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Standard completeness of Hájek basic logic and decompositions of BL-chains

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Abstract The aim of this paper is to survey the tools needed to prove the standard completeness of Hájek Basic Logic with respect to continuous t-norms. In particular, decompositions of totally ordered BL-algebras into simpler components are considered in some detail.

1 Introduction

A propositional formula φ is called a t-*tautology* if whenever we evaluate the formula assigning to the variables in φ real numbers between 0 and 1, and we interpret the conjunction by a continuous t-norm and the implication by the corresponding residuum, we obtain the value 1. The *standard completeness theorem* for Hájek's Basic Logic is the assertion that φ *is deducible in Basic Logic if and only if* φ *is a* t-*tautology*. Our aim in this paper is to summarize the concepts and results needed to prove the standard completeness theorem.

It is shown in [14] that a propositional formula φ is deducible in Basic Logic if and only if the equation $\varphi = \top$ holds in all BL-algebras.[1]

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[1] More precisely, it can be shown that Basic Logic is *strongly algebraizable* in the sense of Blok–Pigozzi (see [3]), with *equivalent variety semantics* the class BL of all BL-algebras, *defining equation* $x = \top$ and *equivalent formula* $(x \to y) * (y \to x)$, that is, for any set $\Sigma \cup \{\varphi, \psi\}$ of BL-formulas, we have:

- $\Sigma \vdash_{\mathrm{BL}} \varphi$ iff $\{\psi = \top : \psi \in \Sigma\} \models_{\mathrm{BL}} \varphi = \top$, (i.e. for each valuation υ on a BL-algebra A, $\upsilon[\Sigma] \subseteq \{\top^A\}$ implies $\upsilon(\varphi) = \top^A$)
- $\varphi = \psi \dashv\models_{\mathrm{BL}} (\varphi \to \psi) * (\psi \to \varphi) = \top$,

The above is equivalent to that for each set of BL-equations Γ:

- $\Gamma \models_{\mathrm{BL}} \varphi = \psi$ iff $\{(\alpha \to \beta) * (\beta \to \alpha) : \alpha = \beta \in \Gamma\} \vdash_{\mathrm{BL}} (\varphi \to \psi) * (\psi \to \varphi)$,
- $\varphi \dashv\vdash_{BL} (\varphi \to \top) * (\top \to \varphi)$.

By a BL-chain we understand a BL-algebra which is totally ordered under the natural order defined by the implication. Since every BL-algebra is a subdirect product of BL-chains, an equation $\varphi = \top$ holds in all BL-algebras if and only if it holds in all BL-chains. A typical example of a BL-chain is provided by the real unit segment [0, 1] equipped with the BL-structure defined by a continuous t-norm. BL-chains of this kind are called t-*algebras* in [14]. In particular, the t-algebras corresponding to the Lukasiewicz, product and min t-norms are called *the standard MV-chain*, *the standard* Π-*chain*, and *the standard G-chain*, respectively. Now the standard completeness theorem for Basic Logic can be rephrased as follows: *an equation holds in all BL-chains if and only if it holds in all* t-*algebras*. The prove of the above statement depends on the structure of BL-chains.

The following classical result of Mostert and Shields [18, Sect. 5.4 Theorem B] gives the structure of t-algebras:

Theorem 1 *Let* $*$ *be a continuous* t-*norm on the real segment* [0, 1], *and let* $E = \{x \in [0, 1] : x * x = x\}$. *Then* E *is closed in the usual topology of* [0, 1] *and if* x, y *belong to* E, *then* $x * y = \min(x, y)$. *The complement of* E *is the union of disjoint intervals. Let* P *be the closure of one of these. Then* P *equipped with the restriction of* $*$ *is a BL-chain isomorphic to either the standard MV-chain or the standard* Π-*chain. Finally, if* $x \in P$ *and* $y \notin P$, *then* $x * y = \min(x, y)$.

Hájek [15] generalized Moster and Shields theorem for the case of BL-chains satisfying some conditions preventing the existence of certain 'pathological triples' (see Sect. 4). It was shown in [9] that these pathological triples cannot exist in any BL-chain. Therefore Hájek version of Moster and Shields theorem holds for arbitrary BL-chains. Then using the existence of partial embeddings from MV-chains, Π-chains and G-chains into the standard MV-chain, standard Π-chain and standard G-chain, respectively, the standard completeness theorem is obtained.

In Sects. 2 and 3, we sketch the path to obtain Hájek decomposition of BL-chains in terms of MV-algebras (corresponding to Lukasiewicz t-norm), Π-algebras (corresponding to the product t-norm) and Gödel – Heyting algebras (corresponding to minimum t-norm). In Sect. 4 we give the notion of partial embeddings and we review the main facts about the representation of MV and Π chains in terms of ordered abelian groups. The standard completeness theorem for Basic Logic is proved in Sect. 5. Finally, in the Appendix we compare Hájek decomposition with an alternative decomposition of BL-chains, in terms of hoops, that was obtained in [1] (see also [5]) and is summarized, together with some of its applications, in Montagna's contribution [17] in the present volume. We assume the reader familiar with Basic Logic and BL-algebras, as exposed in Hájek's monograph [14].

