The Toeplitz algebra on the Bergman space coincides with its commutator ideal.

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Abstract

Let L^2_a be the Bergman space of the unit disk and $\mathfrak{T}(L^2_a)$ be the Banach algebra generated by Toeplitz operators T_f, with $f \in L^\infty$. We prove that the closed bilateral ideal of $\mathfrak{T}(L^2_a)$ generated by operators of the form $T_fT_g - T_gT_f$ coincides with $\mathfrak{T}(L^2_a)$.

1 Introduction

If $0 < p \leq \infty$ let $L^p = L^p(\mathbb{D}, dA)$, where $\mathbb{D}$ is the open unit disk and $dA(z) = (1/\pi)dxdy$ $(z = x + iy)$ is the normalized area measure on $\mathbb{D}$. The Bergman space L^p_a is formed by the analytic functions in L^p. If $1 < p < \infty$ then

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(\omega)}{(1-\overline{\omega}z)^2} dA(\omega)$$

is a bounded projection from L^p onto L^p_a. This is the usual Bergman projection. For $a \in L^\infty$ let $M_a : L^p \to L^p$ be the operator of multiplication by a and $P_a = PM_a$. Then $\|P_a\| \leq C_p\|a\|_\infty$, where C_p is the norm of P acting on L^p. The Toeplitz operator $T_a : L^p_a \to L^p_a$ is the restriction of P_a to the space L^p_a. If B is a Banach space, we will write $\mathfrak{L}(B)$ for the algebra of all bounded operators on B and $\mathfrak{T}(L^p_a)$ for the closed subalgebra of $\mathfrak{L}(L^p_a)$ generated by $\{T_a : a \in L^\infty\}$.

If A is a Banach algebra, its commutator ideal $\mathfrak{C}A$ is the closed bilateral ideal generated by elements of the form $[x, y] \stackrel{\text{def}}{=} xy - yx$, with $x, y \in A$. It is clear that $\mathfrak{C}A$ is the smallest closed ideal of A such that $A/\mathfrak{C}A$ is a commutative Banach algebra. There is an extensive

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literature on commutator ideals and abelianizations of Toeplitz algebras acting on the Hardy space H^2. The book of Nikoslkii [5] contains plentiful information and further references. In contrast with this situation, we only have a handful of results for Toeplitz algebras of operators on L_a^2. Probably the most relevant papers on the subject are [2], [4] and [1].

If H is a Hilbert space of dimension greater than one then $\mathfrak{C}\mathfrak{L}(H) = \mathfrak{L}(H)$. Although this situation is very unusual for Toeplitz algebras, the purpose of this paper is to prove the following

Theorem 1.1 *The Toeplitz algebra on L_a^2 coincides with its commutator ideal.*

In [3] it is shown that if $\phi(z) = \exp(i \log\log |z|^{-2})$ then the semicommutator $T_{\overline{\phi}\phi} - T_{\overline{\phi}} T_\phi$ is a nontrivial scalar multiple of the identity. Analogously, it could happen that there are two simple functions $a, b \in L^\infty$ such that $T_a T_b - T_b T_a$ is easily seen to be invertible. This would immediately prove Theorem 1.1. Since I was unable to find such functions or even prove their existence, the proof here is considerably more complicated.

2 Segmentation

For $z \in \mathbb{D}$ let $\varphi_z(\omega) = (z - \omega)/(1 - \overline{z}\omega)$, the special automorphism of $\mathbb{D}$ that interchanges 0 and z. The pseudo-hyperbolic metric is defined by $\rho(z, \omega) = |\varphi_z(\omega)|$ for $z, \omega \in \mathbb{D}$. It is well known that ρ is invariant under the action of $\mathrm{Aut}(\mathbb{D})$. We will also use that

$$\rho(z, \omega) \geq \frac{\rho(z, u) - \rho(u, \omega)}{1 - \rho(z, u)\rho(u, \omega)} \quad \text{for all } \ z, \omega, u \in \mathbb{D}.$$

If $0 < r < 1$ write $K(z, r) \stackrel{\text{def}}{=} \{\omega \in \mathbb{D} : \rho(\omega, z) \leq r\}$ for the closed ball of center z and radius r with respect to ρ. A sequence $\mathcal{S} = \{z_n\}$ in $\mathbb{D}$ will be called separated if $\inf_{i \neq j} \rho(z_i, z_j) > 0$. Although I have not found the next result in its present form in the literature, it is a well known feature of separated sequences. We sketch here a proof.

