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Representations of copointed Hopf algebras arising from the tetrahedron rack

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Abstract We study the copointed Hopf algebras attached to the Nichols algebra of the affine rack $\mathrm{Aff}(\mathbb{F}_4, \omega)$, also known as tetrahedron rack, and the 2-cocycle -1. We investigate the so-called Verma modules and classify all the simple modules. We conclude that these algebras are of wild representation type and not quasitriangular, also we analyze when these are spherical.

1 Introduction

We work over an algebraically closed field $\Bbbk$ of characteristic zero. Let G be a finite non-abelian group and let $\Bbbk^G$ denote the algebra of functions on G. A Hopf algebra with coradical isomorphic to $\Bbbk^G$ for some G is called *copointed*. Nicolás Andruskiewitsch and the second author began the study of the copointed Hopf algebras by classifying those finite-dimensional with $G = \mathbb{S}_3$ in [AV1] and by analyzing the representation theory of them in [AV2].

Since $\Bbbk^G$ is a commutative semisimple algebra, the representation theory of a copointed Hopf algebra over $\Bbbk^G$ is studied in [AV2] by analogy with the representation theory of semisimple Lie algebras, with $\Bbbk^G$ playing the role of the Cartan subalgebra and the induced modules from the simple one-dimensional $\Bbbk^G$-modules as Verma modules.

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There are few examples of Nichols algebras of finite-dimension over non-abelian groups, see for instance [G2, HLV]. In particular, those arising from affine racks are only seven, including the tetrahedron rack. If X is one of these affine racks, then all the liftings of the Nichols algebra $\mathcal{B}(-1, X)$ over $\Bbbk^G$ were classified in [GIV], where G is any group admitting a principal YD-realization of X with constant 2-cocycle -1. Also the liftings of $\mathcal{B}(X, -1)$ over the group algebra $\Bbbk G$ were classified in [GIV].

The notation used in the following is explained in Section 3. Let G be a finite group and $V \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ a faithful principal YD-realization of the tetrahedron rack with constant 2-cocycle -1. The Nichols algebra $\mathcal{B}(V)$ has dimension 72. The ideal of relations of $\mathcal{B}(V)$ is generated by four quadratic elements and only one of degree six called z. By [GIV], the liftings of $\mathcal{B}(V)$ over $\Bbbk^G$ are the copointed Hopf algebras $\{\mathcal{A}_{G,\lambda}\}_{\lambda\in\Bbbk}$, in which the quadratic relations of $\mathcal{B}(V)$ still hold and the 6-degree relation $z = 0$ deforms to $z = \lambda(1 - \chi_z^{-1}) \in \Bbbk^G$.

The goal of this paper is to investigate the representation theory of the family $\{\mathcal{A}_{G,\lambda}\}_{\lambda\in\Bbbk}$ following the strategy of [AV2]. We conclude that there are essentially two kinds of Verma modules. Here is an account of our main results which apply to any group G admitting a faithful principal YD-realization of the tetrahedron rack with constant 2-cocycle -1:

- *Let $g \in G$. If the element $z = \lambda(1-\chi_z^{-1})$ annihilates the generator of the Verma modules M_g, then M_g inherits a structure of $\mathcal{B}(V)$-module such that it is a free $\mathcal{B}(V)$-module of rank 1, see Lemma 14. Hence M_g has a unique simple quotient of dimension 1 called $\Bbbk_g$.*
- *Otherwise M_g is the direct sum of six 12-dimensional non isomorphic simple projective modules L_i^g, see Lemma 15. Tables 1–6 in the Appendix describe the simple modules L_i^g.*
- *We prove that $\mathcal{A}_{G,\lambda}$ is of wild representation type, Proposition 17.*
- *We give a necessary condition for a copointed Hopf algebra to be quasitriangular, Lemma 8. As a consequence $\mathcal{A}_{G,\lambda}$ is not quasitriangular, Proposition 12.*
- *We characterize those $\mathcal{A}_{G,\lambda}$ which are spherical Hopf algebras, see Proposition 18.*

The other copointed Hopf algebras classified in [GIV] are defined by similar relations to $\mathcal{A}_{G,\lambda}$, roughly speaking a set of quadratic ones and other single relation of bigger degree, but their dimension are much bigger than $\dim \mathcal{A}_{G,\lambda} = 72|G|$. To extend this work to the other copointed Hopf algebras in [GIV], a better understanding of the corresponding Nichols algebras is needed. We hope that our work will be useful for this purpose.

The paper is organized as follows. In Section 2 we analyze the representation theory of copointed Hopf algebras with emphasis in the weight spaces of the modules, we characterize the one-dimensional modules and describe the subalgebra corresponding to the homogeneous elements of degree $e \in G$. In Section 3, we present our main object of study: the algebras $\mathcal{B}(V)$ and $\mathcal{A}_{G,\lambda}$. In Section 4 we concentrate our attention on representations of the algebras $\{\mathcal{A}_{G,\lambda}\}_{\lambda\in\Bbbk}$. A description of the simple $\mathcal{A}_{G,\lambda}$-modules is in the Appendix.

1.1 Conventions and notation

We set $\Bbbk^* = \Bbbk \setminus \{0\}$. If X is a set, $\Bbbk X$ denotes the free vector space over X.

Let A be a Hopf algebra. Then Δ, ε, $\mathcal{S}$ denote respectively the comultiplication, the counit and the antipode. The group of group-like elements is $G(A)$. Let ${}^A_A\mathcal{YD}$ be the category of Yetter-Drinfeld modules over A. The Nichols algebra $\mathcal{B}(V)$ of $V \in {}^A_A\mathcal{YD}$ is the graded quotient $T(V)/\mathcal{J}$ where $\mathcal{J}(V)$ is the largest Hopf ideal of $T(V)$ generated as an ideal by homogeneous elements of degree ≥ 2 [AS, 2.1].

Let $\{A_{[n]}\}_{n\geq 0}$ denote the coradical filtration of A. Assume $A_{[0]} = H$ is a Hopf subalgebra. Let $\mathrm{gr}A$ be the graded Hopf algebra associated to the coradical filtration. Then $\mathrm{gr}A \simeq R\#H$ where $R \in {}^H_H\mathcal{YD}$ is called the *diagram of* A and $V = R_{[1]} \in {}^H_H\mathcal{YD}$ is the *infinitesimal braiding* [AS, Definition 1.15]. If $R = \mathcal{B}(V)$, then A is said to be a *lifting of* $\mathcal{B}(V)$ (*over* H).

Recall that two idempotents $\mathbf{e}, \widetilde{\mathbf{e}} \in A$ are *orthogonal* if $\mathbf{e}\widetilde{\mathbf{e}} = 0 = \widetilde{\mathbf{e}}\mathbf{e}$. An idempotent is *primitive* if it is not possible to express it as the sum of two nonzero orthogonal idempotents. A set $\{\mathbf{e}_i\}_{i\in I}$ of idempotents of A is *complete* if $1 = \sum_{i\in I} e_i$.

Assume $\dim A < \infty$. Then A is a Frobenius algebra, see *e. g.* [FMoS, Lemma 1.5]. Let $\mathbf{e}$ be a primitive idempotent of A. Then $top(A\mathbf{e}) = A\mathbf{e}/rad(A\mathbf{e})$ and the socle $soc(A\mathbf{e})$ of $A\mathbf{e}$ are simple modules [CR, Theorems 54.11 and 58.12]. Moreover, $A\mathbf{e}$ is the injective hull of $soc(A\mathbf{e})$ and the projective cover of $top(A\mathbf{e})$, see *e. g.* [CR, page 400 and Theorem 58.14]. We denote by $\mathrm{Irr}\, A$ a set of representative of simple A-modules.

2 Representations of copointed Hopf algebras

Let G be a finite group, $\Bbbk G$ the group algebra and $\Bbbk^G$ the algebra of functions on G. Let $\{g : g \in G\}$ and $\{\delta_g : g \in G\}$ be the dual basis of $\Bbbk G$ and $\Bbbk^G$, respectively; e denotes the identity element of G.

If M is a $\Bbbk^G$-module, then $M[g] = \delta_g \cdot M$ is the *isotypic component of weight* $g \in G$. We denote by $\Bbbk_g$ the one-dimensional $\Bbbk^G$-module of weight g. We define

$$M^\times = \oplus_{g\neq e} M[g] \quad \text{and} \quad \mathrm{Supp}\, M = \{g \in G : M[g] \neq 0\}.$$

Let A be a finite-dimensional *copointed* Hopf algebra over $\Bbbk^G$, i. e. its coradical is isomorphic to $\Bbbk^G$. We consider A as a left $\Bbbk^G$-module via the left adjoint action

$$\mathrm{ad}\, \delta_t(a) = \sum_{s\in G} \delta_s a \delta_{t^{-1}s} \quad \forall t \in G,\, a \in A.$$

By [AV1, Lemma 3.1], $A = \oplus_{g\in G} A[g]$ is a G-graded algebra and

$$\delta_t a_s = a_s \delta_{s^{-1}t} \qquad \forall a_s \in A[s], s, t \in G. \tag{1}$$

If M is an A-module, then M is a $\Bbbk^G$-module by restriction. Hence

$$A[g] \cdot M[h] \subseteq M[gh] \qquad \forall g, h \in G \text{ by (1).} \tag{2}$$

That is, M is a G-graded A-module.

