HOCHSCHILD (CO)HOMOLOGY OF HOPF CROSSED PRODUCTS

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ABSTRACT. For a general crossed product $E = A\#_f H$, of an algebra A by a Hopf algebra H, we obtain complexes simpler than the canonical ones, giving the Hochschild homology and cohomology of E. These complexes are equipped with natural filtrations. The spectral sequences associated to them is a natural generalization of the one obtained in [H-S] by the direct method. We also get that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex.

INTRODUCTION

Let G be a group, $S = \bigoplus S_g$ a strongly G-graded algebra and V an S-bimodule. In [L] was shown that there is a convergent spectral sequence

$$E_{rs}^2 = \mathrm{H}_r(G, \mathrm{H}_s(S_e, V)) \Rightarrow \mathrm{H}_{r+s}(S, V),$$

where e denotes the identity of G. In [S] was shown that this result remains valid for H-Galois extensions (in his paper the author deals with both the homology and the cohomology of these algebras). An important particular type of H-Galois extensions are the crossed products with convolution invertible cocycle $E = A \#_f H$, of an algebra A by a Hopf algebra H (for the definition see Section one). The purpose of our paper is to construct complexes simpler than the canonical ones, given the Hochschild (co)homology of E with coefficients in an arbitrary E-bimodule. These complexes are equipped with canonical filtrations. We show that the spectral sequences associated to them coincide with the ones obtained using a natural generalization of the direct method introduced in [H-S], and with the ones constructed in [S] (when these are specialize to crossed products). In the case of group extensions these results were proved in [E] and [B].

This paper is organized as follows: in Section 1 a resolution (X_*, d_*) of a crossed product $E = A \#_f H$ is given. To accomplish this construction we do not use the fact that the cocycle is convolution invertible. Moreover, we give a recursive construction of morphisms $\phi_* : (X_*, d_*) \to (E \otimes \overline{E}^* \otimes E, b'_*)$ and $\psi_* : (E \otimes \overline{E}^* \otimes E, b'_*) \to (X_*, d_*)$, where $(E \otimes \overline{E}^* \otimes E, b'_*)$ is the normalized Hochschild resolution, such that $\psi_* \phi_* = id$ and we show that $\phi_* \psi_*$ is homotopically equivalent to the identity map. Consequently our resolution is a direct sum of the normalized Hochschild resolution. We

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also recursively construct an homotopy $\phi_* \psi_* \xrightarrow{\omega_{*+1}} id_*$. Both, the canonical normalized resolution and (X_*, d_*) are equipped with natural filtrations, which are preserved by the maps ϕ_*, ψ_* and ω_{*+1} .

In Section 2, for an *E*-bimodule *M*, we get complexes $\widehat{X}_*(E, M)$ and $\widehat{X}^*(E, M)$, giving the Hochschild homology and cohomology of *E* with coefficients in *M* respectively. The filtration of (X_*, d_*) induces filtrations on $\widehat{X}_*(E, M)$ and $\widehat{X}^*(E, M)$. So, we obtain converging spectral sequences $E_{rs}^1 = \operatorname{Hr}(A, M \otimes \overline{H}^s) \Rightarrow \operatorname{Hr}_{r+s}(E, M)$ and $E_1^{rs} = \operatorname{Hr}(A, \operatorname{Hom}_k(\overline{H}^s, M)) \Rightarrow \operatorname{Hr}^{r+s}(E, M)$. Using the results of Section 1, we get that these spectral sequences are the ones associated to suitable filtrations of the Hochschild normalized chain and cochain complexes $(M \otimes \overline{E}^*, b_*)$ and $(\operatorname{Hom}_k(\overline{E}^*, M), b^*)$. This allows us to give very simple proofs of the main results of [H-S] and [G].

In Section 3, we show that, if the cocycle is convolution invertible, then the complexes $\widehat{X}_*(E, M)$ and $\widehat{X}^*(E, M)$ are isomorphic to simpler complexes $\overline{X}_*(E, M)$ and $\overline{X}^*(E, M)$ respectively. Then, we compute the term E_{rs}^2 and E_2^{rs} of the spectral sequences obtained in Section 2. Moreover, using the above mentioned filtrations, we prove that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex. Finally, as an application we obtain some results about the $\operatorname{Tor}_{*}^{E}$ and $\operatorname{Ext}_{E}^{*}$ functors and an upper bound for the global dimension of E (for group crossed products this bound was obtained in [A-R]).

In addition to the direct method developed in [H-S], there are another two classical methods to obtain spectral sequences converging to $H_*(E, M)$ and with E^2 -term $H_*(H, H_*(A, M))$. Namely the Cartan-Leray and the Grothendieck spectral sequences of a crossed product. In Section 4, we recall these constructions and we prove that these spectral sequences are isomorphic to the one obtained in Section 2. This generalizes the main results of [B].