2 Ordinal sums of BL-chains

An element z of a BL-algebra A is said to be *idempotent* iff $z^2 = z * z = z$. The set of all idempotents of A will be denoted by $Idp\,(A)$.

Lemma 1 *Let A be a BL-algebra. The following conditions are equivalent for each $z \in A$:*

(i) $z \in Idp\,(A)$.
(ii) *For each $x \in A$, $z * x = z \wedge x$.*

Proof Suppose that $z^2 = z$. Then, for each $x \in A$:

$$z * (z \wedge x) = z * (z * (z \to x)) = z^2 * (z \to x) = z \wedge x.$$

Therefore, $z \wedge x = z * (z \wedge x) \le z * x$, and since we always have $z * x \le z \wedge x$, we have proved that (i) implies (ii). The converse implication is trivial. □

Remark 1 As an immediate consequence of the above lemma, one has that *if $z \in Idp\,(A)$, then, for each x, y in A, $z \le x \to y$ iff $z \wedge x \le y$.*

By a Gödel–Heyting chain (G-chain for short) we understand a totally ordered set C with bottom 0 and top 1 endowed with the BL-algebra structure given by the operations

$$x * y = \min(x, y), \quad \text{and} \quad x \to y = \begin{cases} 1 & \text{if } x \le y \\ y & \text{if } x > y \end{cases}.$$

In the light of the above remark, we have that *a BL-chain C is a G-chain iff $C = Idp\,(C)$.*

Let u, v be idempotents of a BL-algebra A such that $u \le v$. It follows from Lemma 1 that the segment $[u, v] := \{x \in A \mid u \le x \le v\}$ is closed under $*$. For all x, y in $[u, v]$, define

$$x \Rightarrow_{uv} y := (v \wedge (x \to y)) \vee u.$$

With this notations, we have the following:

Theorem 2 *Let A be a BL-algebra. If $u, v \in Idp\,(A)$ are such that $u \le v$, then the system $\langle [u, v], *, \Rightarrow_{uv}, u\rangle$ is a BL-algebra, that we shall denote by $[u, v]_A$. Moreover one has that $Idp\,([u, v]_A) = Idp\,(A) \cap [u, v]$.*

A crucial notion to understand the structure of BL-chains is the operation of *ordinal sum*, used to compose different BL-chains to get new BL-chains.

Let $\langle I, \le\rangle$ be a chain, with least element 0 and largest element 1. For each $i \in I$, let

$$i^+ = \begin{cases} \inf\{j \in I : i < j\}, & \text{if it exists;} \\ i, & \text{otherwise.} \end{cases}$$

Let $\{(C_i, *_i, \to_i, \bot_i, \top_i)\}_{i \in I}$ be a family of BL-chains such that $\top_i = \bot_{i^+}$. Then

$$\bigsqcup_{i \in I} C_i = \left(\bigcup_{i \in I} C_i, *, \to, \bot, \top\right) \tag{1}$$

where $\bot = \bot_0$, $\top = \top_1$ and

$$x * y = y * x = \begin{cases} x *_i y & \text{if } x, y \text{ are in } C_i, \\ x & \text{if } x \in C_i, y \in C_j \text{ and } i < j. \end{cases}$$

$$x \to y = \begin{cases} \top, & \text{if } x, y \in C_i, \text{ and } x \le_i y, \\ & \text{or } x \in C_i, y \in C_j \text{ and } i < j, \\ x \to_i y, & \text{if } x, y \in C_i, \text{ and } x > y \\ y, & \text{if } x \in C_i, y \in C_j \text{ and } i > j \end{cases}$$

is a BL-chain called *ordinal sum of the family* $\{C_\alpha\}_{\alpha \in I}$.

If $I = \{0, \ldots, n\}$, then we write $C_0 \sqcup \cdots \sqcup C_n$ in place of $\bigsqcup_{i \le n} C_i$.

Remark 2 In the above definition, if $i = i^+$, then $\top_i = \bot_{i^+} = \bot_i$ and $C_i = \{\top_i\}$. For instance, the real unit interval $[0, 1]$ considered as a G-chain is the ordinal sum $\bigsqcup_{r \in [0,1]} C_r$, where $C_r = [r, r] = \{r\}$ for each $r \in [0, 1]$.

Remark 3 If C, C_1, C_2 are BL-chains such that $C = C_1 \sqcup C_2$, then $\top_1 = \bot_2$ is an idempotent of C, and $C_1 = [\bot, \top_1]_C$, $C_=[\top_1, \top]_C$. Moreover, if $E = Idp\,(C)$, then $C = \bigsqcup_{i \in E} [i, i^+]_C$

Definition 1 A BL-chain C is called *irreducible* if there do not exist BL-chains C_1 and C_2 having at least two elements such that $C = C_1 \sqcup C_2$.

It follows from Remark 3 that *a BL-chain C is irreducible iff $Idp\,(C) = \{\bot, \top\}$*. Hence the standard MV-chain and the standard Π-chain are both irreducible. A G-chain C is irreducible iff C has at most two elements. The following example shows that there are other irreducible BL-chains.