We denote by $A_{\Bbbk^G} = A$ as right $\Bbbk^G$-module via the right multiplication. Its isotypic components are $(A_{\Bbbk^G})[g] = A\delta_g$ for all $g \in G$. Note that A is a $\Bbbk^G$-bimodule with the above actions since $\Bbbk^G \subseteq A[e]$.

Let $R \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ be the diagram of A. Then the multiplication in A induces an isomorphism $R \otimes \Bbbk^G \longrightarrow A$ of $\Bbbk^G$-bimodules [AAGMV, Lemma 4.1]. Hence we can think of R as a left $\Bbbk^G$-submodule of A and therefore

$$A[g] = R[g]\,\Bbbk^G \text{ and } (A_{\Bbbk^G})[g] = R\,\delta_g \qquad \forall g \in G. \tag{3}$$

As in [AV2], we define the *Verma module of A of weight $g \in G$* as the induced module

$$M_g = \operatorname{Ind}_{\Bbbk^G}^{A} \Bbbk_g = A \otimes_{\Bbbk^G} \Bbbk\delta_g.$$

Then M_g is projective, being induced from a module over a semisimple algebra, and hence injective, because A is Frobenius. By (1) and (3), the weight spaces satisfy $M_g[h] = R[hg^{-1}]\delta_g$ for all $h \in G$. Also, $M_g = A\delta_g = R\delta_g$ and $A = \oplus_{g \in G} M_g$.

Notice that if L is a simple A-module and $0 \neq v \in L[g]$, then L is a quotient of M_g via $\delta_g \mapsto \delta_g \cdot v = v$.

Let $\mathbf{e} \in A$ be an idempotent. We say that $\mathbf{e}$ is a *g-idempotent* if $\mathbf{e} \in R[e]\delta_g$. A set $\{\mathbf{e}_i\}_{i \in I}$ of g-idempotents is called *complete* if $\delta_g = \sum_{i \in I} \mathbf{e}_i$. Next lemma ensures that there always exists a complete set of orthogonal primitive g-idempotents.

Lemma 1. *Let $g \in G$, $\mathbf{e}$ be a g-idempotent and $\mathcal{E}_g = \{\mathbf{e}_i\}_{i \in I}$ be a set of orthogonal idempotents of A such that $\delta_g = \sum_{i \in I} \mathbf{e}_i$.*

(a) $\mathcal{E}_g$ is a complete set of g-idempotents.

(b) $\mathbf{e}$ is primitive if and only if it is not possible to express $\mathbf{e}$ as a sum of orthogonal g-idempotents.

(c) There is a complete set of orthogonal primitive g-idempotents in A.

(d) $\mathbf{e} \cdot M = \mathbf{e} \cdot M[g] \subseteq M[g]$ for any A-module M.

(e) If $\#\mathcal{E}_g = \dim R[e]$, then $\mathbf{e}_i$ is primitive for all $i \in I$. Moreover, if $\mathbf{e}$ is primitive, then $\mathbf{e} = \mathbf{e}_i$ for some $i \in I$.

(f) If $\#\mathcal{E}_g = \dim R[e]$, then $A\mathbf{e}_i \not\simeq A\mathbf{e}_j$ if $i \neq j$.

Proof (a) Fix $i \in I$ and set $\alpha = \mathbf{e}_i$ and $\beta = \sum_{i \neq j \in I} \mathbf{e}_j$. If $t \in G$ and $t \neq g$, then $0 = \delta_g\delta_t = \alpha\delta_t + \beta\delta_t$. Since α and β are orthogonal, $\alpha\delta_t = 0$. Hence $\alpha = \alpha\delta_g$ because $1 = \sum_{g \in G} \delta_g$. Similarly $\alpha = \delta_g\alpha$. Let $a_s \in R[s]$ such that $\alpha = \sum_{s \in G} a_s\delta_g$. Then $\alpha = \delta_g\alpha = \sum_{s \in G} \delta_g a_s \delta_g = \sum_{s \in G} a_s \delta_{s^{-1}g}\delta_g = a_e\delta_g$. That is, $\alpha = \mathbf{e}_i$ is a g-idempotent.

(b) The first implication is obvious. For the second implication, we proceed as in (a). (c) follows from (a) and (b). (d) holds because $\mathbf{e} \in R[e]\delta_g$.

(e) is a consequence of the fact that $\mathcal{E}_g$ is a basis of $R[e]\delta_g$. Indeed, pick $\alpha = \mathbf{e}_i \in \mathcal{E}_g$ and suppose $\alpha = a + b$ with a and b orthogonal g-idempotents of A. Then $(Aa)[e] \oplus (Ab)[e] = (A\alpha)[e] = (\Bbbk\mathcal{E}_g)\alpha = \Bbbk\alpha$ and therefore $a = 0$ or $b = 0$. For the second statement, we write $\mathbf{e} = \sum_{i\in I} a_i\mathbf{e}_i$ with $a_i \in \Bbbk$, $i \in I$. Since $\mathbf{e}^2 = \mathbf{e}$, $a_i = 0$ or 1 for all $i \in I$ and hence $\mathbf{e} = \mathbf{e}_i$ for some $i \in I$.

(f) $(A\mathbf{e}_i)[e] = \Bbbk\mathbf{e}_i \neq (A\mathbf{e}_j)[e] = \Bbbk\mathbf{e}_j$ if $i \neq j$. Hence $A\mathbf{e}_i \not\simeq A\mathbf{e}_j$.

Given a set of idempotents $\mathcal{E}$ and an A-module M, we write

$$\operatorname{Supp}_{\mathcal{E}} M = \{\mathbf{e} \in \mathcal{E} : \mathbf{e} \cdot M \neq 0\}.$$

By [CR, Theorem 54.16] if L is a simple A-module and $\mathbf{e} \in \operatorname{Supp}_{\mathcal{E}} L$, then

$$top(A\mathbf{e}) \simeq L.$$

This allows us to analyze the dimension of the weight spaces of the simple A-modules using g-idempotents.

Lemma 2. *Let $g \in G$ and $\mathcal{E}_g = \{\mathbf{e}_i\}_{i\in I}$ be a complete set of orthogonal primitive g-idempotents. Let L be a simple A-module.*

(a) $\dim L[g] = \# \operatorname{Supp}_{\mathcal{E}_g} L$.

(b) If $\#\mathcal{E}_g = \dim R[e]$ *or* 1, *then* $\dim L[g] = 1$ *or* 0.

(c) $\mathcal{E}_g = \bigcup_{L\in \operatorname{Irr} A} \operatorname{Supp}_{\mathcal{E}_g} L$ *is a partition.*

(d) $\dim R[e] \geq \sum_{L\in\operatorname{Irr} A}(\dim L[g])^2 = \sum_{L\in\operatorname{Irr} A}(\# \operatorname{Supp}_{\mathcal{E}_g} L)^2 \geq \#\mathcal{E}_g$.

Proof (a) By [CR, Theorem 54.16], $\dim \mathbf{e}_i \cdot L = 1$ for all $\mathbf{e}_i \in \operatorname{Supp}_{\mathcal{E}_g} L$. Pick $w_i \in \mathbf{e}_i \cdot L - \{0\}$ for each $i \in I$. Then $\{w_i : i \in I\}$ is a basis of $L[g]$ since $v = \delta_g \cdot v = \sum_{\mathbf{e}_i\in\operatorname{Supp}_{\mathcal{E}_g} L} \mathbf{e}_i \cdot v$ for all $v \in L[g]$.

(b) If $\#\mathcal{E}_g = 1$, then $\dim L[g] = 1$ or 0 by (a). If $\#\mathcal{E}_g = \dim R[e]$, the statement follows from (a) and Lemma 1 (f).

(c) is clear. (d) follows from (a) and (c) since

$$R[e]\delta_g = \oplus_{i\in I} R[e]\mathbf{e}_i = \oplus_{L\in\operatorname{Irr} A} \oplus_{\mathbf{e}_i\in\operatorname{Supp}_{\mathcal{E}_g} L} R[e]\mathbf{e}_i.$$

In some cases, the simple A-modules can be distinguished by their weight spaces.

Lemma 3. *Let $g \in G$ and $\mathcal{E}_g = \{\mathbf{e}_i\}_{i\in I}$ be a complete set of orthogonal primitive g-idempotents and assume that $top(A\mathbf{e}_i)$ and $top(A\mathbf{e}_j)$ are not isomorphic as $\Bbbk^G$-modules for all $i \neq j$. Let L be a simple A-module. Then $L \simeq top(A\mathbf{e}_i)$ as A-modules if and only if $L \simeq top(A\mathbf{e}_i)$ as $\Bbbk^G$-modules.*

Proof If $L \simeq top(A\mathbf{e}_i)$ as $\Bbbk^G$-modules, then $g \in \operatorname{Supp} L$. Hence $L \simeq top(A\mathbf{e}_j)$ for some j. Then $i = j$ because $top(A\mathbf{e}_i)$ and $top(A\mathbf{e}_j)$ are not isomorphic as $\Bbbk^G$-modules for $i \neq j$. The other implication is obvious.

