Approximation and symbolic calculus for Toeplitz algebras on the Bergman space

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Abstract

If $f \in L^\infty(\mathbb{D})$ let T_f be the Toeplitz operator on the Bergman space L^2_a of the unit disk $\mathbb{D}$. For a C^*-algebra $A \subset L^\infty(\mathbb{D})$ let $\mathfrak{T}(A)$ denote the closed operator algebra generated by $\{T_f : f \in A\}$. We characterize its commutator ideal $\mathfrak{C}(A)$ and the quotient $\mathfrak{T}(A)/\mathfrak{C}(A)$ for a wide class of algebras A. Also, for $n \geq 0$ integer, we define the n-Berezin transform $B_n S$ of a bounded operator S, and prove that if $f \in L^\infty(\mathbb{D})$ and $f_n = B_n T_f$ then $T_{f_n} \to T_f$.

1 Introduction and preliminaries

Suppose that A is a C^*-algebra with unit. The commutator ideal $\mathfrak{C}$ is the closed bilateral ideal generated by the elements $[x, y] = xy - yx$, with $x, y \in A$. The quotient $A/\mathfrak{C}$ is a commutative C^*-algebra with unit, which by the Gelfand-Naimark Theorem is isometrically isomorphic to $C(M)$, the algebra of continuous functions on some compact Hausdorff space M. Following the arrows

$$A \to A/\mathfrak{C} \xrightarrow{\cong} C(M)$$

we can associate to every $x \in A$ a function $f_x \in C(M)$, which is the 'symbol' referred to in the title of the paper. Since the algebra A is determined by $\mathfrak{C}$ and $C(M)$, the study of these two objects is an important tool for a better understanding of A. The possible advantages of this point of view are that $C(M)$ can be treated by topological methods, since it depends exclusively on the space M, and that $\mathfrak{C}$ is usually much smaller than A. Of course, the first step of this journey is to determine $\mathfrak{C}$ and $C(M)$. The whole process is known as abelianization, and it can be carried out for a much wider class of algebras than

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C^*-algebras. In particular, these ideas have being widely studied in the context of Toeplitz algebras acting on the Hardy space H^2 (see [18, pp. 339-392]). The literature shows some partial attempts to develop a similar scheme for Toeplitz algebras acting on the Bergman space $L_a^2 = L_a^2(dA)$, where dA is the normalized area measure on $\mathbb{D}$ (see [14, Ch. 4] for a general discussion). We give below a brief summary of known results.

Let $\mathfrak{L}(L_a^2)$ be the algebra of bounded operators on L_a^2. If $\mathcal{B} \subset L^\infty(\mathbb{D})$ is a closed subalgebra, let $\mathfrak{T}(\mathcal{B})$ be the closed subalgebra of $\mathfrak{L}(L_a^2)$ generated by the Toeplitz operators $\{T_a : a \in \mathcal{B}\}$ and $\mathfrak{C}(\mathcal{B})$ be the commutator ideal of $\mathfrak{T}(\mathcal{B})$. In [11] Coburn proved that $\mathfrak{C}(C(\overline{\mathbb{D}}))$ is the ideal of compact operators and $\mathfrak{T}(C(\overline{\mathbb{D}}))/\mathfrak{C}(C(\overline{\mathbb{D}}))$ is isomorphic to $C(\partial\mathbb{D})$. In [17] McDonald and Sundberg characterized the quotient $\mathfrak{T}(\mathcal{U})/\mathfrak{C}(\mathcal{U})$, where $\mathcal{U}$ is the C^*-algebra in $L^\infty(\mathbb{D})$ generated by H^∞. Later, the two papers by Axler and Zheng [4], [5] provided additional information on Coburn's and McDonald-Sundberg's theorems by giving characterizations of the respective commutator ideals in terms of the Berezin transform. We give precise statements of these results in Sections 6 and 7. In [20] the author showed that $\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{T}(L^\infty(\mathbb{D}))$. Despite these results, no systematic theory of abelianization has been given so far for Toeplitz algebras on the Bergman space. One of the purposes of this paper is to develop a general theory of abelianization for Toeplitz algebras $\mathfrak{T}(\mathcal{B})$, where $\mathcal{B}$ belongs to a special class of C^*-algebras in $L^\infty(\mathbb{D})$ that we call hyperbolic. Our main goal is to explain the underlying phenomenon that is apparently common to Coburn's and McDonald-Sundberg's theorems, and to apply it to other hyperbolic algebras.

Let $\mathcal{A} \subset L^\infty(\mathbb{D})$ be the algebra of functions on $\mathbb{D}$ that are uniformly continuous with respect to the pseudohyperbolic metric. If n is a nonnegative integer, we define the n-Berezin transform $B_n : \mathfrak{L}(L_a^2) \to \mathcal{A}$. This is a linear operator, and we show that if $a \in L^\infty(\mathbb{D})$ and $a_n = B_n T_a$, then T_{a_n} tends to T_a in operator norm. In particular, the Toeplitz algebras associated to $L^\infty(\mathbb{D})$ and $\mathcal{A}$ coincide. This will allow us to reduce the study of $\mathfrak{T}(\mathcal{B})$ and $\mathfrak{C}(\mathcal{B})$ for some C^*-algebras $\mathcal{B} \subset L^\infty(\mathbb{D})$ that are not hyperbolic, to the case of hyperbolic algebras. Once the reduction is made, we can use the maximal ideal space of $\mathcal{A}$ as a powerful tool to describe $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$. We begin fixing some notation.

For $z \in \mathbb{D}$ denote $\varphi_z(\omega) = (z - \omega)/(1 - \bar{z}\omega)$. The pseudohyperbolic metric on $\mathbb{D}$ is defined as $\rho(z, \omega) = |\varphi_z(\omega)|$. This metric is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. Sometimes, especially in estimates involving the triangle inequality, it will be useful to use the hyperbolic metric

$$h(z, \omega) = \log \frac{1 + \rho(z, \omega)}{1 - \rho(z, \omega)}, \quad z, \omega \in \mathbb{D}$$

instead of ρ. The passage from one metric to the other is justified because $f(x) = \log \frac{1+x}{1-x}$ is a strictly increasing function of $x \in (0, 1)$. For $z \in \mathbb{D}$, $r \in (0, 1)$ and $s \in (0, \infty)$ write

$$K(z, r) = \{\omega \in \mathbb{D} : \rho(z, \omega) \leq r\} \quad \text{and} \quad K_h(z, r) = \{\omega \in \mathbb{D} : h(z, \omega) \leq s\}$$

for the closed pseudohyperbolic (resp. hyperbolic) disk of center z and radius r (resp. s).

Let $\mathcal{B} \subset L^\infty(\mathbb{D})$ be a closed subalgebra, where by algebra we always mean a unitary algebra. The maximal ideal space of $\mathcal{B}$ is

$$M(\mathcal{B}) = \{\alpha : \mathcal{B} \to \mathbb{C} : \ \alpha \text{ is linear, multiplicative and } \alpha(1) = 1\},$$

provided with the weak $*$ topology induced by the dual space of $\mathcal{B}$. It is a compact Hausdorff space. We can look at a function $f \in \mathcal{B}$ as a continuous function on $M(A)$ via the Gelfand transform $\hat{f}(\alpha) = \alpha(f)$ $(\alpha \in M(\mathcal{B}))$. If $\mathcal{B} \subset C(\mathbb{D}) \cap L^\infty(\mathbb{D})$ separates points of $\mathbb{D}$ then evaluations at points of $\mathbb{D}$ are members of $M(\mathcal{B})$. So, $\mathbb{D}$ is naturally imbedded into $M(\mathcal{B})$, and $\hat{f}$ is an extension to the whole maximal space of the function f. Unless the contrary is stated we avoid writing the hat for the Gelfand transform and look at f as a function on $M(\mathcal{B})$. The algebra

$$\mathcal{A} = \{f : \mathbb{D} \to \mathbb{C} : f \text{ is uniformly continuous with respect to } \rho\}$$

will be a major protagonist of this paper. It is C^*-subalgebra of $L^\infty(\mathbb{D})$ such that $\mathbb{D}$ is dense in $M(\mathcal{A})$. Indeed, there cannot be $\alpha \in M(\mathcal{A}) \setminus \mathbb{D}$, because otherwise there is $f \in \mathcal{A}$ with $f(\alpha) = 0$ while $|f| \geq \delta > 0$ on $\mathbb{D}$ (since $\mathcal{A}$ is a C^*-algebra). Since such f is invertible in $\mathcal{A}$, it is not in the maximal ideal Ker α. Further information on $M(\mathcal{A})$ can be found in [8].

If $a \in L^\infty(\mathbb{D})$ let M_a be the multiplication operator on $L^2(\mathbb{D})$ and T_a be the Toeplitz operator on L^2_a. That is, $T_a = P_+ M_a$, where $P_+ : L^2(\mathbb{D}) \to L^2_a$ is the Bergman projection. It is clear that $\|M_a\| = \|a\|_\infty$ and $\|T_a\| \leq \|a\|_\infty$. A big difference with Toeplitz operators on the Hardy space H^2 is that the latter inequality is not always an equality, although we still have that $T_a = 0$ only when $a = 0$. For $z \in \mathbb{D}$, the 'change of variable operator' is given by $U_z f = (f \circ \varphi_z)\varphi_z'$. That is,

$$(U_z f)(\omega) = f(\varphi_z(\omega)) \frac{|z|^2 - 1}{(1 - \overline{z}\omega)^2}.$$

Is easy to prove that $U_z T_a U_z = T_{a \circ \varphi_z}$ for every $a \in L^\infty(\mathbb{D})$, and since U_z is unitary and self-adjoint, then

$$(T_{a_1} \ldots T_{a_n})_z = (U_z T_{a_1} U_z) \ldots (U_z T_{a_n} U_z) = T_{a_1 \circ \varphi_z} \ldots T_{a_n \circ \varphi_z} \tag{1.1}$$

for $a_1, \ldots, a_n \in L^\infty(\mathbb{D})$. We will write $S_z = U_z T_a U_z$ for $S \in \mathfrak{L}(L^2_a)$.

The paper is organized as follows. The main results are Theorems 5.7, 6.4 and 6.5. In Section 2 we introduce the n-Berezin transform of a bounded operator and study its basic properties. If $a \in L^\infty(\mathbb{D})$, $B_n T_a$ coincides with $B_n(a)$, the more familiar n-Berezin transform of a function. In Section 3 we study the maximal ideal space of $\mathcal{A}$ and use some of its features to define the notion of hyperbolic algebra. A characterization of these algebras is obtained in terms of interpolating sequences.

If $S \in \mathfrak{T}(\mathcal{B})$, where $\mathcal{B}$ is a hyperbolic algebra, we construct in Section 4 a continuous map $\Psi_S^{\mathcal{B}}$ from the maximal ideal space of $\mathcal{B}$ into $\mathfrak{T}(\mathcal{B})$, when provided with the strong operator

topology, and study its interaction with the n-Berezin transform. We prove that $\Psi_S^{\mathcal{B}}$ is multiplicative as a function of S, which translates into a kind of asymptotic multiplicative behavior of B_n. This will be a fundamental tool for much of what follows. Theorem 5.7 shows that $T_{B_n(a)}$ tends to T_a for $a \in L^\infty(\mathbb{D})$. As a consequence we obtain that if $B_n(a)$ belongs to a hyperbolic algebra $\mathcal{B}$ for infinitely many values of n then $T_a \in \mathfrak{T}(\mathcal{B})$. This argument will reduce the study of $\mathfrak{T}(C)$ for some non-hyperbolic algebras $C \subset L^\infty(\mathbb{D})$ to the hyperbolic case.

Theorem 6.4 gives a characterization of $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$ when $\mathcal{B}$ is hyperbolic. If S is a finite sum of finite products of Toeplitz operators with symbols in $L^\infty(\mathbb{D})$ and $\mathcal{B}$ is a hyperbolic algebra, Theorem 6.5 provides a necessary and sufficient condition for $S \in \mathfrak{T}(\mathcal{B})$ and $S \in \mathfrak{C}(\mathcal{B})$. Section 7 is devoted to applications of the previous results. It is shown that the theorem of McDonald-Sundberg and part of Coburn's theorem are particular cases of Theorem 6.4. An example will be given to illustrate how Theorems 5.7 and 6.4 can be used to characterize $\mathfrak{C}(C)$ and $\mathfrak{T}(C)/\mathfrak{C}(C)$ for some C^*-algebras $C \subset L^\infty(\mathbb{D})$ that are not hyperbolic. Finally, we give a partial result towards a possible characterization of the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$, where $\mathcal{K}$ denotes the ideal of compact operators. We finish the paper posing some open problems.

2 The n-Berezin transform.

If n is a nonnegative integer and $z \in \mathbb{D}$, the function

$$K_z^{(n)}(\omega) = \frac{1}{(1-\bar{z}\omega)^{2+n}} \quad (\omega \in \mathbb{D})$$

is the reproducing kernel of z in the weighted Bergman space $L_a^2(dA_n)$, where $dA_n(\omega) = (n+1)(1-|\omega|^2)^n dA(\omega)$. The n-Berezin transform of an operator $S \in \mathfrak{L}(L_a^2)$ is defined as

$$(B_n S)(z) \stackrel{\text{def}}{=} (n+1)(1-|z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \, \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)} \rangle. \tag{2.1}$$

It is clear that $B_n S \in C^\infty(\mathbb{D})$ for every $S \in \mathfrak{L}(L_a^2)$. Using that $\sum_{j=0}^n \binom{n}{j} (-1)^j |\omega|^{2j} = (1-|\omega|^2)^n$ we see that if $S = T_a$, with $a \in L^\infty(\mathbb{D})$, then

$$\begin{aligned}
(B_n\, a)(z) \stackrel{\text{def}}{=} (B_n T_a)(z) &= (n+1)(1-|z|^2)^{2+n} \sum_{j=0}^{n} \binom{n}{j} (-1)^j \int_D \frac{a(\omega)|\omega|^{2j}}{|1-\bar{z}\omega|^{2(2+n)}}\, dA(\omega) \\
&= \int_D a(\omega) \frac{(1-|z|^2)^{2+n}}{|1-\bar{z}\omega|^{2(2+n)}} (n+1)(1-|\omega|^2)^n \, dA(\omega) \\
&= \int_D a(\varphi_z(\zeta))(n+1)(1-|\zeta|^2)^n \, dA(\zeta), \qquad (2.2)
\end{aligned}$$

where the last equality comes from the change of variables $\omega = \varphi_z(\zeta)$. Since $dA_n(\xi)$ is a probability measure that tends to concentrate its mass at 0 when $n\to\infty$, then $(B_n\, a)(z)$ is an average of a satisfying $\|B_n(a)\|_\infty \le \|a\|_\infty$ for all $a \in L^\infty(\mathbb{D})$. A straightforward calculation shows that B_n maps $L^\infty(\mathbb{D})$ into $\mathcal{A}$ for every $n \ge 0$, and we will prove in Corollary 4.6 that the same holds for $\mathfrak{L}(L_a^2)$. The last expression in (2.2) clearly shows that $\|B_n(a) - a\|_\infty \to 0$ when $n\to\infty$ for every $a \in \mathcal{A}$. That is, the sequence $\{B_n\}$ works as an approximate identity for $\mathcal{A}$. In particular, $\lim_n \|T_{B_n(a)} - T_a\| = 0$ for $a \in \mathcal{A}$.

The 0-Berezin transform of an operator is the usual Berezin transform, which has been extensively used in recent research (see for instance [2], [4], [5] and [19]). The n-Berezin transforms of functions (not necessarily bounded) were introduced by Berezin in [6]. Many of the results of this section were proved by Ahern, Flores and Rudin [2] for n-Berezin transforms of functions of several variables. However, the results here do not follow immediately from theirs, because there are *a priori* several ways to define B_nS for $n \ge 1$ and $S \in \mathfrak{L}(L_a^2)$ so that (2.2) holds when $S = T_a$. If for instance $S \in \mathfrak{L}(L_a^2) \cap \mathfrak{L}(L_a^2(dA_n))$, then the usual Berezin transform of S with respect to the weighted Bergman space $L_a^2(dA_n)$ is $(1-|z|^2)^{2+n}\langle SK_z^{(n)}, K_z^{(n)}\rangle_{dA_n}$, which differs from our definition of B_nS. It is precisely because of the results of this section (especially Proposition 2.4) that I convinced myself (and hopefully convince the reader) about (2.1) as the right definition of B_nS for $S \in \mathfrak{L}(L_a^2)$.

Lemma 2.1 *Let $S \in \mathfrak{L}(L_a^2)$ and $n \ge 0$. Then*

$$(n+2)(1-|z|^2)B_n(S - T_{\overline{\omega}}ST_\omega)(z) = (n+1)B_{n+1}(T_{1-\overline{\omega}z}ST_{1-\omega\overline{z}})(z) \tag{2.3}$$

for every $z \in \mathbb{D}$.

Proof. A simple rearrangement of terms gives

$$\begin{aligned}
\sum_{j=0}^{n} \tbinom{n}{j}\, (-1)^j\, [\langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)}\rangle &- \langle S(\omega^{j+1} K_z^{(n)}), \omega^{j+1} K_z^{(n)}\rangle] \\
&= \langle SK_z^{(n)}, K_z^{(n)}\rangle + (-1)^{n+1}\langle S(\omega^{n+1}K_z^{(n)}), \omega^{n+1}K_z^{(n)}\rangle \\
&+ \sum_{j=1}^{n} [\tbinom{n}{j} + \tbinom{n}{j-1}](-1)^j\, \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)}\rangle \\
&= \sum_{j=0}^{n+1} \tbinom{n+1}{j}\, (-1)^j\, \langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)}\rangle.
\end{aligned}$$

Multiplying by $(n+2)(n+1)(1-|z|^2)^{3+n}$ and using that $T_{1-\omega\overline{z}}(\omega^j K_z^{(n+1)}) = \omega^j K_z^{(n)}$, the above equality becomes (2.3). □

Lemma 2.2 $B_nS_\alpha = (B_nS)\circ\varphi_\alpha$ *for every $n \ge 0$, $S \in \mathfrak{L}(L_a^2)$ and $\alpha \in \mathbb{D}$.*

Proof. We shall prove the lemma by induction on n. The easy identity

$$(1-\varphi_\alpha(\omega)\overline{z})^{-1} = (1-\alpha\overline{z})^{-1}(1-\overline{\alpha}\omega)(1-\overline{\varphi_\alpha(z)}\omega)^{-1} \tag{2.4}$$

implies that

$$(U_\alpha K_z^{(0)})(\omega) = \frac{(|\alpha|^2-1)}{(1-\overline{\alpha}\omega)^2(1-\varphi_\alpha(\omega)\overline{z})^2} = \frac{(|\alpha|^2-1)}{(1-\alpha\overline{z})^2} K^{(0)}_{\varphi_\alpha(z)}(\omega).$$

Thus $(B_0 S_\alpha)(z) = (1-|\varphi_\alpha(z)|^2)^2 \langle SK^{(0)}_{\varphi_\alpha(z)}, K^{(0)}_{\varphi_\alpha(z)}\rangle = (B_0 S)(\varphi_\alpha(z))$. This takes care of $n=0$. The main tool for the inductive step will be formula (2.3), that we rewrite as

$$(B_{n+1}S)(z) = c_n(1-|z|^2)B_n[T_{(1-\overline{\omega}z)^{-1}}(S - T_{\overline{\omega}}ST_\omega)T_{(1-\omega\overline{z})^{-1}}](z), \tag{2.5}$$

where $c_n = (n+2)/(n+1)$. By (1.1) then

$$\begin{aligned}&T_{(1-\overline{\omega}z)^{-1}}(U_\alpha SU_\alpha - T_{\overline{\omega}}U_\alpha SU_\alpha T_\omega)T_{(1-\omega\overline{z})^{-1}} = \\ &\quad U_\alpha T_{(1-\overline{\varphi_\alpha(\omega)}z)^{-1}}[S - T_{\overline{\varphi_\alpha(\omega)}}ST_{\varphi_\alpha(\omega)}]T_{(1-\varphi_\alpha(\omega)\overline{z})^{-1}}U_\alpha = J.\end{aligned}$$

Then (2.4) yields

$$\begin{aligned} J &= |1-\alpha\overline{z}|^{-2}U_\alpha T_{(1-\varphi_\alpha(z)\overline{\omega})^{-1}}[T_{1-\alpha\overline{\omega}}ST_{1-\overline{\alpha}\omega} - T_{\overline{\alpha}-\overline{\omega}}ST_{\alpha-\omega}]T_{(1-\overline{\varphi_\alpha(z)}\omega)^{-1}}U_\alpha \\ &= \frac{(1-|\alpha|^2)}{|1-\alpha\overline{z}|^2}U_\alpha T_{(1-\varphi_\alpha(z)\overline{\omega})^{-1}}[S - T_{\overline{\omega}}ST_\omega]T_{(1-\overline{\varphi_\alpha(z)}\omega)^{-1}}U_\alpha. \end{aligned} \tag{2.6}$$

Hence,

$$\begin{aligned}(B_{n+1}S_\alpha)(z) &= c_n(1-|z|^2)B_n(J)(z) \\ &= c_n(1-|\varphi_\alpha(z)|^2)B_n(U_\alpha T_{(1-\varphi_\alpha(z)\overline{\omega})^{-1}}[S-T_{\overline{\omega}}ST_\omega]T_{(1-\overline{\varphi_\alpha(z)}\omega)^{-1}}U_\alpha)(z) \\ &= c_n(1-|\varphi_\alpha(z)|^2)B_n(T_{(1-\varphi_\alpha(z)\overline{\omega})^{-1}}[S-T_{\overline{\omega}}ST_\omega]T_{(1-\overline{\varphi_\alpha(z)}\omega)^{-1}})(\varphi_\alpha(z)) \\ &= B_{n+1}(S)(\varphi_\alpha(z)),\end{aligned}$$

where the first equality comes from (2.5) with $U_\alpha SU_\alpha$ instead of S, the second from (2.6), the third by inductive hypothesis and the last one from (2.5) with $\varphi_\alpha(z)$ instead of z. $\square$

Corollary 2.3 *If $S \in \mathfrak{L}(L^2_a)$ and $n \geq 0$ then $\|B_nS\|_\infty \leq (n+1)2^n\|S\|$.*

Proof. Since $\|K_z^{(0)}\| = (1-|z|^2)^{-1}$ then

$$|(B_0S)(z)| = (1-|z|^2)^2|\langle S(K_z^{(0)}), K_z^{(0)}\rangle| \leq \|S\|.$$

Suppose that the corollary holds for n, and we shall see that it holds for $n+1$. By (2.3) $(B_{n+1}S)(0) = (n+2/n+1)B_n(S - T_{\overline{\omega}}ST_\omega)(0)$. Thus

$$\begin{aligned}
|(B_{n+1}S)(0)| &\leq \frac{n+2}{n+1}\left(\|B_nS\|_\infty + \|B_n(T_{\overline{\omega}}ST_\omega)\|_\infty\right) \\
&\leq \frac{n+2}{n+1}\left((n+1)2^n\|S\| + (n+1)2^n\|T_{\overline{\omega}}ST_\omega\|\right) \\
&\leq (n+2)2^{n+1}\|S\|.
\end{aligned}$$

Replacing S by U_zSU_z the result follows from Lemma 2.2. □

The (conformally) invariant Laplacian is $\tilde{\Delta} = (1-|z|^2)^2 4\partial\overline{\partial}$, where ∂ and $\overline{\partial}$ are the traditional Cauchy-Riemann operators. So, when f is analytic on $\mathbb{D}$, $\partial f = f'$, $\partial\overline{f} = 0$, $\overline{\partial}\,\overline{f} = \overline{f'}$ and $\overline{\partial} f = 0$. It is easy to check that $(\tilde{\Delta} f)\circ\psi = \tilde{\Delta}(f\circ\psi)$ for every $\psi \in \mathrm{Aut}(\mathbb{D})$.