Example 1 Let $C = [0, 2]$ with its natural order as a subset of $\boldsymbol{R}$ and endowed with the operations $*$ and $\to$ defined as follows:

$$x * y = \begin{cases} 1 + (x-1)(y-1) & \text{if } x, y \in (1, 2], \\ x & \text{if } x \in [0, 1] \text{ and } y \in (1, 2], \\ (x + y - 1) \vee 0 & \text{if } x, y \in [0, 1]. \end{cases}$$

$$x \to y = \begin{cases} 2 & \text{if } x \leq y, \\ (y - 1/x - 1) + 1 & \text{if } x > y \text{ and } x, y \in (1, 2], \\ y & \text{if } x > y, x \in (1, 2] \\ & \text{and } y \in [0, 1], \\ 1 - x + y & \text{if } x > y \text{ and } x, y \in [0, 1]. \end{cases}$$

It follows that $1 \in Idp\,(C)$ and $C = [0, 1] \sqcup [1, 2]$, where $[0, 1]$ is the standard MV-chain and $[1, 2]$ is a Π-algebra (isomorphic to the standard Π-chain). Let $A := C \setminus \{1\}$. It is easy to see that A is a subalgebra of C, and that $Idp\,(A) = \{0, 2\}$.

In the next section we are going to consider another property that, together with irreducibility, will allow us to characterize the standard MV and Π chains.

3 Saturated BL-chains

A pair (X, Y) is called a *cut* in a BL-chain C if the following conditions hold:

(C_1) $X \cup Y = C$,
(C_2) $x \leq y$, for all $x \in X$ and $y \in Y$,
(C_3) Y is closed under $*$, and
(C_4) $x * y = x$, for all $x \in X$ and $y \in Y$.

Example 2 Let C be a BL-chain, u an idempotent of C such that $\perp \leq u \leq \top$, $X = \{x \in C \mid x < u\}$, $\bar{X} = X \cup \{u\}$, $Y = \{x \in C \mid x > u\}$ and $\bar{Y} = Y \cup \{u\}$. Then $(\bar{X}, \bar{Y})$, $(X, \bar{Y})$ and $(\bar{X}, Y)$ are examples of cuts in C.

A proof of the next lemma can be found in [15].

Lemma 2 *The following properties hold true for any cut (X, Y) in a BL-chain C:*

(C_5) *If $X \cap Y \neq \emptyset$, then $X \cap Y = \{u\}$, where u is an idempotent of C,*
(C_6) *If $x \in X$ and $t \leq x$, then $t \in X$,*
(C_7) *X is closed under $*$: if x, y are in X, then $x * y \in X$,*
(C_8) *If $x \in X \setminus Y$ and $y \in Y \setminus X$, then $y \to x = x$.*

□

Let A be the BL-chain considered in Example 1. It is easy to see that $([0, 1), (1, 2])$ is a cut in A which is not determined by an idempotent of A. This example motivates the following definition.

Definition 2 A BL-chain C is called *saturated* if for each cut (X, Y) in C there is an idempotent $u \in C$ such that $x \leq u \leq y$ for all $x \in X$ and $y \in Y$.

A proof of the next lemma can be found in [9] (see also [15]).

Lemma 3 *If C is a saturated BL-chain and $E = Idp\,(C)$ is the set of idempotent elements for $*$, then:*

(i) *Any subset $A \subseteq E$ has* sup *and* inf *in C, and both of them belong to E.*
(ii) *For any $c \in E$ there exists a greatest closed interval $[a, b] \subseteq E$ such that $c \in [a, b]$.*
(iii) *For any $\alpha \notin E$ there exists a closed interval $[a, b]$ such that $\alpha \in [a, b]$ and $[a, b] \cap E = \{a, b\}$.*

□

We will denote by $I(E) = I(Idp\,(C))$ the set of intervals defined in (iii) of last lemma, that is $I(E) = \{[a, b] \mid a, b \in E, a \leq b, (a, b) \cap E = \emptyset\}$. Moreover, denote by $G(E)$ the set of proper (non singletons) intervals defined from (ii) and let $E_{\text{is}} = E \setminus (I(E) \cup G(E))$, the subscript is coming from *isolated*.

With this notation we have:

Corollary 1 *Let C be a saturated BL-chain and $\langle I, \leq \rangle$ be the totally ordered set defined by*

$$I = \{a \in C \mid a \in E_{\text{is}} \text{ or } \exists\, b \in C \text{ s.t. } [a, b] \in I(E) \cup G(E)\},$$

and $\leq$ being the restriction of the chain order on I. For each $a \in I$, let M_a be either $[a, a]$, if $a \in E_{\text{is}}$, or the corresponding algebra $[a, b]_C$ for each $[a, b] \in I(E) \cup G(E)$, otherwise. Then $C = \bigsqcup_{a \in I} M_a$, where $\bigsqcup$ is as in Eq. 1.

The next example, borrowed from [9], shows that E_{is} may be non-empty.

Example 3 Let, for each natural $n \leq 1$, $a_n = (1/2) - (1/n)$ and $b_n = (1/2) + (1/n)$. Obviously, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = (1/2)$. Consider the BL-chain C on the real unit interval $[0, 1]$ defined as the following ordinal sum

$$\left(\bigsqcup_{n \geq 1} [a_n, a_{n+1}]_C \right) \sqcup \left[\frac{1}{2}, \frac{1}{2} \right] \sqcup \left(\bigsqcup_{n \geq 1} [b_{n+1}, b_n]_C \right)$$

where $[a_n, a_{n+1}]_C$ and $[b_{n+1}, b_n]_C$ are either MV or product chains. In this case, $E_{\text{is}} = \{1/2\}$, while $G(E) = \emptyset$ and $I(E) = \{[a_n, a_{n+1}], [b_{n+1}, b_n] \mid n \geq 1\}$.