For each $g \in G$, let $\mathcal{E}_g$ be a complete set of orthogonal primitive g-idempotents. If $\mathbf{e}, \tilde{\mathbf{e}} \in \mathcal{E}_g$ and $\mathbf{e}A\tilde{\mathbf{e}} \neq 0$, it is said that $\mathbf{e}$ and $\tilde{\mathbf{e}}$ are *linked*. This is an equivalence relation [CR, Definition 55.1]. Let $\mathcal{E}_g = \bigcup_{i \in I_g} B_i$ be the corresponding partition. The subalgebra $A[e] = R[e]\Bbbk^G$ can be used to compute the simple A-modules, see for instance [NaVO, Theorem 2.7.2].

Lemma 4. *Let $\mathcal{E}_g = \bigcup_{i \in I_g} B_i$ be as above. Then $\bigoplus_{\mathbf{e} \in B_i} A[e]\mathbf{e}$ is a subalgebra such that*

$$\{L[g] : L \in \operatorname{Irr} A \ \text{ and } \ B_i \cap \operatorname{Supp}_{\mathcal{E}_g} L \neq \emptyset\}$$

is a set of representative simple modules. Moreover as algebras

$$A[e] = \prod_{g \in G,\, i \in I_g} \bigoplus_{\mathbf{e} \in B_i} A[e]\mathbf{e}$$

Proof By (1), $\mathbf{e}\tilde{\mathbf{e}} = 0 = \tilde{\mathbf{e}}\mathbf{e}$ if either $\mathbf{e} \in \mathcal{E}_g$ and $\tilde{\mathbf{e}} \in \mathcal{E}_h$ with $g \neq h$ or $\mathbf{e}, \tilde{\mathbf{e}} \in \mathcal{E}_g$ but are not linked. Clearly, B_i is a complete set of orthogonal primitive idempotents of $\bigoplus_{\mathbf{e} \in B_i} A[e]\mathbf{e}$. Also $top(A[e]\mathbf{e}) = L[g]$ since $L[g] = top(A\mathbf{e})[g] = \overline{A[e]\mathbf{e}}$ for all $\mathbf{e} \in \mathcal{E}_g$.

For $g \in G$, we define the linear map $\chi_g : A \mapsto \Bbbk$ by

$$\chi_g(rf) = \varepsilon(r)f(g) \quad \forall\, rf \in A = R\Bbbk^G. \tag{4}$$

If χ_g is an algebra map, then $\Bbbk_g$ is also an A-module. Notice that Nichols algebras satisfy the hypothesis of the next lemma by [AV1, Lemma 3.1 (f)].

Lemma 5. *Let G be a finite group, A a finite-dimensional copointed Hopf algebra over $\Bbbk^G$ with diagram $R \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ and $\chi : A \mapsto \Bbbk$ an algebra map. If R is generated by $R^\times$ as an algebra, then $\chi = \chi_g$ for some $g \in G$ and $G(A^*)$ is a subgroup of G via $\chi_g \mapsto g$.*

Proof Let $g \in G$ such that $\chi(f) = f(g)$ for all $f \in \Bbbk^G$. By (1), $\chi(R^\times) = 0$ and then $\chi = \chi_g$. Since $\chi_g * \chi_h$ is an algebra map and $\chi_g * \chi_h(f) = f(gh)$ for all $f \in \Bbbk^G$, the proposition follows.

Example 1 Let $V \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ with finite-dimensional Nichols algebra $\mathcal{B}(V)$. Then $\{\delta_g : g \in G\}$ is a complete set of orthogonal primitive idempotents of $\mathcal{B}(V)\#\Bbbk^G$ and therefore $\{\Bbbk_g : g \in G\}$ are its simple modules.

Let $\int_A^r$ (resp. $\int_A^l$) denote the space of right (resp. left) integrals, see for example [Mo]. If $t \in \int_A^r$, then $\alpha \in G(A^*)$ is said to be *distinguished* whether $at = \alpha(a)t$ for all $a \in A$.

Lemma 6. *Let G be a finite group, A a finite-dimensional copointed Hopf algebra over $\Bbbk^G$ and $\alpha = \chi_g \in G(A^*)$ the distinguished group-like element. If $\mathbf{e}$ is a primitive idempotent, then*

$$\operatorname{Supp}(top(A\mathbf{e})) = g^{-1}\operatorname{Supp}(soc(A\mathbf{e})).$$

In particular, $\int_A^l = soc(A\mathbf{e}_{g^{-1}}) \subset R[g]\mathbf{e}_{g^{-1}}$ where $\mathbf{e}_{g^{-1}}$ is the primitive g^{-1}-idempotent such that $top(A\mathbf{e}_{g^{-1}}) \simeq \Bbbk_{g^{-1}}$.

Proof Let $\eta : A \to A$ be the Nakayama automorphism. If M is an A-module, then $\overline{M}$ denotes the vector space M with action $a \cdot m = \eta^{-1}(a)m$ for all $a \in A$, $m \in M$. Since $\eta^{-1}(a) = \langle \alpha^{-1}, S^2(a)_1 \rangle S^2(a)_2$ for all $a \in A$, see *e. g.* [FMoS, Lemma 1.5], $M[g^{-1}h] = \overline{M}[h]$ for all $h \in G$. Finally, $top(A\mathbf{e}) = soc(A\mathbf{e})$, see *e. g.* [NeSc, Lemma 2], and the lemma follows.

We include the next lemma for completeness.

Lemma 7. *Let A be an algebra and $a_1, ..., a_n$ be idempotents of A such that $a_i a_j = a_j a_i$ for all $i, j = 1, ..., n$. Set*

$$\mathbf{e}_i = a_i + a_i \sum_{\ell=1}^{i-1} (-1)^\ell \sum_{1 \leq j_1 < \cdots < j_\ell \leq i-1} a_{j_1} \cdots a_{j_\ell}.$$

Then $\mathbf{e}_i \mathbf{e}_j = \delta_{j,i} \mathbf{e}_i$ for all $i, j = 1, ..., n$.

Proof For $j < i$, we write

$$\mathbf{e}_i = a_i + a_i \sum_{\ell=1}^{i-1} (-1)^\ell \sum_{\substack{1 \leq j_1 < \cdots < j_\ell \leq i-1 \\ j_s \neq j}} a_{j_1} \cdots a_{j_\ell} + a_i \sum_{\ell=1}^{i-1} (-1)^\ell \sum_{\substack{1 \leq j_1 < \cdots < j_\ell \leq i-1 \\ j_s = j \text{ for some } s}} a_{j_1} \cdots a_{j_\ell}.$$

Then $a_j \mathbf{e}_i = 0$ and hence $\mathbf{e}_j \mathbf{e}_i = \delta_{i,j} \mathbf{e}_i$ for all $i, j = 1, ..., n$.

The order of the set $\{a_i\}$ alters the result of the above lemma. Moreover, it can produce $\mathbf{e}_i = 0$ for some i. For example: $\{1, a\}$ and $\{a, 1\}$ with a an idempotent.

2.1 Quasitriangular copointed Hopf algebras

Let G be a non-abelian group and A be a quasitriangular finite-dimensional copointed Hopf algebra over $\Bbbk^G$ with R-matrix $R \in A \otimes A$. Let (A_R, R) be its unique minimal subquasitriangular Hopf algebra [R]. Then $A_R = HB$ with $H, B \subseteq A$ Hopf subalgebras such that $B \simeq H^{*\mathrm{cop}}$ by [R, Proposition 2 and Theorem 1].

Lemma 8. *H, B and A_R are pointed Hopf algebras over abelian groups. Moreover, A_R is neither a group algebra nor the bosonization of its diagram by $G(A_R)$.*

Proof Since $H_{[0]} = H \cap A_{[0]}$ and $B_{[0]} = B \cap A_{[0]}$, there are group epimorphisms $G \to G_H$ and $G \to G_B$ such that $H_{[0]} = \Bbbk^{G_H}$ and $B_{[0]} = \Bbbk^{G_B}$. Then there is an epimorphism of Hopf algebras $B \xrightarrow{\simeq} H^{*\mathrm{cop}} \longrightarrow \Bbbk G_H$. By [Mo, Corollary 5.3.5], the restriction $B_{[0]} = \Bbbk^{G_B} \to \Bbbk G_H$ is surjective. Thus G_H is an abelian group. *Mutatis mutandi*, we see that G_B is also an abelian group. Hence H and B are generated by skew-primitives and group-likes elements by [An, Theorem 2] and therefore also is $A_R = HB$. Then $A_R = HB$, H and B are pointed Hopf algebras over abelian groups. Set $\Gamma = G(A_R)$.

Now we assume $A_R = \Bbbk\Gamma$ and let $\delta_g \in \Bbbk^G \setminus \Bbbk\Gamma$. By a property of the R-matrix, it must hold $R\Delta(\delta_g) = \Delta^{cop}(\delta_g)R$. However, this is not possible since R is invertible and $\Bbbk^G$ is commutative but not cocommutative. Then $A_R \neq \Bbbk\Gamma$.

Finally, we assume that $A_R = \mathcal{B}(V)\#\Bbbk\Gamma$ where $\mathcal{B}(V)$ is the diagram of A_R which is a Nichols algebra by [An, Theorem 2]. Let $R_0 \in \Bbbk\Gamma\otimes\Bbbk\Gamma$ and $R^+ \in \mathcal{B}(V)^+\#\Bbbk\Gamma\otimes\Bbbk\Gamma + \Bbbk\Gamma\otimes\mathcal{B}(V)^+\#\Bbbk\Gamma$ such that $R = R_0 + R^+$. Then R_0 is invertible since R is so and $\mathcal{B}(V)^+$ is nilpotent. If $\delta_g \in \Bbbk^G \setminus \Bbbk\Gamma$, then it must hold $R_0\Delta(\delta_g) = \Delta^{cop}(\delta_g)R_0$ by a property of the R-matrix. As above, this is not possible. Therefore $A_R \neq \mathcal{B}(V)\#\Bbbk\Gamma$.

