BRZEZIŃSKI'S CROSSED PRODUCTS AND BRAIDED HOPF CROSSED PRODUCTS

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Abstract. Recently, we introduced a notion of braided Hopf crossed product which generalizes the notion of classical Hopf crossed product defined independently by Blattner, Cohen and Montgomery and by Doi and Takeuchi. A very much general concept of crossed product is indebted to Brzeziński. In this paper we give a sufficient condition for a Brzeziński's crossed product be a braided Hopf crossed product. Majid prove that the quantum double of a quasitriangular Hopf algebra is isomorphic to a classical Hopf crossed product. As an application of our result we obtain a generalization of Majid's Theorem.

Introduction

Let A be an associative and unitary algebra and let V be a vector space, endowed with a distinguished element 1. Let $A\#V$ be an associative and unitary algebra, with underlying vector space $A\otimes V$. We let $a\#v$ denote the element $a\otimes v$ of $A\otimes V$ when it is considered as an element of $A\#V$. Following Brzeziński [Br], we say that $A\#V$ is a crossed product of A with V if $(a\#1)(b\#v)=ab\#v$ for all $a,b\in A$ and $v\in V$.

This is a very general definition. When V is a Hopf algebra H, then an important source of examples is given by the classical Hopf crossed products introduced independently in [B-C-M] and [D-T], but there are many Brzeziński's crossed products that do not fit in this setting.

In [G-G2] we began the study of a type of crossed products $A\#_f H$, called braided Hopf crossed products, which seem to have many of the properties of the classical Hopf crossed products. Every one of these algebras $A\#_f H$ is determined by the following data: an algebra A, a braided bialgebra H, and maps $s\colon H\otimes A\to A\otimes H$, $\rho\colon H\otimes A\to A$ and $f\colon H\otimes H\to A$, satisfying suitable hypothesis (see Section 1). The multiplication map of $A\#H$ is defined by

$$\mu_{A\#_f H} := (\mu\otimes H)\,(\mu\otimes f\otimes \mu)\,(A\otimes\rho\otimes\Delta_{H\otimes^c H})\,(A\otimes H\otimes s\otimes H)\,(A\otimes\Delta\otimes A\otimes H),$$

where c is the braid of H and $\Delta_{H\otimes^c H}=(H\otimes c\otimes H)\,(\Delta\otimes\Delta)$. The maps s, ρ and f are called the transposition, the weak s-action and the cocycle of $A\#_f H$, respectively. When H is a standard Hopf algebra and $s(h\otimes a)=a\otimes h$, for all $a\in A$ and $h\in H$, we recover the classical Hopf crossed products.

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From the definitions it is immediate that every braided Hopf crossed product $A\#_f H$ is a Brzeziński's crossed product. At this point it is natural to ask for a characterization of the Brzeziński's crossed products $A\#H$ of this type. In Theorem 2.1 we solve this problem under the hypothesis that the map $h\otimes a \mapsto (1\#h)(a\#1)$ is compatible with the multiplication map of H, in a sense that we precise below. As an immediate corollary, we obtain a characterization of the Brzeziński's crossed products that satisfy the same hypothesis and that are classical Hopf crossed products.

Let H be an standard Hopf algebra. A braided smash product $A\#H$ is a braided crossed product with trivial cocycle. In Section 3 we use Theorem 2.1 to obtain a sufficient condition for the quantum double $\mathrm{D}(H)$ be isomorphic to a braided smash product of H with an algebra $\underline{H}^*$, which is a sort of deformation of H^*. More precisely, in [G-G3], we introduced the notion of semiquasitriangular Hopf algebra (see Definition 3.2). These algebras, which generalize the quasitriangular Hopf algebras, have many of the basic properties of the last ones. For instance they have associated braided categories in a natural way. In Theorem 3.4 we prove that if (H,R) is a semiquasitriangular Hopf algebra, then $\mathrm{D}(H)$ is isomorphic to a braided smash product $\underline{H}^*\#H$. This gives a version for the setting of braided Hopf crossed products of the following result of Majid: if (H,R) is a quasitriangular Hopf algebra, then the quantum double $\mathrm{D}(H)$ is isomorphic to a classical Hopf crossed product. Moreover, it is easy to see that if (H,R) is quasitriangular, then the transposition of $\underline{H}^*\#H$ is the flip $h\otimes a\mapsto a\otimes h$. So, the result of Majid follows from Theorem 3.4.

In this article we work in the category of vector spaces over a field k. Then we assume implicitly that all the maps are k-linear maps. The tensor product over k is denoted by $\otimes$, without any subscript. Given vector spaces U,V,W and a map $f\colon V\to W$ we write $U\otimes f$ for $\mathrm{id}_U\otimes f$ and $f\otimes U$ for $f\otimes \mathrm{id}_U$. We assume that the reader is familiar with the notions of algebra, coalgebra, module and comodule. Unless otherwise explicitly established we assume that the algebras are associative unitary and the coalgebras are coassociative counitary. Given an algebra A and a coalgebra C, we let $\mu\colon A\otimes A\to A$, $\eta\colon k\to A$, $\Delta\colon C\to C\otimes C$ and $\epsilon\colon C\to k$ denote the multiplication, the unit, the comultiplication and the counit, respectively, specified with a subscript if necessary. Moreover, given k-vector spaces V and W, we let $\tau\colon V\otimes W\to W\otimes V$ denote the flip $\tau(v\otimes w)=w\otimes v$.

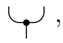
All the results in Section 2 of this paper are valid in the context of monoidal categories. In fact, in the first two sections of this article we use the nowadays well known graphic calculus for monoidal and braided categories. As usual, morphisms will be composed from up to down and tensor products will be represented by horizontal concatenation in the corresponding order. The identity map of a vector space will be represented by a vertical line. Given an algebra A, the diagrams

, ,  and 

stand for the multiplication map, the unit, the action of A on a left A-module and the action of A on a right A-module, respectively. Given a coalgebra C, the comultiplication, the counit, the coaction of C on a right C-comodule and the coaction of C on a left C-module will be represented by the diagrams

, ,  and ,

respectively. The maps c, s, χ, $\mathcal{F}$ and f, which appear in Definitions 1.5, 1.7, 1.2 and 1.9 will be represented by the diagrams

 and ,

respectively. The inverse maps of c and s (when s is bijective) will be represented by

 and .

Any other map $g\colon V \to W$ will be geometrically represented by the diagram

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Let H be a Hopf algebra. In Section 3, for the comultiplication of H we use the Sweedler notation $\Delta(h) = h_1 \otimes h_2$, without summation symbol.

1. Preliminaries

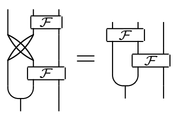
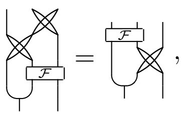
In this section we recall some basic results about crossed products, that we will need later

Brzeziński's crossed products

Definition 1.1. Let V and W be vector spaces and let $c\colon V \otimes W \to W \otimes V$ be a map. If V is an algebra, then we say that c is compatible with the algebra structure of V if $c\,(\eta \otimes W) = W \otimes \eta$ and $c\,(\mu \otimes W) = (W \otimes \mu)\,(c \otimes V)\,(V \otimes c)$. If V is a coalgebra, then we say that c is compatible with the coalgebra structure of V if $(W \otimes \epsilon)\,c = \epsilon \otimes W$ and $(W \otimes \Delta)\,c = (c \otimes V)\,(V \otimes c)\,(\Delta \otimes W)$. Finally, if W is an algebra or a coalgebra, then we introduce the notion that c is compatible with the structure of W in the obvious way.