Lemma 2.1 *Let $\mathcal{S}$ be a separated sequence and $0 < \sigma < 1$. Then there is a finite decomposition $\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N$ such that for every $1 \leq i \leq N$: $\rho(z, \omega) > \sigma$ for all $z \neq \omega$ in $\mathcal{S}_i$.*

Proof. Since $\mathcal{S}$ is separated, there is some positive integer N depending only on σ and the degree of separation of $\mathcal{S}$, such that $K(z, \sigma) \cap \mathcal{S}$ has no more than N points for every $z \in \mathbb{D}$. Let $\mathcal{S}_1 \subset \mathcal{S}$ be a maximal sequence such that $\rho(z, \omega) > \sigma$ for every $z, \omega \in \mathcal{S}_1$ with $z \neq \omega$. The maximality implies that $\mathcal{S} \subset \bigcup_{z \in \mathcal{S}_1} K(z, \sigma)$. If $\mathcal{S} = \mathcal{S}_1$ we are done. Otherwise suppose that $n \geq 2$, $\mathcal{S}_1, \ldots, \mathcal{S}_{n-1}$ are chosen and $\mathcal{S} \setminus (\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{n-1}) \neq \emptyset$. Let $\mathcal{S}_n \subset \mathcal{S} \setminus (\mathcal{S}_1 \cup \ldots \cup \mathcal{S}_{n-1})$ be a maximal sequence such that $\rho(z, \omega) > \sigma$ for every $z, \omega \in \mathcal{S}_n$

with $z \neq \omega$. By the maximality at the previous steps, if $z \in \mathcal{S}_n$ there is some $z_i \in \mathcal{S}_i$ such that $z \in K(z_i, \sigma)$ for every $1 \leq i \leq n-1$. Therefore $\{z, z_1, \dots, z_{n-1}\} \subset K(z,\sigma) \cap \mathcal{S}$, and consequently $n \leq N$. □

Lemma 2.2 *For $1 \leq k \leq m$ let $\{a_j^k\}_{j\geq 1}$ be sequences in the unit ball of L^∞ such that $\operatorname{supp} a_j^k \subset K(\alpha_j, r)$, where $K(\alpha_j, r) \cap K(\alpha_i, r) = \emptyset$ if $i \neq j$. Suppose that $1 < p < \infty$ and $\{R_j\}_{j\geq 1}$ is a bounded sequence in $\mathfrak{L}(L^p)$. If $f \in L^p$ is such that $\sum_{j\geq 1} M_{a_j^m} R_j f \in L^p$ then*

$$\|\sum_{j\geq 1} P_{a_j^1} \dots P_{a_j^m} R_j f\|_p \leq C_p^m \, \|\sum_{j\geq 1} M_{a_j^m} R_j f\|_p \,,$$

where C_p is the norm of the projection P acting on L^p.

Proof. For every $j \geq 1$ write $Q_j = P_{a_j^2} \dots P_{a_j^{m-1}} P$ and $S = \sum_{j\geq 1} M_{a_j^1} Q_j M_{a_j^m} R_j$. Then $\|Q_j\| \leq C_p^{m-1}$ and for $f \in L^p$ we have

$$\begin{aligned}\|Sf\|_p^p &= \|\sum_{j\geq 1} M_{a_j^1} Q_j M_{a_j^m} R_j f\|_p^p \;=\; \sum_{j\geq 1} \|M_{a_j^1} Q_j M_{a_j^m} R_j f\|_p^p \\ &\leq C_p^{(m-1)p} \sum_{j\geq 1} \|M_{a_j^m} R_j f\|_p^p \;=\; C_p^{(m-1)p} \|\sum_{j\geq 1} (M_{a_j^m} R_j) f\|_p^p. \qquad (2.1)\end{aligned}$$

If the last quantity is finite then $Sf \in L^p$ and the partial sums $S_n f = \sum_{j=1}^n M_{a_j^1} Q_j M_{a_j^m} R_j f$ converge to Sf in L^p-norm when $n \to \infty$. Therefore

$$\begin{aligned}\|\sum_{j\geq 1} P_{a_j^1} \dots P_{a_j^m} R_j f\|_p^p &= \lim_n \|\sum_{j=1}^n P_{a_j^1} \dots P_{a_j^m} R_j f\|_p^p \\ &= \lim_n \|P S_n f\|_p^p \;\leq\; C_p \|Sf\|_p^p.\end{aligned}$$

The lemma follows combining this equality with (2.1). □

Corollary 2.3 *Taking $R_j = I$ for every j in Lemma 2.2 we obtain*

$$\|\sum_{j\geq 1} P_{a_j^1} \dots P_{a_j^m}\|_{\mathfrak{L}(L^p)} \leq C_p^m.$$

Proof. By the lemma,

$$\|\sum_{j\geq 1} P_{a_j^1} \dots P_{a_j^m} f\|_p \leq C_p^m \, \|\sum_{j\geq 1} M_{a_j^m} f\|_p \leq C_p^m \, \|M_{(\sum_{j\geq 1} a_j^m)} f\|_p \leq C_p^m \|f\|_p$$

for every $f \in L^p$. □

The next result is a particular case of Lemma 4.2.2 in [6].