3 The affine rack $\mathrm{Aff}(\mathbb{F}_4, \omega)$ and their associated algebras

Let $\mathbb{F}_4$ be the finite field of four elements and $\omega \in \mathbb{F}_4$ such that $\omega^2+\omega+1 = 0$. The affine rack $\mathrm{Aff}(\mathbb{F}_4, \omega)$ is the set $\mathbb{F}_4$ with operation $a \triangleright b = \omega b + \omega^2 a$.

Let $(\cdot, g, \chi_G)$ be a *faithful principal YD-realization* of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$ over a finite group G [AG3, Definition 3.2], that is

- $\cdot$ is an action of G over $\mathbb{F}_4$,
- $g : \mathbb{F}_4 \to G$ is an injective function such that $g_{h\cdot i} = hg_ih^{-1}$ and $g_i \cdot j = i \triangleright j$ for all $i, j \in \mathbb{F}_4$, $h \in G$
- $\chi_G : G \to \Bbbk^*$ is a multiplicative character such that $\chi_G(g_i) = -1$ for all $i \in \mathbb{F}_4$; we can consider such a χ_G by [AG3, Lemma 3.3(d)].

These data define a structure on $V = \Bbbk\{x_i\}_{i\in\mathbb{F}_4}$ of Yetter-Drinfeld module over $\Bbbk^G$ via

$$\delta_t \cdot x_i = \delta_{t,g_i^{-1}} x_i \quad \text{and} \quad \lambda(x_i) = \sum_{t\in G} \chi_G(t^{-1})\delta_t\otimes x_{t^{-1}\cdot i} \;\; \forall t \in G,\, i \in X. \quad (5)$$

We obtain (5) using the fact that the categories ${}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ and ${}_{\Bbbk^G}^{\Bbbk G}\mathcal{YD}$ are braided equivalent [AG1, Proposition 2.2.1], see [GIV, Subsection 3.2] for details.

We denote by G' the subgroup of G generated by $\{g_i\}_{i\in\mathbb{F}_4}$. Then G' is a quotient of the *enveloping group of* $\mathrm{Aff}(\mathbb{F}_4, \omega)$ [EG, J]:

$$G_{\mathrm{Aff}(\mathbb{F}_4,\omega)} = \langle g_i \mid g_ig_j = g_{i\triangleright j}g_i,\; i, j \in \mathbb{F}_4\rangle.$$

Let $m \in \mathbb{N}$. We denote by $C_m = \langle t\rangle$ the cyclic group of order m. The semidirect product group $\mathbb{F}_4 \rtimes_\omega C_{6m}$ is given by $t \cdot i = \omega i$ for all $i \in \mathbb{F}_4$.

Examples 9. (1) Let $k, m \in \mathbb{N}$, $0 \leq k < m$. The (m, k)-*affine realization* of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$ over $\mathbb{F}_4 \rtimes_\omega C_{6m}$ [GIV, Proposition 2.6] is defined by

- $g : \mathbb{F}_4 \to \mathbb{F}_4 \rtimes_\omega C_{6m}$, $i \mapsto g_i = (i, t^{6k+1})$;

– $\cdot : \mathbb{F}_4 \rtimes_\omega C_{6m} \to \mathbb{F}_4$ is $h \cdot i = j$, if $hg_ih^{-1} = g_j$;
– $\chi_{\mathbb{F}_4 \rtimes_\omega C_{6m}} : \mathbb{F}_4 \rtimes_\omega C_{6m} \longmapsto \Bbbk^*$, $(j, t^s) \mapsto (-1)^s$, $\forall i, j \in A$, $s \in \mathbb{N}$.

(2) The next example gives a nontrivial lifting of $\mathcal{B}(V)$, see the next subsection. Suppose that $m \mid 6k+1$. Let G_1 be a finite group with a multiplicative character $\chi_{G_1} : G_1 \to \Bbbk^*$ such that $\chi_{G_1}^6 \neq 1$. Then the (m,k)-affine realization is extended to a principal YD-realization over $G = \mathbb{F}_4 \rtimes_\omega C_{6m} \times G_1$ setting $G_1 \cdot i = i$ and $\chi_G = \chi_{\mathbb{F}_4 \rtimes_\omega C_{6m}} \times \chi_{G_1}$. Note that $z \in T(V)[e]$ and $\chi_G^6 \neq 1$, where z is defined in (7).

(3) Let $(\cdot, g, \chi_G)$ be a faithful principal YD-realization of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$ over a finite group G. If $G' \leq G_1 \leq G$ are subgroups, then $(\cdot, g, (\chi_G)_{|G_1})$ is a faithful principal YD-realization of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$ over G_1. For instance, $G_1 = \ker \chi_G^6$.

3.1 A Nichols algebra over $\mathrm{Aff}(\mathbb{F}_4, \omega)$

From now on, we fix a faithful principal YD-realization $(\cdot, g, \chi_G)$ over a finite group G of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$. Let $V \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ be as in (5).

In [GIV, Subsection 2.2] it was discussed how braided functors modify the Nichols algebras. As a consequence the defining relations of the Nichols algebra $\mathcal{B}(V)$ were calculated [GIV, Proposition 2.10 (b)] using previous results of [G1] for the pointed case.

Namely, $\mathcal{B}(V)$ is the quotient of $T(V)$ by the ideal $\mathcal{J}(V)$ generated by

$$x_i^2, \quad x_j\, x_i + x_i\, x_{(\omega+1)i+\omega j} + x_{(\omega+1)i+\omega j}\, x_j \quad \forall i, j \in \mathbb{F}_4 \text{ and} \tag{6}$$

$$z := (x_\omega x_0 x_1)^2 + (x_1 x_\omega x_0)^2 + (x_0 x_1 x_\omega)^2. \tag{7}$$

We are specially interested in the case where $z \in T(V)[e]$, since otherwise the liftings of $\mathcal{B}(V)$ are trivial, see Theorem 11 (b). In Example 9 (2) this condition is satisfied.

Let $\mathbb{B}$ be the basis of $\mathcal{B}(V)$ consisting of all possible words $m_1m_2m_3m_4m_5$ such that m_i is an element in the ith row of the next list

$$\begin{aligned} &1, x_0,\\ &1, x_1,\ x_1x_0,\\ &1, x_\omega x_0 x_1,\\ &1, x_\omega,\ x_\omega x_0,\\ &1, x_{\omega^2}. \end{aligned}$$

By (5) the weight of a monomial $x_{i_1} \cdots x_{i_\ell} \in T(V)$ is $g_{i_1}^{-1} \cdots g_{i_\ell}^{-1}$. Set $g_{top} = g_0^{-1}g_1^{-1}g_0^{-1}g_\omega^{-1}g_0^{-1}g_1^{-1}g_\omega^{-1}g_0^{-1}g_{\omega^2}^{-1}$. An integral of $\mathcal{B}(V)$ is

$$m_{top} = x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2} \in \mathbb{B}[g_{top}].$$

Lemma 10. *Let G be a finite group with a faithful principal YD-realization $(\cdot, g, \chi_G)$ of $(\mathrm{Aff}(\mathbb{F}_4,\omega),-1)$. Hence*

(a) $\operatorname{Supp}\mathcal{B}(V) = \operatorname{Supp}\mathbb{B} \subset G'$.

(b) $G' \longmapsto \mathbb{F}_4 \rtimes_\omega C_6$, $g_i \mapsto (i,t)$ *is an epimorphism of groups.*

(c) If $z \in T(V)[e]$, *then* $\mathbb{B}[e] = \{1, b_1, b_2, b_3, b_4, b_5\}$ *where*

$$b_1 = x_0x_1x_0x_\omega x_0x_{\omega^2}, \quad b_2 = x_0x_\omega x_0x_1x_\omega x_{\omega^2}, \quad b_3 = x_1x_0x_\omega x_0x_1x_{\omega^2}$$
$$b_4 = x_1x_\omega x_0x_1x_\omega x_0, \quad b_5 = x_0x_1x_\omega x_0x_1x_\omega.$$

(d) Let $y = \sum_{i\in\mathbb{F}_4} x_i$ *and* $U = \Bbbk\{x_0 - x_1, x_0 - x_\omega, x_0 - x_{\omega^2}\}$. *Then* $\Bbbk y$ *and* U *are simple* $\Bbbk^G$*-comodules such that* $V = \Bbbk y \oplus U$.

Proof (a) holds since the elements of $\mathbb{B}$ are $\Bbbk^G$-homogeneous and $\mathcal{B}(V)$ is a $\Bbbk^G$-module algebra.