Let A be an unitary algebra and let V be a vector space equipped with a distinguished element $1 \in V$.

Definition 1.2. Let $\chi\colon V \otimes A \to A \otimes V$ and $\mathcal{F}\colon V \otimes V \to A \otimes V$ be maps. Following [Br] we say that χ is a twisting map if it is compatible with the algebra structure of A and $\chi(1 \otimes a) = a \otimes 1$, that $\mathcal{F}$ is normal if $\mathcal{F}(1 \otimes v) = \mathcal{F}(v \otimes 1) = 1 \otimes v$, and that $\mathcal{F}$ is a cocycle satisfying the twisted module condition if

 and , where  $= \chi$ and  $= \mathcal{F}$.

More precisely, the first equality says that $\mathcal{F}$ is a cocycle and the second one says that $\mathcal{F}$ satisfies the twisted module condition.

Let $A\#V$ be an associative and unitary algebra with underlying vector space $A \otimes V$. As usual, we write $a\#v$ to denote an elementary tensor $a \otimes v \in A\#V$. Recall that $A\#V$ is said to be a Brzeziński's crossed product if it satisfies $(a\#1)(b\#v) = ab\#v$ for all $a, b \in A$ and $v \in V$.

Theorem 1.3. *If $\chi\colon V\otimes A\to A\otimes V$ is a twisting map and $\mathcal{F}\colon V\otimes V\to A\otimes V$ is a normal cocycle that satisfies the twisted module condition, then $A\otimes V$ becomes a Brzeziński's crossed product via the multiplication map*

$$\mu_{A\#V} := (\mu\otimes V)\,(\mu\otimes\mathcal{F})\,(A\otimes\chi\otimes V)$$

Conversely, given a Brzeziński's crossed product $A\#V$, the maps $\chi(v\otimes a) := (1\#v)(a\#1)$ and $\mathcal{F}(v\otimes w) := (1\#v)(1\#w)$ are a twisting map and a normal cocycle that satisfies the twisted module condition, respectively.

Example 1.4. (Twisted tensor products) Let B be an algebra, $\chi\colon B\otimes A\to A\otimes B$ a twisting map and $\mathcal{F}\colon B\otimes B\to A\otimes B$ the *trivial cocycle* $\mathcal{F}(v\otimes v') = 1\otimes vv'$. It is immediate that $\mathcal{F}$ is normal and satisfies the cocycle condition. Moreover, the twisted module condition reduces to $(A\otimes\mu)\,(\chi\otimes B)\,(B\otimes\chi) = \chi\,(\mu\otimes A)$. Hence, χ is a twisting map in the sense of [C-S-V] and (B,A,χ) is called a matched pair of algebras. The crossed products $A\otimes_\chi B$, constructed from these type of data are called twisted tensor products or matched products. These algebras, which are a direct generalization of the tensor products, were introduced in [C-S-V] and [Ta]. Examples of this construction are the Ore extensions $A[X,\alpha,\delta]$, where $\alpha\colon A\to A$ is an endomorphism and $\delta\colon A\to A$ is an α-derivation. In this case $B = k[X]$ and the twisting map χ is determined by the equality $\chi(X\otimes a) = \alpha(a)\otimes X + \delta(a)\otimes 1$.

Braided Hopf crossed products

Braided bialgebras and braided Hopf algebras were introduced by Majid (see his survey [M1]). In this subsection, we make a quick review of this subject following the intrinsic presentation given by Takeuchi in [T]. Then, we review the concept of braided Hopf crossed products introduced in [G-G2]. Let V be a vector space. Recall that a map $c\in \mathrm{End}_k(V\otimes V)$ is called a braiding operator if it satisfies the equality

$$(c\otimes V)\,(V\otimes c)\,(c\otimes V) = (V\otimes c)\,(c\otimes V)\,(V\otimes c).$$

Definition 1.5. A braided bialgebra is a vector space H, endowed with an algebra structure, a coalgebra structure and a bijective braiding operator $c\in\mathrm{End}_k(H\otimes H)$, called the braid of H, such that: c is compatible with the algebra and coalgebra structures of H, η is a coalgebra morphism, ϵ is an algebra morphism and $\Delta\,\mu = (\mu\otimes\mu)\,(H\otimes c\otimes H)\,(\Delta\otimes\Delta)$. Moreover, if there exists a map $S\colon H\to H$, which is the inverse of the identity in the monoid $\mathrm{End}_k(H)$ with the convolution product, then we say that H is a braided Hopf algebra and we call S the antipode of H.

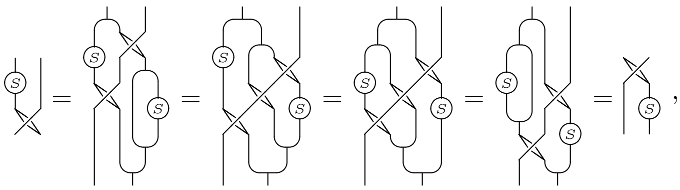
Usually H denotes a braided bialgebra, understanding the structure maps, and c denotes its braid.

Remark 1.6. Assume that H is an algebra and a coalgebra and that $c\in\mathrm{Aut}_k(H\otimes H)$ is a solution of the braiding equation, which is compatible with the algebra and coalgebra structures. Then, $H\otimes H$ is the underlying space of an algebra $H\otimes_c H$, with unit $\eta\otimes\eta$ and multiplication map $(\mu\otimes\mu)\,(H\otimes c\otimes H)$. It is easy to check that H is a braided bialgebra with braid c if and only if $\Delta\colon H\to H\otimes_c H$ and $\epsilon\colon H\to k$ are morphisms of algebras.

Let H be a braided bialgebra and A an algebra.

Definition 1.7. A transposition of H on A is a twisting map $s\colon H\otimes A\to A\otimes H$ which is compatible with bialgebra structure of H. That is, s satisfies the equation $(s\otimes H)\,(H\otimes s)\,(c\otimes A) = (A\otimes c)\,(s\otimes H)\,(H\otimes s)$ (compatibility of s with c) and it is compatible with the algebra and coalgebra structures of H.

In the original definition of transposition of a braided Hopf algebra H on an algebra A it was also required that $s\,(S\otimes A) = (A\otimes S)\,s$. But this condition is automatically satisfied. In fact, if V is a vector space and $s\colon H\otimes V\to V\otimes H$ is a map which is compatible with the algebra and coalgebra structures of H, then



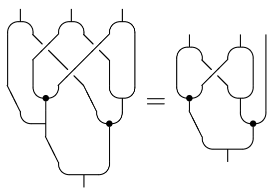
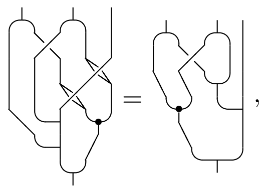
as desired.

Definition 1.8. Let $s\colon H\otimes A\to A\otimes H$ be a transposition. A weak s-action of H on A is a map $\rho\colon H\otimes A\to A$, that satisfies:

(1) $\rho\,(H\otimes\mu) = \mu\,(\rho\otimes\rho)\,(H\otimes s\otimes A)\,(\Delta\otimes A\otimes A)$,
(2) $\rho(h\otimes 1) = \epsilon(h)1$, for all $h\in H$,
(3) $\rho(1\otimes a) = a$, for all $a\in A$,
(4) $s\,(H\otimes\rho) = (\rho\otimes H)\,(H\otimes s)\,(c\otimes A)$.