In a first appendix we give a method to construct (under suitable hypothesis) a projective resolution of the k-algebra E as $E^e = E \otimes E^{\text{op}}$ -bimodule, simpler than the canonical one of Hochschild. This method, which can be considered as a variant of the perturbation lemma, is used to prove the main result of Section 1. The boundary maps of the resolution (X_*, d_*) are recursively defined in Section 1. In a second appendix we give closed formulas for these maps.

1. A resolution for a crossed product

Let A be a k-algebra and H a Hopf algebra. We will use the Sweedler notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$, with the summation understood and superindices instead of subindices. Recall some definitions of [B-C-M] and [D-T]. A weak action of H on A is a bilinear map $(h, a) \mapsto a^h$ from $H \times A$ to A such that, for $h \in H$, $a, b \in A$

1)
$$(ab)^{h} = a^{h^{(1)}}b^{h^{(2)}}$$

2) $1^{h} = \epsilon(h)1,$
3) $a^{1} = a.$

Let A be a k-algebra and H a Hopf algebra with a weak action on A. Given a k-linear map $f: H \otimes H \to A$, let $A \#_f H$ be the k-algebra (in general non associative and without 1) with underlying vector space $A \otimes H$ and multiplication map

$$(a \otimes h)(b \otimes l) = ab^{h^{(1)}} f(h^{(2)}, l^{(1)}) \otimes h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes h$ of $A\#_f H$ will usually be written a#h to remind us H is weakly acting on A. The algebra $A\#_f H$ is called a *crossed* product if it is associative with 1#1 as identity element. It is easy to check that this happens if and only if f and the weak action satisfy the following conditions:

- i) (Normality of f) for all $h \in H$, we have $f(h, 1) = f(1, h) = \epsilon(h)1_A$,
- ii) (Cocycle condition) for all $h, l, m \in H$, we have

$$f(l^{(1)}, m^{(1)})^{h^{(1)}} f(h^{(2)}, l^{(2)}m^{(2)}) = f(h^{(1)}, l^{(1)}) f(h^{(2)}l^{(2)}, m),$$

iii) (Twisted module condition) for all $h, l \in H, a \in A$ we have

$$(a^{l^{(1)}})^{h^{(1)}}f(h^{(2)},l^{(2)}) = f(h^{(1)},l^{(1)})a^{h^{(2)}l^{(2)}}$$

In this section we obtain a resolution (X_*, d_*) of a crossed product $E = A \#_f H$ as an *E*-bimodule, which is simpler than the canonical one of Hochschild. To begin, we fix some notations:

- 1) For each k-algebra B, we put $\overline{B} = B/k$. Moreover, given $b \in B$ we also let b denote the class of b in \overline{B} .
- 2) We write $B^l = B \otimes \cdots \otimes B$, $\overline{B}^l = \overline{B} \otimes \cdots \otimes \overline{B}$ (*l* times) and $B_l(B) = B \otimes \overline{B}^l \otimes B$, for each natural number *l*.
- 3) Given $a_0 \otimes \cdots \otimes a_r \in A^{r+1}$ and $0 \le i < j \le r$, we write $\mathbf{a}_{ij} = a_i \otimes \cdots \otimes a_j \in A^{j-i+1}$.
- 4) Given $h_0 \otimes \cdots \otimes h_s \in H^{s+1}$ and $0 \le i < j \le s$, we write $\mathbf{h}_{ij} = h_i \otimes \cdots \otimes h_j$ and $\mathfrak{h}_{ij} = h_i \cdots h_j \in H$.
- 5) Given $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H^{s+1}$, we let $\mathbf{h}^{(1)} \otimes \mathbf{h}^{(2)}$ denote the comultiplication of \mathbf{h} in H^{s+1} . So, $\mathbf{h}^{(1)} \otimes \mathbf{h}^{(2)} = (h_0^{(1)} \otimes \cdots \otimes h_s^{(1)}) \otimes (h_0^{(2)} \otimes \cdots \otimes h_s^{(2)})$.
- 6) Given $a \in A$, $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in A^r$ and $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H^{s+1}$, we write $a^{\overline{\mathbf{h}}} = (\dots (((a^{h_s})^{h_{s-1}})^{h_{s-2}})^{h_{s-3}} \dots)^{h_0}$ and $\mathbf{a}^{\overline{\mathbf{h}}} = a_1^{\overline{\mathbf{h}_{0s}^{(1)}}} \otimes \cdots \otimes a_r^{\overline{\mathbf{h}_{0s}^{(r)}}}$.