Proposition 2.4 *Let $S \in \mathfrak{L}(L^2_a)$ and $n \geq 0$. Then*

$$\tilde{\Delta} B_n S = 4(n+1)(n+2)(B_nS - B_{n+1}S). \tag{2.7}$$

Proof. By Lemma 2.2 and the conformal invariance of $\tilde{\Delta}$ it is enough to prove that the equality holds at $z = 0$. Using the mentioned properties of ∂ and $\overline{\partial}$, a tedious but straightforward calculation gives

$$\begin{aligned}
&\tilde{\Delta}[(1-|z|^2)^{n+2}\langle S(\omega^j K_z^{(n)}), \omega^j K_z^{(n)}\rangle](0) \\
&\quad = 4(n+2)(-\langle S\omega^j, \omega^j\rangle + (n+2)\langle S\omega^{j+1}, \omega^{j+1}\rangle)
\end{aligned} \tag{2.8}$$

for every $0 \leq j \leq n$. So, writing $X_j = (-1)^j\langle S\omega^j, \omega^j\rangle$, we have

$$\begin{aligned}
\tilde{\Delta}(B_nS)(0) &= 4(n+1)(n+2)\sum_{j=0}^{n}\binom{n}{j}\left[-X_j - (n+2)X_{j+1}\right] \\
&= 4(n+1)(n+2)\left\{-X_0 - (n+2)X_{n+1} - \sum_{j=1}^{n}\left[\binom{n}{j} + (n+2)\binom{n}{j-1}\right]X_j\right\}.
\end{aligned}$$

On the other hand,

$$(B_nS - B_{n+1}S)(0) = -(n+2)X_{n+1} + \sum_{j=0}^{n}\left[(n+1)\binom{n}{j} - (n+2)\binom{n+1}{j}\right]X_j.$$

A comparison of the coefficients for each X_j gives the result. □

Corollary 2.5 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 1$ then*

$$B_n S = \left(1 - \frac{\tilde{\Delta}}{4n(n+1)}\right) B_{n-1} S \tag{2.9}$$

and

$$B_n S = G_n(\tilde{\Delta}) B_0 S, \tag{2.10}$$

where

$$G_n(\lambda) = \prod_{k=1}^{n} \left(1 - \frac{\lambda}{4k(k+1)}\right).$$

Proof. Formula (2.9) is a rewriting of (2.7), while (2.10) follows immediately from (2.9). □

Lemma 2.6 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$ then $\tilde{\Delta} B_0(B_n S) = B_0 \tilde{\Delta}(B_n S)$.*

Proof. If $f = B_n S$, Corollary 2.3 and (2.7) imply that f and $\tilde{\Delta} f$ are bounded. Hence, Lemma 1 of [1] says that $\tilde{\Delta} B_0 f = B_0 \tilde{\Delta} f$. □

Corollary 2.7 *Let $S \in \mathfrak{L}(L_a^2)$ and $k, j \geq 0$. Then $(B_k B_j)(S) = (B_j B_k)(S)$.*

Proof. Combine (2.10) with the previous lemma. □

3 Algebras related to the maximal ideal space of $\mathcal{A}$

For the next two subsections, if $E \subset M(\mathcal{A})$ then $\overline{E}$ denotes the closure of E in the space $M(\mathcal{A})$. Since the $M(\mathcal{A})$-topology agrees with the Euclidean topology on $\mathbb{D}$, $\overline{E}$ has the same meaning in both topologies when $E \subset r\mathbb{D}$ for some $0 < r < 1$. Later on, we will have to distinguish between closures in different spaces. A sequence $\{z_n\} \subset \mathbb{D}$ is separated if $\rho(z_n, z_k) \geq \delta > 0$ for $n \neq k$.

3.1 One-to-one maps from $\mathbb{D}$ into $M(\mathcal{A})$

Lemma 3.1 *Let $E, F \subset \mathbb{D}$. Then $\overline{E} \cap \overline{F} = \emptyset$ if and only if $\rho(E, F) > 0$.*

Proof. If $\overline{E} \cap \overline{F} = \emptyset$ then there is $f \in \mathcal{A}$ such that $f \equiv 1$ on E and $f \equiv 0$ on F. The uniform ρ-continuity of f implies that $\rho(E, F) = \rho(\overline{E} \cap \mathbb{D}, \overline{F} \cap \mathbb{D}) > 0$. Now suppose that $\rho(E, F) \geq \alpha > 0$ and consider the function

$$f(z) = \begin{cases} 1 & \text{if } \rho(z, E) \leq \alpha/2 \\ 0 & \text{if } \rho(z, E) > \alpha/2 \end{cases}$$

Simple estimates show that $B_n(f) \to 1$ uniformly on $\{z : \rho(z,E) < \alpha/4\}$ and $B_n(f) \to 0$ uniformly on $\{z : \rho(z,F) < \alpha/4\}$. Since $B_n(f) \in \mathcal{A}$, it separates $\overline{E}$ from $\overline{F}$ for n big enough, showing that they are disjoint. $\square$

Let $x \in M(\mathcal{A})$ and suppose that (z_α) is a net in $\mathbb{D}$ that tends to x. We can think of (φ_{z_α}) as a net in the product space $M(\mathcal{A})^{\mathbb{D}}$. By compactness there is a convergent subnet $(\varphi_{z_{\alpha_\beta}})$, meaning that there is some function $\varphi : \mathbb{D} \to M(\mathcal{A})$ such that $f \circ \varphi_{z_{\alpha_\beta}} \to f \circ \varphi$ pointwise on $\mathbb{D}$ for every $f \in \mathcal{A}$. We aim to show that the whole net (z_α) tends to φ and that φ does not depend on the net. So, suppose that (ω_γ) is another net in $\mathbb{D}$ converging to x such that φ_{ω_γ} tends to some $\psi \in M(\mathcal{A})^{\mathbb{D}}$. If $\varphi \neq \psi$ there is some $\xi \in \mathbb{D}$ such that $\varphi(\xi) \neq \psi(\xi)$. Then there are closed disjoint neighborhoods $U, V \subset M(\mathcal{A})$ of $\varphi(\xi)$ and $\psi(\xi)$, respectively. Since $\varphi_{z_{\alpha_\beta}}(\xi) \to \varphi(\xi)$ and $\varphi_{\omega_\gamma}(\xi) \to \psi(\xi)$, there are tails of both nets satisfying

$$E = \{\varphi_{z_{\alpha_\beta}}(\xi) : \beta \geq \beta_0\} \subset U \quad \text{and} \quad F = \{\varphi_{\omega_\gamma}(\xi) : \gamma \geq \gamma_0\} \subset V.$$

By Lemma 3.1 then $\rho(E,F) \geq \rho(U \cap \mathbb{D}, V \cap \mathbb{D}) > 0$. Since for every $z, \omega \in \mathbb{D}$ there is a constant $c_\xi > 0$ such that

$$\rho(\varphi_z(\xi), \varphi_\omega(\xi)) < c_\xi \rho(z, \omega),$$

then

$$\rho(E,F) \leq c_\xi \inf\{\rho(z_{\alpha_\beta}, \omega_\gamma) : \beta \geq \beta_0,\ \gamma \geq \gamma_0\} = 0,$$

where the last equality holds because both nets (z_{α_β}) and (ω_γ) tend to x. We obtain a contradiction and consequently $\varphi = \psi$. The map φ will be denoted φ_x, and notice that $\varphi_x(0) = \lim \varphi_{z_\alpha}(0) = \lim z_\alpha = x$. The following lemma is in [20, Lemma 2.1].

Lemma 3.2 *Let $\mathcal{S}$ be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N$ such that for every $1 \leq j \leq N$: $\rho(z,\omega) > \sigma$ for all $z \neq \omega$ in $\mathcal{S}_j$.*

Lemma 3.3 *Every $x \in M(\mathcal{A})$ is in the closure of some separated sequence.*

Proof. Suppose that $x \in M(\mathcal{A})$ and let (ω_α) be a net in $\mathbb{D}$ such that $\omega_\alpha \to x$. Take a separated sequence $\mathcal{S}$ such that $\rho(z, \mathcal{S}) < 1/8$ for every $z \in \mathbb{D}$, and for each ω_α pick some z_α in $\mathcal{S}$ such that $\rho(z_\alpha, \omega_\alpha) < 1/8$ for every α. Therefore there is $\xi_\alpha \in 8^{-1}\mathbb{D}$ so that $\omega_\alpha = \varphi_{z_\alpha}(\xi_\alpha)$. Taking subnets we can assume that $\xi_\alpha \to \xi$ with $|\xi| \leq 1/8$. We claim that $\varphi_{z_\alpha}(\xi)$ tends to x. Indeed, if $f \in \mathcal{A}$ then

$$|f(\varphi_{z_\alpha}(\xi)) - f(x)| \leq |f(\varphi_{z_\alpha}(\xi)) - f(\varphi_{z_\alpha}(\xi_\alpha))| + |f(\omega_\alpha) - f(x)|,$$

where the first summand tends to 0 because $\rho(\varphi_{z_\alpha}(\xi), \varphi_{z_\alpha}(\xi_\alpha)) = \rho(\xi, \xi_\alpha) \to 0$, and the second summand tends to 0 because $\omega_\alpha \to x$. Thus, x is in the closure of the sequence $\mathcal{T} = \{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}$. By Lemma 3.2 we can split $\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N$, where for each $1 \leq j \leq N$,

$\rho(z_1, z_2) > 1/2$ when $z_1, z_2 \in \mathcal{S}_j$ are different. We also have the corresponding decomposition $\mathcal{T} = \mathcal{T}_1 \cup \ldots \cup \mathcal{T}_N$, where $\mathcal{T}_j = \{\varphi_z(\xi) : z \in \mathcal{S}_j\}$. Hence, there is at least one j_0 such that x is in the closure of $\mathcal{T}_{j_0}$. The lemma will follow if we show that $\mathcal{T}_{j_0}$ is a separated sequence. If $z_1, z_2 \in \mathcal{S}_{j_0}$ are different then

$$\begin{aligned}\rho(z_1, z_2) &\leq \rho(z_1, \varphi_{z_1}(\xi)) + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) + \rho(\varphi_{z_2}(\xi), z_2)\\ &= 2|\xi| + \rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)).\end{aligned}$$

So, $\rho(\varphi_{z_1}(\xi), \varphi_{z_2}(\xi)) \geq (1/2) - 2|\xi| \geq 1/4$, proving our claim. $\square$

Lemma 3.4 *Let (z_α) be a net in $\mathbb{D}$ converging to $x \in M(\mathcal{A})$. Then*

(i) *φ_x is a continuous one-to-one map from $\mathbb{D}$ into $M(\mathcal{A})$,*

(ii) *$f \circ \varphi_x \in \mathcal{A}$ for every $f \in \mathcal{A}$,*

(iii) *$f \circ \varphi_{z_\alpha} \to f \circ \varphi_x$ uniformly on compact sets of $\mathbb{D}$ for every $f \in \mathcal{A}$.*

Proof. Suppose that $\omega \in \mathbb{D}$ and $f \in \mathcal{A}$. Given $\varepsilon > 0$ there is $\delta > 0$ such that $|f(u) - f(v)| < \varepsilon$ if $\rho(u, v) < \delta$. Take $\omega_1 \in K(\omega, \delta)$. Since $\rho(\varphi_{z_\alpha}(\omega_1), \varphi_{z_\alpha}(\omega)) = \rho(\omega_1, \omega) < \delta$ then $|f(\varphi_{z_\alpha}(\omega_1)) - f(\varphi_{z_\alpha}(\omega))| < \varepsilon$ for every α. Then

$$\begin{aligned}&|f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))|\\ &\quad\leq |f(\varphi_x(\omega_1)) - f(\varphi_{z_\alpha}(\omega_1))| + |f(\varphi_{z_\alpha}(\omega_1)) - f(\varphi_{z_\alpha}(\omega))| + |f(\varphi_{z_\alpha}(\omega)) - f(\varphi_x(\omega))|\\ &\quad\leq |f(\varphi_x(\omega_1)) - f(\varphi_{z_\alpha}(\omega_1))| + |f(\varphi_{z_\alpha}(\omega)) - f(\varphi_x(\omega))| + \varepsilon\end{aligned}$$

for every α. Taking limits in α we get $|f(\varphi_x(\omega_1)) - f(\varphi_x(\omega))| \leq \varepsilon$ when $\rho(\omega_1, \omega) < \delta$. This proves the continuity of φ_x and (ii).

To prove that φ_x is one-to-one, for an arbitrary $0 < r < 1$ we will construct a function $f \in \mathcal{A}$ (depending on r) such that $(f \circ \varphi_x)(\omega) = \omega$ when $|\omega| < r$. It is convenient to deal with the hyperbolic metric h instead of ρ. Write $s = \log \frac{1+r}{1-r}$. By Lemma 3.2 there is a sequence $\{z_n\}$ in $\mathbb{D}$ whose closure contains x and such that $h(z_n, z_m) > 5s$ if $n \neq m$. Therefore

$$h(K_h(z_n, 2s), K_h(z_m, 2s)) \geq s \quad \text{if} \quad n \neq m. \tag{3.1}$$

Take $g \in C(\mathbb{D})$ such that $g(\omega) = \omega$ if $h(\omega, 0) < s$ (i.e.: if $|\omega| < r$) and $g(\omega) = 0$ if $h(\omega, 0) > 2s$. Thus $g \circ \varphi_{z_n}$ is supported in $K_h(z_n, 2s)$ and

$$f = \sum_{n \geq 1} (g \circ \varphi_{z_n}) \in C(\mathbb{D}).$$

Since g is uniformly continuous with respect to the Euclidean metric then it is h-uniformly continuous. Hence, given $\varepsilon > 0$ there is δ, with $0 < \delta < s/2$, such that

$$|g(\xi_1) - g(\xi_2)| < \varepsilon \quad \text{if} \quad h(\xi_1, \xi_2) < \delta. \tag{3.2}$$

Let $\omega_1, \omega_2 \in \mathbb{D}$ such that $h(\omega_1, \omega_2) < \delta$. By (3.1) $K_h(\omega_1, \delta)$ cuts at most one of the disks $K_h(z_n, 2s)$. If it doesn't cut any, then $f(\omega_1) = f(\omega_2) = 0$. If it cuts $K_h(z_{n_0}, 2s)$ then $f(\omega_1) - f(\omega_2) = g(\varphi_{z_{n_0}}(\omega_1)) - g(\varphi_{z_{n_0}}(\omega_2))$, and since $h(\varphi_{z_{n_0}}(\omega_1), \varphi_{z_{n_0}}(\omega_2)) = h(\omega_1, \omega_2) < \delta$ then (3.2) says that $|f(\omega_1) - f(\omega_2)| < \varepsilon$. Thus $f \in \mathcal{A}$.

If k is any positive integer and $|\omega| < r$ then $h(0, \omega) < s$ and $\varphi_{z_k}(\omega) \in K_h(z_k, s)$. So, (3.1) and the inclusion: $\operatorname{supp}(g \circ \varphi_{z_n}) \subset K_h(z_n, 2s)$ imply that $(g \circ \varphi_{z_n})(\varphi_{z_k}(\omega)) = 0$ for $n \neq k$. Consequently

$$f(\varphi_{z_k}(\omega)) = (g \circ \varphi_{z_k})(\varphi_{z_k}(\omega)) = g(\omega) = \omega.$$

Thus, if (z_α) is a net of points in $\{z_n\}$ that tends to x then $(f \circ \varphi_{z_\alpha})(\omega) = \omega$ for every α and every $\omega \in r\mathbb{D}$. Therefore $(f \circ \varphi_x)(\omega) = \omega$ when $\omega \in r\mathbb{D}$.

Suppose that (iii) fails. This means that there are $f \in \mathcal{A}$, $0 < r < 1$ and $\varepsilon > 0$ such that $|(f \circ \varphi_{z_\alpha})(\xi_\alpha) - (f \circ \varphi_x)(\xi_\alpha)| > \varepsilon$ for some points $\xi_\alpha \in r\mathbb{D}$. We can also assume that $\xi_\alpha \to \xi$. Since $(f \circ \varphi_{z_\alpha})(\xi) \to (f \circ \varphi_x)(\xi)$, this contradicts the uniform ρ-continuity of f. □

3.2 The hyperbolic parts

Definition. If $x, y \in M(\mathcal{A})$ define $\rho(x, y) = \sup \rho(\mathcal{S}, \mathcal{T})$, where $\mathcal{S}$ and $\mathcal{T}$ run over all the separated sequences in $\mathbb{D}$ so that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. Defining $h(x, y)$ in analogous fashion, we have $h(x, y) = \log \frac{1-\rho(x,y)}{1+\rho(x,y)}$.

Lemma 3.5 *Let $x, y \in M(\mathcal{A}) \setminus \mathbb{D}$. Then*

(1) $\rho(x, y) = a < 1$ *if and only if* $y = \varphi_x(\omega)$ *for some* ω *with* $|\omega| = a$.

(2) $y = \varphi_x(\xi)$ *with* $\xi \in \mathbb{D}$ *if and only if every separated sequences* $\mathcal{S}, \mathcal{T}$ *such that* $x \in \overline{\mathcal{S}}$ *and* $y \in \overline{\mathcal{T}}$ *satisfy* $\rho(\mathcal{T}, \{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}) = 0$.

(3) $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) = h(\xi_1, \xi_2)$ *for every* $\xi_1, \xi_2 \in \mathbb{D}$.

(4) h *is a* $[0, +\infty]$*-valued metric on* $M(\mathcal{A})$.

Proof. (1). Suppose that $\rho(x, y) = a < 1$ and take $b \in (a, 1)$. The continuity of φ_x implies that $\varphi_x(\overline{b\mathbb{D}})$ is compact. So, if $y \notin \varphi_x(\overline{b\mathbb{D}})$ there are closed disjoint neighborhoods U of $\varphi_x(\overline{b\mathbb{D}})$ and V of y. Let $\mathcal{S}$ and $\mathcal{T}$ be separated sequences in $\mathbb{D}$ such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{T}$. If (z_α) is a net in $\mathcal{S}$ that tends to x then $\varphi_{z_\alpha}(\xi) \to \varphi_x(\xi)$ for every $\xi \in \overline{b\mathbb{D}}$. By a compactness argument $\varphi_{z_\alpha}(\overline{b\mathbb{D}}) \subset U$ for a tail $(z_\alpha)_{\alpha \geq \alpha_0}$ of the original net. Let $\mathcal{S}_1 = \{z_n \in \mathcal{S} : z_n = z_\alpha \text{ for some } \alpha \geq \alpha_0\}$. Then $x \in \overline{\mathcal{S}}_1$ and $\varphi_{z_n}(\overline{b\mathbb{D}}) \subset U$ for every $z_n \in \mathcal{S}_1$. This means that

$$K(z_n, b) \subset U \quad \text{for every} \quad z_n \in \mathcal{S}_1. \tag{3.3}$$

On the other hand, since V is a neighborhood of y then

$$y \in \overline{\mathcal{T}}_1, \quad \text{where} \quad \mathcal{T}_1 = \{z \in \mathcal{T} : z \in V\}. \tag{3.4}$$

Since U and V are disjoint, (3.3) and (3.4) say that $\rho(\mathcal{S}_1, \mathcal{T}_1) \geq b > a$, contradicting the definition of $\rho(x, y) = a$. Since $b \in (a, 1)$ is arbitrary then $y \in \varphi_x(\overline{a\mathbb{D}})$, so $y = \varphi_x(\omega)$ with $|\omega| \leq a$.