(b) By [AG2, Lemma 1.9 (1)], the quotient of G' by its center $\mathcal{Z}(G')$ is isomorphic to $\mathrm{Inn}_\triangleright \mathrm{Aff}(\mathbb{F}_4,\omega) = \mathbb{F}_4 \rtimes_\omega C_3$ via $\overline{g_i} \mapsto (i,t)$, $i \in \mathbb{F}_4$. Then $G'/(\mathcal{Z}(G') \cap \ker \chi_G) \simeq \mathbb{F}_4 \rtimes_\omega C_3 \times C_2 \simeq \mathbb{F}_4 \rtimes_\omega C_6$.

(c) If $z \in \mathbb{B}[e]$, then $\{1, b_1, b_2, b_3, b_4, b_5\} \subseteq \mathbb{B}[e]$ since $g_ig_j = g_{i\triangleright j}g_i$. The other inclusion follows using (b).

(d) is equivalent to prove that $\Bbbk y$ and U are simple $\Bbbk G$-modules via the action $g \cdot x_i = \chi_G(g)\, x_{g\cdot i}$, $i \in \mathbb{F}_4$. Clearly, $\Bbbk y$ and U are $\Bbbk G$-submodules and $\Bbbk y$ is $\Bbbk G$-simple. Moreover, it is an straightforward computation to show that U is $\Bbbk G'$-simple and therefore $\Bbbk G$-simple.

3.2 Copointed Hopf algebras over $\mathrm{Aff}(\mathbb{F}_4,\omega)$

The copointed Hopf algebras over $\Bbbk^G$ whose infinitesimal braiding arises from a principal YD-realization of the affine rack $\mathrm{Aff}(\mathbb{F}_4,\omega)$ with the constant 2-cocycle -1 are classified in [GIV] as follows.

By (5) the smash product Hopf algebra $T(V)\#\Bbbk^G$ is defined by

$$\delta_t x_i = x_i \delta_{g_i\, t} \quad \text{and}$$
$$\Delta(x_i) = x_i \otimes 1 + \sum_{t\in G} \chi_G(t)\delta_{t^{-1}} \otimes x_{t\cdot i} \quad \forall t \in G,\ i \in X. \tag{8}$$

Definition 1 Let $\lambda \in \Bbbk$ and assume $z \in T(V)[e]$. The Hopf algebra $\mathcal{A}_{G,\lambda}$ is the quotient of $T(V)\#\Bbbk^G$ by the ideal generated by (6) and $z - f$ where

$$f = \lambda(1 - \chi_z^{-1}) \quad \text{and} \quad \chi_z = \chi_G^6.$$

Notice that if either $\lambda = 0$ or $\chi_z = 1$, then $\mathcal{A}_{G,\lambda} = \mathcal{B}(V)\#\Bbbk^G$.

The next theorem is [GIV, Main theorem 2 and Theorem 4.5].

Theorem 11. *Let H be a copointed Hopf algebra over $\Bbbk^G$ whose infinitesimal braiding arises from a principal YD-realization of the affine rack* $\mathrm{Aff}(\mathbb{F}_4,\omega)$ *with the constant 2-cocycle* -1.

(a) If $G = G'$, then $H \simeq \mathcal{B}(V)\#\Bbbk^G$.

(b) If $z \in T(V)^\times$, then $H \simeq \mathcal{B}(V)\#\Bbbk^G$.

(c) If $z \in T(V)[e]$, then $H \simeq \mathcal{A}_{G,\lambda}$ for some $\lambda \in \Bbbk$.

(d) $\mathcal{A}_{G,\lambda}$ is a cocycle deformation of $\mathcal{A}_{G,\lambda'}$, for all $\lambda, \lambda' \in \Bbbk$.

(e) $\mathcal{A}_{G,\lambda}$ is a lifting of $\mathcal{B}(V)$ over $\Bbbk^G$ for all $\lambda, \lambda' \in \Bbbk$.

(f) $\mathcal{A}_{G,\lambda} \simeq \mathcal{A}_{G,1} \not\simeq \mathcal{A}_{G,0}$ for all $\lambda \in \Bbbk^*$. □

We think of $\mathcal{A}_{G,\lambda}$ as an algebra presented by generators $\{x_i, \delta_g : i \in \mathbb{F}_4,\ g \in G\}$ and relations:

$$\begin{aligned} \delta_g x_i = x_i \delta_{g_i g}, \quad x_i^2 = 0, \quad \delta_g \delta_h = \delta_g(h)\delta_g, \quad 1 = \sum_{g \in G} \delta_g, \\ x_0 x_\omega + x_\omega x_1 + x_1 x_0 = 0 = x_0 x_{\omega^2} + x_{\omega^2} x_\omega + x_\omega x_0, \\ x_1 x_{\omega^2} + x_0 x_1 + x_{\omega^2} x_0 = 0 = x_\omega x_{\omega^2} + x_1 x_\omega + x_{\omega^2} x_1 \quad \text{and} \\ x_\omega x_0 x_1 x_\omega x_0 x_1 + x_1 x_\omega x_0 x_1 x_\omega x_0 + x_0 x_1 x_\omega x_0 x_1 x_\omega = f, \end{aligned} \tag{9}$$

for all $i \in \mathbb{F}_4$ and $g \in G$. Since $\chi_z(g_i) = 1$, it holds that

$$f\, x_i = x_i\, f \quad \forall i \in \mathbb{F}_4. \tag{10}$$

A basis for $\mathcal{A}_{G,\lambda}$ is $\mathbb{A} = \{x\delta_g | x \in \mathbb{B}, g \in G\}$ and a basis for the Verma module M_g is $\mathbb{M} = \{x_{i_1} \cdots x_{i_s} \delta_g \in \mathbb{B}\delta_g\}$.

Proposition 12. *$\mathcal{A}_{G,\lambda}$ is not quasitriangular.*

Proof Let A be a pointed Hopf subalgebra of $\mathcal{A}_{G,\lambda}$ with abelian group of group-like elements. Then A is generated by skew-primitives and group-likes elements by [An, Theorem 2].

Let $y = \sum_{i \in \mathbb{F}_4} x_i$. The space of skew-primitives of $\mathcal{A}_{G,\lambda}$ is $\Bbbk G(\mathcal{A}_{G,\lambda}) \oplus \Bbbk y\, \Bbbk G(\mathcal{A}_{G,\lambda})$ by Lemma 10 (d). Then A is generated by y and $G(A)$. By (9), $y^2 = 0$ and hence $A \subseteq (\Bbbk[y]/\langle y^2 \rangle)\#\Bbbk G(A)$. Therefore $\mathcal{A}_{G,\lambda}$ is not quasitriangular by Lemma 8.

4 Representation theory of $\mathcal{A}_{G,\lambda}$

Let $(\cdot, g, \chi_G)$ be a faithful principal YD-realization of $(\mathrm{Aff}(\mathbb{F}_4, \omega), -1)$ over a fixed finite group G. Let $V \in {}^{\Bbbk^G}_{\Bbbk^G}\mathcal{YD}$ be as in (5).

Also we fix $\lambda \in \Bbbk^*$ and assume $z \in T(V)[e]$ and $\chi_z \neq 1$. In this section we study the Hopf algebra $\mathcal{A}_{G,\lambda}$, Definition 1.

For $g \in G \setminus \ker \chi_z$, we define

$$\mathbf{e}_1^g = -\frac{1}{f(g)} b_1 \delta_g, \qquad \mathbf{e}_2^g = -\frac{1}{f(g)} b_2 \delta_g, \qquad \mathbf{e}_3^g = \frac{1}{f(g)} b_3 \delta_g,$$

$$\mathbf{e}_4^g = \frac{1}{f(g)} (b_4 - b_3) \delta_g, \qquad \mathbf{e}_5^g = \frac{1}{f(g)} (b_5 + b_1) \delta_g \qquad \text{and}$$

$$\mathbf{e}_6^g = \delta_g + \frac{1}{f(g)}(b_2 - b_4 - b_5)\delta_g,$$

where $b_1, b_2, b_3, b_4, b_5 \in \mathcal{A}_{G,\lambda}$ are as in Lemma 10 (c).

Lemma 13. *A complete set of orthogonal primitive idempotents of* $\mathcal{A}_{G,\lambda}$ *is*

$$\mathcal{E} := \{\delta_h,\, \mathbf{e}_1^g,\, \mathbf{e}_2^g,\, \mathbf{e}_3^g,\, \mathbf{e}_4^g,\, \mathbf{e}_5^g,\, \mathbf{e}_6^g \,|\, h \in \ker\chi_z,\, g \in G \setminus \ker\chi_z\}.$$

Proof By Lemma 10 (c), $\{b_i\delta_g | 1 \le i \le 6\}$ is a basis of $\mathcal{B}(V)[e]\delta_g$ for all $g \in G$. By (9) and (10), it holds that:

$$\begin{array}{lllll} b_1^2 = -b_1 f, & b_1b_2 = 0, & b_1b_3 = 0, & b_1b_4 = 0, & b_1b_5 = b_1 f, \\ b_2b_1 = 0, & b_2^2 = -b_2 f, & b_2b_3 = 0, & b_2b_4 = 0, & b_2b_5 = 0, \\ b_3b_1 = 0, & b_3b_2 = 0, & b_3^2 = b_3 f, & b_3b_4 = b_3 f, & b_3b_5 = 0, \\ b_4b_1 = 0, & b_4b_2 = 0, & b_4b_3 = b_3 f, & b_4^2 = b_4 f, & b_4b_5 = 0, \\ b_5b_1 = b_1 f, & b_5b_2 = 0, & b_5b_3 = 0, & b_5b_4 = 0, & b_5^2 = b_5 f. \end{array} \tag{11}$$

Therefore $\mathcal{E}_h = \{\delta_h\}$ is a complete set of orthogonal primitive h-idempotent for all $h \in \ker\chi_z$. If $g \in G \setminus \ker\chi_z$, we apply Lemma 7 to the ordered set

$$\left\{-\frac{1}{f(g)}b_1\delta_g,\, -\frac{1}{f(g)}b_2\delta_g,\, \frac{1}{f(g)}b_3\delta_g,\, \frac{1}{f(g)}b_4\delta_g,\, \frac{1}{f(g)}b_5\delta_g,\, \delta_g\right\}$$

and hence $\mathcal{E}_g = \{\mathbf{e}_i^g | 1 \le i \le 6\}$ is a complete set of orthogonal primitive g-idempotent. Then $\mathcal{E} = \cup_{g\in G}\mathcal{E}_g$.