An s-action is a weak s-action which satisfies $\rho\,(H\otimes\rho) = \rho\,(\mu\otimes A)$.

Definition 1.9. Let $f\colon H\otimes H\to A$ be a map. We say that f is normal if $f(1\otimes x) = f(x\otimes 1) = \epsilon(x)$ for all $x\in H$, and that f is a cocycle that satisfies the twisted module condition if

 and , where  $= f$.

More precisely, the first equality is the cocycle condition and the second one is the twisted module condition. Finally, we say that f is compatible with s if

$$(f\otimes H)\,(H\otimes c)\,(c\otimes H) = s\,(H\otimes f).$$

Let $s\colon H\otimes A\to A\otimes H$ be a transposition, $\rho\colon H\otimes A\to A$ a weak s-action and $f\colon H\otimes H\to A$ a normal cocycle compatible with s, satisfying the twisted module condition. Let $\chi\colon H\otimes A\to A\otimes H$ and $\mathcal{F}\colon H\otimes H\to A\otimes H$ be the maps defined by $\chi := (\rho\otimes H)\,(H\otimes s)\,(\Delta\otimes A)$ and $\mathcal{F} := (f\otimes\mu)\,(H\otimes c\otimes H)\,(\Delta\otimes\Delta)$. In [G-G2] was proved that χ is a twisting map and $\mathcal{F}$ is a normal cocycle satisfying the twisted module condition.

Definition 1.10. The crossed product $A\#_f H$, associated with (s, ρ, f), is the algebra constructed from χ and $\mathcal{F}$ as in Theorem 1.3.

Let $H \otimes^c H$ be the coalgebra with underlying space $H \otimes H$, comultiplication map $\Delta_{H\otimes^c H} := (H \otimes c \otimes H)\,(\Delta \otimes \Delta)$ and counit $\epsilon_{H\otimes^c H} := \epsilon_H \otimes \epsilon_H$. An important class of braided Hopf crossed products are those whose cocycle $f\colon H \otimes^c H \to A$ is convolution invertible. They are the cleft extensions (see [G-G1, Section 10]). A particular case are the smash products $A\#H$ associated with (s, ρ), that is, the crossed products with trivial cocycle $f(h \otimes l) = \epsilon(h)\epsilon(l)$. Note that in this case ρ is an s-action.

2. Brzeziński's crossed products which are Braided Hopf crossed products

Let H be a braided Hopf algebra, A an algebra, s a transposition of H on A, $\rho\colon H \otimes A \to A$ a weak s-action and $f\colon H \otimes H \to A$ a normal cocycle compatible with s, satisfying the twisted module condition. Let $A\#_f H$ be the braided Hopf crossed product associated with (s, ρ, f). Let $\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A)$ be the twisting map associated with ρ. In [G-G1, Prop. 6.5] was proved that ρ is an s-action iff the twisting map χ satisfies the equation $\chi\,(\mu \otimes A) = (A \otimes \mu)\,(\chi \otimes H)\,(H \otimes \chi)$. In this section we characterize the Brzeziński's crossed products $A\#H$ that can be constructed from a triple (s, ρ, f), consisting of a transposition s, an s-action ρ and a normal cocycle f compatible with s. By the above discussion the twisting map χ of $A\#H$ must be compatible with the multiplication of H.

Let $A\#_f H$ be a braided Hopf crossed product. In [G-G1, Prop. 9.1] was proved that the map $\mathcal{F} = (f \otimes \mu)\,\Delta_{H\otimes^c H}$ satisfies the equality $(\mathcal{F} \otimes \mu)\,\Delta_{H\otimes^c H} = (A \otimes \Delta)\,\mathcal{F}$. So, it is natural to assume this condition among the hypothesis of the next theorem.

Theorem 2.1. *Let $\chi\colon H \otimes A \to A \otimes H$ be a twisting map and $\mathcal{F}\colon H \otimes H \to A \otimes H$ a normal cocycle that satisfies the twisted module condition. Assume that χ is compatible with the multiplication of H and that $(\mathcal{F} \otimes \mu)\,\Delta_{H\otimes^c H} = (A \otimes \Delta)\,\mathcal{F}$. Let $s\colon H \otimes A \to A \otimes H$, $\rho\colon H \otimes A \to A$ and $f\colon H \otimes H \to A$ be the maps defined by*

$$s = (\rho \otimes H)\,(S \otimes \chi)\,(\Delta \otimes A), \qquad \rho = (A \otimes \epsilon)\,\chi \qquad \text{and} \qquad f = (A \otimes \epsilon)\,\mathcal{F}.$$

If the following conditions are satisfied:

1. $(s \otimes H)\,(H \otimes s)\,(c \otimes A) = (A \otimes c)\,(s \otimes H)\,(H \otimes s)$,
2. $(s \otimes H)\,(H \otimes s)\,(\Delta \otimes A) = (A \otimes \Delta)\,s$,
3. $(\rho \otimes H)\,(H \otimes s)\,(c \otimes A) = s\,(H \otimes \rho)$,
4. $(f \otimes H)\,(H \otimes c)\,(c \otimes H) = s\,(H \otimes f)$,

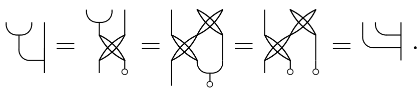
then, s is a transposition, ρ is an s-action, f is a normal cocycle compatible with s which satisfies the twisted module condition,

$$\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A) \qquad \text{and} \qquad \mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}.$$

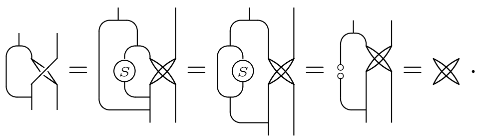
So, the Brzeziński's crossed product constructed from $(\chi, \mathcal{F})$ is the braided Hopf crossed product constructed from (s, ρ, f).

Proof. We must prove that s, ρ and f satisfy all the conditions established in Definitions 1.7, 1.8 and 1.9, with exception of the ones assumed in the hypothesis, and we must check the formulas for χ and $\mathcal{F}$. Using that $\chi(1 \otimes a) = a \otimes 1$ and $\chi(h \otimes 1) = 1 \otimes h$ for all $a \in A$ and $h \in H$, it is easy to see that $\rho(1 \otimes a) = a$ and $\rho(h \otimes 1) = \epsilon(h)1$. From these facts it follows immediately that $s(1 \otimes a) = a \otimes 1$ and $s(h \otimes 1) = 1 \otimes h$. Moreover, f is normal, since $\mathcal{F}$ is. Next, we check the remaining properties.