Reciprocally, suppose that $y = \varphi_x(\omega)$ with $|\omega| = a$, and let $\mathcal{S}, \mathcal{T}$ be separated sequence in $\mathbb{D}$ such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. If (z_α) is a net in $\mathcal{S}$ that tends to x then $\varphi_{z_\alpha}(\omega) \to y$. Thus $y \in \overline{\mathcal{T}}_1$, where $\mathcal{T}_1 = \{\varphi_{z_n}(\omega) : z_n \in \mathcal{S}\}$. So, $y \in \overline{\mathcal{T}}_1 \cap \overline{\mathcal{T}} \neq \emptyset$ and by Lemma 3.1, $\rho(\mathcal{T}_1, \mathcal{T}) = 0$. That is, given $\varepsilon > 0$ there are $z_n \in \mathcal{S}$ and $\omega_n \in \mathcal{T}$ such that $\rho(\varphi_{z_n}(\omega), \omega_n) < \varepsilon$, which yields

$$\rho(z_n, \omega_n) \leq \rho(z_n, \varphi_{z_n}(\omega)) + \rho(\varphi_{z_n}(\omega), \omega_n) < |\omega| + \varepsilon = a + \varepsilon.$$

So, $\rho(\mathcal{S}, \mathcal{T}) \leq a$ and by definition $\rho(x, y) \leq a$.

(2). The necessity follows from Lemma 3.1. If $y \neq \varphi_x(\xi)$ then $\rho(y, \varphi_x(\xi)) \neq 0$ and there are separated sequences $\mathcal{T}_1, \mathcal{T}_2$ such that $\varphi_x(\xi) \in \overline{\mathcal{T}}_1$, $y \in \overline{\mathcal{T}}_2$ and $\rho(\mathcal{T}_1, \mathcal{T}_2) \geq \delta > 0$. Let $\mathcal{S}$ be a separated sequence such that $x \in \overline{\mathcal{S}}$. Therefore x is in the closure of $\mathcal{S}_1 = \{z_n : \rho(\varphi_{z_n}(\xi), \mathcal{T}_1) < \delta/2\}$, because if $x \in \overline{\mathcal{S} \setminus \mathcal{S}_1}$ then

$$\varphi_x(\xi) \in \overline{\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus \mathcal{S}_1\}} \cap \overline{\mathcal{T}}_1$$

while

$$\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S} \setminus \mathcal{S}_1\}, \mathcal{T}_1) \geq \delta/2,$$

which contradicts Lemma 3.1. So, for $z_n \in \mathcal{S}_1$, $\rho(\varphi_{z_n}(\xi), \mathcal{T}_2) \geq \delta/2$.

(3). Fix $\xi_1, \xi_2 \in \mathbb{D}$. By Lemma 3.2 there is a separated sequence $\mathcal{S} = \{z_k\}$ such that $x \in \overline{\mathcal{S}}$ and $h(z_n, z_m) \geq h(\xi_1, \xi_2) + h(0, \xi_1) + h(0, \xi_2)$ if $n \neq m$. Since

$$\begin{aligned} h(z_n, z_m) &\leq h(z_n, \varphi_{z_n}(\xi_1)) + h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)) + h(\varphi_{z_m}(\xi_2), z_m) \\ &= h(0, \xi_1) + h(0, \xi_2) + h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)), \end{aligned}$$

then $h(\varphi_{z_n}(\xi_1), \varphi_{z_m}(\xi_2)) \geq h(\xi_1, \xi_2)$ if $n \neq m$. Therefore

$$h(\{\varphi_{z_n}(\xi_1)\}_{n \geq 1}, \{\varphi_{z_m}(\xi_2)\}_{m \geq 1}) = h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) = h(\xi_1, \xi_2),$$

implying that $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \geq h(\xi_1, \xi_2)$. For the other inequality let $\mathcal{T}_1, \mathcal{T}_2$ be separated sequences such that $\varphi_x(\xi_1) \in \overline{\mathcal{T}}_1$ and $\varphi_x(\xi_2) \in \overline{\mathcal{T}}_2$. For a separated sequence $\mathcal{S}$ such that $x \in \overline{\mathcal{S}}$ and $\varepsilon > 0$ write

$$\mathcal{S}' = \{z_n \in \mathcal{S} : h(\varphi_{z_n}(\xi_1), \mathcal{T}_1) < \varepsilon,\ h(\varphi_{z_n}(\xi_2), \mathcal{T}_2) < \varepsilon\}$$

and $\mathcal{S}'' = \mathcal{S} \setminus \mathcal{S}'$. By (2) $x \notin \overline{\mathcal{S}''}$. So, $x \in \overline{\mathcal{S}'}$ and

$$h(\mathcal{T}_1, \mathcal{T}_2) \leq h(\varphi_{z_n}(\xi_1), \varphi_{z_n}(\xi_2)) + 2\varepsilon = h(\xi_1, \xi_2) + 2\varepsilon.$$

That is, $h(\varphi_x(\xi_1), \varphi_x(\xi_2)) \le h(\xi_1, \xi_2) + 2\varepsilon$.

(4). We must prove only that given $x, y, z \in M(\mathcal{A})$,

$$h(x,y) \le h(x,z) + h(z,y) \tag{3.5}$$

The inequality is obvious if its right member is infinite. Otherwise (1) says that $x = \varphi_z(\xi_1)$ and $y = \varphi_z(\xi_2)$ for some $\xi_1, \xi_2 \in \mathbb{D}$. Then (3.5) becomes $h(\varphi_z(\xi_1), \varphi_z(\xi_2)) \le h(\varphi_z(\xi_1), \varphi_z(0)) + h(\varphi_z(0), \varphi_z(\xi_2))$, which holds by (3). $\square$

Definition. If $x \in M(\mathcal{A})$ define the hyperbolic part of x as $H(x) = \{\varphi_x(\omega) : \omega \in \mathbb{D}\}$. Observe that (1) of Lemma 3.5 implies that

$$H(x) = \{y \in M(\mathcal{A}) : \rho(x,y) < 1\} = \{y \in M(\mathcal{A}) : h(x,y) < \infty\}$$

and by (4) of the same lemma, $\{H(x) : x \in M(\mathcal{A})\}$ is a partition of $M(\mathcal{A})$. In fact if $z \in H(x) \cap H(y)$ then for any $u \in H(x)$, $h(u,y) \le h(u,x) + h(x,z) + h(z,y) < \infty$. So, $H(x) \subset H(y)$ and by symmetry they coincide.

Lemma 3.6 *The map $x \mapsto \varphi_x$ from $M(\mathcal{A})$ into $M(\mathcal{A})^{\mathbb{D}}$ is continuous.*

Proof. Let (x_α) be a net in $M(\mathcal{A})$ that tends to x and $\xi \in \mathbb{D}$. We must show that if (x_β) is a subnet such that $\varphi_{x_\beta}(\xi) \to y$ then $y = \varphi_x(\xi)$. Let $\mathcal{S} = \{z_n\}$ and $\mathcal{T} = \{\omega_n\}$ be separated sequences such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$. For $\delta > 0$ write

$$U = \bigcup_{n\ge1} K(z_n, \delta) \quad \text{and} \quad V = \bigcup_{n\ge1} K(\omega_n, \delta).$$

Since there is $f \in \mathcal{A}$ such that $f(z_n) = 0$ for all n and $f \equiv 1$ on $\mathbb{D} \setminus U$ then $\overline{U} \supset \{m \in M(\mathcal{A}) : |f(m)| < 1/2\}$, a neighborhood of x. So, $\overline{U}$ is a neighborhood of x and by the same reason $\overline{V}$ is a neighborhood of y. Since $x_\beta \to x$ and $\varphi_{x_\beta}(\xi) \to y$, there is β_0 such that for every $\beta \ge \beta_0$,

(i) $\varphi_{x_\beta}(\xi) \in \overline{V}$, and

(ii) $x_\beta \in \overline{\mathcal{S}}_\beta$, where $\mathcal{S}_\beta = \{z_n(\beta)\}_{n\ge1}$ is a separated sequence in U.

Assume that $\beta \ge \beta_0$. Since $\varphi_{x_\beta}(\xi) \in \overline{\{\varphi_{z_n(\beta)}(\xi)\}_{n\ge1}} \cap \overline{(\cup_n K(\omega_n, \delta))}$ then Lemma 3.1 says that $\rho(\{\varphi_{z_n(\beta)}(\xi)\}, \mathcal{T}) \le \delta$. So, there is n_0 such that $\rho(\varphi_{z_{n_0}(\beta)}(\xi), \mathcal{T}) < 2\delta$. On the other hand, by definition of U and (ii) there is some $z_{k_0} \in \mathcal{S}$ such that $\rho(z_{k_0}, z_{n_0}(\beta)) \le \delta$. Since there is $c_\xi > 0$ such that

$$\rho(\varphi_{z_{k_0}}(\xi), \varphi_{z_{n_0}(\beta)}(\xi)) \le c_\xi \rho(z_{k_0}, z_{n_0}(\beta)) \le c_\xi \delta,$$

then $\rho(\varphi_{z_{k_0}}(\xi), \mathcal{T}) \le (c_\xi + 2)\delta$. This shows that $\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}, \mathcal{T}) \le (c_\xi + 2)\delta$, and since $\delta > 0$ is arbitrary, $\rho(\{\varphi_{z_n}(\xi) : z_n \in \mathcal{S}\}, \mathcal{T}) = 0$. Since $\mathcal{S}$ and $\mathcal{T}$ are arbitrary separated sequences such that $x \in \overline{\mathcal{S}}$ and $y \in \overline{\mathcal{T}}$ then (2) of Lemma 3.5 tells us that $y = \varphi_x(\xi)$. $\square$

3.3 Hyperbolic algebras

A closed self-adjoint subalgebra $\mathcal{B}$ of $\mathcal{A}$ that separates the points of $\mathbb{D}$ and contains the constants will be called a *prehyperbolic* algebra. For such $\mathcal{B}$, Theorem 4.28 of [13] implies that whenever $b \in \mathcal{B}$ is invertible in $\mathcal{A}$ then the inverse belongs to $\mathcal{B}$. Hence, the disk is dense in $M(\mathcal{B})$, because if there exists $y \in M(\mathcal{B})$ that is not in the closure of $\mathbb{D}$ then there is $f \in \mathcal{B}$ such that $f(y) = 0$ and $|f| \geq \delta > 0$ on $\mathbb{D}$. Since clearly f is invertible in $\mathcal{A}$, then so is in $\mathcal{B}$ and consequently f cannot vanish anywhere in $M(\mathcal{B})$, a contradiction. The inclusion of $\mathcal{B}$ in $\mathcal{A}$ induces by transposition a projection $\pi : M(\mathcal{A}) \to M(\mathcal{B})$. Since $\pi(\mathbb{D}) = \mathbb{D}$ is dense in $M(\mathcal{B})$ then π is onto. For a set $E \subset \mathbb{D}$ we write $\overline{E}^M$, with $M = M(\mathcal{A})$ or $M(\mathcal{B})$, to distinguish between closures in the corresponding space. No distinction will be made for the closure of sets in $\mathbb{C}$.

A closed set $F \subset M(\mathcal{A})$ will be called saturated if $H(x) \subset F$ whenever $x \in F$. If $\pi : M(\mathcal{A}) \to M(\mathcal{B})$ is the natural projection, write

$$G_{\mathcal{B}} = \{y \in M(\mathcal{B}) : \pi^{-1}(y) \text{ is a singleton}\}$$

and

$$\Gamma_{\mathcal{B}} = \{y \in M(\mathcal{B}) : \pi^{-1}(y) \text{ is saturated}\}.$$

That is, if $y \in M(\mathcal{B})$ then $y \in G_{\mathcal{B}}$ if and only if $\mathcal{B}$ separates every $x \in \pi^{-1}(y)$ from any other point of $M(\mathcal{A})$ (so $\pi^{-1}(y) = \{x\}$), and $y \in \Gamma_{\mathcal{B}}$ if and only if $b \circ \varphi_x$ is constant for all $x \in \pi^{-1}(y)$ and $b \in \mathcal{B}$. Since no single point is a saturated set then $G_{\mathcal{B}} \cap \Gamma_{\mathcal{B}} = \emptyset$. In addition, there could be points in $M(\mathcal{B})$ that are not in $G_{\mathcal{B}} \cup \Gamma_{\mathcal{B}}$. We will be interested in the cases that exclude the last possibility.

Definition. A prehyperbolic algebra $\mathcal{B}$ will be called hyperbolic if $M(\mathcal{B}) = G_{\mathcal{B}} \cup \Gamma_{\mathcal{B}}$. That is, if $\pi^{-1}(\pi(x)) = \{x\}$ or contains $H(x)$ for every $x \in M(\mathcal{A})$.

Lemma 3.7 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra. Then*

(1) $\Gamma_{\mathcal{B}}$ *is closed,*

(2) *the restriction $\pi_0 : \pi^{-1}(G_{\mathcal{B}}) \to G_{\mathcal{B}}$ of π is an onto homeomorphism.*

Proof. (1). If x is in the closure of $\pi^{-1}(\Gamma_{\mathcal{B}})$ take a net (x_α) in $\pi^{-1}(\Gamma_{\mathcal{B}})$ that tends to x. By definition of $\Gamma_{\mathcal{B}}$, $f \circ \varphi_{x_\alpha}$ is constant for every $f \in \mathcal{B}$. Hence, if $\omega \in \mathbb{D}$ and $f \in \mathcal{B}$, Lemma 3.6 gives

$$f(x) - f(\varphi_x(\omega)) = \lim f(x_\alpha) - f(\varphi_{x_\alpha}(\omega)) = 0,$$

implying that $f \circ \varphi_x \equiv f(x)$, so $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. That is, $\pi^{-1}(\Gamma_{\mathcal{B}})$ is closed in $M(\mathcal{A})$ and then $\pi(\pi^{-1}(\Gamma_{\mathcal{B}}))$ is closed in $M(\mathcal{B})$.

(2). By definition of $G_{\mathcal{B}}$, π_0 is one-to-one and onto, so we must show that $\pi_0^{-1} : G_{\mathcal{B}} \to \pi^{-1}(G_{\mathcal{B}})$ is continuous. Let (y_α) be a net in $G_{\mathcal{B}}$ such that $y_\alpha \to y \in G_{\mathcal{B}}$ and let

$x_\alpha \in \pi^{-1}(G_{\mathcal{B}})$ such that $\pi(x_\alpha) = y_\alpha$. If (x_{α_β}) is a convergent subnet of (x_α), say to x, then $y_{\alpha_\beta} = \pi(x_{\alpha_\beta}) \to \pi(x) = y$. So, $x \in \pi^{-1}(y)$, but since $y \in G_{\mathcal{B}}$ then $\pi^{-1}(y) = \{x\}$. Hence every convergent subnet of (x_α) tends to x, and then $x_\alpha \to x$. $\square$

Proposition 3.8 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra and $y \in M(\mathcal{B})$. The following conditions are equivalent*

- (a$_1$) $y \in \Gamma_{\mathcal{B}}$.
- (a$_2$) $f \circ \varphi_{z_\alpha} \to c \in \mathbb{C}$ *uniformly on compact sets for every net (z_α) in $\mathbb{D}$ tending to y and every $f \in \mathcal{B}$.*
- (a$_3$) *For every separated sequence $\mathcal{S}$ such that $y \in \overline{\mathcal{S}}^{M(\mathcal{B})}$ and every $f \in \mathcal{B}$ there is a subsequence $\{z_n\}$ of $\mathcal{S}$ (depending on f) such that $f \circ \varphi_{z_n} \to c \in \mathbb{C}$ pointwise on $\mathbb{D}$.*

Proof. (a$_1$)$\Rightarrow$(a$_2$). If $y \in \Gamma_{\mathcal{B}}$ then $\pi^{-1}(y)$ is saturated. Let (z_α) be a net in $\mathbb{D}$ such that $z_\alpha \to y$ in $M(\mathcal{B})$ and take a subnet (z_{α_β}) that converges in $M(\mathcal{A})$, say to x. Thus $\pi(z_{\alpha_\beta}) \to \pi(x) = y$ in $M(\mathcal{B})$, saying that $x \in \pi^{-1}(y)$. Since $H(x) \subset \pi^{-1}(y)$ (because it is saturated) then

$$f(\varphi_x(\xi)) = \lim f(\varphi_{z_{\alpha_\beta}}(\xi)) = \text{ const. } = \lim f(\varphi_{z_{\alpha_\beta}}(0)) = \lim f(z_{\alpha_\beta}) = f(y)$$

for every $f \in \mathcal{B}$ and $\xi \in \mathbb{D}$. This proves that whenever (z_{α_β}) is a subnet of (z_α) that converges in $M(\mathcal{A})$ then $f \circ \varphi_{z_{\alpha_\beta}} \to f(y)$ pointwise. By Lemma 3.4 the convergence is also uniform on compact sets, and consequently $f \circ \varphi_{z_\alpha} \to f(y)$ in that way.

(a$_2$)$\Rightarrow$(a$_3$). If $y \in \overline{\mathcal{S}}^{M(\mathcal{B})}$ there is a net (z_α) in $\mathcal{S}$ such that $z_\alpha \to y$ in $M(\mathcal{B})$. If $f \in \mathcal{B}$ then by (a$_2$), $f \circ \varphi_{z_\alpha} \to c \in \mathbb{C}$ uniformly on compact sets. Therefore for any positive integer n there is some z_α (that we rename as z_n) such that

$$\sup\{\, |(f \circ \varphi_{z_n})(\omega) - c| : \ |\omega| \le 1 - n^{-1}\} \le n^{-1}.$$

Therefore $\{z_n\}$ is a subsequence of $\mathcal{S}$ that satisfies (a$_3$).

(a$_3$)$\Rightarrow$(a$_1$). We will show that (a$_3$) fails when (a$_1$) fails. If $y \notin \Gamma_{\mathcal{B}}$ there is $x \in \pi^{-1}(y)$ such that $H(x) \not\subset \pi^{-1}(y)$. Therefore there is $f \in \mathcal{B}$ such that $f \circ \varphi_x \neq$ const., or what is the same, $(f \circ \varphi_x)(\omega) \neq f(x)$ for some $\omega \in \mathbb{D}$. Put $\eta = |(f \circ \varphi_x)(\omega) - f(x)| > 0$. If $\mathcal{S}$ is any separated sequence such that $x \in \overline{\mathcal{S}}^{M(\mathcal{A})}$ and we take

$$\mathcal{S}_1 = \{z \in \mathcal{S} : |(f \circ \varphi_{z_n})(\omega) - f(z_n)| \ge \eta/2\}$$

then $x \in \overline{\mathcal{S}}_1^{M(\mathcal{A})}$. Hence $y = \pi(x) \in \overline{\mathcal{S}}_1^{M(\mathcal{B})}$ and (a$_3$) fails for $\mathcal{S}_1$ and f. $\square$

Suppose that f is a continuous function from $M(\mathcal{A})$ into a topological space T. If $\mathcal{B}$ is a hyperbolic algebra, the restriction $f|_{\mathbb{D}}$ admits a continuous extension from $M(\mathcal{B})$ into T if

and only if $f(\pi^{-1}(y)) =$ const. for every $y \in \Gamma_{\mathcal{B}}$. In particular, for $T = \mathbb{C}$ we obtain that $f \in \mathcal{A}$ belongs to $\mathcal{B}$ if and only if $f(\pi^{-1}(y)) =$ const. for every $y \in \Gamma_{\mathcal{B}}$.

Let $\mathcal{B} \subset L^\infty(\mathbb{D})$ be a closed algebra. A sequence $\{z_n\} \subset \mathbb{D}$ is called interpolating for $\mathcal{B}$ if for every $\{\eta_n\} \in \ell^\infty$ there exists $f \in \mathcal{B}$ such that $f(z_n) = \eta_n$ for every n. It is clear that if $\mathcal{B}$ is a subalgebra of $\mathcal{A}$ then every interpolating sequence for $\mathcal{B}$ must be separated and that every separated sequence is interpolating for $\mathcal{A}$. We say that $f \in \mathcal{A}$ separates two sets $E, F \subset M(\mathcal{A})$ when $\overline{f(E)} \cap \overline{f(F)} = \emptyset$.

Proposition 3.9 *Let $\mathcal{B} \subset \mathcal{A}$ be a prehyperbolic algebra. For $y \in M(\mathcal{B})$ consider the following conditions*

- (b$_1$) *$y \in G_{\mathcal{B}}$.*
- (b$_2$) *There is an interpolating sequence $\mathcal{S} = \{z_n\}$ for $\mathcal{B}$, whose closure in $M(\mathcal{B})$ contains y, such that for every $\delta > 0$ sufficiently small there exists $f \in \mathcal{B}$ that separates $\{z_n\}$ from $\mathbb{D} \setminus \bigcup_n K(z_n, \delta)$.*

Then (b$_2$) *implies* (b$_1$)*, and if $\mathcal{B}$ is hyperbolic,* (b$_1$) *implies* (b$_2$).