It is clear that for each $[a, b] \in G(E)$, $[a, b]_C$ is a G-chain. By decomposing it into its irreducible components, we have the following Hájek decomposition theorem of saturated BL-chains [15, Theorem 4]:

Corollary 2 *Each saturated BL-chain C is an ordinal sum of saturated irreducible BL-chains. Namely, $C = \bigsqcup_{\alpha \in Idp\,(C)} [\alpha, \alpha^+]_C$.*

Notice that the decompositions given in Corollaries 1 and 2 coincide in the components from $I(E)$ and E_{is}, but differ in the components from $G(E)$. Indeed, every component of $G(E)$ appears in Hájek decomposition as a sum of singletons, corresponding to points that have no upper neighbor, and of two element BL-chains corresponding to points with upper neighbor. When $[a, b] \in I(E)$, then $[a, b]_C$ is a saturated an irreducible BL-chain. In the next section we are going to investigate the structure of these chains.

4 Saturated and irreducible BL-chains

Remark 4 Let a, b be elements of a BL-chain. If $b \to a = a \leq \top$, then $b * a = a$. Indeed, $a = a \wedge b = b * (b \to a) = b * a$.

Given x, y, z elements of a BL-chain C, (x, y, z) is called *pathological triple* provided the following conditions are satisfied:

$x \leq z \leq y$

$x * y = x,$

$x * z \leq x,$

$z * y \leq z.$

Lemma 4 [9, Theorem 3.1] *A BL-chain cannot have pathological triples.*

Proof Suppose (x, y, z) is a pathological triple and let $s = x \to x * z$. Note that $s \leq y$ (because $s \geq y$ would imply $s * x \geq y * x = x \geq x * z$, a contradiction). Moreover:

$$y \to s = y \to (x \to x * z) = (y * x) \to x * z$$
$$= x \to x * z = s.$$

Hence, by Remark 4, one obtains that $y * s = s$. On the other hand, by residuation, $z \leq s$. Therefore:

$$z = s \wedge z = s * (s \to z) = (y * s) * (s \to z) = y * z \leq z,$$

a contradiction. □

From the above theorem and Lemmas 7 and 8 of [15] we obtain:

Lemma 5 *Each saturated and irreducible BL-chain satisfies the following properties:*

(i) $x * y = x$ *implies* $y = \top$ *or* $x = \bot$,
(ii) $x * z = y * z > \bot$ *implies* $x = y$.

We say that an element $x > \bot$ of a BL-chain C is a *zero divisor* provided that there is $y \in C$, $y > \bot$ such that $x * y = \bot$.

Lemma 6 *If C is a saturated and irreducible BL-chain with a zero divisor, then:*

(i) *All elements* $x \in C \setminus \{\bot, \top\}$ *are zero divisors,*
(ii) $\neg\neg x = x$ *for all* $x \in C$.

Theorem 3 *A saturated and irreducible BL-chain is either*

(i) *An MV-chain if it has a zero divisor, or*
(ii) *A* Π*-chain, otherwise.*

Proof Let C be an irreducible and saturated BL-chain. If C has a zero divisor, then by Lemma 6 C is an MV-algebra. Suppose now that C has no zero divisors. For each $x \in C$, $\bot < x$ implies $\neg x = \bot$, hence $\neg x \wedge x = \bot$. Besides, if $\bot < y \in C$, then since

$$x * (x \to (x * y)) = x \wedge (x * y) = x * y > \bot,$$

(ii) of Lemma 5 and $\neg\neg x = \top$ imply

$$\neg\neg x * ((x \to (x * y) \to y) = \top * \top = \top.$$

Hence C is a Π-chain. □

From Corollary 1 and the above theorem we finally obtain the structure of saturated BL-chains:

Theorem 4 *Each saturated chain C can be written as*

$$C = \bigsqcup_{i \in I} M_i,$$

where I is a bounded totally ordered set and each M_i *is either a G-chain, an MV-chain or a* Π*-chain.* □

Clearly, the above theorem is a far-reaching generalization of the Mostert and Shields Theorem mentioned in the Introduction.

Taking into account that all finite BL-chains are saturated and that the only non-trivial finite Π-chain is the two-element chain $\{\bot, \top\}$, we have the following:

Corollary 3 *Any finite BL-chain is a finite ordinal sum of MV and G-chains.* □

Despite the fact that Theorem 4 applies only to *saturated* BL-chains, it gives information on the structure of BL-chains in general, due to the following theorem, announced in [15, Theorem 3]. For the reader convenience we sketch a proof.

Theorem 5 *Each BL-chain C can be isomorphically embedded into a saturated BL-chain* $\bar{C}$*. Moreover, C is* dense *in* $\bar{C}$*, i.e., for any two new idempotents* $u \leq u'$ *in* $\bar{C} \setminus C$*, there is an* $x \in C$ *such that* $u \leq x \leq u'$.