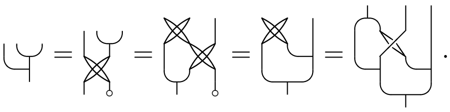
1) $\rho\,(H \otimes \rho) = \rho\,(\mu \otimes A)$: Since χ is compatible with the multiplication map of H,



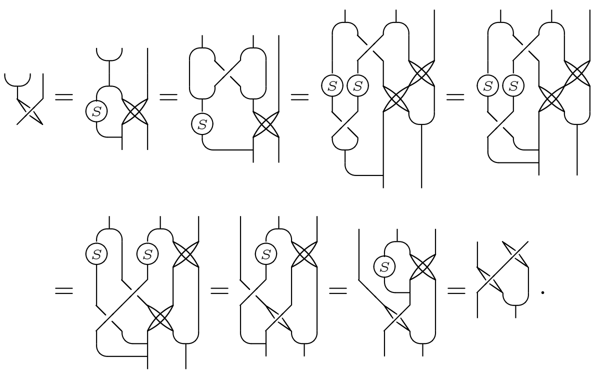
2) $\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A)$: Since $\rho\,(H \otimes \rho) = \rho\,(\mu \otimes A)$ and $\rho(1 \otimes a) = a$,



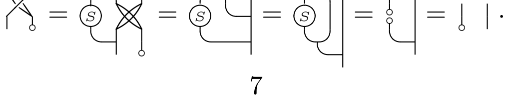
3) Item (1) of Definition 1.8: Since $\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A)$ and χ is compatible with the multiplication of A, we have



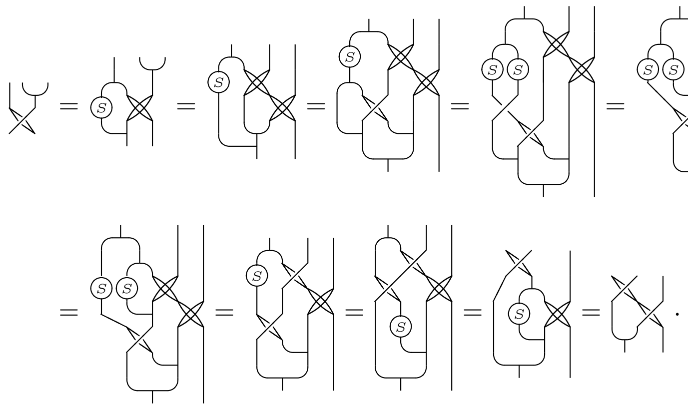
4) Compatibility of s with μ_H: By the compatibility of χ with the multiplication map of H, the fact that $\rho\,(H \otimes \rho) = \rho\,(\mu \otimes A)$, and item (3) of the hypothesis, we have



5) Compatibility of s with ϵ: Since $\rho\,(H \otimes \rho) = \rho\,(\mu \otimes A)$ and $\rho(1 \otimes a) = a$,



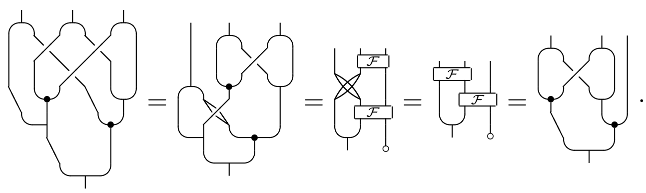
6) Compatibility of s with μ_A: First note that, by the discussion following Definition 1.7, s is compatible with S. Using this, the fact that χ is compatible with the multiplication of A, item (1) of Definition 1.8, and items (2) and (3) of the hypothesis, we obtain



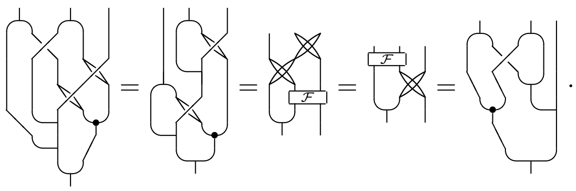
7) $\mathcal{F} = (f \otimes \mu)\Delta_{H \otimes^c H}$: Since $(\mathcal{F} \otimes \mu)\,\Delta_{H\otimes^c H} = (A \otimes \Delta)\,\mathcal{F}$,

$$(f \otimes \mu)\,\Delta_{H\otimes^c H} = ((A \otimes \epsilon)\,\mathcal{F} \otimes \mu)\,\Delta_{H\otimes^c H} = (A \otimes (\epsilon \otimes H)\,\Delta)\,\mathcal{F} = \mathcal{F}.$$

8) f is a cocycle: By item (4) of the hypothesis, and the facts that $\mathcal{F}$ is a cocycle, $\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A)$ and $\mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}$, we have



9) f satisfies the twisted module condition: By item (3) of the hypothesis, the fact that $\mathcal{F}$ satisfies the twisted module condition, $\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A)$ and $\mathcal{F} = (f \otimes \mu)\Delta_{H\otimes^c H}$, we have



This completes the proof. $\square$

Corollary 2.2. *Let H be a standard Hopf algebra. Let $\chi\colon H \otimes A \to A \otimes H$ be a twisting map and $\mathcal{F}\colon H \otimes H \to A \otimes H$ a normal cocycle that satisfies the twisted module condition. Assume that χ is compatible with the multiplication of H and* *that* $(\mathcal{F}\otimes\mu)\,\Delta_{H\otimes H}=(A\otimes\Delta)\,\mathcal{F}$. *Let* $\rho\colon H\otimes A\to A$ *and* $f\colon H\otimes H\to A$ *be the maps defined by*

$$\rho=(A\otimes\epsilon)\,\chi \qquad \text{and} \qquad f=(A\otimes\epsilon)\,\mathcal{F}.$$

If $(\rho\otimes H)\,(S\otimes\chi)\,(\Delta\otimes A)$ *is the flip, then* ρ *is an action in the sense of* [B-C-M], f *is a normal cocycle which satisfies the twisted module condition and*

$$\chi=(\rho\otimes H)\,(H\otimes\tau)\,(\Delta\otimes A) \qquad \text{and} \qquad \mathcal{F}=(f\otimes\mu)\Delta_{H\otimes H}.$$

So, the Brzeziński's crossed product constructed from $(\chi,\mathcal{F})$ *is the standard crossed product constructed from* (ρ,f).

In the next section we will apply Theorem 2.1 in order to obtain a property of the quantum double of semiquasitriangular Hopf algebras (see Definition 3.2). In the following examples we give more immediate applications.

Example 2.3. Let $A[X,\alpha,\delta]$ be an Ore extension (see Example 1.4). Consider the polynomial ring $k[X]$ endowed with the usual structure of Hopf algebra. Applying Theorem 2.1 it is easy to check that $A[X,\alpha,\delta]$ is an smash product $A\#k[X]$ if and only if $\alpha\,\delta=\delta\,\alpha$. This is the most simple way that we know to prove the main assertion of [G-G1, Example 3.10].

Example 2.4. Let H be the algebra $k[X]/\langle X^2\rangle$, endowed with the braided Hopf algebra structure given by $\Delta(X)=1\otimes X+X\otimes 1$ and $c(X\otimes X)=-X\otimes X$. Let A be an algebra. Consider maps $\chi\colon H\otimes A\to A\otimes H$ and $\mathcal{F}\colon H\otimes H\to A\otimes H$ and write $\chi(X\otimes a)=\alpha(a)\otimes X+\delta(a)\otimes 1$ and $\mathcal{F}(X\otimes X)=a_X\otimes 1+b_X\otimes X$. A direct computation shows that χ is a bijective twisting map compatible with the multiplication of H if and only if

(1) $\alpha\colon A\to A$ is an automorphism,

(2) $\delta\colon A\to A$ is an α-derivation,

(3) $\delta\,\alpha+\alpha\,\delta=0$,

(4) $\delta^2=0$,

and that in this case $\mathcal{F}$ is a normal cocycle satisfying the twisting module condition if and only if

(5) $a_X=\alpha(a_X)+\delta(b_X)$,

(6) $b_Xa_X=\delta(a_X)+\alpha(b_X)a_X$,

(7) $\alpha^2(a)a_X=a_Xa+b_X\delta(a)$,

(8) $b_Xa=\alpha(a)b_X$.