Proof. (b$_2$)$\Rightarrow$(b$_1$). Let $y \in M(\mathcal{B})$ and $\mathcal{S}$ as in (b$_2$). We claim that $\pi^{-1}(y) \subset \overline{\mathcal{S}}^{M(\mathcal{A})}$, because otherwise there is $x \in \pi^{-1}(y)$ and a separated sequence $\mathcal{T} \subset \mathbb{D}$, with $x \in \overline{\mathcal{T}}^{M(\mathcal{A})}$, such that $\rho(\mathcal{S}, \mathcal{T}) \geq \alpha > 0$. The continuity of π implies that $y = \pi(x) \in \overline{\mathcal{T}}^{M(\mathcal{B})}$, but this is not possible because by hypothesis there is $f \in \mathcal{B}$ such that $\overline{f(\mathcal{S})} \cap \overline{f(\mathcal{T})} = \emptyset$, which contradicts $y \in \overline{\mathcal{S}}^{M(\mathcal{B})} \cap \overline{\mathcal{T}}^{M(\mathcal{B})}$.

Now suppose that there are two different points $x_1, x_2 \in \pi^{-1}(y)$. Then there is a disjoint decomposition $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2$, where $x_1 \in \overline{\mathcal{S}}_1^{M(\mathcal{A})}$ and $x_2 \in \overline{\mathcal{S}}_2^{M(\mathcal{A})}$. Since $\mathcal{S}$ is interpolating for $\mathcal{B}$ there exists $f \in \mathcal{B}$ that separates $\mathcal{S}_1$ from $\mathcal{S}_2$, leading to the same contradiction obtained before. Hence, $\pi^{-1}(y)$ is a single point.

(b$_1$)$\Rightarrow$(b$_2$) for $\mathcal{B}$ hyperbolic. If $y \in G_{\mathcal{B}}$ then $\pi^{-1}(y) = \{x\}$ for some $x \in M(\mathcal{A})$. Since $\pi^{-1}(\Gamma_{\mathcal{B}})$ is closed in $M(\mathcal{A})$ (by Lemma 3.7) and $x \notin \pi^{-1}(\Gamma_{\mathcal{B}})$ then there is a closed neighborhood F of x in $M(\mathcal{A})$ such that $F \cap \pi^{-1}(\Gamma_{\mathcal{B}}) = \emptyset$. Hence there is $f \in \mathcal{A}$ such that $f \equiv 1$ on F and $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$.

Let $\mathcal{T} \subset \mathbb{D}$ be a separated sequence such that $x \in \overline{\mathcal{T}}^{M(\mathcal{A})}$. Since $f \equiv 1$ on a neighborhood of x then $x \in \overline{\mathcal{S}}^{M(\mathcal{A})}$, where $\mathcal{S} = \{z \in \mathcal{T} : f(z) = 1\} = \{z_n\}$. Hence, $y = \pi(x) \in \overline{\mathcal{S}}^{M(\mathcal{B})}$. Observe also that $\overline{\mathcal{S}}^{M(\mathcal{A})} \subset F \subset \pi^{-1}(G_{\mathcal{B}})$.

Let $\{\eta_n\}$ be an arbitrary sequence in ℓ^∞ and take $g \in \mathcal{A}$ such that $g(z_n) = \eta_n$ for every n. Since $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then so is $h = fg \in \mathcal{A}$, and consequently $h \in \mathcal{B}$. In addition, $h(z_n) = f(z_n)g(z_n) = \eta_n$ for every n, which shows that $\mathcal{S}$ is interpolating for $\mathcal{B}$. Since f is ρ-uniformly continuous and $f(z_n) = 1$ for all n then $\bigcup K(z_n, \delta) \subset \{z : |f(z)| > 1/2\}$ when $\delta > 0$ is small enough. Take $a \in \mathcal{A}$ such that

$$a(z_n) = 1 \ \text{ for all } \ n, \quad \text{and} \quad a \equiv 0 \ \text{ on } \ \mathbb{D} \setminus \bigcup_n K(z_n, \delta). \tag{3.6}$$

Since $f \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then

$$\pi^{-1}(\Gamma_{\mathcal{B}}) \subset \overline{\{z : |f(z)| < 1/4\}}^{M(\mathcal{A})} \subset \overline{\mathbb{D} \setminus \bigcup_n K(z_n, \delta)}^{M(\mathcal{A})},$$

implying that $a \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$. Hence $a \in \mathcal{B}$ and (3.6) says that it separates $\mathcal{S}$ from $\mathbb{D} \setminus \bigcup_n K(z_n, \delta)$. So (b$_2$) holds. □

Propositions 3.8 and 3.9 provide criteria to decide whether a given prehyperbolic algebra is hyperbolic or not. Let us summarize these criteria in the next corollary.

Corollary 3.10 *A prehyperbolic algebra $\mathcal{B}$ is hyperbolic if and only if every $y \in M(\mathcal{B})$ satisfies some of the conditions* (a$_1$), (a$_2$) (a$_3$) *or some of the conditions* (b$_1$), (b$_2$).

4 Operator-valued compact maps

We recall that if $S \in \mathfrak{L}(L^2_a)$ and $z \in \mathbb{D}$ then $S_z = U_z S U_z$, where $U_z f = (f \circ \varphi_z)\varphi'_z$. Consider the map $\Psi_S : \mathbb{D} \to \mathfrak{L}(L^2_a)$ given by $\Psi_S(z) = S_z$. We will study the possibility to extend Ψ_S continuously to $M(\mathcal{A})$ when $\mathfrak{L}(L^2_a)$ is provided with the weak or the strong operator topology (WOT and SOT, respectively). We will also look for a possible extension to $M(\mathcal{B})$, where $\mathcal{B}$ is an arbitrary hyperbolic algebra.

Theorem 4.1 *Let (E, d) be a metric space and $f : \mathbb{D} \to E$ be a continuous map. Then f admits a continuous extension from $M(\mathcal{A})$ into E if and only if f is uniformly (ρ, d) continuous and $\overline{f(\mathbb{D})}$ is compact.*

Proof. Suppose that $f \in C(M(\mathcal{A}), E)$. Since $\mathbb{D}$ is dense in the compact space $M(\mathcal{A})$ then $\overline{f(\mathbb{D})} = f(M(\mathcal{A}))$ is compact. If f is not uniformly (ρ, d) continuous there are two sequences $z_n, \omega_n \in \mathbb{D}$ such that $\rho(z_n, \omega_n) \to 0$ and $d(f(z_n), f(\omega_n)) \geq \delta > 0$ for every n. By the continuity of f on $\mathbb{D}$ the sequence does not accumulate on $\mathbb{D}$. Let $x \in \overline{\{z_n\}}^{M(\mathcal{A})} \setminus \mathbb{D}$ and (z_α) be a subnet of $\{z_n\}$ that tends to x. Since every z_α is some $z_{n(\alpha)}$, writing $\omega_\alpha = \omega_{n(\alpha)}$ we have a subnet (ω_α) of the sequence $\{\omega_n\}$ such that

$$\rho(z_\alpha, \omega_\alpha) \to 0 \quad \text{and} \quad d(f(z_\alpha), f(\omega_\alpha)) \geq \delta \quad \text{for all} \quad \alpha. \tag{4.1}$$

The first condition in (4.1) implies that $g(\omega_\alpha) \to g(x)$ for every $g \in \mathcal{A}$, meaning that $\omega_\alpha \to x$ in $M(\mathcal{A})$. Since f is continuous on $M(\mathcal{A})$ then $\lim f(\omega_\alpha) = f(x) = \lim f(z_\alpha)$, contradicting (4.1).

Now assume that f is uniformly (ρ, d) continuous on $\mathbb{D}$ and $\overline{f(\mathbb{D})}$ is compact. For $x \in M(\mathcal{A})$ write

$$F(x) \stackrel{\text{def}}{=} \{\lambda \in E : f(z_\alpha) \to \lambda, \text{ for some net } z_\alpha \to x,\ z_\alpha \in \mathbb{D}\}.$$

The compactness of $\overline{f(\mathbb{D})}$ assures that $F(x)$ is nonempty. Then F is a multivalued function defined on $M(\mathcal{A})$, and a standard diagonal argument shows that f can be extended continuously to $M(\mathcal{A})$ if and only if $F(x)$ is single-valued for every $x \in M(\mathcal{A})$. So, let $x \in M(\mathcal{A})$ and assume that there are $\lambda_1, \lambda_2 \in F(x)$ such that $d(\lambda_1, \lambda_2) = \alpha > 0$. Let $B(\lambda, r)$ denote the open ball in E of center $\lambda \in E$ and radius $r > 0$, and consider the sets

$$V_i = \{z \in \mathbb{D} : f(z) \in B(\lambda_i, \alpha/4)\}, \quad i = 1, 2.$$

Since $\lambda_i \in F(x)$ then $x \in \overline{V}_i^{M(\mathcal{A})}$ for $i = 1, 2$. Lemma 3.1 then tells us that $\rho(V_1, V_2) = 0$. On the other hand,

$$d(f(V_1), f(V_2)) \geq d(B(\lambda_1, \alpha/4), B(\lambda_2, \alpha/4)) \geq \frac{\alpha}{2}.$$

By the uniform (ρ, d)-continuity of f, the last inequality implies that $\rho(V_1, V_2) > 0$, a contradiction. □

Lemma 4.2 *For $z, \alpha \in \mathbb{D}$ put $\lambda = \lambda(z, \alpha) = (\alpha\overline{z} - 1)/(1 - z\overline{\alpha})$. Then $U_{\varphi_z(\alpha)} U_z = V_\lambda U_\alpha$, where $(V_\lambda f)(\omega) = \lambda f(\lambda\omega)$ for $f \in L_a^2$.*

Proof. Since the function $\varphi_{\varphi_z(\alpha)} \circ \varphi_z \circ \varphi_\alpha$ is an automorphism that fixes the origin, it must be a rotation. A little bit of algebra shows that this function maps λ to 1. Since $\varphi_{\varphi_z(\alpha)}$ is its own inverse then $\varphi_z \circ \varphi_\alpha(\lambda\omega) = \varphi_{\varphi_z(\alpha)}(\omega)$. Therefore

$$\begin{aligned}
(U_{\varphi_z(\alpha)} U_z f)(\omega) &= (f \circ \varphi_z \circ \varphi_{\varphi_z(\alpha)})(\omega)\, \varphi_z'(\varphi_{\varphi_z(\alpha)}(\omega))\, \varphi_{\varphi_z(\alpha)}'(\omega) \\
&= (f \circ \varphi_z \circ \varphi_z \circ \varphi_\alpha)(\lambda\omega)\, \varphi_z'(\varphi_z \circ \varphi_\alpha(\lambda\omega))\, \varphi_z'(\varphi_\alpha(\lambda\omega))\, \varphi_\alpha'(\lambda\omega)\, \lambda \\
&= (f \circ \varphi_\alpha)(\lambda\omega)\, \varphi_\alpha'(\lambda\omega)\, \lambda \;=\; (V_\lambda U_\alpha f)(\omega),
\end{aligned}$$

where the third equality holds because since $\varphi_z \circ \varphi_z = id$ then $(\varphi_z' \circ \varphi_z)\varphi_z' = 1$. □

Lemma 4.3 *Let $f \in L_a^2$ and $\varepsilon > 0$. Then there is $\delta = \delta(f, \varepsilon) > 0$ such that*

$$\rho(z_1, z_2) < \delta \;\Rightarrow\; \|U_{z_1} f - U_{z_2} f\| < \varepsilon.$$

Proof. Since the polynomials are dense in L_a^2 and $\|U_z\| = 1$ for every $z \in \mathbb{D}$, it is enough to assume that f is a polynomial. If $\rho(z_1, z_2) < \delta$ then $z_2 = \varphi_{z_1}(\alpha)$ with $|\alpha| < \delta$. By the previous lemma,

$$(I - U_{\varphi_{z_1}(\alpha)} U_{z_1}) f(\omega) = f(\omega) - f\left(\frac{\alpha - \lambda\omega}{1 - \overline{\alpha}\lambda\omega}\right)\left(\frac{|\alpha|^2 - 1}{1 - \overline{\alpha}\lambda\omega}\right)\lambda,$$

where λ comes from the lemma. When $\alpha \to 0$ we have $\lambda(z_1, \alpha) \to -1$ uniformly in z_1, so the above expression tends to 0 uniformly in z_1 and ω. Hence,

$$\|U_{z_1} f - U_{\varphi_{z_1}(\alpha)} f\| = \|(U_{\varphi_{z_1}(\alpha)} U_{z_1} - I) f\| < \varepsilon$$

if $|\alpha|$ is small enough. That is, if δ is small enough. □

Proposition 4.4 *Let $S \in \mathfrak{L}(L_a^2)$. Then the map $\Psi_S : \mathbb{D} \to (\mathfrak{L}(L_a^2), \mathrm{WOT})$ extends continuously to $M(\mathcal{A})$.*

Proof. The closed the ball $B(0, \|S\|) \subset \mathfrak{L}(L_a^2)$ of center 0 and radius $\|S\|$ is compact and metrizable with the WOT-topology. Since $\Psi_S(\mathbb{D})$ is contained in $B(0, \|S\|)$, Theorem 4.1 reduces the problem to show that Ψ_S is uniformly continuous from the disk with the pseudo-hyperbolic metric into $B(0, \|S\|)$ with the weak operator topology. This amounts to see that for every $f, g \in L_a^2$, the function $z \mapsto \langle S_z f, g\rangle$ is uniformly continuous from $(\mathbb{D}, \rho)$ into $(\mathbb{C}, |\ |)$. For $z_1, z_2 \in \mathbb{D}$ we have

$$U_{z_1} S U_{z_1} - U_{z_2} S U_{z_2} = U_{z_1} S (U_{z_1} - U_{z_2}) + (U_{z_1} - U_{z_2}) S U_{z_2} = A + B.$$

If $f, g \in L_a^2$ then $|\langle Af, g\rangle| \le \|U_{z_1} S\| \, \|(U_{z_1} - U_{z_2}) f\|_2 \, \|g\|_2$ and $|\langle Bf, g\rangle| = |\langle f, B^* g\rangle| \le \|f\|_2 \, \|U_{z_2} S^*\| \, \|(U_{z_1} - U_{z_2}) g\|_2$. By Lemma 4.3 both expressions can be made small if we take $\rho(z_1, z_2)$ small enough. $\square$

Theorem 4.5 *Let $S \in \mathfrak{T}(\mathcal{A})$. Then the map $\Psi_S : \mathbb{D} \to (\mathfrak{L}(L_a^2), \mathrm{SOT})$ extends continuously to $M(\mathcal{A})$. In addition, $\Psi_S(M(\mathcal{A})) \subset \mathfrak{T}(\mathcal{A})$.*

Proof. First suppose that $S = T_a$, with $a \in \mathcal{A}$. If $z \in \mathbb{D}$ tends to $x \in M(\mathcal{A})$, Lemma 3.4 says that $a \circ \varphi_z \to a \circ \varphi_x$ uniformly on compact sets. Thus, if $f \in L_a^2$ and $0 < r < 1$,

$$\|(T_{a\circ\varphi_z} - T_{a\circ\varphi_x}) f\|^2 \le \sup_{rD} |a \circ \varphi_z - a \circ \varphi_x|^2 \; \|f\|^2 + 2\|a\|_\infty^2 \int_{D \setminus rD} |f|^2 \, dA.$$

We can choose some $r = r(f, \|a\|_\infty)$ close enough to 1 so that the second term is smaller than a given $\varepsilon > 0$, and for such r the first term tends to 0 as $z \to x$. Since $\Psi_{S+T} = \Psi_S + \Psi_T$, the case of a polynomial in Toeplitz operators reduces to the case $S = T_{a_1} \ldots T_{a_k}$, where $a_j \in \mathcal{A}$ and $\|a_j\|_\infty \le 1$ for $j = 1, \ldots, k$. Consider the operators

$$S_j = \begin{cases} T_{a_1\circ\varphi_z} \ldots T_{a_{j-1}\circ\varphi_z} T_{a_j\circ\varphi_x} \ldots T_{a_k\circ\varphi_x} & \text{if } 1 \le j \le k \\ T_{a_1\circ\varphi_z} \ldots T_{a_k\circ\varphi_z} & \text{if } j = k+1 \end{cases}$$

If $f \in L_a^2$ then $\|(S_{k+1} - S_1) f\| \le \sum_{j=1}^k \|(S_{j+1} - S_j) f\|$, and since we have proved that $T_{a_j\circ\varphi_z} - T_{a_j\circ\varphi_x} \to 0$ in the strong operator topology as $z \to x$, then

$$\begin{aligned} \|(S_{j+1} - S_j) f\| &= \|T_{a_1\circ\varphi_z} \ldots T_{a_{j-1}\circ\varphi_z} (T_{a_j\circ\varphi_z} - T_{a_j\circ\varphi_x}) T_{a_{j+1}\circ\varphi_x} \ldots T_{a_k\circ\varphi_x} f\| \\ &\le \|(T_{a_j\circ\varphi_z} - T_{a_j\circ\varphi_x}) T_{a_{j+1}\circ\varphi_x} \ldots T_{a_k\circ\varphi_x} f\| \to 0 \end{aligned}$$

when $z \to x$. Finally, if $S \in \mathfrak{T}(\mathcal{A})$ is arbitrary, given $\varepsilon > 0$ there is a polynomial in Toeplitz operators with symbols in $\mathcal{A}$, say T, such that $\|S - T\| < \varepsilon$. By Proposition 4.4 there is some $S_x \in \mathfrak{L}(L_a^2)$ such that $S_z - T_z \to S_x - T_x$ weakly when $z \to x$. Weak limits do not increase norms, so $\|S_x - T_x\| \le \varepsilon$. The result follows because $\|S_z - T_z\| < \varepsilon$ for all $z \in \mathbb{D}$ and $T_z \to T_x$ strongly when $z \to x$. $\square$

Corollary 4.6 *If $S \in \mathfrak{L}(L_a^2)$ and $n \geq 0$ is an integer then $B_n S \in \mathcal{A}$. Besides, $B_n S_x = (B_n S) \circ \varphi_x$ for every $x \in M(\mathcal{A})$.*

Proof. By (2.1) and Lemma 2.2

$$(B_n S)(z) = ((B_n S) \circ \varphi_z)(0) = (B_n S_z)(0) = (n+1) \sum_{j=0}^{n} \binom{n}{j} (-1)^j \langle S_z \omega^j, \omega^j \rangle.$$

Since by Proposition 4.4 the map $z \mapsto \langle S_z \omega^j, \omega^j \rangle$ extends continuously to $M(\mathcal{A})$, it belongs to $\mathcal{A}$ for every $0 \leq j \leq n$. For the second assertion take a net (z_α) in $\mathbb{D}$ that tends to x and then take limits in the equality $(B_n S_{z_\alpha})(\xi) = (B_n S)(\varphi_{z_\alpha}(\xi))$ for each fixed $\xi \in \mathbb{D}$. The first term tends to $(B_n S_x)(\xi)$ because Proposition 4.4 says that $z \mapsto \langle S_z \omega^j K_\xi^{(n)}, \omega^j K_\xi^{(n)} \rangle$ extends continuously to $M(\mathcal{A})$, and the second term tends to $(B_n S)(\varphi_x(\xi))$ because $B_n S \in \mathcal{A}$. □

Corollary 4.7 *If $S \in \mathfrak{L}(L_a^2)$ and $x \in M(\mathcal{A})$ the following conditions are equivalent*

(i) *$S_u = \lambda I$ for every $u \in H(x)$*

(ii) *$S_u = \lambda I$ for some $u \in H(x)$*

(iii) *$B_0 S \equiv \lambda$ on $H(x)$.*

Proof. Since $H(u) = H(x)$ when $u \in H(x)$ then every $v \in H(x)$ has the form $v = \varphi_u(\omega)$ for some $\omega \in \mathbb{D}$. By the previous corollary

$$(B_0 S)(v) = (B_0 S)(\varphi_u(\omega)) = (B_0 S_u)(\omega).$$

This identity and the fact that B_0 acts in a one-to-one fashion on $\mathfrak{L}(L_a^2)$ give all the equivalences. □

Since for $a \in \mathcal{A}$ we have $(T_a)_z^* = T_{\overline{a} \circ \varphi_z} \to T_{\overline{a} \circ \varphi_x} = (T_a)_x^*$ in the SOT-topology when $z \to x$, then also $(T_z)^* \to (T_x)^*$ in the SOT-topology for all $T \in \mathfrak{T}(\mathcal{A})$. Also, since the product of a WOT-convergent and a SOT-convergent net in $\mathfrak{L}(L_a^2)$ tends weakly to the product of the limits, Proposition 4.4 and Theorems 4.5 imply that

$$S_x T_x = (ST)_x \quad \text{and} \quad T_x S_x = (TS)_x \tag{4.2}$$

for every $S \in \mathfrak{L}(L_a^2)$, $T \in \mathfrak{T}(\mathcal{A})$ and $x \in M(\mathcal{A})$. This fails if we only assume $S, T \in \mathfrak{L}(L_a^2)$. Indeed, consider the operator defined by $Sf(\omega) = f(-\omega)$. Since $S^2 = I$ then $(S^2)_x = I$ for every $x \in M(\mathcal{A})$. On the other hand, since $SK_z^{(0)} = K_{-z}^{(0)}$ then

$$(B_0 S)(z) = (1 - |z|^2)^2 \langle K_{-z}^{(0)}, K_z^{(0)} \rangle = \frac{(1 - |z|^2)^2}{(1 + |z|^2)^2}.$$

So $(B_0 S)(z) \to 0$ when $|z| \to 1$, and then $(B_0 S)(x) = 0$ for all $x \in M(\mathcal{A}) \setminus \mathbb{D}$. Corollary 4.7 then tells us that $S_x = 0$ for $x \in M(\mathcal{A}) \setminus \mathbb{D}$, making (4.2) impossible for this choice of S and $T = S$.