Let M be an $\mathcal{A}_{G,\lambda}$-module. Since $\mathcal{A}_{G,\lambda}$ is a quotient of $T(V)\#\Bbbk^G$, M also is a $T(V)\#\Bbbk^G$-module. Moreover, M is a $T(V)\#\Bbbk^{\ker\chi_z}$-module if $\operatorname{Supp} M \subseteq \ker\chi_z$ since $T(V)\#\Bbbk^{\ker\chi_z}$ is a subalgebra of $T(V)\#\Bbbk^G$, cf. Example 9 (3).

Lemma 14. *Let* $h \in \ker\chi_z$.

(a) If M *is an* $\mathcal{A}_{G,\lambda}$*-module with* $\operatorname{Supp} M \subseteq \ker\chi_z$, *then* M *is a module over* $\mathcal{B}(V)\#\Bbbk^{\ker\chi_z}$.

(b) M_h *is a free* $\mathcal{B}(V)$*-module of rank* 1 *generated by* δ_h.

(c) $\chi_h : \mathcal{A}_{G,\lambda} \to \Bbbk$ *is an algebra map.*

(d) $top(M_h) \simeq \Bbbk_h$ *and* $soc(M_h) \simeq \Bbbk_{g_{top}h}$.

(e) $\int^l_{\mathcal{A}_{G,\lambda}} = soc(M_{g_{top}^{-1}})$ *and* $\chi_{g_{top}}$ *is the distinguished group-like element.*

Proof (a) Since M is a $T(V)\#\Bbbk^{\ker\chi_z}$-module, we have to see that the elements in (6) and z act by zero over M. This is true for the first elements because they are zero in $\mathcal{A}_{G,\lambda}$. If $h \in \ker\chi_z$, then $f\delta_h = 0$ and hence $z\cdot M[h] = f\cdot(\delta_h\cdot M) = 0$. (b) follows from (a). (c) is clear. (d) and (e) follows from (b) and Lemma 6.

For each $\mathbf{e}_i^g \in \mathcal{E}$, we set $L_i^g = \mathcal{A}_{G,\lambda}\mathbf{e}_i^g$.

Lemma 15. *(a)* L_i^g *is an injective and projective simple module of dimension* 12 *for all* $\mathbf{e}_i^g \in \mathcal{E}$.

(b) There exist $\Bbbk^G$*-submodules* $L_1, \dots, L_6 \subset \mathcal{B}(V)$ *such that* $\mathcal{B}(V) = L_1 \oplus \cdots \oplus L_6$ *and* $L_i^g = L_i \delta_g$ *for all* $i = 1, \dots, 6$ *and* $g \in G$.

(c) $\operatorname{Supp} L_i \neq \operatorname{Supp} L_j$ *and* $\operatorname{Supp} L_i^g = (\operatorname{Supp} L_i)g$ *for all* $1 \leq i, j \leq 6$ *and* $g \in G$.

(d) $L_i^g \simeq L_j^h$ *if and only if* $(\operatorname{Supp} L_i)g = (\operatorname{Supp} L_j)h$.

Proof (a) Let $v = \overline{\mathbf{e}_i^g} \in top(L_i^g)$. Since $f(g)v = z \cdot v = (x_\omega x_0 x_1)^2 \cdot v + b_4 \cdot v + b_5 \cdot v \neq 0$, there are $x_{i_6}, \dots, x_{i_1} \in \mathcal{A}_{G,\lambda}$ such that $x_{i_\ell} \cdots x_{i_1} \cdot v \neq 0$ for all $\ell = 1, ..., 6$.

We claim that $\dim top(L_i^g) \geq 11$. In fact, if $1 \leq \ell < 6$, then by (6)

$$x_{i_{\ell+1}} x_{i_\ell} \cdots x_{i_1} \cdot v = -x_{i_\ell} x_{(\omega+1)i_\ell + \omega i_{\ell+1}} \cdots x_{i_1} \cdot v - x_{(\omega+1)i_\ell + \omega i_{\ell+1}} x_{i_{\ell+1}} \cdots x_{i_1} \cdot v \neq 0$$

and hence $x_{(\omega+1)i_\ell + \omega i_{\ell+1}} \cdots x_{i_1} \cdot v \neq 0$ or $x_{i_{\ell+1}} \cdots x_{i_1} \cdot v \neq 0$. Therefore using Lemma 10 (b), we see that $\# \operatorname{Supp} top(L_i^g) \geq 11$.

Now, we show that $L_i^g = soc(L_i^g) = top(L_i^g)$ and (a) follows. Otherwise, $\dim L_i^g \geq 22$ since $\dim top(L_i^g) = \dim soc(L_i^g)$ by [CR, Lemma 58.4]. But the above claim holds for all i and hence $72 = \dim M_g \geq 22 + 5 \cdot 11$, a contradiction.

(b) follows from Tables 1–6 in Appendix. (c) $\operatorname{Supp} L_i^g = (\operatorname{Supp} L_i)g$ follows from (b). If $G' = \mathbb{F}_4 \rtimes C_6$, then $\operatorname{Supp} L_i \neq \operatorname{Supp} L_j$ by Table 7 in Appendix and therefore for any G' by Lemma 10 (b). (d) follows from (c) and Lemma 3.

We consider the product set $\{1, 2, 3, 4, 5, 6\} \times G$ with the equivalence relation $i \times g \sim j \times h$ if and only if $(\operatorname{Supp} L_i)g = (\operatorname{Supp} L_j)h$. Let $\mathfrak{X}$ be the set of equivalence classes of $\sim$. We denote by $[i, g]$ the equivalence class of $i \times g$. By Lemma 15 (d), we can define $L_{[i,g]} = L_i^g$.

Theorem 16. *Every simple* $\mathcal{A}_{G,\lambda}$*-module is isomorphic to either*

$$\begin{array}{rll} \Bbbk_g & \textit{for a unique} & g \in \ker \chi_z \quad \textit{or} \\ L_{[i,g]} & \textit{for a unique} & [i,g] \in \mathfrak{X}. \end{array}$$

In particular, (up to isomorphism) there are $|\ker \chi_z|$ *one-dimensional simple* $\mathcal{A}_{G,\lambda}$*-modules and* $\frac{(|G| - |\ker \chi_z|)}{2}$ *12-dimensional simple* $\mathcal{A}_{G,\lambda}$*-modules.*

Proof It follows from Lemmata 13, 14 and 15.

Example 2 Assume $G' = \mathbb{F}_4 \rtimes C_6$ and let $g \in G \backslash \ker \chi_z$. The set $\mathfrak{X}$ is completely defined by the equivalence class $[1, g]$ which is

$$\Big\{ 1 \times g, 2 \times (1, t^2)g, (3, tg), 4 \times (\omega, t^2)g,\ 5 \times (1, t)g,\ 6 \times (\omega, 1)g,\ 1 \times (0, t^3)g$$
$$2 \times (1, t^5)g,\ 3 \times (0, t^4)g,\ 4 \times (\omega, t^5)g,\ 5 \times (1, t^4)g,\ 6 \times (\omega, t^3)g \Big\}.$$

Hence

$$L_{[1,g]} = L_1^g \simeq L_2^{(1,t^2)g} \simeq L_3^{(0,t)g} \simeq L_4^{(\omega,t^2)g} \simeq L_5^{(1,t)g} \simeq L_6^{(\omega,1)g} \simeq$$

$$L_1^{(0,t^3)g} \simeq L_2^{(1,t^5)g} \simeq L_3^{(0,t^4)g} \simeq L_4^{(\omega,t^5)g} \simeq L_5^{(1,t^4)g} \simeq L_6^{(\omega,t^3)g}.$$

Note that $i \times g \sim i \times (0, t^3)g$ for all i, then $L_i^g \simeq L_i^{(0,t^3)g}$.

In fact, $(\operatorname{Supp} L_2)(1, t^2) = \operatorname{Supp} L_1$, see Tables 1 and 2. Then $L_1^g \simeq L_2^{(1,t^2)g}$ by Lemma 15 (d). The other isomorphisms are obtained in the same way.

4.1 Decomposition of the category of $\mathcal{A}_{G,\lambda}$-modules

Fix $\lambda \in \Bbbk^*$ and assume $z \in T(V)[e]$ and $\chi_z \neq 1$. Let $I \subset \{1, 2, 3, 4, 5, 6\} \times G$ be a set of representative of the equivalence classes of $\sim$. Let M be an $\mathcal{A}_{G,\lambda}$-module.