Lemma 2.4 *If $t > -1$, c is real and*

$$F_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|\omega|^2)^t}{|1-z\overline{\omega}|^{2+t+c}}\, dA(\omega) \quad (z \in \mathbb{D}),$$

then $F_{c,t}$ is bounded when $c < 0$ and $|F_{c,t}(z)| \leq C(1-|z|^2)^{-c}$ when $c > 0$.

Lemma 2.5 *Let $0 < r < 1$ and $\{\alpha_j\}_{j\geq 1} \subset \mathbb{D}$ such that $K(\alpha_j, r) \cap K(\alpha_i, r) = \emptyset$ if $i \neq j$. If $r < R < 1$ and $0 < \beta < 1$ then*

$$\int_{\mathbb{D}} \sum_j \left[\chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,R)}(\omega)\right] \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2}\, dA(\omega) \leq c_\beta(R)(1-|z|^2)^{-\beta}, \tag{2.2}$$

where $c_\beta(R) \to 0$ when $R \to 1$.

Proof. If $z \in K(\alpha_j, r)$ and $\omega \in \mathbb{D} \setminus K(\alpha_j, R)$ then

$$\rho(\omega, z) \geq \frac{\rho(\omega,\alpha_j) - \rho(\alpha_j, z)}{1-\rho(\alpha_j,z)\rho(\omega,\alpha_j)} > \frac{R-r}{1-Rr} = \delta,$$

where $\delta = \delta(R) \to 1$ when $R \to 1$. Therefore $\mathbb{D} \setminus K(\alpha_j, R) \subset \mathbb{D} \setminus K(z, \delta)$ and

$$\sum_j \chi_{K(\alpha_j,r)}(z)\chi_{D\setminus K(\alpha_j,R)}(\omega) \leq \sum_j \chi_{K(\alpha_j,r)}(z)\, \chi_{D\setminus K(z,\delta)}(\omega).$$

Hence, the integral in (2.2) is bounded by

$$\begin{aligned} &\sum_j \chi_{K(\alpha_j,r)}(z) \int_{\mathbb{D}} \chi_{D\setminus K(z,\delta)}(\omega) \frac{(1-|\omega|^2)^{-\beta}}{|1-z\overline{\omega}|^2}\, dA(\omega) \\ &\qquad = \sum_j \chi_{K(\alpha_j,r)}(z) \int_{|v|>\delta} \frac{(1-|\varphi_z(v)|^2)^{-\beta}}{|1-z\overline{v}|^2}\, dA(v) \\ &\qquad \leq \int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}} (1-|z|^2)^{-\beta}\, dA(v), \end{aligned} \tag{2.3}$$

where the equality comes from the change of variables $v = \varphi_z(\omega)$ and the inequality because $K(\alpha_j, r)$ are pairwise disjoint. Pick some $p = p(\beta) > 1$ satisfying simultaneously the conditions $p\beta < 1$ and $p(2-\beta) < 2$. If $p^{-1} + q^{-1} = 1$, Holder's inequality gives

$$\int_{|v|>\delta} \frac{(1-|v|^2)^{-\beta}}{|1-z\overline{v}|^{2-2\beta}}\, dA(v) \leq \left(\int_{\mathbb{D}} \frac{(1-|v|^2)^{-p\beta}}{|1-z\overline{v}|^{2p(1-\beta)}}\, dA(v)\right)^{1/p} (1-\delta^2)^{1/q}.$$

Since $2p(1-\beta) = 2 - p\beta + [p(2-\beta) - 2] < 2 - p\beta$, then Lemma 2.4 says that the last expression is bounded by $C_\beta(1-\delta^2)^{1/q}$, where C_β depends only on β. Going back to (2.3) we see that the integral in (2.2) is bounded by

$$C_\beta(1-\delta(R)^2)^{1/q(\beta)}(1-|z|^2)^{-\beta},$$

proving the lemma. □

Lemma 2.6 *Let $0 < r < 1$ and $\alpha_j \in \mathbb{D}$ (for $j \geq 1$) such that $K(\alpha_j, r)$ are pairwise disjoint. Suppose that $R \in (r, 1)$ and $a_j, A_j \in L^\infty$ are functions of norm ≤ 1 such that*

$$\operatorname{supp} a_j \subset K(\alpha_j, r) \quad \textit{and} \quad \operatorname{supp} A_j \subset \mathbb{D} \setminus K(\alpha_j, R).$$

Then $\sum_{j\geq 1} M_{a_j} P M_{A_j}$ is bounded on L^p for every $1 < p < \infty$, with norm bounded by some constant $k_p(R) \to 0$ when $R \to 1$.