Lemma 4.8 *Let $S \in \mathfrak{L}(L^2_a)$ and $x \in M(\mathcal{A})$. Suppose that there is some $n_0 \geq 0$ such that $(B_{n_0}S) \circ \varphi_x = g$ harmonic. Then $(B_nS) \circ \varphi_x = g$ for every $n \geq 0$.*

Proof. By Corollary 4.6, $\tilde{\Delta}(B_{n_0}S_x) = \tilde{\Delta}g = 0$, which together with (2.7) yields $B_{n_0+1}S_x = B_{n_0}S_x = g$. Then $B_nS_x = g$ for every $n \geq n_0$. Thus $B_0(B_nS_x) = B_0g = g$ for $n \geq n_0$, implying that

$$g = \lim_{n \longrightarrow \infty} B_0 B_n S_x = \lim_{n \longrightarrow \infty} B_n B_0 S = B_0 S,$$

where the second equality follows from Corollary 2.7 and the last one because since $B_0S \in \mathcal{A}$ by Corollary 4.6, then $B_n(B_0S) \rightarrow B_0S$ uniformly. Taking $n_0 = 0$, we have proved above that $B_nS = g$ for every $n \geq 0$. □

By the lemma we can add two more equivalences to Corollary 4.7, saying that $B_nS \equiv \lambda$ on $H(x)$ for every (or for some) $n \geq 0$.

Theorem 4.9 *Let $S \in \mathfrak{T}(\mathcal{A})$ and $\mathcal{B}$ be a hyperbolic algebra. Then the following conditions are equivalent,*

(1) *$S_x = \lambda I$ when $x \in \pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$, where $\lambda \in \mathbb{C}$ depends only on y,*

(2) *there is a continuous map $\Psi^{\mathcal{B}}_S : M(\mathcal{B}) \rightarrow (\mathfrak{T}(\mathcal{A}), SOT)$ such that $\Psi^{\mathcal{B}}_S \circ \pi = \Psi_S$,*

(3) *$B_nS \in \mathcal{B}$ for some $n \geq 0$,*

(4) *$B_nS \in \mathcal{B}$ for all $n \geq 0$.*

If $S \in \mathfrak{L}(L^2_a)$ the theorem holds replacing $(\mathfrak{T}(\mathcal{A}), SOT)$ by $(\mathfrak{L}(L^2_a), WOT)$ in (2).

Proof. If (1) holds then for every $y \in M(\mathcal{B})$ and $x \in \pi^{-1}(y)$, S_x is an operator that only depends on y. Hence $\Psi^{\mathcal{B}}_S(y) = S_x$ is well defined and satisfies the equation in (2). The continuity of $\Psi^{\mathcal{B}}_S$ from $M(\mathcal{B})$ into any of the metric spaces $(\mathfrak{T}(\mathcal{A}), \mathrm{SOT})$ or $(\mathfrak{L}(L^2_a), \mathrm{WOT})$ (according to the hypothesis) follows from the respective continuity of Ψ_S, which is given by Theorem 4.5 and Proposition 4.4.

Now suppose that (2) holds. This means that if $y \in M(\mathcal{B})$ then S_x is the same operator T for every $x \in \pi^{-1}(y)$. Since $\varphi_x(\mathbb{D}) \subset \pi^{-1}(y)$ for $y \in \Gamma_{\mathcal{B}}$, then $S_{\varphi_x(\omega)} = T$ for every $\omega \in \mathbb{D}$. Corollary 4.6 then says that

$$(B_0S)(\varphi_x(\omega)) = (B_0S_{\varphi_x(\omega)})(0) = (B_0T)(0)$$

for every $x \in \pi^{-1}(y)$ and $\omega \in \mathbb{D}$. Writing $\lambda = (B_0T)(0)$, we obtain that $B_0S \equiv \lambda$ on $H(x)$ for every $x \in \pi^{-1}(y)$. Hence B_0S is constant on $\pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$, meaning that $(B_0S)|_D$ extends continuously to $M(\mathcal{B})$. Since the Gelfand-Naimark Theorem identifies $\mathcal{B}$ with $C(M(\mathcal{B}))$ then $B_0S \in \mathcal{B}$. This proves (3) for $n = 0$. If (3) holds for some $n_0 \geq 0$ then

$B_{n_0}S = \lambda_y \in \mathbb{C}$ on $\pi^{-1}(y)$ for every $y \in \Gamma_{\mathcal{B}}$. Lemma 4.8 then implies that the same happens with B_nS for all $n \geq 0$. This proves (4). Finally, if (4) holds then $(B_0S)|_{\pi^{-1}(y)} = \lambda_y \in \mathbb{C}$ for $y \in \Gamma_{\mathcal{B}}$. In particular, $B_0S \equiv \lambda_y$ on $H(x)$ for every $x \in \pi^{-1}(y)$. Then (1) follows from Corollary 4.7. □

If $S \in \mathfrak{L}(L_a^2)$ satisfies the conditions of the theorem then the map $z \mapsto S_z$ admits a continuous extension to $M(\mathcal{B})$ given by $\Psi_S^{\mathcal{B}}$. Write $\Psi_S^{\mathcal{B}}(y) = \widehat{S}_y^{\mathcal{B}}$ when $y \in M(\mathcal{B})$. If $\mathcal{B} = \mathcal{A}$ we keep the previous notation $\Psi_S(y) = S_y$ for $y \in M(\mathcal{A})$. Also, since it is clear that we can identify $\widehat{S}_z^{\mathcal{B}}$ with S_z when $z \in \mathbb{D}$, we do not make this notation distinction for $z \in \mathbb{D}$. Observe that if $y \in M(\mathcal{B})$ and (z_α) is a net in $\mathbb{D}$ that tends to y in $M(\mathcal{B})$, then $\widehat{S}_y^{\mathcal{B}}$ admits the two alternative and equivalent expressions

$$\widehat{S}_y^{\mathcal{B}} = \lim_{\alpha} S_{z_\alpha},$$

a WOT-limit in general and a SOT-limit if $S \in \mathfrak{T}(\mathcal{A})$, or

$$\widehat{S}_y^{\mathcal{B}} = S_x \text{ for some (or all) } x \in \pi^{-1}(y),$$

where $\pi : M(\mathcal{A}) \to M(\mathcal{B})$ is the natural projection. Also, if $b \in \mathcal{B}$ we can look at b as a continuous function on $M(\mathcal{B})$ or on $M(\mathcal{A})$. If $\mathcal{B} \neq \mathcal{A}$ we write $\widehat{b}^{\mathcal{B}}$ when we need to distinguish the domain of the function, otherwise b will be looked as a function on $M(\mathcal{A})$. Of course, if $z \in \mathbb{D}$ then $b(z)$ has the same meaning either way.

If $\mathcal{B}$ is a hyperbolic algebra, $b \in \mathcal{B}$ and $y \in \Gamma_{\mathcal{B}}$, then for every $x \in \pi^{-1}(y)$ we have $(T_b)_x = T_{b\circ\varphi_x} = \lambda I$ with $\lambda \in \mathbb{C}$ depending only on y (actually $\lambda = \widehat{b}^{\mathcal{B}}(y)$). Since $\mathfrak{T}(\mathcal{B})$ is generated by these Toeplitz operators, the same holds for every $S \in \mathfrak{T}(\mathcal{B})$. Theorem 4.9 then says that $B_nS \in \mathcal{B}$ when $S \in \mathfrak{T}(\mathcal{B})$, for every nonnegative integer n.

5 Approximation and truncation by Toeplitz operators

If $A \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $\mathfrak{T}_0(A)$ for the algebra generated by the Toeplitz operators T_a, with $a \in A$, without taking closure. In [4] Axler and Zheng found simple but very ingenious estimates for the norm of operators in $\mathfrak{T}_0(L^\infty(\mathbb{D}))$. The present section (especially Lemmas 5.2 and 5.5) makes heavy use of their method.

5.1 Norm estimates and truncation

The following lemma is a particular case of Lemma 4.2.2 in [21].

Lemma 5.1 *If* $c < 0$ *and* $t > -1$ *then*

$$J_{c,t}(z) = \int_D \frac{(1-|\omega|^2)^t}{|1-z\overline{\omega}|^{2+t+c}}\, dA(\omega), \quad z \in \mathbb{D},$$

is bounded.

The next result appeared in [4] for $p = 6$. The proof sketched here is a standard modification of that proof involving Lemma 5.1.

Lemma 5.2 *Let* $p > 4$. *Then there is a constant* $C_p < \infty$ *such that if* $S \in \mathfrak{L}(L_a^2)$, *then*

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1-|w|^2}}\, dA(w) \le \frac{C_p \|S_z 1\|_p}{\sqrt{1-|z|^2}} \tag{5.1}$$

for all $z \in \mathbb{D}$ *and*

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1-|z|^2}}\, dA(z) \le \frac{C_p \|S_w^* 1\|_p}{\sqrt{1-|w|^2}} \tag{5.2}$$

for all $w \in \mathbb{D}$.

Proof. To prove (5.1) let $S \in \mathfrak{L}(L_a^2)$ and fix $z \in \mathbb{D}$. Since $U_z 1 = (|z|^2 - 1)K_z^{(0)}$ and $U_z U_z = I$ then $U_z S_z 1 = (|z|^2 - 1)SK_z^{(0)}$. Thus

$$\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1-|w|^2}}\, dA(w) = \frac{1}{1-|z|^2} \int_D \frac{|(S_z 1)(\varphi_z(w))|\, |\varphi_z{}'(w)|}{\sqrt{1-|w|^2}}\, dA(w).$$

Making the substitution $w = \varphi_z(\lambda)$ in the last integral and using Holder's inequality with $1/p + 1/q = 1$, we obtain

$$\begin{aligned}
\int_D \frac{|(SK_z^{(0)})(w)|}{\sqrt{1-|w|^2}}\, dA(w) &= \frac{1}{\sqrt{1-|z|^2}} \int_D \frac{|(S_z 1)(\lambda)|}{|1-\bar{z}\lambda|\sqrt{1-|\lambda|^2}}\, dA(\lambda) \\
&\le \frac{\|S_z 1\|_p}{\sqrt{1-|z|^2}} \left(\int_D \frac{dA(\lambda)}{|1-\bar{z}\lambda|^q (1-|\lambda|^2)^{q/2}} \right)^{1/q} \\
&= \frac{\|S_z 1\|_p}{\sqrt{1-|z|^2}}\; J(z)^{1/q},
\end{aligned}$$

where

$$J(z) = \int_D \frac{(1-|\lambda|^2)^{-q/2}}{|1-\bar{z}\lambda|^{2-(q/2)+(3/2)q-2}}\, dA(\lambda).$$

Since $p > 4$ then $q < 4/3$, which yields $q/2 < 2/3 < 1$ and $(3/2)q - 2 < 0$. By Lemma 5.1 there is $J_q > 0$ such that $J(z) \leq J_q$ for every $z \in \mathbb{D}$. This proves (5.1) with $C_p = J_q^{1/q}$. Replace S with S^* and interchange the roles of w and z in (5.1) to obtain

$$\int_D \frac{|(S^* K_w^{(0)})(z)|}{\sqrt{1-|z|^2}}\, dA(z) \leq \frac{C_p \|S_w^* 1\|_p}{\sqrt{1-|w|^2}}.$$

Then use the equality $(S^* K_w^{(0)})(z) = \overline{(SK_z^{(0)})(w)}$ to obtain (5.2). □

Lemma 5.3 *Let $S \in \mathfrak{L}(L_a^2)$, $a, b \in L^\infty(\mathbb{D})$ and $p > 4$. Then*

$$\|T_b S T_a\|_{\mathfrak{L}(L_a^2)} \leq C_p \left(\|a\|_\infty \|b\|_\infty\right)^{1/2} \sup_{z\in D}\{\|S_z 1\|_p\, |a(z)|\}^{1/2} \sup_{\omega\in D}\{\|S_w^* 1\|_p\, |b(w)|\}^{1/2},$$

where C_p is the constant of Lemma 5.2.

Proof. For $f \in L_a^2$ and $w \in D$, we have

$$\begin{aligned}(ST_a f)(w) &= \langle ST_a f, K_w^{(0)}\rangle = \langle af, S^* K_w^{(0)}\rangle \\ &= \int_D f(z)a(z)\overline{(S^* K_w^{(0)})(z)}\, dA(z) \\ &= \int_D f(z)a(z)(SK_z^{(0)})(w)\, dA(z).\end{aligned}$$

Thus, if M_b denotes the multiplication operator,

$$(M_b S T_a)f(w) = \int_D f(z)a(z)b(w)(SK_z^{(0)})(w)\, dA(z).$$

If $\Phi(z,w) = |a(z)b(w)(SK_z^{(0)})(w)|$ and $h(z) = (1-|z|^2)^{-1/2}$ then (5.1) yields

$$\begin{aligned}\int_D \Phi(z,w)h(w)\, dA(w) &\leq C_p \|b\|_\infty \|S_z 1\|_p\, |a(z)|\, h(z) \\ &\leq C_p \|b\|_\infty \sup_{z\in D}\{\|S_z 1\|_p\, |a(z)|\} h(z),\end{aligned}$$

and by (5.2)

$$\begin{aligned}\int_D \Phi(z,w)h(z)\, dA(w) &\leq C_p \|a\|_\infty \|S_w^* 1\|_p\, |b(w)|\, h(w) \\ &\leq C_p \|a\|_\infty \sup_{\omega\in D}\{\|S_w^* 1\|_p\, |b(w)|\} h(w).\end{aligned}$$

By Schur's theorem (see the proof in [21, p. 42]) the operator $M_b S T_a$ satisfies an inequality as in the lemma. The result follows because $\|(T_b S T_a)f\|_{L^2} \leq \|(M_b S T_a)f\|_{L^2}$ for every $f \in L_a^2$. □

Suppose that $1 < p < p' < \infty$, $f \in L^p(\mathbb{D})$ and $0 < r < 1$. Split the integral $\|f\|_p^p = \|f\chi_{D\setminus rD}\|_p^p + \|f\chi_{rD}\|_p^p$, where χ_E denotes the characteristic function of the set E. Taking $\alpha = p'/p$ and $\beta = p'/(p'-p)$ we have $\alpha^{-1} + \beta^{-1} = 1$. By Holder's inequality

$$\|f\chi_{D\setminus rD}\|_p^p \leq \|f\|_{\alpha p}^p \; \|\chi_{D\setminus rD}\|_\beta = \|f\|_{p'}^p (1-r^2)^{1-\frac{p}{p'}},$$

and consequently

$$\|f\|_p^p \leq \|f\|_{p'}^p (1-r^2)^{1-\frac{p}{p'}} + \|f\chi_{rD}\|_p^p. \tag{5.3}$$

Proposition 5.4 *Suppose that $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$ and $F \subset M(\mathcal{A})$ is a closed saturated set such that $B_0 S \equiv 0$ on F. Given $\varepsilon > 0$ there is an open neighborhood Ω of F in $M(\mathcal{A})$ such that if $U \subset \Omega \cap \mathbb{D}$ is measurable, then*

$$\|T_{a\chi_U} S\|_{\mathfrak{L}(L_a^2)} < \varepsilon \quad \textit{and} \quad \|S T_{a\chi_U}\|_{\mathfrak{L}(L_a^2)} < \varepsilon \tag{5.4}$$

for every $a \in L^\infty(\mathbb{D})$ with $\|a\|_\infty \leq 1$.

Proof. Since F is saturated and $B_0 S \equiv 0$ on F, Proposition 4.4 and Corollary 4.7 say that $S_z \overset{\text{WOT}}{\to} S_x = 0$ when $z \to x \in F$, with $z \in \mathbb{D}$. Thus $S_z 1 \to 0$ weakly in L_a^2 and consequently

$$S_z 1 \to 0 \;\; \text{uniformly on compact sets as} \;\; z \to x \;\; (z \in \mathbb{D}) \tag{5.5}$$

for every $x \in F$. Write $S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}$, with $a_j^i \in L^\infty(\mathbb{D})$, and fix p, p' with $4 < p < p'$. Then

$$\|S_z 1\|_{p'} = \|P_+ \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \circ \varphi_z}\|_{p'} \leq c_{p'} \sum_{i=1}^m \prod_{j=1}^{n_i} \|a_j^i\|_\infty = c, \tag{5.6}$$

where $c_{p'}$ is the norm of the analytic projection P_+ acting on $L^{p'}(\mathbb{D})$. For $0 < r < 1$ (5.3) yields

$$\|S_z 1\|_p^p \leq \|S_z 1\|_{p'}^p (1-r)^{1-\frac{p}{p'}} + \|(S_z 1)\chi_{rD}\|_p^p.$$

By (5.6) there is r close enough to 1 so that the first member of the sum is smaller than $\varepsilon/2$, while (5.5) and the compactness of F imply that there is a neighborhood Ω of F so that the second member is smaller than $\varepsilon/2$ for every $z \in \Omega \cap \mathbb{D}$. In particular, if $U \subset \Omega \cap \mathbb{D}$ this holds for every $z \in U$. Since $\|a\|_\infty \leq 1$, Lemma 5.3 gives

$$\|S T_{a\chi_U}\|^2 \leq C_p \sup\{\|S_z 1\|_p : z \in U\} \; \sup_D \|S_\omega^* 1\|_p \leq C_p \, c\varepsilon^{1/p},$$

where c comes from (5.6) with S^* instead of S, and C_p is the constant of Lemma 5.3. To prove the first inequality of (5.4) observe that $B_0 S^* = \overline{B_0 S}$ also satisfies the hypothesis of the proposition and $\|T_{a\chi_U} S\| = \|S^* T_{\overline{a}\chi_U}\|$. □

5.2 Approximation properties of the k-Berezin transforms

Lemma 5.5 *Suppose that $\{S_k\}$ is a bounded sequence in $\mathfrak{L}(L_a^2)$ such that $\|B_0 S_k\|_\infty \to 0$ when $k\to\infty$. Then*

$$\sup_{z\in D} |(S_k)_z 1| \to 0$$

uniformly on compact subsets of $\mathbb{D}$ when $k\to\infty$.

Proof. Since there is a constant C such that $\|S_k\| \leq C$ for every k, then it is enough to prove that for every $S \in \mathfrak{L}(L_a^2)$, $\eta > 0$ and $r \in (0,1)$, there is a function $c(r,\eta) > 0$, independent of S, such that

$$\sup_{z\in D} |(S_z 1)(u)| \leq c(r,\eta)\|B_0 S\|_\infty + \eta\|S\| \tag{5.7}$$

when $u \in r\mathbb{D}$. Since

$$K_z^{(0)}(w) = \sum_{m=0}^{\infty}(m+1)\bar{z}^m\omega^m, \tag{5.8}$$

then for $z, \lambda \in \mathbb{D}$ we have

$$(B_0 S)(\varphi_z(\lambda)) = (B_0 S_z)(\lambda) = (1-|\lambda|^2)^2 \sum_{j,m=0}^{\infty}(j+1)(m+1)\langle S_z\omega^j, \omega^m\rangle \bar{\lambda}^j\lambda^m,$$

where the first equality comes from Lemma 2.2. Then, for $0 < \delta < 1/2$ (to be chosen later) we obtain

$$\begin{aligned}
\int_{\delta D} \frac{(B_0 S)(\varphi_z(\lambda))\bar{\lambda}^n}{(1-|\lambda|^2)^2}\, dA(\lambda) &= \sum_{j,m=0}^{\infty}(j+1)(m+1)\langle S_z\omega^j, \omega^m\rangle \int_{\delta D} \bar{\lambda}^{j+n}\lambda^m\, dA(\lambda) \\
&= \sum_{j=0}^{\infty}(j+1)\langle S_z\omega^j, \omega^{j+n}\rangle \delta^{2j+2n+2} \\
&= \delta^{2n+2}\left(\langle S_z 1, \omega^n\rangle + \sum_{j=1}^{\infty}(j+1)\langle S_z\omega^j, \omega^{j+n}\rangle\delta^{2j}\right).
\end{aligned}$$

Since $0 < \delta < 1/2$ and $\|\omega^j\| = (j+1)^{-1/2}$ then

$$\begin{aligned}
|\langle S_z 1, \omega^n\rangle| &\leq \frac{1}{\delta^{2n+2}}\|B_0 S\|_\infty \int_{\delta D}\frac{\delta^n\, dA(\lambda)}{(1-|\lambda|^2)^2} + \|S\|\sum_{j=1}^{\infty}(j+1)\|\omega^j\|\,\|\omega^{j+n}\|\delta^{2j} \\
&\leq 2\delta^{-n}\|B_0 S\|_\infty + \delta\,\|S\|,
\end{aligned} \tag{5.9}$$

where the last inequality holds because $\sum_{j=1}^{\infty}\delta^{2j} \leq \delta$ when $0 < \delta < 1/2$. By (5.8)

$$(S_z 1)(u) = \langle S_z 1, K_u^{(0)}\rangle = \sum_{n\geq 0}(n+1)\langle S_z 1, \omega^n\rangle u^n,$$

implying that

$$|(S_z1)(u)| \le \sum_{0\le n\le N-1}(n+1)|\langle S_z1,\omega^n\rangle| + \sum_{n\ge N}(n+1)^{1/2}\|S_z\|r^n \tag{5.10}$$

for $z \in \mathbb{D}$, $u \in r\mathbb{D}$ and $N \ge 1$. Since $r \in (0,1)$ we can fix some integer $N = N(r,\eta)$ big enough so that the second sum is bounded by $(\eta/2)\|S\|$. Using (5.9) in (5.10) we get

$$\begin{aligned}|(S_z1)(u)| &\le N\sum_{0\le n\le N-1}|\langle S_z1,\omega^n\rangle| + (\eta/2)\|S\| \\ &\le 2N^2\delta^{-N}\|B_0S\|_\infty + N^2\delta\|S\| + (\eta/2)\|S\|\end{aligned}$$

for $z \in \mathbb{D}$ and $u \in r\mathbb{D}$. Choosing $\delta = \delta(r,\eta) < \min\{\eta/2N^2, 1/2\}$ we obtain (5.7) with $c(r,\eta) = 2N^2\delta^{-N}$. □

Lemma 5.6 *Let $\{S_k\}$ be a sequence in $\mathfrak{L}(L^2_a)$ such that for some $p' > 4$,*

$$\|B_0S_k\|_\infty \to 0, \quad \textit{when} \quad k\to\infty, \tag{5.11}$$

$$\sup_{z\in D}\|(S_k)_z1\|_{p'} \le C \quad \textit{and} \quad \sup_{\omega\in D}\|(S_k^*)_\omega 1\|_{p'} \le C, \tag{5.12}$$

where $C > 0$ does not depend on k. Then $\|S_k\|_{\mathfrak{L}(L^2_a)}\to 0$ when $k\to\infty$.

