Neumann algebra generated by B and the Jones projection e (which is the map E regarded as an element on $\mathcal{L}_A(B)$). As shown in [1], if $A \subset B$ are of type II$_1$, $\pi_1(S_1^E(B))$ is isomorphic to the additive group

$$Z(A)_{sa}/\{xTr(1-e) : x \in x \in Z(A)_{sa}\}$$

where $Z(A)_{sa}$ denotes the set of selfadjoint elements of the centre $Z(A)$ of A. Note that $Tr(e)$ equals the inverse of $E([E])$, where $[E]$ is the center valued index (in the centre of B) of the expectation E (see [5]).

Remark 4.11 Suppose now that $A \subset B$ is a finite index inclusion of factors of type II_1. As in the remark above, regard B as a (selfdual) C*-module over A. In this case the first homotopy group of $\mathcal{P}_A(B)$ is trivial, and the second is $I\!R$. One recovers the index of the inclusion as the generator of $\partial(\pi_2(\mathcal{P}_A(B))) \subset I\!R^2$. Indeed, in the proof of 5.9 it was shown that $\partial(\pi_2(\mathcal{P}_A(B))) = \{(s,t) \in I\!R^2 : Tr(se + t(1-e)) = 0\}$. Here $Tr(e) = [B:A]^{-1}$, where $[B:A]$ is the Jones index of the inclusion. A generator for $\partial(\pi_2(\mathcal{P}_A(B)))$ is then the pair $(1, 1-[B:A])$

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