Proof. Write

$$\Phi(z, \omega) = \sum_{j\geq 1} \chi_{K(\alpha_j, r)}(z) \chi_{D\setminus K(\alpha_j, R)}(\omega) \frac{1}{|1 - \overline{\omega} z|^2}.$$

Let $f \in L^p$. Since $\|a_j\|_\infty, \|A_j\|_\infty \leq 1$ for all j, then

$$\left| \left(\sum_{j\geq 1} M_{a_j} P M_{A_j} f \right)(z) \right| = \left| \sum_{j\geq 1} a_j(z) \int_{\mathbb{D}} A_j(\omega) f(\omega) \frac{dA(\omega)}{(1 - \overline{\omega} z)^2} \right|$$
$$\leq \int_{\mathbb{D}} \Phi(z, \omega) |f(\omega)| \, dA(\omega).$$

Taking $h(z) = (1 - |z|^2)^{-1/pq}$, where $p^{-1} + q^{-1} = 1$, Lemma 2.5 asserts that

$$\int_{\mathbb{D}} \Phi(z, \omega) h(\omega)^q \, dA(\omega) \leq c_{p^{-1}}(R) h(z)^q$$

and Lemma 2.4 implies that there is some $C > 0$ such that

$$\int_{\mathbb{D}} \Phi(z, \omega) h(z)^p \, dA(z) \leq C h(\omega)^p.$$

By Schur's theorem [6, p. 42] the integral operator with kernel $\Phi(z, \omega)$ is bounded on L^p and its norm is bounded by $(c_{p^{-1}}(R))^{\frac{1}{q}} C^{\frac{1}{p}} \to 0$ as $R \to 1$. □

Let $a_j, b_j \in L^\infty$ ($j \geq 1$) be functions of norm at most 1 supported on $K(\alpha_j, r)$, where the pseudo-hyperbolic disks are pairwise disjoint. By Lemma 2.1 for any $\sigma \in (r, 1)$ there is some $n = n(\sigma) \geq 1$ and a partition of the positive integers $\mathbb{N} = N_1 \cup \ldots \cup N_n$ such that

$$\rho(\alpha_i, \alpha_j) > \sigma \quad \text{for} \quad i \neq j, \;\; i, j \in N_k \;\; (1 \leq k \leq n).$$

Lemma 2.7 *If $1 < p < \infty$ then*

$$\sum_{1\leq k\leq n} \left[\left(\sum_{j\in N_k} P_{a_j^1} \ldots P_{a_j^m} \right) P_{(\sum_{i\in N_k} b_i)} \right] \to \sum_{j\geq 1} P_{a_j^1} \ldots P_{a_j^m} P_{b_j} \tag{2.4}$$

in operator norm when $\sigma \to 1$.

Proof. Write $B_j = \sum_{i\in N_k, i\neq j} b_i$ when $j \in N_k$ for some $1 \le k \le n$. Since $P_{(\sum_{i\in N_k} b_i)} = P_{b_j} + P_{B_j}$, the first term in (2.4) can be decomposed as

$$\sum_{k=1}^{n} \left[\sum_{j\in N_k} P_{a_j^1} \dots P_{a_j^m} P_{b_j} + \sum_{j\in N_k} P_{a_j^1} \dots P_{a_j^m} P_{B_j} \right] = S_1 + S_2,$$

where

$$S_1 = \sum_{k=1}^{n} \sum_{j\in N_k} P_{a_j^1} \dots P_{a_j^m} P_{b_j} = \sum_{j\ge 1} P_{a_j^1} \dots P_{a_j^m} P_{b_j}$$

and

$$S_2 = \sum_{k=1}^{n} \sum_{j\in N_k} P_{a_j^1} \dots P_{a_j^m} P_{B_j} = \sum_{j\ge 1} P_{a_j^1} \dots P_{a_j^m} P_{B_j}.$$