Proof Let C be a BL-chain and S be the set of all non-saturated cuts of C. For each $\alpha \in S$, the upper part of α will be denoted by α^+, and the lower part, by α^-. On $\bar{C} = C \cup S$ define the binary relation $\preceq$ by the following rules, where $\leq$ denotes the order relation on C, and x, y are elements of $\bar{C}$:

$$x \preceq y \text{ iff } \begin{cases} x, y \text{ are both in } C \text{ and } x \leq y, \\ x, y \text{ are both in } S \text{ and } x^- \subset y^-, \\ x \in C, \ \ y \in S \text{ and } x \in y^-, \\ x \in S, \ \ y \in C \text{ and } y \in x^+. \end{cases}$$

It follows that $\preceq$ is a total order on $\bar{C}$, bounded by $\bot_C$ and $\top_C$. Extend the operations $*$ and $\to$ from C to $\bar{C}$ as follows, where $x \in \bar{C}$, $\alpha \in S$ and $c \in C$:

$$\alpha * x = x * \alpha = \min(\alpha, x),$$

where min is with respect to the total order $\preceq$,

$$\alpha \to x = \begin{cases} \top & \text{if } \alpha \preceq x, \\ x & \text{if } \alpha \succ x. \end{cases}$$

$$c \to \alpha = \begin{cases} \top & \text{if } c \in \alpha^-, \\ \alpha & \text{if } c \in \alpha^+. \end{cases}$$

It can be shown that with these operations $\bar{C}$ becomes a BL-chain. We are going to prove that it is saturated. Let $\bar{\alpha}$ be a cut in $\bar{C}$. It is clear that $\alpha = \bar{\alpha} \cap C$ is a cut in C, and, moreover, that $\alpha^+ = \bar{\alpha}^+ \cap C$ and $\alpha^- = \bar{\alpha}^- \cap C$. We have two possibilities:

(i) α is determined by an idempotent $u \in C$,
(ii) $\alpha \in S$.

In case (i) $\bar{\alpha}$ is determined by u, and in case (ii) $\bar{\alpha}$ is determined by α, which is an idempotent in $\bar{C}$. Indeed, suppose we are in case (i), and let $x \in \bar{\alpha}^+$. If $x \in C$, then $x \in \alpha^+$ and hence $u \preceq x$. Let $x = \beta \in S$ and suppose (absurdum hypothesis) that $u \in \beta^+$. Since $\beta \in S$, there is $b \in \beta^+$ such that $b \leq u$. Then $b \in \alpha^- \subseteq \bar{\alpha}^-$ and $\beta \prec b$, a contradiction. Hence if $x \in \bar{\alpha}^+$ then $x \succeq u$. Dual arguments show that $x \in \bar{\alpha}^-$ implies that $x \preceq u$. Therefore $\bar{\alpha}$ is determined by u. Suppose now that we are in case (ii), and let $x \in \bar{\alpha}^+$. If $x \in C$, then $x \in \alpha^+$ and consequently, $\alpha \prec x$. Let $x = \beta \in S$ and suppose (absurdum hypothesis) that $\beta^- \subset \alpha^-$. If $a \in \alpha^- \setminus \beta^-$, then $a \in \bar{\alpha}$ and $a \in \beta^+$, i.e., $\beta \prec a$. This contradiction shows that $x \in \bar{\alpha}^+$ implies $x \succeq \alpha$. Dual arguments show that $\bar{\alpha}$ is determined by α. Consequently, we have shown that $\bar{C}$ is saturated. By construction, C is isomorphically embedded in $\bar{C}$, and the density condition is satisfied. □

Note that if a BL-chain C is not MV, Π or G, then $\overline{C}$ contains a closed segment $[\alpha, \beta]$ such that α, β are idempotents in $\overline{C}$, $(\alpha, \beta) \subset C$, $C \setminus ((\alpha, \beta) \cup \{0, 1\}) \neq \emptyset$ and $[\alpha, \beta]_{\overline{C}}$ is an MV or a product chain.

5 Partial embeddings

Let A, B be BL-chains, and let S be a *finite* subset of A such that $\{\bot, \top\} \subseteq S$. A *partial embedding of A into B with domain S* is a function $f : S \to B$ that fulfills the following conditions, where x, y, z denote elements of S:

PE_1 $x \leq y$ iff $f(x) \leq f(y)$,
PE_2 $x * y = z$ implies $f(x) * f(y) = f(z)$,
PE_3 $x \to y = z$ implies $f(x) \to f(y) = f(z)$,
PE_4 $f(\bot) = \bot$, $f(\top) = \top$.

We say that a BL-chain A is partially embeddable in a BL-chain C provided that each *finite* subset of A containing $\bot$ and $\top$ is the domain of a partial embedding of A into C. We say that A is partially embeddable in a family $\{C_i\}_{i \in I}$ of BL-chains provided that each *finite* subset of A containing $\bot$ and $\top$ is the domain of a partial embedding of A into C_i, for some $i \in I$.