It is easy to see that $(\mathcal{F}\otimes\mu)\,\Delta_{H\otimes^c H}=(A\otimes\Delta)\,\mathcal{F}$ if and only if $b_X=0$. Under this condition, items (5)–(8) become

(5') $a_X=\alpha(a_X)$,

(6') $\delta(a_X)=0$,

(7') $\alpha^2(a)a_X=a_Xa$.

Now, let $s\colon H\otimes A\to A\otimes H$, $\rho\colon H\otimes A\to A$ and $f\colon H\otimes H\to A$ as in Theorem 2.1. A direct computation shows that $s(X\otimes a)=\alpha(a)\otimes X$, $\rho(X\otimes a)=\delta(a)$ and $f(X\otimes X)=a_X$. It is easy to see that under conditions (1)–(4), (5')–(7') and $b_X=0$, the hypothesis of Theorem 2.1 are satisfied. Hence, s is a transposition, ρ

is an s-action, f is a normal cocycle compatible with s which satisfies the twisted module condition,

$$\chi = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes A) \qquad \text{and} \qquad \mathcal{F} = (f \otimes \mu)\Delta_{H \otimes^c H}.$$

So, the Brzeziński's crossed product constructed from $(\chi, \mathcal{F})$ is the braided Hopf crossed product constructed from (s, ρ, f).

A concrete example of this situation is obtained taking $A = k[Y]$, $\alpha(Y^n) = (-1)^n Y^n$,

$$\delta(Y^n) = \begin{cases} Y^{n-1} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even,} \end{cases}$$

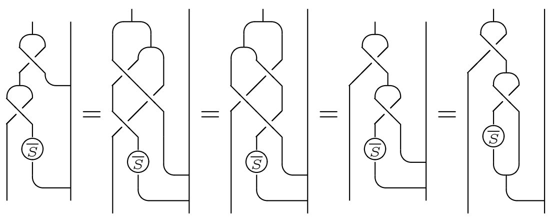
and $f(X \otimes X) = P(Y^2)$, where P is an arbitrary polinomial.

An immediate corollary of the next result is that s is bijective if and only if χ is.

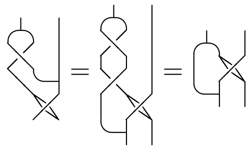
Proposition 2.5. *Let H be a braided Hopf algebra and V a left H-module. Let $\theta\colon H \otimes V \to H \otimes V$ be the map defined by $\theta := (H \otimes \rho)\,(c^{-1} \otimes V)\,(\Delta \otimes V)$, where $\rho\colon H \otimes V \to V$ denotes the action of H on V. Then, θ is bijective. Moreover, for each map $s\colon H \otimes V \to V \otimes H$ satisfying $s\,(H \otimes \rho) = (\rho \otimes H)\,(H \otimes s)\,(c \otimes V)$, it is true that*

$$s\,\theta = (\rho \otimes H)\,(H \otimes s)\,(\Delta \otimes V).$$

Proof. Let $\widetilde{\theta} := (H \otimes \rho)\,(H \otimes \overline{S} \otimes V)\,(c^{-1} \otimes V)\,(\Delta \otimes V)$, where $\overline{S}$ denotes the composition inverse of S. We claim that $\widetilde{\theta}$ is the composition inverse of θ. In fact, since c is compatible with the coalgebra structure of H, c is a solution of the braiding equation and V is an H-module, we have

$\widetilde{\theta}\,\theta =$  $= \mathrm{id}_{H \otimes A}$,

and in a similar way we can check that $\theta\,\widetilde{\theta} = \mathrm{id}_{H \otimes A}$. Finally, by the hypothesis about s,

$s\,\theta =$  .

This completes the proof. $\square$

3. Quantum double of semiquasitriangular Hopf algebras

In [M2] it has been proven that if (H, R) is a quasitriangular Hopf algebra, then the quantum double $\mathrm{D}(H)$ is isomorphic to a classical Hopf crossed product. In this section we recall the notion of semiquasitriangular Hopf algebra introduced in

[G-G3] and we prove that if (H,R) is a finite-dimensional semiquasitriangular Hopf algebra, then the quantum double $\mathrm{D}(H)$ is isomorphic to a Hopf crossed product in the sense of [G-G2]. This is our version for this setting of the above mentioned theorem.

Before beginning we establish some notations. Let H be a Hopf algebra and $R=\sum_i R_i^{(1)}\otimes R_i^{(2)}$ an invertible element of $H\otimes H$. We will write $R=R^{(1)}\otimes R^{(2)}$, understanding the summation symbol and the index i. Similarly $R^{-(1)}\otimes R^{-(2)}$ denotes R^{-1}. When it is necessary we let $R'^{(1)}\otimes R'^{(2)}$, $\overline{R}^{(1)}\otimes\overline{R}^{(2)}$, etcetera denote copies of R.

Definition 3.1. A Hopf algebra H is called semicocommutative if the right adjoint coaction factorizes through $H\otimes \mathrm{Z}(H)$, where $\mathrm{Z}(H)$ denotes the center of H. That is, if $h_2\otimes S(h_1)h_3\in H\otimes \mathrm{Z}(H)$ for all $h\in H$.

For instance, the commutative and cocommutative Hopf algebras are semicocommutative Hopf algebras. Moreover, the class of these algebras is closed under the operations of taking tensor products, subHopfalgebras and quotients.

Definition 3.2. A semiquasitriangular Hopf algebra is a pair (H,R), where H is a Hopf algebra with bijective antipode and $R\in H\otimes H$ is an invertible element satisfying

1. $R_1^{(1)}\otimes R_2^{(1)}\otimes R^{(2)}=R^{(1)}\otimes R'^{(1)}\otimes R^{(2)}R'^{(2)}$,
2. $R^{(1)}\otimes R_1^{(2)}\otimes R_2^{(2)}=R^{(1)}R'^{(1)}\otimes R'^{(2)}\otimes R^{(2)}$,
3. $R^{(1)}\otimes R_2^{(2)}R'^{(1)}\otimes R_1^{(2)}R'^{(2)}=R^{(1)}\otimes R'^{(1)}R_1^{(2)}\otimes R'^{(2)}R_2^{(2)}$,
4. $R_2^{(1)}R'^{(1)}\otimes R_1^{(1)}R'^{(2)}\otimes R^{(2)}=R'^{(1)}R_1^{(1)}\otimes R'^{(2)}R_2^{(1)}\otimes R^{(2)}$,
5. $\nu(h):=R^{(2)}h_2R'^{(2)}\otimes S(h_1)S(R^{(1)})h_3R'^{(1)}\in H\otimes \mathrm{Z}(H)$ for all $h\in H$,

If (H,R) is a semiquasitriangular Hopf algebra, we say that R is a semiquasitriangular structure for H.

For instance, every quasitriangular Hopf algebra is semiquasitriangular. Moreover $(H,1_{H\otimes H})$ is a semiquasitriangular Hopf algebra if and only if H is semicocommutative.