If $i \times g \in I$, then $d^M_{[i,g]} = \dim(\mathbf{e}_i^g \cdot M)$ is the number of composition factors of M which are isomorphic to $L_{[i,g]}$ [CR, Theorem 54.16]. The number $d^M_{[i,g]}$ can be calculated keeping in mind Lemma 1 (d). Since $L_{[i,g]}$ is projective and injective by Lemma 15, there is a submodule $N \subseteq M$ such that $\operatorname{Supp} N \subseteq \ker \chi_z$ and

$$M = N \oplus \bigoplus_{j \in I} (L_j)^{d^M_{[i,g]}}.$$

Moreover, N is a $\mathcal{B}(V)\#\Bbbk^{\ker \chi_z}$-module by Lemma 14 (a).

4.2 Representation type of $\mathcal{A}_{G,\lambda}$

Now, we do not make any assumptions on z and λ can be zero. Let $\Bbbk_g$ and $\Bbbk_h$ be one-dimensional $\mathcal{A}_{G,\lambda}$-modules such that $g = g_i^{-1}h \in \ker \chi_z$ for some $i \in \mathbb{F}_4$. We define the $\mathcal{A}_{G,\lambda}$-module $M_{g,h} = \Bbbk\{w_h, w_g\}$ by $\Bbbk w_g \simeq \Bbbk_g$ as $\mathcal{A}_{G,\lambda}$-modules, $w_h \in M[h]$ and $x_j w_h = \delta_{j,i} w_g$ for all $j \in \mathbb{F}_4$.

Proposition 17. *The extensions of one-dimensional $\mathcal{A}_{G,\lambda}$-modules are either trivial or isomorphic to $M_{g,h}$ for some $g, h \in \ker \chi_z$. Hence $\mathcal{A}_{G,\lambda}$ is of wild representation type.*

Proof Let M be an extension of $\Bbbk_h$ by $\Bbbk_g$. Then $M = M[g] \oplus M[h]$ as $\Bbbk^G$-modules and $M[g] \simeq \Bbbk_g$ as $\mathcal{A}_{G,\lambda}$-modules. Since $x_i \cdot M[h] \subset M[g_i^{-1}h]$, the first part follows.

For the second part we can easily see that $\operatorname{Ext}^1_{\mathcal{A}_{G,\lambda}}(\Bbbk_g, \Bbbk_h)$ is either 1 or 0 for all $g, h \in \ker \chi_z$. Then the separated quiver of $\mathcal{A}_{G,\lambda}$ is wild. The details for this proof are similar to [AV2, Proposition 26].

4.3 Is $\mathcal{A}_{G,\lambda}$ spherical?

A Hopf algebra H is *spherical* [BaW1] if there is $\omega \in G(H)$ such that

$$\mathcal{S}^2(x) = \omega x \omega^{-1} \quad \forall x \in H \text{ and} \tag{12}$$

$$\operatorname{tr}_V(\omega) = \operatorname{tr}_V(\omega^{-1}) \quad \forall V \in \operatorname{Irr} H \quad \text{by [AAGTV, Proposition 2.1].} \tag{13}$$

Proposition 18. *$\mathcal{B}(V)\#\Bbbk^G$ is spherical iff $\chi_G^2 = 1$. Moreover, $(\mathcal{A}_{G,\lambda}, \chi_G)$ with $\lambda \neq 0$ is spherical iff $(\chi_{G|\ker\chi_z})^2 = 1$.*

Proof It is a straightforward computation to see that χ_G satisfies (12) using (8). Let $V \in \operatorname{Irr} \mathcal{A}_{G,\lambda}$. If $\dim V = 12$, then V is projective and therefore $\operatorname{tr}_V(\chi_G^{\pm 1}) = 0$ [BaW2, Proposition 6.10]. If $V = \Bbbk_h$ with $h \in \ker \chi_z$, then (13) holds iff $\chi_G(h) = \pm 1$.

Example 3 Let $(\cdot, g, \chi_G)$ be the faithful principal YD-realization in Example 9 (2). Then $(\mathcal{A}_{G,\lambda}, \chi_G)$ is a spherical Hopf algebra with non involutory pivot.

Any spherical Hopf algebra H has an associated tensor category $\underline{\operatorname{Rep}}(H)$ which is a quotient of $\operatorname{Rep}(H)$, see [AAGMV, BaW1, BaW2] for the background of this subject. Moreover, $\underline{\operatorname{Rep}}(H)$ is semisimple but rarely is a fusion category in the sense of [ENO], *i. e.* $\underline{\operatorname{Rep}}(H)$ rarely has a finite number of irreducibles. One hopes to find new examples of fusion categories as tensor subcategories of $\underline{\operatorname{Rep}}(H)$ for a suitable H. However, this is not possible for $H = \mathcal{A}_{G,\lambda}$, see below.

Remark 19. Assume that $(\mathcal{A}_{G,\lambda}, \chi_G)$ is spherical. Then only the one-dimensional simple modules survive in $\underline{\operatorname{Rep}}(\mathcal{A}_{G,\lambda})$ since the other simple modules are projective. Then $\underline{\operatorname{Rep}}(\mathcal{A}_{G,\lambda})$ is equivalent to $\underline{\operatorname{Rep}}(\mathcal{B}(V)\#\Bbbk^{\ker\chi_z})$ by Subsection 4.1, where the pivot $\chi_{G|\ker\chi_z}$ is involutory. Hence any fusion subcategory of $\underline{\operatorname{Rep}}(\mathcal{A}_{G,\lambda})$ is equivalent to $\operatorname{Rep}(K)$, with K a semisimple quasi-Hopf algebra, by [AAGTV, Proposition 2.12].

Appendix

The next tables describe the structure of the 12-dimensional simple modules of $\mathcal{A}_{G,\lambda}$. These were used in Lemma 15.

Table 1 Action of the generators x_i on $L_1^g = \mathcal{A}_{G,\lambda}\mathbf{e}_1^g$

Linear basis of L_1^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	0	0	$-f(g)c_6$	$-f(g)c_{10}$
$c_2 = x_0x_1x_0x_\omega x_0x_{\omega^2}\delta_g = -f(g)\mathbf{e}_1^g$	0	0	$-c_5$	$-c_9$
$c_3 = x_0x_1x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	0	c_1	$f(g)c_{12}$	0
$c_4 = x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	0	c_2	c_{11}	0
$c_5 = x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	0	c_7	0	$-c_3$
$c_6 = x_0x_\omega x_0x_{\omega^2}\delta_g$	0	c_8	0	$-c_4$
$c_7 = x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	c_1	0	0	$-f(g)c_{12}$
$c_8 = x_1x_0x_\omega x_0x_{\omega^2}\delta_g$	c_2	0	0	c_{11}
$c_9 = x_1x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	c_3	0	$-c_7$	0
$c_{10} = x_1x_\omega x_0x_{\omega^2}\delta_g$	c_4	0	$-c_8$	0
$c_{11} = x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$	c_5	c_9	0	0
$c_{12} = x_\omega x_0x_{\omega^2}\delta_g$	c_6	c_{10}	0	0

Table 2 Action of the generators x_i on $L_2^g = \mathcal{A}_{G,\lambda}\mathbf{e}_2^g$

Linear basis of L_2^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$	0	0	c_6	$-f(g)c_{10}$
$c_2 = x_0x_1x_0x_\omega x_{\omega^2}\delta_g$	0	0	$-c_5$	$-c_9$
$c_3 = x_0x_1x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$	0	c_1	$-c_{12}$	0
$c_4 = x_0x_1x_\omega x_{\omega^2}\delta_g$	0	c_2	c_{11}	0
$c_5 = x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g = f(g)\mathbf{e}_2^g$	0	c_7	0	$-c_3$
$c_6 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $-x_0x_\omega x_{\omega^2}\delta_g$	0	$-f(g)c_8$	0	$f(g)c_4$
$c_7 = x_1x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$	c_1	0	0	$-c_{12}$
$c_8 = x_1x_0x_\omega x_{\omega^2}\delta_g$	c_2	0	0	c_{11}
$c_9 = x_1x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$	c_3	0	$-c_7$	0
$c_{10} = x_1x_\omega x_{\omega^2}\delta_g$	c_4	0	$-c_8$	0
$c_{11} = x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$	c_5	c_9	0	0
$c_{12} = x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g - x_\omega x_{\omega^2}\delta_g$	c_6	$-f(g)c_{10}$	0	0

Table 3 Action of the generators x_i on $L_3^g = \mathcal{A}_{G,\lambda}\mathbf{e}_3^g$