Lemma 7 *Let A be a BL-chain partially embeddable in the family $\{B_i\}_{i \in I}$ of BL-chains. If a BL-equation fails in A, then there is $i \in I$ such that the equation also fails in B_i.*

Proof Suppose that an equation does not hold in A. We can safely assume that the equation is of the form $\tau(x_1, \ldots, x_n) = \top$, where τ is a BL-term in the variables x_1, ..., x_n (see footnote 1). Then there are elements $a_1, \ldots, a_n$ in A such that $\tau^A(a_1, \ldots, a_n) < \top$. Let S be the set formed by all subterms of τ evaluated in A by assigning the value a_i to the variable x_i, $i = 1, \ldots, n$, together with $\bot$ and $\top$. There is $i \in I$ such that S is the domain of a partial embedding f of A into B_i. By induction on the complexity of terms, we can prove that for each subterm $\sigma(x_1, \ldots, x_n)$ of $\tau(x_1, \ldots, x_n)$ we have that $f(\sigma^A(a_1, \ldots, a_n)) = \sigma^{B_i}(f(a_1), \ldots, f(a_n))$. Hence taking into account PE_1 and PE_4 we can conclude that $\tau^{B_i}(f(a_1), \ldots, f(a_n)) = f(\tau^A(a_1, \ldots, a_n)) < \top$. □

By an *o-group* we shall understand an *abelian* group $\langle G, +, -, 0\rangle$ endowed with a total order relation $\leq$ that is compatible with addition; in other words, $\leq$ has the following translation invariance property, for all x, y, t in G: if $x \leq y$ then $t + x \leq t + y$. The sets $G^+ = \{x \in G : 0 \leq x\}$ and $G^- = \{x \in G : x \leq 0\}$ are called, respectively, the *positive cone* and the *negative cone* of G. We denote by $\mathbf{R}$ the additive group of real numbers with the usual order. The following result is well known. An elementary self-contained proof can be found in [7]:

Theorem 6 *Let G be an o-group. For each finite subset S of G there is a function $f : S \to \mathbf{R}$ fulfilling the following properties, where a, b, c denote arbitrary elements of S:*

PG_1 *$a \leq b$ iff $f(a) \leq f(b)$,*
PG_2 *$a + b = c$ iff $f(a) + f(b) = f(c)$,*
PG_3 *If $0 \in S$, then $f(0) = 0$,*
PG_4 *$b = -a$ iff $f(b) = -f(a)$.*

Let G be an o-group. If $0 < u \in G$, then the segment $[0, u] = \{x \in G : 0 \leq u\}$ becomes an MV-chain under the operations $x * y = \max(0, x + y - u)$, $x \to y = \min(u, u - x + y)$ and $\bot = 0$. This MV-chain will be denoted by $\mathbf{\Gamma}(G, u)$. Chang [6] showed that for every MV-chain C there is an o-group G and $0 < u \in G$ such that C is isomorphic to $\mathbf{\Gamma}(G, u)$ (see [8, Chapt. 2] and the references given there for a far reaching extension of this result). Note that the standard MV-chain coincides with $\mathbf{\Gamma}(\mathbf{R}, 1)$.

Theorem 7 *Every MV-chain C is partially embeddable in the standard MV-chain.*

Proof There is an o-group G and $0 < u \in G$ such that C is isomorphic to $\mathbf{\Gamma}(G, u)$. Hence, to simplify the notations, we suppose that $S = \{0 = a_0, a_1, \ldots, a_{n-1}, a_n = u\} \subseteq G$. Let $S^+ = \{a_i + a_j : 0 \leq i, j \leq n\}$. Since $0 = a_0$, we have that $S \subseteq S^+$. By Theorem 6, there is a function $f : S^+ \to \mathbf{R}$ satisfying properties $PG_1 - PG_4$. Let g denote the restriction of f to S. We shall prove that $h = (1/g(u))g$ is a partial embedding of C into $\mathbf{\Gamma}(\mathbf{R}, 1)$. Since properties PE_1 and PE_4 obviously hold, we need to prove conditions PE_2 and PE_3. Suppose $a_i * a_j = a_k$. If $k > 0$, then $a_i + a_j - u = a_k$, i.e., $a_i + a_j = u + a_k$. Since $a_i, a_j, a_k, u = a_n, a_i + a_j$ and $u + a_k$ are in S^+, we have that $f(a_i) + f(a_j) = f(a_i + a_j)$ and $f(u) + f(a_k) = f(u + a_k)$. Therefore $h(a_i) + h(a_j) - 1 = h(a_k)$, i.e., $h(a_i) * h(a_j) = h(a_k)$. Suppose now that $k = 0$. Then $a_i + a_j \leq u$ and $f(a_i) + f(a_j) = f(a_i + a_j) \leq f(u)$. Hence $h(a_i) * h(a_j) = 0$. This shows that h satisfies property PE_2. Suppose $a_i \to a_j = a_k$. If $k = n$, then $a_i \leq a_j$. Hence $f(a_i) \leq f(a_j)$ and $h(a_i) \to h(a_j) = 1$. If $k < n$, then $u - a_i + a_j = a_k$, i. e., $u + a_j = a_i + a_k$. This implies that $f(u) + f(a_j) = f(u + a_j) = f(a_i + a_k) = f(a_i) + f(a_k)$. Therefore $h(a_i) \to h(a_j) = h(a_k)$, and we have shown that h also satisfies PE_3. □