Let $f \in L^p$. By Lemma 2.2

$$\|S_2 f\|_p \le C_p^m \| \sum_{j\ge 1} M_{a_j^m} P_{B_j} f\|_p. \tag{2.5}$$

If $\omega \in \operatorname{supp} B_j$ for $j \in N_k$ with $1 \le k \le n$, then there is $i \neq j$ in N_k such that $\omega \in K(\alpha_i, r)$. Then

$$\rho(\omega, \alpha_j) \ge \frac{\rho(\alpha_j, \alpha_i) - \rho(\omega, \alpha_i)}{1 - \rho(\alpha_j, \alpha_i)\rho(\omega, \alpha_i)} > \frac{\sigma - r}{1 - \sigma r} = R(\sigma),$$

meaning that $\operatorname{supp} B_j \subset \mathbb{D} \setminus K(\alpha_j, R(\sigma))$. Since $R(\sigma)\to 1$ when $\sigma\to 1$, (2.5) and Lemma 2.6 prove (2.4). □

Corollary 2.8 *Under the conditions of Lemma 2.7,*

$$\sum_{1\le k\le n} \left[\left(\sum_{j\in N_k} T_{a_j^1} \dots T_{a_j^m} \right) T_{(\sum_{i\in N_k} b_i)} \right] \to \sum_{j\ge 1} T_{a_j^1} \dots T_{a_j^m} T_{b_j} \tag{2.6}$$

and

$$\sum_{1\le k\le n} \left[T_{(\sum_{i\in N_k} b_i)} \left(\sum_{j\in N_k} T_{a_j^1} \dots T_{a_j^m} \right) \right] \to \sum_{j\ge 1} T_{b_j} T_{a_j^1} \dots T_{a_j^m} \tag{2.7}$$

in operator norm when $\sigma\to 1$.

Proof. We obtain (2.6) by restricting the operators of (2.4) to L_a^p. To prove (2.7) use (2.6) with

$$\sum_{1\le k\le n} \left[\left(\sum_{j\in N_k} T_{\overline{a}_j^m} \dots T_{\overline{a}_j^1} \right) T_{(\sum_{i\in N_k} \overline{b}_i)} \right]$$

and then take adjoints. □

Proposition 2.9 *Let $1 < p < \infty$ and $c_j^1, \ldots, c_j^l, a_j, b_j, d_j^1, \ldots, d_j^m \in L^\infty$ be functions of norm ≤ 1 supported on $K(\alpha_j, r)$ for $j \geq 1$, where $K(\alpha_j, r) \cap K(\alpha_i, r) = \emptyset$ if $i \neq j$. Then*

$$\sum_{j\geq 1} T_{c_j^1} \ldots T_{c_j^l}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})T_{d_j^1} \ldots T_{d_j^m} \in \mathfrak{CT}(L_a^p).$$

Proof. For $r < \sigma < 1$ decompose $\mathbb{N} = N_1 \cup \ldots \cup N_n$ as in the paragraph that precedes Lemma 2.7. By Corollary 2.8,

$$\sum_{1\leq k\leq n} \left[T_{(\sum_{j\in N_k} a_j)}T_{(\sum_{i\in N_k} b_i)} - T_{(\sum_{i\in N_k} b_i)}T_{(\sum_{j\in N_k} a_j)}\right] \rightarrow \sum_{j\geq 1}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})$$

when $\sigma \rightarrow 1$. Since the first operators belong to the commutator ideal, so does their limit. Thus,

$$\sum_{j\in F}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j}) \in \mathfrak{CT}(L_a^p)$$

for any subset $F \subset \mathbb{N}$. In particular, this hold for $F = N_k$ $(1 \leq k \leq n)$. Then

$$\sum_{1\leq k\leq n} \left[\left(\sum_{j\in N_k}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})\right) T_{(\sum_{i\in N_k} d_i^1)}\right] \in \mathfrak{CT}(L_a^p),$$

and since (2.6) says that the above operators converge to

$$\sum_{j\geq 1}(T_{a_j}T_{b_j} - T_{b_j}T_{a_j})T_{d_j^1}$$

when $\sigma \rightarrow 1$, this operator is also in $\mathfrak{CT}(L_a^p)$. Clearly, the same holds if the sum is over any set $F \subset \mathbb{N}$. We can repeat this process $m - 1$ more times using (2.6) and then l times using (2.7) to obtain the desired result. □

3 An invertible operator in $\mathfrak{CT}(L_a^2)$

Let $a \in L^\infty$ be a real-valued function such that $a(\omega) \geq \delta > 0$ for every $\omega \in \mathbb{D}$. Then T_a is self-adjoint and

$$\langle T_a f, f\rangle = \int_{\mathbb{D}} a|f|^2\, dA \geq \delta \int_{\mathbb{D}} |f|^2\, dA = \delta\|f\|_2^2$$

for every $f \in L_a^2$. Therefore T_a is invertible. Theorem 1.1 will be proved by constructing a function a as above such that $T_a \in \mathfrak{CT}(L_a^2)$.