Recall that the quantum double $D(H)$, of a finite-dimensional Hopf algebra H, is the tensor product $H^*\otimes H$, endowed with the multiplication

$$(\phi\otimes h)(\varphi\otimes l)=\varphi_2\phi\otimes h_2l\langle\varphi_1,S(h_1)\rangle\langle\varphi_3,h_3\rangle,$$

and the codiagonal comultiplication. This is a Hopf algebra with unit $\epsilon\otimes 1$, counit $\epsilon(\phi\otimes h)=\phi(1)\epsilon(h)$ and antipode $S(\phi\otimes h)=S^{-1}(\phi_2)\otimes S(h_2)\langle\phi_1,h_1\rangle\langle\phi_3,S(h_3)\rangle$.

Let (H,R) be a finite-dimensional semiquasitriangular Hopf algebra and let $f\colon \mathrm{D}(H)\to H^*\otimes H$ be the map

$$f(\phi\otimes h)=\phi_1\langle\phi_2,R^{-(1)}\rangle\otimes R^{-(2)}h.$$

Note that f is bijective, with inverse given by $f^{-1}(\phi\otimes h)=\phi_1\langle\phi_2,R^{(1)}\rangle\otimes R^{(2)}h$. In this section B denotes $H^*\otimes H$, endowed with the unique Hopf algebra structure making f an isomorphism of Hopf algebras.

Lemma 3.3. *If (H,R) is a finite-dimensional semiquasitriangular Hopf algebra, then the vector space H^* is an associative algebra with multiplication*

$$\phi \centerdot \varphi = \varphi_2\phi_1\langle\phi_3, R^{(1)}\rangle\langle\varphi_1, S(R^{(2)})\rangle\langle\phi_2, \overline{R}^{(1)}\rangle\langle\varphi_3, \overline{R}^{(2)}\rangle$$

and unit ϵ. Let $\underline{H}^*$ denote this algebra. The algebra B introduced above is a twisted tensor product $\underline{H}^* \otimes_\chi H$, of $\underline{H}^*$ and H, in the sense of Example 1.4. The map χ is given by

$$\chi(h\otimes\phi) = \phi_2 \otimes R^{-(2)}h_2R^{(2)}\langle\phi_3, R^{-(1)}\rangle\langle\phi_5, R^{(1)}\rangle\langle\phi_1, S(h_1)\rangle\langle\phi_4, h_3\rangle.$$

Proof. It is immediate that ϵ is the unit of $\underline{H}^*$ and that $(\epsilon\otimes h)(\epsilon\otimes l) = \epsilon\otimes hl$, for all $h,l\in H$, where the multiplication in the left side of this equality is made out in B. Moreover, the multiplication $(\phi\otimes 1)(\varphi\otimes 1)$ in B, is given by

$$\begin{aligned}
(\phi\otimes 1)(\varphi\otimes 1) &= f\big((\phi_1\langle\phi_2, R^{(1)}\rangle\otimes R^{(2)})(\varphi_1\langle\varphi_2, \overline{R}^{(1)}\rangle\otimes\overline{R}^{(2)})\big)\\
&= f\big(\varphi_2\phi_1\otimes R_2^{(2)}\overline{R}^{(2)}\langle\varphi_1, S(R_1^{(2)})\rangle\langle\varphi_3, R_3^{(2)}\rangle\langle\phi_2, R^{(1)}\rangle\langle\varphi_4, \overline{R}^{(1)}\rangle\big)\\
&= f\big(\varphi_2\phi_1\otimes R'^{(2)}\overline{R}^{(2)}\langle\varphi_1, S(R''^{(2)})\rangle\langle\varphi_3, R^{(2)}\rangle\langle\phi_2, R^{(1)}R'^{(1)}R''^{(1)}\rangle\langle\varphi_4, \overline{R}^{(1)}\rangle\big)\\
&= f\big(\varphi_2\phi_1\otimes R'^{(2)}\overline{R}^{(2)}\langle\varphi_1, S(R''^{(2)})\rangle\langle\phi_2, R^{(1)}R'^{(1)}\rangle\langle\phi_3, R''^{(1)}\rangle\langle\varphi_3, R^{(2)}\overline{R}^{(1)}\rangle\big)\\
&= f\big(\varphi_2\phi_1\otimes R^{(2)}R'^{(2)}\langle\varphi_1, S(R''^{(2)})\rangle\langle\phi_2, R'^{(1)}\overline{R}^{(1)}\rangle\langle\phi_3, R''^{(1)}\rangle\langle\varphi_3, R^{(1)}\overline{R}^{(2)}\rangle\big)\\
&= f\big(\varphi_2\phi_1\otimes R^{(2)}\langle\varphi_1, S(R''^{(2)})\rangle\langle\phi_2, R_2^{(1)}\overline{R}^{(1)}\rangle\langle\phi_3, R''^{(1)}\rangle\langle\varphi_3, R_1^{(1)}\overline{R}^{(2)}\rangle\big)\\
&= f\big(\varphi_2\phi_1\langle\varphi_3\phi_2, R^{(1)}\rangle\otimes R^{(2)}\big)\langle\varphi_1, S(R''^{(2)})\rangle\langle\phi_3, \overline{R}^{(1)}\rangle\langle\phi_4, R''^{(1)}\rangle\langle\varphi_4, \overline{R}^{(2)}\rangle\\
&= \phi\centerdot\varphi\otimes 1,
\end{aligned}$$

where the third equality follows from [G-G2, Prop. 1.2 (2)], the fifth one follows from [G-G2, Prop. 1.4], the sixth one follows from item (1) of Definition 3.2 and the other equalities follow by direct computations. Hence, H and $\underline{H}^*$ are subalgebras of B, which implies that $\underline{H}^*$ is an associative algebra and B is a twisted tensor product $\underline{H}^*\otimes_\chi H$ of $\underline{H}^*$ and H. It remains to check the formula of χ. But,

$$\begin{aligned}
\chi(h\otimes\phi) &= (\epsilon\otimes h)(\phi\otimes 1)\\
&= f\big((\epsilon\otimes h)(\phi_1\langle R^{(1)}, \phi_2\rangle\otimes R^{(2)})\big)\\
&= f\big(\phi_2\otimes h_2R^{(2)}\big)\langle\phi_4, R^{(1)}\rangle\langle\phi_1, S(h_1)\rangle\langle\phi_3, h_3\rangle\\
&= f\big(\phi_2\otimes R^{-(2)}h_2R^{(2)}\big)\langle\phi_3, R^{-(1)}\rangle\langle\phi_5, R^{(1)}\rangle\langle\phi_1, S(h_1)\rangle\langle\phi_4, h_3\rangle,
\end{aligned}$$

as we claim. $\square$

Note that if (H,R) is a finite-dimensional semiquasitriangular Hopf algebra, then H is a left H^*-module via $h^\phi := \phi(S(h_1)R^{-(1)}h_3R^{(1)})R^{-(2)}h_2R^{(2)}$, for all $h\in H$ and $\phi\in H^*$. In fact, this is the left action associated to the right coaction ν of [G-G2, Prop. 2.3].

Theorem 3.4. *If (H,R) is a finite-dimensional semiquasitriangular Hopf algebra, then the map $s\colon H\otimes \underline{H}^* \to \underline{H}^*\otimes H$, defined by $s(h\otimes\phi)=\phi_1\otimes h^{\phi_2}$ is a transposition of H on $\underline{H}^*$, the map $\rho: H\otimes \underline{H}^*\to \underline{H}^*$, given by*

$$\rho(h\otimes\phi)=h_2\rightharpoonup\phi\leftharpoonup S(h_1),$$

is an s-action, and the algebra B introduced before Lemma 3.3 is the smash product associated with (s,ρ).