Linear basis of L_3^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0x_\omega x_0x_1x_{\omega^2}\delta_g$	0	0	c_6	$-c_{10}$
$c_2 = x_0x_1x_0x_{\omega^2}\delta_g$	0	0	$-c_5$	$-c_9$
$c_3 = x_0x_1x_\omega x_0x_1x_{\omega^2}\delta_g$	0	c_1	c_{12}	0
$c_4 = x_0x_1x_{\omega^2}\delta_g$	0	c_2	c_{11}	0
$c_5 = x_0x_\omega x_0x_1x_{\omega^2}\delta_g$	0	c_7	0	$-c_3$
$c_6 = x_0x_1x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $-f(g)x_0x_{\omega^2}\delta_g$	0	c_8	0	$f(g)c_4$
$c_7 = x_1x_0x_\omega x_0x_1x_{\omega^2}\delta_g = f(g)\mathbf{e}_3^g$	c_1	0	0	c_{12}
$c_8 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $-f(g)x_1x_0x_{\omega^2}\delta_g$	$-f(g)c_2$	0	0	$-f(g)c_{11}$
$c_9 = x_1x_\omega x_0x_1x_{\omega^2}\delta_g$	c_3	0	$-c_7$	0
$c_{10} = x_0x_1x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g$ $-f(g)x_1x_{\omega^2}\delta_g$	$-f(g)c_4$	0	c_8	0
$c_{11} = x_\omega x_0x_1x_{\omega^2}\delta_g$	c_5	c_9	0	0
$c_{12} = x_1x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $+x_0x_1x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g - f(g)x_{\omega^2}\delta_g$	$-c_6$	$-c_{10}$	0	0

Table 4 Action of the generators x_i on $L_4^g = \mathcal{A}_{G,\lambda}\mathbf{e}_4^g$

Linear basis of L_4^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0x_\omega x_0\delta_g$	0	0	$-c_6$	$-c_{10}$
$c_2 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g$	0	0	$-f(g)c_5$	$-c_9$
$c_3 = x_0x_1x_\omega x_0\delta_g - x_0x_1x_0x_{\omega^2}\delta_g$	0	c_1	c_{12}	0
$c_4 = x_0x_1x_\omega x_0x_1x_\omega x_0\delta_g - x_0x_1x_0x_\omega x_0x_1x_{\omega^2}\delta_g$	0	c_2	c_{11}	0
$c_5 = x_0x_\omega x_0\delta_g$	0	c_7	0	$-c_3$
$c_6 = x_0x_\omega x_0x_1x_\omega x_0\delta_g$	0	c_8	0	$-c_4$
$c_7 = x_1x_0x_\omega x_0\delta_g$	c_1	0	0	$-c_{12}$
$c_8 = x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g$	c_2	0	0	$-c_{11}$
$c_9 = x_1x_\omega x_0\delta_g - x_1x_0x_{\omega^2}\delta_g$	c_3	0	$-c_7$	0
$c_{10} = x_1x_\omega x_0x_1x_\omega x_0\delta_g - x_1x_0x_\omega x_0x_1x_{\omega^2}\delta_g$ $= f(g)\mathbf{e}_4^g$	c_4	0	$-c_8$	0
$c_{11} = x_0x_1x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g - f(g)x_0x_{\omega^2}\delta_g$ $+f(g)x_\omega x_0\delta_g$	c_5	c_9	0	0
$c_{12} = -x_0x_\omega x_0x_1x_{\omega^2}\delta_g + x_\omega x_0x_1x_\omega x_0\delta_g$	c_6	c_{10}	0	0

Table 5 Action of the generators x_i on $L_5^g = \mathcal{A}_{G,\lambda}\mathbf{e}_5^g$

Linear basis of L_5^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0x_\omega\delta_g$	0	0	$-c_6$	c_{10}
$c_2 = x_0x_1x_0x_\omega x_0x_1x_\omega\delta_g$	0	0	$-c_5$	c_9
$c_3 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $+f(g)x_0x_1x_\omega\delta_g$	0	$f(g)c_1$	$-f(g)c_{12}$	0
$c_4 = x_0x_1x_\omega x_0x_1x_\omega\delta_g - x_0x_1x_0x_\omega x_0x_{\omega^2}\delta_g$ $= f(g)\mathbf{e}_5^g$	0	c_2	c_{11}	0
$c_5 = x_0x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g + f(g)x_0x_\omega\delta_g$	0	$f(g)c_7$	0	c_3
$c_6 = x_0x_\omega x_0x_1x_\omega\delta_g - f(g)x_0x_{\omega^2}\delta_g$	0	c_8	0	c_4
$c_7 = x_1x_0x_\omega\delta_g$	c_1	0	0	c_{12}
$c_8 = x_1x_0x_\omega x_0x_1x_\omega\delta_g$	c_2	0	0	c_{11}
$c_9 = x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g + f(g)x_1x_\omega\delta_g$	c_3	0	$-f(g)c_7$	0
$c_{10} = x_1x_\omega x_0x_1x_\omega\delta_g - x_1x_0x_\omega x_0x_{\omega^2}\delta_g$	c_4	0	$-c_8$	0
$c_{11} = x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $+x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g + f(g)x_\omega\delta_g$	c_5	c_9	0	0
$c_{12} = x_\omega x_0x_1x_\omega\delta_g - x_0x_\omega x_0x_{\omega^2}\delta_g$	c_6	c_{10}	0	0

Table 6 Action of the generators x_i on $L_6^g = \mathcal{A}_{G,\lambda}\mathbf{e}_6^g$

Linear basis of L_6^g	$x_0\cdot$	$x_1\cdot$	$x_\omega\cdot$	$x_{\omega^2}\cdot$
$c_1 = x_0x_1x_0\delta_g$	0	0	$-c_6$	$-c_{10}$
$c_2 = x_0x_1x_0x_{\omega^2}x_0x_1\delta_g$	0	0	$-c_5$	c_9
$c_3 = x_0x_1x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g + f(g)x_0x_1\delta_g$	0	$f(g)c_1$	c_{12}	0
$c_4 = x_0x_1x_\omega x_0x_1\delta_g - x_0x_1x_0x_\omega x_{\omega^2}\delta_g$	0	c_2	c_{11}	0
$c_5 = -x_0x_1x_\omega x_0x_1x_\omega x_0\delta_g + f(g)x_0\delta_g$	0	c_7	0	c_3
$c_6 = x_0x_\omega x_0x_1\delta_g$	0	c_8	0	$-c_4$
$c_7 = -x_0x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g + f(g)x_1x_0\delta_g$	$f(g)c_1$	0	0	c_{12}
$c_8 = x_1x_0x_\omega x_0x_1\delta_g$	c_2	0	0	c_{11}
$c_9 = x_1x_0x_\omega x_0x_1x_\omega x_0\delta_g$ $-x_0x_1x_0x_\omega x_0x_1x_\omega\delta_g + f(g)x_1\delta_g$	c_3	0	$-c_7$	0
$c_{10} = x_1x_\omega x_0x_1\delta_g - x_1x_0x_\omega x_{\omega^2}\delta_g$	c_4	0	$-c_8$	0
$c_{11} = x_0x_\omega x_0x_1x_\omega x_{\omega^2}\delta_g - x_1x_\omega x_0x_1x_\omega x_0\delta_g$ $-x_0x_1x_\omega x_0x_1x_\omega\delta_g + f(g)\delta_g = f(g)\mathbf{e}_6^g$	c_5	c_9	0	0
$c_{12} = -x_0x_1x_0x_\omega x_0x_1x_\omega x_0x_{\omega^2}\delta_g$ $+f(g)x_\omega x_0x_1\delta_g - f(g)x_0x_\omega x_{\omega^2}\delta_g$	$f(g)c_6$	c_{10}	0	0

Table 7 Weight of the vectors c_i in the case $G' = \mathbb{F}_4 \rtimes C_6$

	L_1^g	L_2^g	L_3^g	L_4^g	L_5^g	L_6^g
c_1	$(0,t^3)g$	$(\omega,t^4)g$	$(0,t^5)g$	$(\omega^2,t)g$	$(\omega^2,t^2)g$	$(\omega,t^3)g$
c_2	g	$(\omega,t)g$	$(0,t^2)g$	$(\omega^2,t^4)g$	$(\omega^2,t^5)g$	$(\omega,1)g$
c_3	$(1,t^4)g$	$(\omega,t^5)g$	$(1,1)g$	$(0,t^2)g$	$(0,t^3)g$	$(\omega,t^4)g$
c_4	$(1,t)g$	$(\omega,t^2)g$	$(1,t^3)g$	$(0,t^5)g$	g	$(\omega,t)g$
c_5	$(1,t^5)g$	g	$(1,t)g$	$(\omega^2,t^3)g$	$(\omega^2,t^4)g$	$(0,t^5)g$
c_6	$(1,t^2)g$	$(0,t^3)g$	$(1,t^4)g$	$(\omega^2,1)g$	$(\omega^2,t)g$	$(0,t^2)g$
c_7	$(0,t^4)g$	$(\omega^2,t^5)g$	g	$(1,t^2)g$	$(1,t^3)g$	$(\omega^2,t^4)g$
c_8	$(0,t)g$	$(\omega^2,t^2)g$	$(0,t^3)g$	$(1,t^5)g$	$(1,1)g$	$(\omega^2,t)g$
c_9	$(\omega,t^5)g$	$(\omega^2,1)g$	$(\omega,t)g$	$(0,t^3)g$	$(0,t^4)g$	$(\omega^2,t^5)g$
c_{10}	$(\omega,t^2)g$	$(\omega^2,t^3)g$	$(\omega,t^4)g$	g	$(0,t)g$	$(\omega^2,t^2)g$
c_{11}	$(\omega,1)g$	$(0,t)g$	$(\omega,t^2)g$	$(1,t^4)g$	$(1,t^5)g$	g
c_{12}	$(\omega,t^3)g$	$(0,t^4)g$	$(\omega,t^5)g$	$(1,t)g$	$(1,t^2)g$	$(0,t^3)g$

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