We need to summarize several basic features of Toeplitz operators. If $a, b \in L^\infty$ then $T_aT_b = T_{ab}$ when $\overline{a} \in H^\infty$ or $b \in H^\infty$. If $z \in \mathbb{D}$ then $U_z f = (f \circ \varphi_z)\varphi_z'$ defines a unitary self-adjoint operator on L_a^2. Therefore, if $a \in L^\infty$ and $f, g \in L_a^2$,

$$\langle U_zT_aU_zf, g\rangle = \langle T_aU_zf, U_zg\rangle = \langle a(f \circ \varphi_z)\varphi_z', (g \circ \varphi_z)\varphi_z'\rangle = \langle (a \circ \varphi_z)f, g\rangle,$$

where the last equality comes from changing variables inside the integral. Thus

$$U_z T_{a_1} \dots T_{a_n} U_z = U_z T_{a_1} U_z \dots U_z T_{a_n} U_z = T_{a_1 \circ \varphi_z} \dots T_{a_n \circ \varphi_z} \tag{3.1}$$

for $a_j \in L^\infty$, $1 \le j \le n$. By diagonal operator we always mean diagonal with respect to the orthonormal basis $\{\sqrt{n+1}\; z^n\}_{n \ge 0}$

A straightforward calculation with polar coordinates shows that if $a \in L^\infty$ is a radial function (i.e.: $a(z) = a(|z|)$), then T_a is diagonal with n-entry

$$\lambda_n(a) = \int_0^1 a(t^{1/2})(n+1)t^n \, dt. \tag{3.2}$$

If χ_r denotes the characteristic function of the ball $\{|\omega| \le r\}$, where $0 < r < 1$, then (3.2) yields $T_{\chi_r} \omega^n = r^{2(n+1)} \omega^n$.

Lemma 3.1 *Let $a \in L^\infty$ be a radial function and $0 < r < 1$. Then*

$$T_{\chi_r} T_a = T_{\chi_r(\omega)\, a(\omega/r)}.$$

Proof. The operator $T_{\chi_r(\omega)\, a(\omega/r)}$ is diagonal, and its n-entry is

$$\begin{aligned} \int_0^1 \chi_{[0,r]}(t^{1/2})\, a\left(\frac{t^{1/2}}{r}\right)(n+1)t^n \, dt &= \int_0^{r^2} a\left(\frac{t^{1/2}}{r}\right)(n+1)t^n \, dt \\ &= r^{2n+2} \int_0^1 a(u^{1/2})(n+1)u^n \, du, \end{aligned}$$

where the last equality comes from the change of variables $u = t/r^2$. By (3.2) $T_{\chi_r} T_a$ is also diagonal and has the same entries. □

A simple calculation shows that $\langle T_{\overline{\omega}} \omega^n, \omega^k \rangle = \langle \omega^n, \omega^{k+1} \rangle = \langle (n/n+1)\omega^{n-1}, \omega^k \rangle$ if $n \ge 1$. A recursive argument then gives that for every nonnegative integer k,

$$T_{\overline{\omega}^k} \omega^n = \left(\frac{n+1-k}{n+1}\right) \omega^{n-k} \quad \text{if} \quad n \ge k$$

and $T_{\overline{\omega}^k} \omega^n = 0$ if $n < k$. Thus

$$T_{\overline{\omega}^k} T_{\chi_r} \omega^n = r^{2(n+1)} \left(\frac{n+1-k}{n+1}\right) \omega^{n-k} \quad \text{if} \quad n \ge k,$$

and since $T_{\chi_r} T_{\omega^k} \omega^n = r^{2(n+k+1)} \omega^{n+k}$ then

$$(T_{\overline{\omega}^k} T_{\chi_r})(T_{\chi_r} T_{\omega^k}) \omega^n = r^{4(n+k+1)} \left(\frac{n+1}{n+k+1}\right) \omega^n = r^{4k} T_{\chi_{r^2}} T_{\overline{\omega}^k} T_{\omega^k} \omega^n, \tag{3.3}$$

where the second equality comes from the limit case $r = 1$ in the first equality and from $T_{\chi_{r^2}}\omega^n = r^{4(n+1)}\omega^n$. Since T_{χ_r} and $T_{\omega^k}T_{\overline{\omega}^k}$ are diagonal, they commute, and since $T_{\chi_r}^2 = T_{\chi_{r^2}}$ then

$$T_{\chi_r}T_{\omega^k}T_{\overline{\omega}^k}T_{\chi_r} = T_{\chi_r}^2 T_{\omega^k}T_{\overline{\omega}^k} = T_{\chi_{r^2}}T_{\omega^k}T_{\overline{\omega}^k}. \tag{3.4}$$