Proof. By (5.12) and Lemma 5.3 with $a = b = 1$,

$$\|S_k\|_{\mathfrak{L}(L^2_a)} \le C_{p'}\sup_{z\in D}\{\|(S_k)_z1\|_{p'}\}^{1/2}\sup_{\omega\in D}\{\|(S_k^*)_w1\|_{p'}\}^{1/2} \le C_{p'}C.$$

Hence, $\{S_k\}$ is a bounded sequence in $\mathfrak{L}(L^2_a)$ that satisfies (5.11). Under these conditions Lemma 5.5 says that

$$\sup_{z\in D}|(S_k)_z1| \to 0 \ \text{ uniformly on compact sets of } \mathbb{D}. \tag{5.13}$$

Let p with $4 < p < p'$. By (5.3)

$$\sup_{z\in D}\|(S_k)_z1\|_p^p \le \sup_{z\in D}\|(S_k)_z1\|_{p'}^p(1-r)^{1-\frac{p}{p'}} + \sup_{z\in D}\|[(S_k)_z1]\chi_{rD}\|_p^p$$

for every $0 < r < 1$. By (5.12) the first member of the sum is bounded by $C^p(1-r)^{1-\frac{p}{p'}}$, which can be made small by taking r close to 1, and by (5.13) the second member of the sum tends to 0 as $k\to\infty$. Therefore, $\sup_{z\in D}\|(S_k)_z1\|_p\to 0$ when $k\to\infty$ for every $p \in (4,p')$. Using again Lemma 5.3, this time with p instead of p', we obtain

$$\begin{aligned}\|S_k\|_{\mathfrak{L}(L^2_a)} &\le C_p\sup_{z\in D}\{\|(S_k)_z1\|_p\}^{1/2}\sup_{\omega\in D}\{\|(S_k^*)_w1\|_p\}^{1/2} \\ &\le C_p\sup_{z\in D}\{\|(S_k)_z1\|_p\}^{1/2}\,C^{1/2} \to 0\end{aligned}$$

when $k\to\infty$, where the last inequality holds by (5.12), since $\|\ \|_p \le \|\ \|_{p'}$. □

Theorem 5.7 *If $a \in L^\infty(\mathbb{D})$ then $T_{B_k(a)} \to T_a$ in operator norm when $k \to \infty$. In particular, $\mathfrak{T}(\mathcal{A}) = \mathfrak{T}(L^\infty(\mathbb{D}))$.*

Proof. Write $S_k = T_{B_k(a)} - T_a$. Since Corollary 2.7 says that $B_0 B_k = B_k B_0$ on $\mathfrak{L}(L^2_a)$ then

$$B_0 S_k = B_0 T_{B_k(a)} - B_0 T_a = B_0 B_k(a) - B_0(a) = B_k B_0(a) - B_0(a),$$

which tends uniformly to 0 when $k \to \infty$ because $B_0(a) \in \mathcal{A}$. That is, $\{S_k\}$ satisfies (5.11). On the other hand, if $p' > 4$ then

$$\|(S_k)_z 1\|_{p'} = \|P_+ M_{(B_k(a)-a)\circ\varphi_z} 1\|_{p'} \le c_{p'}(\|B_k(a)\|_\infty + \|a\|_\infty) \le 2c_{p'}\|a\|_\infty,$$

where $c_{p'}$ is the norm of the analytic projection P_+ acting on $L^{p'}(\mathbb{D})$ (see [21, p. 54]). Since $(S_k^*)_z = P_+ M_{\overline{(B_k(a)-a)}\circ\varphi_z}$ then also

$$\|(S_k^*)_z 1\|_{p'} \le 2c_{p'}\|a\|_\infty.$$

So, $\{S_k\}$ satisfies (5.12) and Lemma 5.6 then says that $\|S_k\|_{\mathfrak{L}(L^2_a)} \to 0$ as $k \to \infty$. $\square$

Remark 5.8 An obvious consequence of the theorem is that Theorems 4.5 and 4.9 hold for $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$. The argument of Theorem 5.7 works word by word for any $S \in \mathfrak{L}(L^2_a)$ such that $T_{B_k S} - S$ satisfies (5.12) for some $p' > 4$. So, $T_{B_k S} \to S$ for such S. Maybe this holds for every $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$, which would imply that $\mathfrak{T}(L^\infty(\mathbb{D}))$ is the closure of $\{T_a : a \in \mathcal{A}\}$.

6 Abelianization

Lemma 6.1 *Let $F \subset M(\mathcal{A}) \setminus \mathbb{D}$ be a closed saturated set, $\Omega \subset M(\mathcal{A})$ an open neighborhood of F and $k \ge 0$ an integer. Write $U = \Omega \cap \mathbb{D}$ and $\mathfrak{F} = \{a \in L^\infty(\mathbb{D}) : a \equiv 0 \text{ on } U\}$. Then*

$$B_k a \equiv 0 \;\text{ on } F \;\text{ for every } \; a \in \mathfrak{F}.$$

In particular, if $\mathcal{B}$ is a hyperbolic algebra and $F = \pi^{-1}(\Gamma_{\mathcal{B}})$ then $B_k a \in \mathcal{B}$ and $T_a \in \mathfrak{T}(\mathcal{B})$.

Proof. By Lemma 4.8 it is enough to prove the lemma for $k = 0$. Let $x \in F$ and take a net (z_α) in $\mathbb{D}$ such that $z_\alpha \to x$. We claim that for every $r \in (0,1)$ there is $\alpha_0 = \alpha_0(r)$ such that $\varphi_{z_\alpha}(r\mathbb{D}) \subset \Omega$ for $\alpha \ge \alpha_0$. Otherwise there is a subnet (z_{α_β}) and points $\xi_\beta \in r\mathbb{D}$ such that $\varphi_{z_{\alpha_\beta}}(r\mathbb{D}) \notin \Omega$ for all β. We can assume that $\xi_\beta \to \xi_0$, with $|\xi_0| \le r$. If $f \in \mathcal{A}$, the inequality

$$|f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) - f(\varphi_x(\xi_\beta))| \le |f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) - f(\varphi_{z_{\alpha_\beta}}(\xi_0))| + |f(\varphi_{z_{\alpha_\beta}}(\xi_0)) - f(\varphi_x(\xi_0))|$$

and the uniform ρ-continuity of f imply that $f(\varphi_{z_{\alpha_\beta}}(\xi_\beta)) \to f(\varphi_x(\xi_0))$. Therefore

$$\varphi_{z_{\alpha_\beta}}(\xi_\beta) \to \varphi_x(\xi_0) \in H(x) \subset F,$$

and since Ω is a neighborhood of F then $\varphi_{z_{\alpha_\beta}}(\xi_\beta) \in \Omega$ for $\beta \geq \beta_0$, a contradiction. So, if $a \in \mathfrak{F}$ and $0 < r < 1$, there is α_0 such that $(a \circ \varphi_{z_\alpha})(\omega) = 0$ for $|\omega| < r$ and $\alpha \geq \alpha_0$. Hence for $\alpha \geq \alpha_0$,

$$|(B_0 a)(z_\alpha)| \leq \int_D |(a \circ \varphi_{z_\alpha})(\omega)|\, dA(\omega) = \int_{D \setminus rD} |(a \circ \varphi_{z_\alpha})(\omega)|\, dA(\omega) \leq \|a\|_\infty (1 - r^2),$$

which can be made arbitrarily small by taking r close enough to 1. Therefore $(B_0 a)(z_\alpha) \to 0$, but since also $(B_0 a)(z_\alpha) \to (B_0 a)(x)$ then $(B_0 a)(x) = 0$, and this happens for all $x \in F$.

Now suppose that $F = \pi^{-1}(\Gamma_{\mathcal{B}})$, with $\mathcal{B}$ a hyperbolic algebra. Since $B_k a \in \mathcal{A}$ identically vanishes on $\pi^{-1}(\Gamma_{\mathcal{B}})$ then $B_k a \in \mathcal{B}$. Consequently $T_{B_k a} \in \mathfrak{T}(\mathcal{B})$, and since by Theorem 5.7, $T_{B_k a} \to T_a$ as $k \to \infty$, then so is T_a. $\square$

Let $F \subset M(\mathcal{A})$ be a closed set. A set $U \subset \mathbb{D}$ will be called a relative neighborhood of F if there is some open neighborhood $\Omega \subset M(\mathcal{A})$ of F such that $U = \Omega \cap \mathbb{D}$. Since the disk is dense in $M(\mathcal{A})$ and Ω is open, it is clear that $\overline{U}^{M(\mathcal{A})}$ contains Ω, and consequently it is a neighborhood of F. Also, for $V \subset \mathbb{D}$ we will denote $V^c = \mathbb{D} \setminus V$.

Lemma 6.2 *Let $S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}$, with $a_j^i \in L^\infty(\mathbb{D})$ for $1 \leq i \leq m$ and $1 \leq j \leq n_i$, and $F \subset M(\mathcal{A})$ a closed saturated set such that $B_0 S \equiv 0$ on F. Then given $\varepsilon > 0$ there exist relative neighborhoods U, V of F such that*

$$\left\| S - \left(\sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i \chi_{V^c}} \right) T_{\chi_{U^c}} \right\| < \varepsilon.$$

Proof. Without loss of generality we can assume that $\|a_j^i\|_\infty \leq 1$ for every i, j. By Proposition 5.4 there is a relative neighborhood U of F such that

$$\|S - S T_{\chi_{U^c}}\| = \|S T_{\chi_U}\| < \varepsilon. \tag{6.1}$$

By Lemma 6.1 and (4.2), for $1 \leq i \leq m$ each of the operators

$$S_k^i \stackrel{\text{def}}{=} \left(\prod_{j=k}^{n_i} T_{a_j^i} \right) T_{\chi_{U^c}}, \ 1 \leq k \leq n_i, \quad S_{n_i+1}^i = T_{\chi_{U^c}}$$

satisfies $B_0 S_k^i = 0$ on F. Hence, a new use of Proposition 5.4 provides a relative neighborhood V of F such that

$$\|T_{a_k^i \chi_V} S_{k+1}^i\| \leq \varepsilon$$

for every $1 \leq i \leq m$ and $1 \leq k \leq n_i$. Indeed, the proposition says that there are relative neighborhoods V_k^i of F that satisfy the inequality for each i and k, but it also says that

their intersection satisfies the inequality. Therefore

$$\begin{aligned}
&\|T_{a_1^i\chi_{V^c}}\dots T_{a_{k-1}^i\chi_{V^c}}S_k^i - T_{a_1^i\chi_{V^c}}\dots T_{a_k^i\chi_{V^c}}S_{k+1}^i\| \\
&\qquad\qquad\qquad = \|T_{a_1^i\chi_{V^c}}\dots T_{a_{k-1}^i\chi_{V^c}}T_{a_k^i}S_{k+1}^i - T_{a_1^i\chi_{V^c}}\dots T_{a_k^i\chi_{V^c}}S_{k+1}^i\| \\
&\qquad\qquad\qquad \le \|T_{a_1^i\chi_{V^c}}\dots T_{a_{k-1}^i\chi_{V^c}}\|\,\|(T_{a_k^i} - T_{a_k^i\chi_{V^c}})S_{k+1}^i\| \\
&\qquad\qquad\qquad \le \|T_{a_k^i\chi_V}S_{k+1}^i\| < \varepsilon,
\end{aligned}$$

which leads to

$$\begin{aligned}
&\|T_{a_1^i}\dots T_{a_{n_i}^i}T_{\chi_{U^c}} - T_{a_1^i\chi_{V^c}}\dots T_{a_{n_i}^i\chi_{V^c}}T_{\chi_{U^c}}\| \\
&\qquad\qquad \le \sum_{k=1}^{n_i}\|T_{a_1^i\chi_{V^c}}\dots T_{a_{k-1}^i\chi_{V^c}}S_k^i - T_{a_1^i\chi_{V^c}}\dots T_{a_k^i\chi_{V^c}}S_{k+1}^i\| \;<\; n_i\,\varepsilon.
\end{aligned}$$

Therefore

$$\left\|\left(\sum_{i=1}^m\prod_{j=1}^{n_i}T_{a_j^i}\right)T_{\chi_{U^c}} - \left(\sum_{i=1}^m\prod_{j=1}^{n_i}T_{a_j^i\chi_{V^c}}\right)T_{\chi_{U^c}}\right\| \le \sum_{i=1}^m n_i\,\varepsilon.$$

Since $ST_{\chi_{U^c}} = (\sum_{i=1}^m\prod_{j=1}^{n_i}T_{a_j^i})T_{\chi_{U^c}}$ and $\varepsilon > 0$ is arbitrary, the lemma follows from (6.1) and the above inequality. $\square$

If $\mathcal{B} \subset L^\infty(\mathbb{D})$ is a subalgebra, we write $\mathfrak{C}_0(\mathcal{B})$ for the bilateral ideal of $\mathfrak{T}_0(\mathcal{B})$ generated by commutators $[T_a, T_b] = T_aT_b - T_bT_a$, with $a, b \in \mathcal{B}$. Therefore, $\mathfrak{C}(\mathcal{B})$ is the closure of $\mathfrak{C}_0(\mathcal{B})$ in $\mathfrak{L}(L_a^2)$.

Lemma 6.3 *Let $\mathcal{B}$ be a hyperbolic algebra. If $S \in \mathfrak{C}_0(L^\infty(\mathbb{D}))$ is such that $B_0S \in \mathcal{B}$ and $\widehat{B_0S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ then $S \in \mathfrak{C}(\mathcal{B})$.*

Proof. By hypothesis

$$S = \sum_{i=1}^m T_{b_1^i}\dots T_{b_{n_i}^i}\,[T_{a_1^i}, T_{a_2^i}]\,T_{c_1^i}\dots T_{c_{k_i}^i},$$

where n_i, k_i and m are some positive integers and all the symbols are in $L^\infty(\mathbb{D})$. If $\widehat{B_0S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ Lemma 6.2 says that given $\varepsilon > 0$ there are relative neighborhoods U, V of $\Gamma_{\mathcal{B}}$ such that if

$$R = \sum_{i=1}^m T_{b_1^i\chi_{V^c}}\dots T_{b_{n_i}^i\chi_{V^c}}\,[T_{a_1^i\chi_{V^c}}, T_{a_2^i\chi_{V^c}}]\,T_{c_1^i\chi_{V^c}}\dots T_{c_{k_i}^i\chi_{V^c}}T_{\chi_{U^c}}$$

then $\|S - R\| < \varepsilon$. By Lemma 6.1 every Toeplitz operator involved in the last expression is in $\mathfrak{T}(\mathcal{B})$. So, $R \in \mathfrak{C}(\mathcal{B})$ and then so is S. $\square$

It is well known that if $\mathcal{B}$, $\mathcal{D}$ are C^*-algebras and ϕ is a $*$-homomorphism from $\mathcal{B}$ to $\mathcal{D}$, then $\|\phi\| \le 1$ and ϕ is an isometry if and only if ϕ is one-to-one [13, p. 100].

Theorem 6.4 *If $\mathcal{B}$ is a hyperbolic algebra then*

(1) $\mathfrak{C}(\mathcal{B}) = \{S \in \mathfrak{T}(\mathcal{B}) : \widehat{B_0S}^{\mathcal{B}} \equiv 0$ *on* $\Gamma_{\mathcal{B}}\} = \{S \in \mathfrak{T}(\mathcal{B}) : \widehat{S}^{\mathcal{B}}_y = 0$ *for all* $y \in \Gamma_{\mathcal{B}}\}$.

(2) $S - T_{B_0S} \in \mathfrak{C}(\mathcal{B})$ *for every* $S \in \mathfrak{T}(\mathcal{B})$.

(3) *The C^*-algebras $\mathfrak{T}(\mathcal{B})/\mathfrak{C}(\mathcal{B})$ and $C(\Gamma_{\mathcal{B}})$ are isomorphic via* $\phi : S + \mathfrak{C}(\mathcal{B}) \mapsto \widehat{B_0S}^{\mathcal{B}}|_{\Gamma_{\mathcal{B}}}$.

Proof. (1). The equality of the last two sets follows from Corollary 4.7. Suppose first that $S \in \mathfrak{C}_0(\mathcal{B})$, so $S = \sum_{1\le i\le n} A_i[T_{a_i}, T_{b_i}]B_i$, where $a_i, b_i \in \mathcal{B}$ and $A_i, B_i \in \mathfrak{T}_0(\mathcal{B})$. If $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$ then $a_i \circ \varphi_x$ and $b_i \circ \varphi_x$ are constant functions for all $1 \le i \le n$. By (4.2) then

$$S_x = \sum_{1\le i\le n} (A_i)_x \,[T_{a_i\circ\varphi_x}, T_{b_i\circ\varphi_x}]\,(B_i)_x = 0.$$

Since every $S \in \mathfrak{C}(\mathcal{B})$ can be approximated by operators of this form, then $S_x = 0$ for every $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. By Corollary 4.7 then $B_0S \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$, which is another way to say that $\widehat{B_0S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$. This proves the inclusion of the first set into the second one.

Suppose now that $S \in \mathfrak{T}(\mathcal{B})$ and $\widehat{B_0S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$. We can assume that $\|S\| = 1$. Let $0 < \varepsilon < 1$ and take $Q \in \mathfrak{T}_0(\mathcal{B})$ such that $\|Q - S\| < \varepsilon$. Since $Q \in \mathfrak{T}(\mathcal{B})$ then $\widehat{Q}^{\mathcal{B}}_y = \lambda I$ and $\widehat{(B_0Q)}^{\mathcal{B}}(y) = \lambda$ for every $y \in \Gamma_{\mathcal{B}}$, where $\lambda \in \mathbb{C}$ depends on y. Thus

$$(\widehat{T}^{\mathcal{B}}_{B_0Q})_y = \lim_{z\longrightarrow y} T_{(B_0Q)\circ\varphi_z} = T_{\widehat{(B_0Q)}^{\mathcal{B}}(y)} = \lambda I.$$

Then $B_0(Q - T_{B_0Q})^{\widehat{}\,\mathcal{B}} \equiv 0$ on Γ_B by Corollary 4.7, and since $\widehat{B_0S}^{\mathcal{B}} \equiv 0$ on Γ_B then $\widehat{B_0(T_{B_0S})}^{\mathcal{B}} \equiv 0$ on Γ_B by the same corollary. So, if $S_1 = Q - T_{B_0Q} + T_{B_0S}$ then $\widehat{B_0S_1}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ and

$$\|S_1 - S\| \le \|Q - S\| + \|T_{B_0S} - T_{B_0Q}\| \le 2\|Q - S\| < 2\varepsilon. \tag{6.2}$$

In [20, Thm. 1.1] it is proved that $\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{T}(L^\infty(\mathbb{D}))$, so it contains the identity I. Since Theorem 5.7 implies that $\mathfrak{C}(L^\infty(\mathbb{D})) = \mathfrak{C}(\mathcal{A})$ then $I \in \mathfrak{C}(\mathcal{A})$. Consequently there is $R \in \mathfrak{C}_0(\mathcal{A})$ such that $\|R - I\| < \varepsilon$. Thus

$$\|RS_1 - S_1\| \le \|R - I\|\,\|S_1\| < \varepsilon(\|S\| + 2\varepsilon) < 3\varepsilon. \tag{6.3}$$

Since $B_0S_1 \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$, Corollary 4.7 says that $S_x = 0$ for all $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$. By (4.2) then $(RS_1)_x = R_x(S_1)_x = 0$ for all $x \in \pi^{-1}(\Gamma_{\mathcal{B}})$, which means that $B_0(RS_1) \in \mathcal{B}$ and $\widehat{B_0(RS_1)}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$. But since $R \in \mathfrak{C}_0(\mathcal{A})$ and $S_1 \in \mathfrak{T}_0(\mathcal{A})$ then $RS_1 \in \mathfrak{C}_0(\mathcal{A})$, which together with Lemma 6.3 gives $RS_1 \in \mathfrak{C}(\mathcal{B})$. By (6.2) and (6.3), $\|RS_1 - S\| < 5\varepsilon$ and (1) follows.

(2). Let $y \in \Gamma_{\mathcal{B}}$. Since $S \in \mathfrak{T}(\mathcal{B})$ then $\widehat{S}^{\mathcal{B}}_y = \lambda I$. Thus $\widehat{(B_0S)}^{\mathcal{B}}(y) = \lambda$ and $(\widehat{T}^{\mathcal{B}}_{B_0S})_y = T_{\widehat{(B_0S)}^{\mathcal{B}}(y)} = \lambda I$. The result then follows from (1).