Taking into account Lemma 7, from the above theorem we obtain Chang's completeness theorem for MV-algebras [6]:

Corollary 4 *An equation holds in all MV-algebras iff it holds in the standard MV-chain.* □

Let G be an o-group and $\perp$ be an element not belonging to G. On the set $G^- \cup \{\perp\}$ define the binary operations $*$ and $\to$ as follows:

$$x * y = \begin{cases} x + y & \text{if } x, y \in G^-, \\ \perp & \text{otherwise,} \end{cases}$$

and

$$x \to y = \begin{cases} \min(0, y - x) & \text{if } x, y \in G^-, \\ 0 & \text{if } x = \perp, \\ \perp & \text{if } x \in G^- \text{ and } y = \perp. \end{cases}$$

It is easy to check that $\langle G \cup \{\perp\}, *, \to, \perp, 0\rangle$ is a Π-algebra, that will be denoted by $\mathfrak{P}(G)$. It is shown in [16] (see [10] for a more general result) that given a Π-chain C, there is an o-group G, unique up to isomorphisms, such that C is isomorphic to $\mathfrak{P}(G)$. Notice that the function $f\colon [0,1] \to \mathbf{R}^- \cup \{\infty\}$ be defined by:

$$f(x) = \begin{cases} \log(x) & \text{if } x > 0, \\ \infty & \text{if } x = 0, \end{cases}$$

is an isomorphism from the standard Π-chain onto the Π-chain $\mathfrak{P}(\mathbf{R})$.

Theorem 8 *Every Π-chain C is partially embeddable into the standard Π-chain.*

Proof Without loosing generality, we can assume that $C = \mathfrak{P}(G)$ for some o-group G. Moreover, it is sufficient to prove that $\mathfrak{P}(G)$ is partially embeddable into $\mathfrak{P}(\mathbf{R})$. Let $S = \{\perp = a_0, a_1, \ldots, a_n = 0\} \subseteq G^- \cup \{\perp\}$, and suppose that $a_i < a_j$ for $0 \le i < j \le n$. Take $S' = \{a_1, \ldots, a_n, -a_{n-1}, \ldots, -a_1\}$. By Theorem 6 there is a function $f\colon S' \to \mathbf{R}$ satisfying properties $PG_1 - PG_4$. Define $g\colon S \to \mathfrak{P}(\mathbf{R})$ as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in S \setminus \{\perp\}, \\ \infty & \text{if } x = \perp. \end{cases}$$

It follows immediately from the definition of the operations that $a_i + a_j = a_k$ implies $g(a_i) + g(a_j) = g(a_k)$, for $i, j, k = 0, \ldots, n$. If $a_i \to a_j = a_n$, then $a_i \le a_j$. Hence $g(a_i) \le g(a_j)$ and $g(a_i) \to g(a_j) = 0 = g(a_n)$. Suppose that $a_i \to a_j = a_k$, with $0 < k < n$. Then $i > 0$, $j > 0$ and $a_j - a_i = a_k$. Since $-a_i \in S'$, $f(a_j) - f(a_i) = f(a_k)$, and then $g(a_i) \to g(a_j) = g(a_k)$. Finally, suppose that $a_i \to a_j = a_0$. In this case $i > 0$ and $j = 0$. Hence we also have $g(a_i) \to g(a_j) = g(a_k)$. □

Corollary 5 **([16])** *An equation holds in all Π-algebras iff it holds in the standard Π-chain.*

The next result, stated for further reference, is obvious.

Lemma 8 *Every G-chain is partially embeddable into the standard G-chain.* □

6 Standard completeness

Theorem 9 *Every BL-chain C is partially embeddable in the family of all t-algebras.*

Proof By Theorem 5 we can assume that C is a saturated BL-chain, and hence, by Theorem 4 that $C = \bigsqcup_{i \in I} M_i$, where I is a bounded totally ordered set and each M_i is either a G-chain, an MV-chain or a Π-chain. If S is a nonempty finite subset of C, then there is a finite subset $\{i_0, i_1, \ldots, i_n\} \subseteq I$ such that $S \subseteq \bigcup_{k \le n} M_{i_k}$ with $\perp^{M_{i_0}} = \perp^C$. For every $k \le n$ there are $a_k, b_k \in C$ such that $M_{i_k} = [a_k, b_k]_C$, thus we define $S_k = S \cap (a_k, b_k)$, $k \le n$, and $S_{n+1} = S \setminus \bigcup_{k \le n} S_k$. Let $0 = r_0 < s_0 \le r_1 < s_1 \le \cdots \le r_n < s_n = 1$ be rational numbers such that $r_k = s_{k+1}$ iff $a_k = b_k + 1$, and