By (3.3), (3.4) and Lemma 3.1,

$$\begin{aligned} S_k \stackrel{\text{def}}{=} [T_{\omega^k\chi_r}, T_{\overline{\omega}^k\chi_r}] &= T_{\chi_{r^2}}(T_{\omega^k}T_{\overline{\omega}^k} - r^{4k}T_{\overline{\omega}^k}T_{\omega^k}) \\ &= T_{\chi_{r^2}}T_{\omega^k}T_{\overline{\omega}^k} - T_{\chi_{r^2}|\omega|^{2k}}. \end{aligned} \tag{3.5}$$

Let $P_0 \in \mathfrak{L}(L_a^2)$ be the operator $P_0 f = f(0)$. Straightforward evaluations on the basis $\{z^n\}_{n\geq 0}$ give the following identities

$$T_\omega T_{\overline{\omega}} = T_{1+\log|\omega|^2}, \quad T_{\omega^2}T_{\overline{\omega}^2} = T_{1+2\log|\omega|^2} + P_0 \quad \text{and} \quad T_{\chi_{r^2}}P_0 = r^4 P_0. \tag{3.6}$$

Then

$$\begin{aligned} 2S_1 - S_2 &\stackrel{\text{by (3.5)}}{=} T_{\chi_{r^2}}(2T_\omega T_{\overline{\omega}} - T_{\omega^2}T_{\overline{\omega}^2}) + T_{\chi_{r^2}(|\omega|^4 - 2|\omega|^2)} \\ &\stackrel{\text{by (3.6)}}{=} T_{\chi_{r^2}(1+|\omega|^4-2|\omega|^2)} - r^4 P_0 \\ &= T_{\chi_{r^2}(1-|\omega|^2)^2} - r^4 P_0. \end{aligned} \tag{3.7}$$

Since $2S_1 - S_2$, T_{χ_r} and P_0 are diagonal operators, they commute. Consequently

$$P_0 T_{\chi_r} T_\omega = T_{\chi_r} P_0 T_\omega = 0,$$

which together with Lemma 3.1 and (3.7) gives

$$T_{\chi_r\overline{\omega}}(2S_1 - S_2)T_{\chi_r\omega} = T_{\overline{\omega}}T_{\chi_r}(2S_1 - S_2)T_{\chi_r}T_\omega = T_{\chi_{r^4}(1-|\omega|^2/r^4)^2|\omega|^2}. \tag{3.8}$$

If $\alpha \in \mathbb{D}$ then (3.1), (3.5) and (3.8) yield

$$\begin{aligned} &T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}(2[T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha}] - [T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha^2}, T_{(\chi_r\circ\varphi_\alpha)\overline{\varphi}_\alpha^2}])\, T_{(\chi_r\circ\varphi_\alpha)\varphi_\alpha} \\ &\qquad = U_\alpha T_{\chi_r\overline{\omega}}(2S_1 - S_2)T_{\chi_r\omega}U_\alpha \\ &\qquad = T_{(\chi_{r^4}\circ\varphi_\alpha)(1-|\varphi_\alpha|^2/r^4)^2|\varphi_\alpha|^2}. \end{aligned} \tag{3.9}$$

Suppose that $0 < r < 1$ and $\{\alpha_j\} \subset \mathbb{D}$ is a sequence such that $K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ for $i \neq j$. Since $(\chi_{r^4} \circ \varphi_\alpha)(\omega) = \chi_{K(\alpha,r^4)}(\omega)$, the characteristic function of $K(\alpha, r^4)$, then

$$A(\omega) \stackrel{\text{def}}{=} \sum_{j\geq 1} \chi_{r^4}(\varphi_{\alpha_j}(\omega))(1 - |\varphi_{\alpha_j}(\omega)|^2/r^4)^2|\varphi_{\alpha_j}(\omega)|^2$$

is in L^∞ with $\|A\|_\infty \leq 1$. In conjunction with (3.9), Proposition 2.9 tells us that

$$T_A = \sum_{j\geq 1} T_{(\chi_{r^4}\circ\varphi_{\alpha_j})(1-|\varphi_{\alpha_j}|^2/r^4)^2|\varphi_{\alpha_j}|^2} \in \mathfrak{CT}(L_a^2). \tag{3.10}$$