Proof. We are going to apply Theorem 2.1 with χ as in Lemma 3.3 and $\mathcal{F}$ the trivial cocycle. By Lemma 3.3 we know that χ is compatible with the multiplication of H and, from the fact that $\mathcal{F}$ is trivial, it follows immediately that $(\mathcal{F}\otimes\mu)\,\Delta_{H\otimes H}=(\underline{H}^*\otimes\Delta)\,\mathcal{F}$. It is clear that $f:=(\underline{H}^*\otimes\epsilon)\,\mathcal{F}$ is the trivial cocycle and so Condition (4) of Theorem 2.1 is clearly satisfied. Hence, it suffices to check that $\rho=(\underline{H}^*\otimes\epsilon)\,\chi$, $s=(\rho\otimes H)\,(S\otimes\chi)\,(\Delta\otimes\underline{H}^*)$ and Conditions (1)–(3) of that theorem are satisfied. Since $(H\otimes\epsilon)(R)=(H\otimes\epsilon)(R^{-1})=1$, we have

$$\begin{aligned}(\underline{H}^*\otimes\epsilon)\,\chi(h\otimes\phi)&=\phi_2\langle\phi_3,1\rangle\langle\phi_5,1\rangle\langle\phi_1,S(h_1)\rangle\langle\phi_4,h_2\rangle\\&=h_2\rightharpoonup\phi\leftharpoonup S(h_1)\end{aligned}$$

and

$$\begin{aligned}(\rho\otimes H)\,(S\otimes\chi)\,(\Delta\otimes\underline{H}^*)(h\otimes\phi)&=\big(S(h_1)\rightharpoonup\phi_2\leftharpoonup S^2(h_2)\big)\,\langle\phi_1,S(h_3)\rangle\\&\quad\otimes R^{-(2)}h_4R^{(2)}\langle\phi_3,R^{-(1)}\rangle\langle\phi_5,R^{(1)}\rangle\langle\phi_4,h_5\rangle\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_3R^{(1)}\big)R^{-(2)}h_4R^{(2)}\\&=\phi_1\otimes h^{\phi_2},\end{aligned}$$

for all $h\in H$ and $\phi\in H^*$, as we need. Next, we check Conditions (1)–(3) of Theorem 2.1.

Condition (1): Let $h,l\in H$ and $\phi\in\underline{H}^*$. By item (5) of Definition 3.2,

$$\begin{aligned}\phi_1\otimes l^{\phi_2}\otimes h^{\phi_3}&=\phi_1\otimes\phi_2\big(S(l_1)R^{-(1)}l_3R^{(1)}S(h_1)\overline{R}^{-(1)}h_3\overline{R}^{(1)}\big)T\\&=\phi_1\otimes\phi_2\big(S(h_1)\overline{R}^{-(1)}h_3\overline{R}^{(1)}S(l_1)R^{-(1)}l_3R^{(1)}\big)T\\&=\phi_1\otimes l^{\phi_3}\otimes h^{\phi_2},\end{aligned}$$

where $T=R^{-(2)}l_2R^{(2)}\otimes\overline{R}^{-(2)}h_2\overline{R}^{(2)}$.

Condition (2): By items (2) and (5) of Definition 3.2,

$$\begin{aligned}(\underline{H}^*\otimes\Delta)\,s(h\otimes\phi)&=(\underline{H}^*\otimes\Delta)\big(\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_3R^{(1)}\big)R^{-(2)}h_2R^{(2)}\big)\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_4R^{(1)}\big)R_1^{-(2)}h_2R_1^{(2)}\otimes R_2^{-(2)}h_3R_2^{(2)}\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}\widetilde{R}^{-(1)}h_4\widetilde{R}^{(1)}R^{(1)}\big)R^{-(2)}h_2R^{(2)}\otimes\widetilde{R}^{-(2)}h_3\widetilde{R}^{(2)}\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_3S(h_4)\widetilde{R}^{-(1)}h_6\widetilde{R}^{(1)}R^{(1)}\big)R^{-(2)}h_2R^{(2)}\otimes\widetilde{R}_2^{-(2)}h_5\widetilde{R}^{(2)}\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_3R^{(1)}S(h_4)\widetilde{R}^{-(1)}h_6\widetilde{R}^{(1)}\big)R^{-(2)}h_2R^{(2)}\otimes\widetilde{R}^{-(2)}h_5\widetilde{R}^{(2)}\\&=\phi_1\otimes\phi_2\big(S(h_1)R^{-(1)}h_3R^{(1)}\big)R^{-(2)}h_2R^{(2)}\otimes\phi_3\big(S(h_4)\widetilde{R}^{-(1)}h_6\widetilde{R}^{(1)}\big)\widetilde{R}^{-(2)}h_5\widetilde{R}^{(2)}\\&=\phi_1\otimes h_1^{\phi_2}\otimes h_2^{\phi_3}\\&=(s\otimes H)\,(H\otimes s)(\Delta\otimes\underline{H}^*)(h\otimes\phi),\end{aligned}$$

for all $h \in H$ and $\phi \in \underline{H}^*$.

Condition (3): Let $c\colon H \otimes H \to H \otimes H$ be the flip. By item (5) of Definition 3.2,

$$
\begin{aligned}
s\,(H \otimes \rho)(h \otimes l \otimes \phi) &= s\big(h \otimes l_2 \rightharpoonup \phi \leftharpoonup S(l_1)\big)\\
&= (l_2 \rightharpoonup \phi \leftharpoonup S(l_1)_1 \otimes h^{(l_2 \rightharpoonup \phi \leftharpoonup S(l_1))_2}\\
&= l_2 \rightharpoonup \phi_1 \leftharpoonup S(l_1) \otimes h^{l_4 \rightharpoonup \phi_2 \leftharpoonup S(l_3)}\\
&= l_2 \rightharpoonup \phi_1 \leftharpoonup S(l_1) \otimes \phi_2\big(S(l_3)S(h_1)R^{-(1)}h_3R^{(1)}l_4\big)R^{-(2)}h_2R^{(2)}\\
&= l_2 \rightharpoonup \phi_1 \leftharpoonup S(l_1) \otimes \phi_2\big(S(l_3)l_4S(h_1)R^{-(1)}h_3R^{(1)}\big)R^{-(2)}h_2R^{(2)}\\
&= l_2 \rightharpoonup \phi_1 \leftharpoonup S(l_1) \otimes \phi_2\big(S(h_1)R^{-(1)}h_3R^{(1)}\big)R^{-(2)}h_2R^{(2)}\\
&= (\rho \otimes H)\,(H \otimes s)\,(c \otimes \underline{H}^*)(h \otimes l \otimes \phi),
\end{aligned}
$$

for all $h, l \in H$ and $\phi \in \underline{H}^*$. $\square$

Let $T\colon H \to H$ be the map defined by $T(h) = R^{(2)}h_2R'^{(2)}S(h_1)S(R^{(1)})h_3R'^{(1)}$ and let $u = S(R^{(2)})R^{(2)}$ the Drinfeld element of (H, R). In [G-G2, Prop. 3.2] was proved that u is invertible and that $S^2(h)u = uT(h)$ for all $h \in H$. We will need the following result

Lemma 3.5. *The following equality holds:*

$$R_1^{(1)} \otimes S^{-1}(R_2^{(1)}) \otimes R^{(2)} = R_1^{(1)} \otimes T(S^{-1}(R_2^{(1)})) \otimes R^{(2)}.$$