(3). By (1) the map ϕ is well-defined and one-to-one. It is clear that ϕ is $*$-linear. Suppose that $S, T \in \mathfrak{T}(\mathcal{B})$ and $y \in \Gamma_{\mathcal{B}}$. Then $\widehat{S}_y^{\mathcal{B}} = \lambda_S I$ and $\widehat{T}_y^{\mathcal{B}} = \lambda_T I$ for some $\lambda_S, \lambda_T \in \mathbb{C}$ that depend on y. Hence

$$\begin{aligned} \widehat{B_0(ST)}^{\mathcal{B}}(y) &= \lim_{z \longrightarrow y} \langle S_z T_z 1, 1 \rangle \;=\; \langle \widehat{S}_y^{\mathcal{B}} \widehat{T}_y^{\mathcal{B}} 1, 1 \rangle \\ &= \langle \lambda_S \lambda_T 1, 1 \rangle \;=\; \lambda_S \lambda_T \;=\; \widehat{(B_0 S)}^{\mathcal{B}}(y)\, \widehat{(B_0 T)}^{\mathcal{B}}(y), \end{aligned}$$

and ϕ is multiplicative. If $f \in C(\Gamma_{\mathcal{B}})$ we can extend f to a continuous function F on $M(\mathcal{B})$. Therefore $F \in \mathcal{B}$ and $\phi(T_F + \mathfrak{C}(\mathcal{B})) = \widehat{B_0 F}^{\mathcal{B}}|_{\Gamma_{\mathcal{B}}} = f$. So, ϕ is onto. $\square$

Theorem 6.5 *Let $\mathcal{B}$ be a hyperbolic algebra and $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$. Then*

(1) *$S \in \mathfrak{T}(\mathcal{B})$ if and only if $B_0 S \in \mathcal{B}$.*

(2) *$S \in \mathfrak{C}(\mathcal{B})$ if and only if $\widehat{B_0 S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$.*

Proof. (1). We know the necessity from Theorem 4.9. Suppose that $S = \sum_{i=1}^m \prod_{j=1}^{n_i} T_{a_j^i}$, where all $a_j^i \in L^\infty(\mathbb{D})$, and $B_0 S \in \mathcal{B}$. Then $T_{B_0 S} \in \mathfrak{T}(\mathcal{B})$ and $B_0(S - T_{B_0 S})^{\widehat{}\mathcal{B}} \equiv 0$ on Γ_B. Consequently Lemma 6.2 tells us that given $\varepsilon > 0$ there are relative neighborhoods U, V of $\Gamma_{\mathcal{B}}$ such that

$$\|S - T_{B_0 S} - \sum_{i=1}^{m} \prod_{j=1}^{n_i} T_{a_j^i \chi_{V^c}} T_{\chi_{U^c}} + T_{(B_0 S)\chi_{V^c}} T_{\chi_{U^c}}\| < \varepsilon.$$

By Lemma 6.1, $T_{a_j^i \chi_{V^c}}$, $T_{\chi_{U^c}}$, $T_{(B_0 S)\chi_{V^c}} \in \mathfrak{T}(\mathcal{B})$ for all $1 \le i \le m$ and $1 \le j \le n_i$. Therefore $S \in \mathfrak{T}(\mathcal{B})$.

(2). The necessity follows from (1) of Theorem 6.4. For the sufficiency, observe that it is implicit in the condition $\widehat{B_0 S}^{\mathcal{B}} \equiv 0$ on $\Gamma_{\mathcal{B}}$ that $B_0 S \in \mathcal{B}$. By the previous assertion then $S \in \mathfrak{T}(\mathcal{B})$. So, (1) of Theorem 6.4 says that $S \in \mathfrak{C}(\mathcal{B})$. $\square$

If $\mathcal{B}$ is a hyperbolic algebra and $a \in \mathcal{A}$, then $a \in \mathcal{B}$ if and only if $B_0 a \in \mathcal{B}$. Therefore the theorem says that $T_a \in \mathfrak{T}(\mathcal{B})$ if and only if $a \in \mathcal{B}$ and that $T_a \in \mathfrak{C}(\mathcal{B})$ if and only if $a \equiv 0$ on $\pi^{-1}(\Gamma_{\mathcal{B}})$.

The algebra $C(\overline{\mathbb{D}})$, of continuous functions on the closed disk is hyperbolic, its maximal ideal space identifies with $\overline{\mathbb{D}}$, and it is immediate that $\Gamma_{C(\overline{\mathbb{D}})} = \partial \mathbb{D}$ via this identification. Since by Coburn's theorem $\mathfrak{C}(C(\overline{\mathbb{D}}))$ is the ideal of compact operators, then part (2) of the theorem says that $S \in \mathfrak{T}_0(L^\infty(\mathbb{D}))$ is compact if and only if $(B_0 S)(z) \to 0$ as $|z| \to 1^-$. That is, we recover the theorem of Axler and Zheng [4, Thm. 2.2]. It is clear that the above condition is equivalent to $S_x = 0$ for all $x \in M(\mathcal{A}) \setminus \mathbb{D}$, or what is the same, $S_z \to 0$ in the SOT-topology when $|z| \to 1$.

7 Applications

7.1 Continuous functions up to a boundary set

Suppose that $E \subset \partial\mathbb{D}$ is a closed set and consider the algebra C_E formed by the functions of $\mathcal{A}$ that extend continuously to E. Then C_E is a hyperbolic algebra. If $id \in \mathcal{A}$ denotes the function $id(z) = z$ and for $\lambda \in \partial\mathbb{D}$ we write

$$M_\lambda = \{x \in M(\mathcal{A}) : id(x) = \lambda\}$$

for the fiber of λ over $M(\mathcal{A})$, then $M(C_E)$ consists of $M(\mathcal{A})/\sim$, where $\sim$ is the equivalence relation that collapses M_λ to a single point (depending on λ) for each $\lambda \in E$. Thus, Γ_{C_E} can be identified with E. Theorem 6.4 then says that

$$\mathfrak{C}(C_E) = \{S \in \mathfrak{T}(C_E) : \lim_{z \longrightarrow E} (B_0 S)(z) = 0\} \quad \text{and} \quad \mathfrak{T}(C_E)/\mathfrak{C}(C_E) \simeq C(E).$$

As mentioned before, when $E = \partial\mathbb{D}$, the above isomorphism is part of Coburn's theorem. Now consider the algebra CL^∞_E formed by the functions in $L^\infty(\mathbb{D})$ that extend continuously to E. Since $CL^\infty_E \not\subset \mathcal{A}$, it is not a hyperbolic algebra. So, at a first sight it is not possible to apply our results to this algebra. Fortunately, Theorem 5.7 gives us a way to overcome this apparent difficulty. In fact, it is easy to prove that if $f \in CL^\infty_E$ then $B_k f \in C_E$ for every $k \geq 0$ and $(B_k f)(\lambda) = f(\lambda)$ for $\lambda \in E$. By Theorem 5.7 then $\mathfrak{T}(C_E) = \mathfrak{T}(CL^\infty_E)$ and $\mathfrak{C}(C_E) = \mathfrak{C}(CL^\infty_E)$.

7.2 The McDonald-Sundberg Theorem

Let $\mathcal{U}$ be the C^*-subalgebra of $L^\infty(\mathbb{D})$ generated by $H^\infty = \{f \in L^\infty(\mathbb{D}) : f \text{ is analytic}\}$. The celebrated corona theorem of Carleson [10] states that $\mathbb{D}$ is dense in $M(H^\infty)$, the maximal ideal space of H^∞. This translates into the alternative description of $\mathcal{U}$ as $C(M(H^\infty))$. Since Schwarz Lemma implies that $H^\infty \subset \mathcal{A}$ then $\mathcal{U} \subset \mathcal{A}$. Therefore $\mathcal{U}$ is a prehyperbolic algebra and we aim to prove that it is hyperbolic.

Clearly, every interpolating sequence for H^∞ is interpolating for $\mathcal{U}$. The interpolating sequences for H^∞ were characterized by Carleson in [9]. Suppose that $x \in M(H^\infty) \setminus \mathbb{D}$ is in the closure of some interpolating sequence $\{z_n\}$ for H^∞, where we can assume that $z_n \neq 0$ for all $n \geq 1$. It is known that the infinite product

$$b(\omega) = \prod_{n \geq 1} \frac{|z_n|}{z_n} \varphi_{z_n}(\omega)$$

represents a function $b \in H^\infty$ such that $b(z_n) = 0$ for all $n \geq 1$. This b is called an interpolating Blaschke product. We also know (see [15, p. 404]) that if $\delta \in (0, 1)$ then there

is $\varepsilon(\delta) > 0$ such that

$$|b(\omega)| \geq \varepsilon(\delta) \quad \text{for every} \quad \omega \in \mathbb{D} \setminus \bigcup_{n \geq 1} K(z_n, \delta).$$

Thus x satisfies condition (b_2) of Proposition 3.9. On the other hand, if $x \in M(H^\infty) \setminus \mathbb{D}$ is not in the closure of any interpolating sequence for H^∞, it is known that for every net (z_α) in $\mathbb{D}$ that tends to x,

$$f \circ \varphi_{z_\alpha} \to \lambda \in \mathbb{C}$$

uniformly on compact sets for every $f \in H^\infty$ (see [15, Ch. X]). Since $\mathcal{U}$ is the C^*-algebra generated by H^∞ the same holds for every $f \in \mathcal{U}$. Thus x satisfies (a_2) of Proposition 3.8. Consequently Corollary 3.10 tells us that $\mathcal{U}$ is hyperbolic and that $\Gamma_{\mathcal{U}}$ is formed by the points $x \in M(H^\infty)$ that are not in the closure of any interpolating sequence for H^∞. Such points are usually called 'trivial points' because they can be characterized as the $x \in M(H^\infty)$ whose Gleason part (with respect to H^∞) is just $\{x\}$. For the definition and further information on Gleason parts the reader may consult the original paper of Hoffman [16] or Garnett's book [15, Ch. X].

Theorem 6.4 now tells us that $\mathfrak{T}(\mathcal{U})/\mathfrak{C}(\mathcal{U}) \simeq C(\Gamma_{\mathcal{U}})$, a result obtained by McDonald and Sundberg in [17]. Theorem 6.4 also says that $\mathfrak{C}(\mathcal{U}) = \{S \in \mathfrak{T}(\mathcal{U}) : \widehat{B_0 S}^{\mathcal{U}} \equiv 0 \text{ on } \Gamma_{\mathcal{U}}\}$ and $S - T_{B_0 S} \in \mathfrak{C}(\mathcal{U})$, which are recent additions to the McDonald-Sundberg Theorem discovered by Axler and Zheng [5].

7.3 The algebra of nontangential limits

Consider the algebra $\mathcal{N} = \{f \in \mathcal{A} : f \text{ has nontangential limits a.e. on } \partial\mathbb{D}\}$. It is clear that $\mathcal{N}$ is prehyperbolic, and we are going to use Corollary 3.10 to show that it is hyperbolic. To do so we need to characterize the interpolating sequences for $\mathcal{N}$. For $u \in \partial\mathbb{D}$ and $0 < \alpha < \pi/2$ let $\Lambda_\alpha(u) = \{u - \omega : |\arg\omega - \arg u| < \alpha, \text{ and } 0 < |u - \omega| < 1\}$ be an angular region with vertex u of total opening 2α. If $V \subset \mathbb{D}$ set

$$\mathrm{NT}_\alpha(V) = \{u \in \partial\mathbb{D} : u \in \overline{V \cap \Lambda_\alpha(u)}\} \quad \text{and} \quad \mathrm{NT}(V) = \bigcup_{0<\alpha<\pi/2} \mathrm{NT}_\alpha(V).$$

Geometrically, $\mathrm{NT}(V)$ is the subset of $\partial\mathbb{D}$ that can be approached nontangentially by points of V. If $u \in \partial\mathbb{D}$, $0 < r < 1$ and $0 < \alpha < \pi/2$, there is some $0 < \beta < \pi/2$ depending on α and r such that the r-pseudohyperbolic neighborhood of $\Lambda_\alpha(u)$ is contained in $\Lambda_\beta(u)$. Thus

$$\mathrm{NT}(V) = \mathrm{NT}(\{z \in \mathbb{D} : \rho(z, V) \leq r\}). \tag{7.1}$$

We write $|E|$ for the one-dimensional Lebesgue measure of $E \subset \partial\mathbb{D}$.

Lemma 7.1 *A separated sequence $\mathcal{S} = \{z_n\}$ is interpolating for $\mathcal{N}$ if and only if $|NT(\mathcal{S})| = 0$. If that is the case, for any $r > 0$ sufficiently small there exists $f \in \mathcal{N}$ that separates $\mathcal{S}$ from $\mathbb{D} \setminus \cup_{n\geq 1} K(z_n, r)$.*

Proof. Suppose that $|NT(\mathcal{S})| = 0$ and $\rho(z_n, z_m) \geq \delta > 0$ for $n \neq m$. By (7.1) then $|\mathrm{NT}(\bigcup_{n\geq 1} K(z_n, \delta/4)| = 0$. Take $f \in \mathcal{A}$ such that $f(z_n) = 1$ for all n and $f \equiv 0$ on $\mathbb{D} \setminus \bigcup_{n\geq 1} K(z_n, \delta/4)$. So, f has null nontangential limit a.e. on $\partial\mathbb{D}$. Thus $f \in \mathcal{N}$ and separates $\mathcal{S}$ from $\mathbb{D} \setminus \bigcup_{n\geq 1} K(z_n, \delta/4)$. If $\{\eta_n\}$ is an arbitrary sequence and we take $g \in \mathcal{A}$ such that $g(z_n) = \eta_n$ for every n then $fg \in \mathcal{N}$ and $f(z_n)g(z_n) = \eta_n$ for every n. So, $\mathcal{S}$ is interpolating for $\mathcal{N}$.

Now suppose that $|\mathrm{NT}(\mathcal{S})| > 0$. If $0 < \alpha_k < \alpha_{k+1} \to \pi/2$ is a strictly increasing sequence, then $\mathrm{NT}(\mathcal{S}) = \bigcup_k \mathrm{NT}_{\alpha_k}(\mathcal{S})$. So, there is some $\alpha_k = \alpha$ such that $|\mathrm{NT}_\alpha(\mathcal{S})| > 0$, and consequently there exists a compact set $E \subset \mathrm{NT}_\alpha(\mathcal{S})$ of positive measure. That is, $u \in \overline{\Lambda_\alpha(u) \cap \mathcal{S}}$ for every $u \in E$. So, if $u \in E$ there is some $z_n \in \Lambda_\alpha(u) \cap \mathcal{S}$. Since $\Lambda_\alpha(u)$ is open, it is geometrically clear that there is a an open neighborhood I_u of u in $\partial\mathbb{D}$ such that $z_n \in \Lambda_\alpha(v) \cap \mathcal{S}$ for every $v \in I_u$. By the compactness of E there is a finite set $\mathcal{R}_1$ in $\mathcal{S}$ such that $\Lambda_\alpha(u) \cap \mathcal{R}_1 \neq \emptyset$ for every $u \in E$. If $r_1 = \max\{|z| : z \in \mathcal{R}_1\}$ and $\mathcal{S}_1 = \{z \in \mathcal{S} : |z| \leq r_1\}$ then we also have $\Lambda_\alpha(u) \cap \mathcal{S}_1 \neq \emptyset$ for every $u \in E$. We can repeat this process with $\mathcal{S} \setminus \mathcal{S}_1$ instead of $\mathcal{S}$ to obtain $r_2 \in (r_1, 1)$ such that if $\mathcal{S}_2 = \{z \in \mathcal{S} : r_1 < |z| \leq r_2\}$ then $\Lambda_\alpha(u) \cap \mathcal{S}_2 \neq \emptyset$ for every $u \in E$. We keep going to construct a sequence $0 < r_1 < \ldots < r_n < \ldots < 1$ such that if $\mathcal{S}_n = \{z \in \mathcal{S} : r_{n-1} < |z| \leq r_n\}$ then

$$\Lambda_\alpha(u) \cap \mathcal{S}_n \neq \emptyset \quad \text{for every} \quad u \in E. \tag{7.2}$$

The sequence $\{r_n\}$ must tend to 1 because if $r_n \leq r < 1$ for every n then $\{z : |z| \leq r\} \cap \mathcal{S}$ is infinite, which is not possible because $\mathcal{S}$ is separated. Now take

$$\mathcal{T}_1 = \bigcup_{j \text{ odd}} \mathcal{S}_j \quad \text{and} \quad \mathcal{T}_2 = \bigcup_{j \text{ even}} \mathcal{S}_j.$$

Since (7.2) holds for all $n \geq 1$ then $E \subset \mathrm{NT}_\alpha(\mathcal{T}_1) \cap \mathrm{NT}_\alpha(\mathcal{T}_2)$, and since $|E| > 0$, the interpolation problem

$$f(z_n) = \begin{cases} 1 & \text{for } z_n \in \mathcal{T}_1 \\ 0 & \text{for } z_n \in \mathcal{T}_2 \end{cases}$$

cannot be solved by a function with nontangential limits almost everywhere on E. □

Theorem 7.2 *The algebra $\mathcal{N}$ is hyperbolic. In addition, $y \in M(\mathcal{N})$ is in $G_\mathcal{N}$ if and only if y is in the closure of some interpolating sequence for $\mathcal{N}$.*

Proof. Let $y \in M(\mathcal{N})$. If y is in the closure of an interpolating sequence for $\mathcal{N}$ the previous lemma says that y satisfies condition (b_2) of Proposition 3.9, so $y \in G_\mathcal{N}$.

If y is not in the closure of an interpolating sequence for $\mathcal{N}$ and $\mathcal{S}$ is a separated sequence with $y \in \overline{\mathcal{S}}^{M(\mathcal{N})}$ then the lemma says that $|\mathrm{NT}(\mathcal{S})| > 0$. So, if $f \in \mathcal{N}$ there must be some point $u \in \mathrm{NT}(\mathcal{S})$ such that f has nontangential limit λ at u, and for some $\alpha \in (0, \pi/2)$, $u \in \overline{\Lambda_\alpha(u) \cap \mathcal{S}}$. Let $\{z_n\}$ be a subsequence in $\Lambda_\alpha(u) \cap \mathcal{S}$ that tends to u. If $0 < r < 1$ then the argument preceding (7.1) says that there is some $\beta = \beta(\alpha, r) \in (0, \pi/2)$ such that $\bigcup_n K(z_n, r) \subset \Lambda_\beta(u)$. So, $f(\varphi_{z_n}(\omega)) \to \lambda$ for $|\omega| \le r$ when $n \to \infty$. Thus y satisfies (a_3) of Proposition 3.8, and consequently $y \in \Gamma_{\mathcal{N}}$. By Corollary 3.10 then $\mathcal{N}$ is hyperbolic. □

The nontangential limit function of $f \in \mathcal{N}$ will be denoted $\tilde{f}$. So, $\tilde{f} \in L^\infty(\partial\mathbb{D})$. Also, we write $z \xrightarrow{\mathrm{nt}} u$ to indicate that z tends nontangentially to $u \in \partial\mathbb{D}$.

Lemma 7.3 *Let $f \in \mathcal{N}$. Then $\widehat{f}^{\mathcal{N}} \equiv 0$ on $\Gamma_{\mathcal{N}}$ if and only if $\tilde{f} = 0$.*

Proof. If there is $y \in \Gamma_{\mathcal{N}}$ such that $|\widehat{f}^{\mathcal{N}}(y)| = \delta > 0$ and $\mathcal{S}$ is a separated sequence such that $y \in \overline{\mathcal{S}}^{M(\mathcal{N})}$, then y is in the $M(\mathcal{N})$-closure of

$$\mathcal{S}_1 = \{z \in \mathcal{S} : |f(z)| > \delta/2\}.$$

Since $y \in \Gamma_{\mathcal{N}}$ then Theorem 7.2 and Lemma 7.1 imply that $|\mathrm{NT}(\mathcal{S}_1)| > 0$, and since $|\tilde{f}| \ge \delta/2$ for almost every point of $\mathrm{NT}(\mathcal{S}_1)$, the sufficiency holds.