When $\omega \in \mathbb{D}$ satisfies $r^4/4 < \rho(\omega, \alpha_j) \leq (3/4)r^4$ for some α_j we have

$$\left(1 - \frac{|\varphi_{\alpha_j}(\omega)|^2}{r^4}\right)^2 |\varphi_{\alpha_j}(\omega)|^2 \geq \left(1 - \frac{3^2 r^8}{4^2 r^4}\right)^2 \frac{r^8}{4^2} \geq \frac{r^8}{2^8},$$

meaning that

$$A(\omega) \geq (r/2)^8 \text{ when } \omega \in K(\alpha_j, (3/4)r^4) \setminus K(\alpha_j, r^4/4) \text{ for some } \alpha_j. \tag{3.11}$$

Lemma 3.2 *Given $0 < \sigma < 1$ there is a separated sequence $\{\alpha_j\}$ in $\mathbb{D}$ such that every $z \in \mathbb{D}$ is in $K(\alpha_j, \frac{3\sigma}{4}) \setminus K(\alpha_j, \frac{\sigma}{4})$ for some α_j.*

Proof. Take a sequence $\{\alpha_j\} \subset \mathbb{D}$ such that $\rho(\alpha_i, \alpha_j) > \sigma/100$ if $i \neq j$ and

$$\rho(\{\alpha_j\}_{j\geq 1}, \omega) \leq \sigma/8 \text{ for every } \omega \in \mathbb{D}. \tag{3.12}$$

For an arbitrary $z \in \mathbb{D}$ write $\beta_j = \varphi_z(\alpha_j)$. The conformal invariance of ρ implies that $\{\beta_j\}_{j\geq 1}$ satisfies (3.12). We claim that there is some β_j such that $\sigma/4 < |\beta_j| \leq (3/4)\sigma$. Otherwise

$$\begin{aligned}\rho(\sigma/2, \{\beta_j\}_{j\geq 1}) &\geq \rho(\sigma/2, \mathbb{D} \setminus \{\sigma/4 < |\omega| \leq (3/4)\sigma\}) \\ &= \rho(\sigma/2, \{\sigma/4, 3\sigma/4\}) \\ &\geq \frac{(\sigma/4)}{1 - (\sigma/4)(\sigma/2)} > \sigma/4.\end{aligned}$$

This contradicts (3.12) with respect to $\{\beta_j\}_{j\geq 1}$ for $\omega = \sigma/2$. If $\sigma/4 < |\beta_{j_0}| \leq (3/4)\sigma$ then

$$\rho(\alpha_{j_0}, z) = \rho(\varphi_z(\alpha_{j_0}), \varphi_z(z)) = \rho(\beta_{j_0}, 0) = |\beta_{j_0}| \in (\sigma/4, 3\sigma/4],$$

and since $z \in \mathbb{D}$ is arbitrary, the lemma follows. □

Returning to our construction, fix $0 < r < 1$ and suppose that $\mathcal{S} = \{\alpha_j\}_{j\geq 1}$ is a sequence satisfying Lemma 3.2 for $\sigma = r^4$. Since $\mathcal{S}$ is separated, by Lemma 2.1 we can decompose $\mathcal{S} = \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_N$, where for each $1 \leq k \leq N$, $K(\alpha_i, r) \cap K(\alpha_j, r) = \emptyset$ if $\alpha_i, \alpha_j \in \mathcal{S}_k$ with $i \neq j$. For $1 \leq k \leq N$ write

$$A_k(\omega) = \sum_{\alpha_j \in \mathcal{S}_k} \chi_{r^4}(\varphi_{\alpha_j}(\omega))(1 - |\varphi_{\alpha_j}(\omega)|^2/r^4)^2 |\varphi_{\alpha_j}(\omega)|^2.$$

Then $\|A_k\|_\infty \leq 1$ and (3.10) says that $T_{A_k} \in \mathfrak{CT}(L_a^2)$. Consequently

$$\sum_{k=1}^N T_{A_k} = T_{(\sum_{k=1}^N A_k)} \in \mathfrak{CT}(L_a^2).$$

In addition, (3.11) says that for every $1 \le k \le N$,

$$A_k(\omega) \ge (r/2)^8 \quad \text{when} \quad \omega \in K(\alpha_j, (3/4)r^4) \setminus K(\alpha_j, r^4/4) \quad \text{for some} \quad \alpha_j \in \mathcal{S}_k,$$

and since Lemma 3.2 asserts that

$$\mathbb{D} = \bigcup_{1 \le k \le N} \bigcup_{\alpha_j \in \mathcal{S}_k} K(\alpha_j, (3/4)r^4) \setminus K(\alpha_j, r^4/4)$$

then $\sum_{k=1}^{N} A_k(\omega) \ge (r/2)^8$ for every $\omega \in \mathbb{D}$. This completes the construction and proves Theorem 1.1.

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