Proof. By Propositions 1.2(4) and 1.3 of [G-G3], we have

$$
\begin{aligned}
&R_1^{(1)} \otimes T(S^{-1}(R_2^{(1)})) \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes R''^{(2)}S^{-1}(R_3^{(1)})R'^{(2)}R_4^{(1)}S(R''^{(1)})S^{-1}(R_2^{(1)})R'^{(1)} \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes R''^{(2)}S^{-1}(R'^{(2)}R_3^{(1)})R_4^{(1)}S(R''^{(1)})S^{-1}(R'^{(1)}R_2^{(1)}) \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes R''^{(2)}S^{-1}(R_2^{(1)}R'^{(2)})R_4^{(1)}S(R''^{(1)})S^{-1}(R_3^{(1)}R'^{(1)}) \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes R''^{(2)}R'^{(2)}S^{-1}(R_2^{(1)})R_4^{(1)}S(R''^{(1)})R'^{(1)}S^{-1}(R_3^{(1)}) \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes S^{-1}(R_2^{(1)})R_4^{(1)}S^{-1}(R_3^{(1)}) \otimes R^{(2)}\\
&\quad = R_1^{(1)} \otimes S^{-1}(R_2^{(1)}) \otimes R^{(2)},
\end{aligned}
$$

as we want. $\square$

Proposition 3.6. *The comultiplication and the antipode of $B = \underline{H}^* \# H$ are given by*

$$\Delta(\phi\#h) = \Big(\phi_1\#R^{(2)}h_1\Big) \otimes \Big((R^{(1)} \triangleright \phi_2)\#h_2\Big)$$

and

$$S(\phi\#h) = (\epsilon\#S(h))\,\Big(S^{-1}(\phi_3) \otimes \langle\phi_2, R'^{(1)}_1\rangle\langle\phi_1, R''^{(2)}\rangle\langle\phi_4, S(R'^{(1)}_2)uR''^{(1)}\rangle R'^{(2)}\Big),$$

where $\triangleright\colon H\otimes H^*\to H^*$ *is the left coadjoint action* $h\triangleright\phi := \phi_2\langle S(\phi_1)\phi_3, h\rangle$.

Proof. We prove the second formula and leave the first one to the reader. Let $h\in H$ and $\phi\in H^*$. It is sufficient to see that $S(\epsilon\# h)=\epsilon\# S(h)$ and

$$S(\phi\otimes 1)=S^{-1}(\phi_3)\otimes\langle\phi_2, R'^{(1)}_1\rangle\langle\phi_1, R''^{(2)}\rangle\langle\phi_4, S(R'^{(1)}_2)uR''^{(1)}\rangle R'^{(2)}.$$

The first equality is immediate and for the second one we have

$$\begin{aligned}
&S(\phi\otimes 1)=f(S(f^{-1}(\phi\otimes 1)))\\
&=f(S(\phi_1\otimes\langle\phi_2,R^{(1)}\rangle R^{(2)}))\\
&=f(S^{-1}(\phi_2)\otimes\langle\phi_1,R^{(2)}_1\rangle\langle\phi_3,S(R^{(2)}_3)\rangle\langle\phi_4,R^{(1)}\rangle S(R^{(2)}_2))\\
&=S^{-1}(\phi_3)\otimes\langle S^{-1}(\phi_2),\overline{R}^{-(1)}\rangle\langle\phi_1,R^{(2)}_1\rangle\langle\phi_4,S(R^{(2)}_3)\rangle\langle\phi_5,R^{(1)}\rangle\overline{R}^{-(2)}S(R^{(2)}_2)\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,\overline{R}^{(1)}\rangle\langle\phi_1,R^{(2)}_1\rangle\langle\phi_4,S(R^{(2)}_3)\rangle\langle\phi_5,R^{(1)}\rangle\overline{R}^{(2)}S(R^{(2)}_2)\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,\overline{R}^{(1)}\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,S(R^{(2)})\rangle\langle\phi_5,R^{(1)}R'^{(1)}R''^{(1)}\rangle\overline{R}^{(2)}S(R'^{(2)})\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,\overline{R}^{(1)}\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,S(R^{(2)})\rangle\langle\phi_5,R^{(1)}S^{-1}(R'^{(1)})R''^{(1)}\rangle\overline{R}^{(2)}R'^{(2)}\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,R'^{(1)}_1\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,S(R^{(2)})\rangle\langle\phi_5,R^{(1)}S^{-1}(R'^{(1)}_2)R''^{(1)}\rangle R'^{(2)}\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,R'^{(1)}_1\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,uS^{-1}(R'^{(1)}_2)R''^{(1)}\rangle R'^{(2)}\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,R'^{(1)}_1\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,uT(S^{-1}(R'^{(1)}_2))R''^{(1)}\rangle R'^{(2)}\\
&=S^{-1}(\phi_3)\otimes\langle\phi_2,R'^{(1)}_1\rangle\langle\phi_1,R''^{(2)}\rangle\langle\phi_4,S(R'^{(1)}_2)uR''^{(1)}\rangle R'^{(2)},
\end{aligned}$$

where the fifth and seventh equalities follow from [G-G2, Prop. 1.3], the sixth and eighth ones follow from [G-G2, Prop. 1.2], the tenth equality follows from Lemma 3.5 and the last one follows from [G-G2, Prop. 3.2].

Remark 3.7. The Hopf algebra B is a bialgebra cross product $\underline{H}^*\ {}^{\nu_l}_{\mu_l}{\bowtie}^{\nu_r}_{\mu_r}\ H$ in the sense of [B-D1, Definition and Proposition 2.15], where

$$\mu_l(h\otimes\phi)=\phi_2\langle\phi_1,S(h_1)\rangle\langle\phi_3,h_3\rangle,\quad \mu_r(h\otimes\phi)=h_2\langle\phi_1,S(h_1)\rangle\langle\phi_2,h_3\rangle,$$
$$\nu_l(\phi)=R^{(2)}\otimes\phi_2\langle S(\phi_1)\phi_3,R^{(1)}\rangle,\quad \nu_r(h)=h\otimes\epsilon_H.$$

However, if the map $s\colon H\otimes\underline{H}^*\to\underline{H}^*\otimes H$ is not the flip, then the smash product associated with (s,ρ) is not the algebra cross product underlying to a Hopf product bialgebra in the sense of [B-D1] and [B-D2]. Indeed, assume that there exists a braided category $\mathcal{C}$ containing H and $\underline{H}^*$ and such that:

(1) H is an algebra and a coalgebra in $\mathcal{C}$,
(2) $\underline{H}^*$ is an algebra and a coalgebra in $\mathcal{C}$,
(3) $c_{H\underline{H}^*}\colon H\otimes\underline{H}^*\to\underline{H}^*\otimes H$ is s.

Then $s(h\otimes\phi)=\phi\otimes h$ for all $h\in H$ and $\phi\in\underline{H}^*$, since

$$c_{H\underline{H}^*}(h\otimes\phi)=(\underline{H}^*\otimes\epsilon_H\otimes\epsilon_{\underline{H}^*}\otimes H)\,c_{BB}((\epsilon_H\# h)\otimes(\phi\# 1_H))$$

and c_{BB} is the flip.

Remark 3.8. It is easy to check that if (H,R) is a finite-dimensional quasitriangular Hopf algebra, then the transposition s of Theorem 3.4 is the flip. So, this result implies [M2, Theorem 7.4.5].

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