Now suppose that $\tilde{f} \ne 0$, so there is some $\delta > 0$ such that $|\tilde{f}| > \delta$ on a set of positive measure. It is easy then to construct a separated sequence $\mathcal{S}$ such that $|\mathrm{NT}(\mathcal{S})| > 0$ and $|f(z)| > \delta/2$ for every $z \in \mathcal{S}$. The necessity will follow if we show that $\overline{\mathcal{S}}^{M(\mathcal{N})} \cap \Gamma_{\mathcal{N}} \ne \emptyset$, because for any y in the intersection we would have $|\widehat{f}^{\mathcal{N}}(y)| \ge \delta/2$. Since $\mathcal{N}$ is hyperbolic, if $\overline{\mathcal{S}}^{M(\mathcal{N})} \cap \Gamma_{\mathcal{N}} = \emptyset$ then $\overline{\mathcal{S}}^{M(\mathcal{N})} \subset G_{\mathcal{N}}$. So, Proposition 3.9 says that for every $y \in \overline{\mathcal{S}}^{M(\mathcal{N})} \setminus \mathcal{S}$ there is an interpolating sequence $\mathcal{T}_y$ for $\mathcal{N}$, such that $y \in \overline{\mathcal{T}_y}^{M(\mathcal{N})}$. Hence, for every $0 < r < 1$ the $M(\mathcal{N})$-closure of $\bigcup_{z \in \mathcal{T}_y} K(z, r)$ is a neighborhood of y (by Lemma 7.1). By the compactness of $\overline{\mathcal{S}}^{M(\mathcal{N})} \setminus \mathcal{S}$ there are finitely many interpolating sequences $\mathcal{T}_1, \ldots, \mathcal{T}_N$ for $\mathcal{N}$ such that the closure of

$$U \stackrel{\text{def}}{=} \bigcup_{1 \le j \le N} \bigcup_{z \in \mathcal{T}_j} K(z, r)$$

is a neighborhood of $\overline{\mathcal{S}}^{M(\mathcal{N})} \setminus \mathcal{S}$. Thus there is $0 < \varrho < 1$ so that $\mathcal{S} \cap \{z \in \mathbb{D} : |z| \ge \varrho\}$ is contained in U. Together with (7.1) this yields

$$\mathrm{NT}(\mathcal{S}) \subset \bigcup_{1 \le j \le N} \mathrm{NT}\Big(\bigcup_{z \in \mathcal{T}_j} K(z, r)\Big) = \bigcup_{1 \le j \le N} \mathrm{NT}(\mathcal{T}_j),$$

which is impossible because $|\mathrm{NT}(\mathcal{S})| > 0$ while $|\mathrm{NT}(\mathcal{T}_j)| = 0$ for $j = 1, \ldots, N$. □

Lemma 7.4 *If $S \in \mathfrak{T}(\mathcal{N})$ then for almost every $u \in \partial\mathbb{D}$ there is $\lambda(u) \in \mathbb{C}$ such that $S_z \xrightarrow{SOT} \lambda(u) I$ when $z \xrightarrow{nt} u$.*

Proof. Let $a \in \mathcal{N}$ and suppose that $u \in \partial\mathbb{D}$ is such that $a(z) \to \lambda \in \mathbb{C}$ when $z \stackrel{\rm nt}{\to} u$. If $0 < \alpha < \pi/2$ and $0 < r < 1$ there is $\beta = \beta(\alpha, r)$ in $(\alpha, \pi/2)$ such that $\varphi_z(\omega) \in \Lambda_\beta(u)$ when $z \in \Lambda_\alpha(u)$ and $|\omega| \le r$. Therefore $a \circ \varphi_z \to \lambda$ uniformly on $r\mathbb{D}$ when $z \to u$ inside $\Lambda_\alpha(u)$. Since r is arbitrary the convergence is uniform on compact sets, implying that $(T_a)_z = T_{a\circ\varphi_z} \to \lambda I$ in the $\mathrm{S}OT$-topology when $z \to u$ inside $\Lambda_\alpha(u)$. Since α is arbitrary and the product of operators is continuous with respect to the $\mathrm{S}OT$-topology, the lemma holds for every $S \in \mathfrak{T}_0(\mathcal{N})$. If $S \in \mathfrak{T}(\mathcal{N})$ take a sequence $\{S_n\}$ in $\mathfrak{T}_0(\mathcal{N})$ that converges to S. So, for every $n \ge 1$ there is a set $E_n \subset \partial\mathbb{D}$ of full measure such that $(S_n)_z \stackrel{\rm SOT}{\to} \lambda_n(u) I$ when $z \stackrel{\rm nt}{\to} u \in E_n$. Therefore the set $E = \cap E_n$ has full measure, and given $\varepsilon > 0$ there is $n_0 = n_0(\varepsilon)$ such that if $u \in E$,

$$|\lambda_n(u) - \lambda_m(u)| \le \lim_{z \stackrel{\rm nt}{\longrightarrow} u} \|(S_n)_z - (S_m)_z\| = \|S_n - S_m\| < \varepsilon \tag{7.3}$$

for all $n, m \ge n_0$. This implies that there is some $\lambda(u) \in \mathbb{C}$ such that $\lambda_n(u) \to \lambda(u)$ for every $u \in E$. If $f \in L^2_a$ has norm 1, $u \in E$ and $n \ge n_0$, (7.3) yields

$$\begin{aligned} \|S_z f - \lambda(u) f\| &\le \|S_z f - (S_n)_z f\| + \|(S_n)_z f - \lambda_n(u) f\| + |\lambda_n(u) - \lambda(u)|\,\|f\| \\ &\le \|S - S_n\| + |\lambda_n(u) - \lambda(u)| + \|(S_n)_z f - \lambda_n(u) f\| \\ &\le 2\varepsilon + \|(S_n)_z f - \lambda_n(u) f\| \to 2\varepsilon \end{aligned}$$

when $z \stackrel{\rm nt}{\to} u$. Thus $S_z f \to \lambda(u) f$ in L^2_a when $z \stackrel{\rm nt}{\to} u \in E$ and the lemma holds for S. $\square$

Theorem 7.5 $\mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \simeq L^\infty(\partial\mathbb{D})$ *and*

$$\begin{aligned} \mathfrak{C}(\mathcal{N}) &= \{S \in \mathfrak{T}(\mathcal{N}) : \widetilde{B_0 S} = 0\} &(7.4) \\ &= \{S \in \mathfrak{T}(\mathcal{N}) : S_z \stackrel{SOT}{\to} 0,\ \textit{when } z \stackrel{nt}{\to} u \textit{ for a.e. } u \in \partial\mathbb{D}\}. &(7.5) \end{aligned}$$

Proof. Equality (7.4) follows immediately from Theorem 6.4 and Lemma 7.3. By Lemma 7.4, for every $S \in \mathfrak{T}(\mathcal{N})$ there is a set $E_S \subset \partial\mathbb{D}$ of full measure and $\lambda_S : E_S \to \mathbb{C}$ such that

$$S_z \stackrel{\rm SOT}{\to} \lambda_S(u) I \quad \text{when} \quad z \stackrel{\rm nt}{\to} u \in E_S. \tag{7.6}$$

Then $(B_0 S)(z) = (B_0 S_z)(0) = \langle S_z 1, 1\rangle \to \lambda_S(u)$ when $z \stackrel{\rm nt}{\to} u \in E_S$, which means that $(\widetilde{B_0 S})(u) = \lambda_S(u)$ for every $u \in E_S$. This proves (7.5).

Let $\Phi : \mathfrak{T}(\mathcal{N})/\mathfrak{C}(\mathcal{N}) \to L^\infty(\partial\mathbb{D})$ given by $\Phi(S + \mathfrak{C}(\mathcal{N})) = \widetilde{B_0 S}$. By (7.4) Φ is well-defined and one-to-one. It is also clear that Φ is $*$-linear. To prove that Φ is multiplicative let $S, T \in \mathfrak{T}(\mathcal{N})$ and use (7.6) to obtain

$$\widetilde{B_0(ST)}(u) = \lim_{z \stackrel{\rm nt}{\longrightarrow} u} \langle S_z T_z 1, 1\rangle = \lambda_S(u)\lambda_T(u) = (\widetilde{B_0 S})(u)\,(\widetilde{B_0 T})(u)$$

for every $u \in E_S \cap E_T$. Hence ϕ is a $*$-homomorphism and we only need to show that it is onto. Let $a \in L^\infty(\partial\mathbb{D})$ and consider the Poisson integral

$$A(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 - |z|^2}{|1 - ze^{-it}|^2}\, a(e^{it})\, dt.$$

So, A is a bounded harmonic function such that $\tilde{A} = a$. Since A is uniformly continuous with respect to ρ then $A \in \mathcal{N}$. So, $T_A \in \mathfrak{T}(\mathcal{N})$ and $\Phi(T_A + \mathfrak{C}(\mathcal{N})) = \widetilde{B_0 T_A} = \widetilde{B_0 A} = \tilde{A} = a$. $\square$

Let $\mathcal{U}$ be the algebra of the McDonald-Sundberg Theorem. Since every $f \in H^\infty$ has nontangential limits a.e. then $\mathcal{U} \subset \mathcal{N} \subset \mathcal{A}$. Therefore

$$\mathfrak{C}(\mathcal{U}) \subset \mathfrak{C}(\mathcal{N}) \subset \mathfrak{C}(\mathcal{A}).$$

We shall show that both inclusions are proper. The function $a = \sin\left(\log \frac{1+|z|}{1-|z|}\right)$ is in $\mathcal{A}$ but has no nontangential limit at any point of $\partial\mathbb{D}$ [8]. Hence, $T_a \in \mathfrak{C}(\mathcal{A}) \setminus \mathfrak{T}(\mathcal{N})$.

The Shilov boundary of H^∞, denoted ∂H^∞, is the smallest closed set $F \subset M(H^\infty)$ such that $\|f\|_\infty = \sup_{x \in F} |\widehat{f}^{\mathcal{U}}(x)|$ for every $f \in H^\infty$. It is known that ∂H^∞ is properly contained in $\Gamma_{\mathcal{U}}$ [15, p. 438], and that a function $f \in \mathcal{U}$ satisfies $\widehat{f}^{\mathcal{U}} \equiv 0$ on ∂H^∞ if and only if its nontangential function vanishes a.e. on $\partial\mathbb{D}$ (see [3, Thm. 7] and [7, Coro. 1.3]). So, take $y \in \Gamma_{\mathcal{U}} \setminus \partial H^\infty$ and $f \in \mathcal{U}$ such that $\widehat{f}^{\mathcal{U}} \equiv 0$ on ∂H^∞ and $\widehat{f}^{\mathcal{U}}(y) = 1$. Since $f(z)$ has trivial nontangential limits almost everywhere then $T_f \in \mathfrak{C}(\mathcal{N})$ but since $\widehat{f}^{\mathcal{U}} \not\equiv 0$ on $\Gamma_{\mathcal{U}}$ then $T_f \notin \mathfrak{C}(\mathcal{U})$.

Let $\mathcal{N}L^\infty$ be the algebra of functions in $L^\infty(\mathbb{D})$ that have nontangential limits a.e. on $\partial\mathbb{D}$. From the paragraph preceding (7.1) easily follows that if $f \in \mathcal{N}L^\infty$ then $B_k f$ has the same nontangential limits as f a.e. on $\partial\mathbb{D}$ for every $k \geq 0$. Thus Theorem 5.7 tells us that $\mathfrak{T}(\mathcal{N}) = \mathfrak{T}(\mathcal{N}L^\infty)$ and $\mathfrak{C}(\mathcal{N}) = \mathfrak{C}(\mathcal{N}L^\infty)$. Moreover, let $E \subset \mathbb{D}$ be a set of positive measure. Then all of the above can be generalized (with similar proofs) for the algebras $\mathcal{N}L^\infty_E = \{f \in L^\infty(\mathbb{D}) : f \text{ has nontangential limits a.e. on } E\}$ and $\mathcal{N}_E = \mathcal{N}L^\infty_E \cap \mathcal{A}$. Hence, we obtain a version of Theorem 7.5, where $\mathcal{N}$ is replaced by $\mathcal{N}_E$ or $\mathcal{N}L^\infty_E$ and $\partial\mathbb{D}$ is replaced by E.

7.4 Constant on hyperbolic parts

Definition. If $F \subset M(\mathcal{A}) \setminus \mathbb{D}$ is a closed saturated set, define

$$\mathrm{CO}(F) = \{f \in \mathcal{A} : \ f|_F = \text{const.}\}.$$

and

$$\mathrm{COH}(F) = \{f \in \mathcal{A} : \ f|_{H(x)} = \text{const. for every } x \in F\}.$$

These notations stand for 'constant on F' and 'constant on hyperbolic parts of F', respectively. It is clear that $\mathrm{CO}(F)$ and $\mathrm{COH}(F)$ are hyperbolic algebras and that

$$F = \pi_1^{-1}(\Gamma_{\mathrm{CO}(F)}) = \pi_2^{-1}(\Gamma_{\mathrm{COH}(F)}),$$

where π_1 and π_2 are the projections from $M(\mathcal{A})$ onto the respective maximal ideal spaces. If $\mathcal{B}$ is a hyperbolic algebra and $\pi : M(\mathcal{A}) \to M(\mathcal{B})$ is the usual projection then

$$\{S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\} \subset \mathfrak{C}(\mathcal{B}) \subset \{S \in \mathfrak{T}(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\}, \tag{7.7}$$

where the first inclusion follows from Theorem 6.5 and the second from Theorem 6.4. Observe that since the first set contains $\mathfrak{C}_0(\mathcal{B})$, it is dense in $\mathfrak{C}(\mathcal{B})$. The significance of $\mathrm{CO}(F)$ and $\mathrm{COH}(F)$ is given by the following

Proposition 7.6 *Let $\mathcal{B}$ be a hyperbolic algebra and $F \subset M(\mathcal{A})$ be a closed saturated set. Then the following conditions are equivalent*

(1) $F = \pi^{-1}(\Gamma_{\mathcal{B}})$,

(2) $\mathfrak{C}(\mathcal{B}) = \mathfrak{C}(\mathrm{COH}(F))$,

(3) $\mathrm{CO}(F) \subset \mathcal{B} \subset \mathrm{COH}(F)$.

Proof. We prove first the equivalence between (1) and (2). If (1) holds then the comment following (7.7) says that $\{S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_F = 0\}$ is dense in both $\mathfrak{C}(\mathcal{B})$ and $\mathfrak{C}(\mathrm{COH}(F))$, so they must coincide. If (2) holds (7.7) implies that

$$\{S \in \mathfrak{T}_0(\mathcal{A}) : B_0 S|_{\pi^{-1}(\Gamma_{\mathcal{B}})} = 0\} \subset \{S \in \mathfrak{T}(\mathcal{A}) : B_0 S|_F = 0\}.$$

Therefore $F \subset \pi^{-1}(\Gamma_{\mathcal{B}})$, and a symmetrical argument gives the other inclusion, so (1) holds.

If (1) holds the functions of $\mathrm{CO}(F)$ are continuous on $M(\mathcal{B})$ and the functions of $\mathcal{B}$ are continuous on $M(\mathrm{COH}(F))$. Since these are all C^*-algebras, (3) holds. If (3) holds then

$$\mathfrak{C}(\mathrm{CO}(F)) \subset \mathfrak{C}(\mathcal{B}) \subset \mathfrak{C}(\mathrm{COH}(F)),$$

so the proof of (2) reduces to show that $\mathfrak{C}(\mathrm{CO}(F)) = \mathfrak{C}(\mathrm{COH}(F))$. But this equality is a special case of the equivalence between (1) and (2). □

Let us write COH for $\mathrm{COH}(M(\mathcal{A})\backslash\mathbb{D})$. In this case the last proposition says that $\mathfrak{C}(\mathrm{COH}) = \mathfrak{C}(C(\overline{\mathbb{D}}))$, and this is the ideal of compact operators $\mathcal{K}$. Then Theorem 6.4 tells us that $S - T_{B_0 S} \in \mathcal{K}$ for every $S \in \mathfrak{T}(\mathrm{COH})$. In particular, $\mathfrak{T}(\mathrm{COH})/\mathcal{K} = \{T_b + \mathcal{K} : b \in \mathrm{COH}\}$. The center of an algebra $\mathcal{B}$ is formed by the elements that commute with all the members of $\mathcal{B}$. Our next result relates $\mathfrak{T}(\mathrm{COH})/\mathcal{K}$ with the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$.

Suppose that $S \in \mathcal{K}$ and for $z \in \mathbb{D}$ let $k_z^{(0)} = (1 - |z|^2) K_z^{(0)}$. Since $\|k_z^{(0)}\| = 1$ and $k_z^{(0)} \to 0$ weakly as $|z| \to 1$, then $|(B_0 S)(z)| \le \|S k_z^{(0)}\| \to 0$ when $|z| \to 1$. Therefore $S_x = 0$ for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$.

Theorem 7.7 *Let $\mathfrak{J} = \{S \in \mathfrak{T}(L^\infty(\mathbb{D})) : S_x = 0$ for $x \in M(\mathcal{A}) \setminus \mathbb{D}\}$. Then*

$$\{T_b + \mathcal{K} : b \in \mathrm{COH}\} \subset \mathrm{Center}(\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}) \subset \{T_b + \mathfrak{J} : b \in \mathrm{COH}\}$$

Proof. We prove first that if $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$ and $b \in \mathrm{COH}$ then $[S, T_b] \in \mathcal{K}$. Let $S_n \in \mathfrak{T}_0(\mathcal{A})$ such that $S_n \to S$. Since $(S_n T_b - T_b S_n) \to (S T_b - T_b S)$ we can assume that $S \in \mathfrak{T}_0(\mathcal{A})$. By (4.2),

$$(ST_b - T_bS)_x = S_x(T_b)_x - (T_b)_x S_x \quad \text{for every} \quad x \in M(\mathcal{A}),$$

and since $(T_b)_x$ is a constant operator for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$, then $[S, T_b]_x = 0$ for $x \in M(\mathcal{A}) \setminus \mathbb{D}$. The comment after Theorem 6.5 then says that $[S, T_b]$ is compact. This proves that $\{T_b + \mathcal{K} : b \in \mathrm{COH}\}$ is contained in the center of $\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K}$.

Now suppose that $S \in \mathfrak{T}(L^\infty(\mathbb{D}))$ is such that $S + \mathcal{K} \subset \mathrm{Center}(\mathfrak{T}(L^\infty(\mathbb{D}))/\mathcal{K})$. This means that $ST_a - T_aS \in \mathcal{K}$ for every $a \in L^\infty(\mathbb{D})$. So, $S_x(T_a)_x - (T_a)_x S_x = 0$ for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$, or equivalently,

$$S_z(T_a)_z - (T_a)_z S_z \overset{\mathrm{SOT}}{\to} 0 \quad \text{as } |z| \to 1. \tag{7.8}$$

Let $x \in M(\mathcal{A}) \setminus \mathbb{D}$ and take a net (z_α) in $\mathbb{D}$ converging to x. The closed ball of center 0 and radius $\|S\|$ in $\mathfrak{L}(L^2_a)$ admits a metric d with the SOT-topology. Since $S_{z_\alpha} \overset{\mathrm{SOT}}{\to} S_x$ then for every integer $n \geq 1$ there is some point of the net, that we rename as z_n, such that $d(S_{z_n}, S_x) < 1/n$. So,

$$S_{z_n} \overset{\mathrm{SOT}}{\to} S_x. \tag{7.9}$$

If $\{r_n\}$ is a sequence in $(0,1)$ that tends to 1, we can assume (taking a subsequence of $\{z_n\}$ if needed) that $K(z_n, r_n) \cap K(z_j, r_j) = \emptyset$ if $n \neq j$. For an arbitrary $a \in L^\infty(\mathbb{D})$ consider the function

$$b(\omega) = \sum_{j \geq 1} (a \circ \varphi_{z_j})(\omega) \chi_{K(z_j, r_j)}(\omega).$$

Hence $(T_b)_{z_n} = T_{b \circ \varphi_{z_n}}$, where

$$\begin{aligned} (b \circ \varphi_{z_n})(\omega) &= a(\omega)\chi_{K(0,r_n)}(\omega) + \sum_{j:\, j \neq n} (a \circ \varphi_{z_j})(\varphi_{z_n}(\omega))\, \chi_{K(\varphi_{z_n}(z_j), r_j)}(\omega) \\ &= g_n(\omega) + h_n(\omega). \end{aligned}$$

Since the support of h_n is disjoint from $K(0, r_n) = r_n\mathbb{D}$ then $|h_n(\omega)| \leq \|a\|_\infty \chi_{D \setminus r_n D}(\omega)$ for all $\omega \in \mathbb{D}$. Since $r_n \to 1$, it is clear that $T_{h_n} \overset{\mathrm{SOT}}{\to} 0$ and $T_{g_n} \overset{\mathrm{SOT}}{\to} T_a$. Thus

$$(T_b)_{z_n} = T_{g_n} + T_{h_n} \overset{\mathrm{SOT}}{\to} T_a. \tag{7.10}$$

By (7.8) $S_{z_n}(T_a)_{z_n} - (T_a)_{z_n} S_{z_n} \overset{\mathrm{SOT}}{\to} 0$, which together with (7.9) and (7.10) gives $S_xT_a - T_aS_x = 0$. This means that S_x commutes with every Toeplitz operator with symbol in $L^\infty(\mathbb{D})$. By

[12, Thm. 10.28] then $S_x = \lambda I$ for some $\lambda \in \mathbb{C}$, and consequently $B_0 S \equiv \lambda$ on $H(x)$ by Corollary 4.7. Since $x \in M(\mathcal{A}) \setminus \mathbb{D}$ is arbitrary then $B_0 S \in \mathrm{COH}$ and

$$(S - T_{B_0 S})_x = S_x - T_{(B_0 S)\circ\varphi_x} = \lambda I - \lambda I = 0$$

for every $x \in M(\mathcal{A}) \setminus \mathbb{D}$. That is, $S - T_{B_0 S} \in \mathfrak{I}$. $\square$

The concept of center plays an important role when studying localizations of C^*-algebras (see [13, Th. 7.47]). I believe that the ideal $\mathfrak{I}$ in Theorem 7.7 is $\mathcal{K}$, so the inclusions of the theorem should be equalities. If $S \in \mathfrak{L}(L_a^2)$, the essential spectrum $\sigma_e(S)$ is the spectrum of $S + \mathcal{K}$ in the Calkin algebra $\mathfrak{L}(L_a^2)/\mathcal{K}$. Let $\sigma(S)$ denote the usual spectrum of S. Is it true that

$$\sigma_e(S) = \bigcup_{x \in M(\mathcal{A})\setminus\mathbb{D}} \sigma(S_x) \quad \text{for every } S \in \mathfrak{T}(L^\infty(\mathbb{D}))?$$

There is strong evidence to support an affirmative answer. This holds for $S \in \mathfrak{T}(\mathrm{COH})$, while the example preceding Lemma 4.8 shows that this fails for a general $S \in \mathfrak{L}(L_a^2)$. This example appeared in [4], where it is also shown that there is an infinite dimensional orthogonal projection P such that $B_0 P(z) \to 0$ when $|z| \to 1$. We do not know the answer even for a general Toeplitz operator with bounded symbol.

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