$$[0,1] = \bigcup_{k \le n} [r_k, s_k] \cup \bigcup_{k < n} [s_k, r_{k+1}].$$

Put $E = Idp(C)$. If either $[a_k, b_k]_C \in G(E)$ or $a_k = b_k$, then endow the segment $[r_k, s_k]$ with a BL-algebra structure isomorphic to the standard G-chain. If $[a_k, b_k]_C$ is an MV-algebra or a Π-algebra, make $[r_k, s_k]$ isomorphic to the standard MV-chain or the standard Π-chain, respectively. If $b_k < a_{k+1}$, make $[s_k, r_{k+1}]$ isomorphic to the standard G-chain. Let T_C be the t-algebra ordinal sum of these segments (disregarding the segments $[s_k, r_{k+1}]$ when $s_k = r_{k+1}$). If $S_k \ne \emptyset$, then, by Lemmas 7, 8, there is a partial embedding $f_k\colon S_k \to [r_k, s_k]$. The function $f_{n+1}\colon S_{n+1} \to [0,1]$ such that $f_{n+1}(a_k) = r_k$ for $a_k \in S_{n+1}$ and $f_{n+1}(b_k) = s_k$ for $b_k \in S_{n+1}$ gives a partial embedding of S_{n+1} into T_C. Therefore the function $g\colon S \to [0,1]$ defined by $g(x) = f_i(x)$ for $x \in S_i$, $i = 0, \ldots, n+1$ defines a partial embedding of S into the t-algebra T_C. □

Corollary 6 *An equation holds in all BL-algebras iff it holds in all t-algebras.* □

Remark 5 Despite the fact that t-algebras are built from G-chains, MV-chains and Π-chains, the variety generated by G-algebras, MV-algebras and Π-algebras is a *proper* subvariety of the variety of BL-algebras, characterized by the equation (see [9]).

$$\begin{aligned}&(x \to x * y)\\ &\quad \to [(x \to \perp) \vee y \vee ((x \to x * x) \wedge (y \to y * y))] = \top\end{aligned}$$

Appendix: Hoop decompositions of BL-algebras

As already noted, the decomposition given in Theorem 4 applies only to *saturated* BL-chains. This is due to the requirements that each component M_i of the ordinal sum $\bigsqcup_{i \in I} M_i$ be a BL-chain, hence having bottom and top elements, and that the top element of M_i be less or equal than the bottom element of M_j for $i < j$. By relaxing these requirements Aglianò and Montagna [1] gave a decomposition which applies to arbitrary BL-chains. The building blocks of their decomposition are *basic hoops* (also known as *generalized BL-algebras* [4, 5, 11, 12]). Basic hoops are obtained from BL-algebras by deleting the necessity of having a bottom element. Given a totally ordered set I and a family $\{A_i\}_{i\in I}$ of basic hoops, the ordinal sum $\bigoplus_{i\in I} A_i$ is defined in a similar way as we defined ordinal sums of BL-chains in Sect. 2, with the main difference that *it must have a top element which is the common top element of all the components* A_i. The ordinal sum is a BL-algebra iff I has a bottom element i_0 and A_{i_0} is a BL-algebra, i.e., A_{i_0} has a bottom element. We refer to Montagna's contribution [17] in the present volume for the precise definitions and details. The sum irreducible basic hoops are the *totally ordered Wajsberg hoops*.

There are two classes of totally ordered Wajsberg hoops: cancellative Wajsberg hoops and bounded Wajsberg hoops. Given a Π-chain C, the set $C^+ = \{x \in C : x > \bot\}$, with the operations $*$ and $\to$ inherited from C, becomes a cancellative Wajsberg hoop, and all totally ordered cancellative Wajsberg hoops can be obtained in this way. Bounded Wajsberg hoops coincide with MV-algebras. Hence all MV-chains (and, in particular, the two-element boolean algebra) are sum irreducible according to both, Hájek and Aglianò–Montagna definitions. But the Π-chains are irreducible according to Hájek and reducible according to Aglianò–Montagna, because it follows from their definition that every Π-chain C is the ordinal sum of the two-element boolean algebra and the cancellative Wajsberg hoop C^+. The two-element boolean algebra is a sum irreducible G-chain according to both definitions. According to Aglianò–Montagna definition, each nontrivial G-chain is an ordinal sum of two-element boolean algebras, in contrast, according to Hájek definition, it can be considered as an ordinal sum of singletons. Moreover, the components of Aglianò–Montagna decomposition of a BL-chain C are subhoops of C, but the components of Hájek decomposition in general are not substructures of C. For instance, the BL-chain A considered in Example 1, that is irreducible in the sense of Hájek, according to Aglianò–Montagna is the ordinal sum of the standard MV-chain and the Wajsberg hoop obtained by deleting 0 from the standard Π-chain.

Since totally ordered cancellative hoops are precisely the negative cones of totally ordered abelian groups, the Standard Completeness Theorem for Basic Logic can be proved from Aglianò–Montagna decomposition of BL-chains following lines similar to those of the proof of Theorem 9.

Some applications of the decomposition of BL-chains in ordinal sums of totally ordered Wajsberg hoops are mentioned in [17]. For instance, it is shown that *the variety of BL-algebras is generated by a BL-chain which is the ordinal sum of infinitely many copies of the standard MV-chain.*

We close this paper recalling the following result from [4], that plays a fundamental role to describe free algebras in varieties of BL-algebras generated by a BL-chain. Given a BL-chain C, $MV(C) = \{x \in C : \neg\neg x = x\}$ is an MV-chain, and $D(C) = \{x \in C : \neg\neg x = \top\}$ is a totally ordered basic hoop. *Every nontrivial BL-chain C is the ordinal sum, in the sense of Aglianò–Montagna, of* $MV(C)$ *and* $D(C)$.

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