1.1. The resolution (X_*, d_*)

Let $Y_s = E \otimes \overline{H}^s \otimes H$ $(s \ge 0)$ and $X_{rs} = E \otimes \overline{H}^s \otimes \overline{A}^r \otimes E$ $(r, s \ge 0)$. The groups X_{rs} are *E*-bimodules in an obvious way and the groups Y_s are *E*-bimodules via the left canonical action and the right action

$$(a_0 \otimes \mathbf{h})(a \# h) = a_0 a^{\overline{\mathbf{h}^{(1)}}} f(h_{s+1}^{(2)}, h^{(1)})^{\overline{\mathbf{h}_{0s}^{(2)}}} \otimes \left(\mathbf{h}_{0s}^{(3)} \otimes h_{s+1}^{(3)} h^{(2)}\right)$$

where $\mathbf{h} = h_0 \otimes \cdots \otimes h_{s+1}$. Let us consider the diagram of *E*-bimodules and *E*-bimodule maps

$$\begin{array}{c} \vdots \\ \downarrow \partial_2 \\ Y_1 \xleftarrow{\mu_1} X_{01} \xleftarrow{d_{11}^0} X_{11} \xleftarrow{d_{21}^0} \dots \\ \downarrow \partial_1 \\ Y_0 \xleftarrow{\mu_0} X_{00} \xleftarrow{d_{10}^0} X_{10} \xleftarrow{d_{20}^0} \dots, \end{array}$$

where $\mu_* \colon X_{0*} \to Y_*, \ d_{**}^0 \colon X_{**} \to X_{*-1,*} \text{ and } \partial_* \colon Y_* \to Y_{*-1} \text{ are defined by:}$ $\mu_s(a_0 \otimes \mathbf{h}_{0s} \otimes a_1 \otimes h_{s+1}) = a_0 a_1^{\overline{\mathbf{h}_{0s}^{(1)}}} \otimes \mathbf{h}_{0s}^{(2)} \otimes h_{s+1},$ $d_{rs}^0(a_0 \otimes \mathbf{h}_{0s} \otimes \mathbf{a} \otimes h_{s+1}) = a_0 a_1^{\overline{\mathbf{h}_{0s}^{(1)}}} \otimes \mathbf{h}_{0s}^{(2)} \otimes \mathbf{a}_{2,r+1} \otimes h_{s+1}$ $+ \sum_{i=1}^r (-1)^i a_0 \otimes \mathbf{h}_{0s} \otimes \mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+2,r+1} \otimes h_{s+1},$ $\partial_s(a \otimes \mathbf{h}) = \sum_{i=0}^s (-1)^{i+1} a_i (h_i^{(1)}, h_{i+1}^{(1)})^{\overline{\mathbf{h}_{i-1}^{(1)}}} \otimes \mathbf{h}_{0,i-1}^{(2)} \otimes h_i^{(2)} h_{i+1}^{(2)} \otimes \mathbf{h}_{i+2,s+1},$

where $\mathbf{a} = a_1 \otimes \cdots \otimes a_{r+1}$ and $\mathbf{h} = h_0 \otimes \cdots \otimes h_{s+1}$. We have left *E*-module maps $\sigma_{0*}^0 \colon Y_* \to X_{0*}$ and $\sigma_{*+1,*}^0 \colon X_{**} \to X_{*+1,*}$, given by $\sigma_{r+1,s}^0(a_0 \otimes \mathbf{h}_{0s} \otimes \mathbf{a} \otimes h_{s+1}) = (-1)^{r+1}a_0 \otimes \mathbf{h}_{0s} \otimes \mathbf{a} \otimes 1 \# h_{s+1}$ for $r \geq -1$. Clearly (Y_*, ∂_*) is a complex and $\sigma_{*+1,s}^0$ is a contracting homotopy of

$$Y_s \xleftarrow{\mu_s} X_{0s} \xleftarrow{d_{1s}^0} X_{1s} \xleftarrow{d_{2s}^0} X_{2s} \xleftarrow{d_{2s}^0} X_{3s} \xleftarrow{d_{3s}^0} X_{4s} \xleftarrow{d_{4s}^0} X_{5s} \xleftarrow{d_{5s}^0} \dots$$

So, we are in the situation considered in Appendix A. We define *E*-bimodule maps $d_{rs}^l: X_{rs} \to X_{r+l-1,s-l}$ $(r \ge 0 \text{ and } 1 \le l \le s)$ recursively, by:

$$d_{rs}^{l}(\mathbf{x}) = \begin{cases} -\sigma_{0,s-1}^{0} \partial_{s} \mu_{s}(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1, \\ -\sum_{j=1}^{l-1} \sigma_{l-1,s-l}^{0} d_{j-1,s-j}^{l-j} d_{0s}^{j}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \le s \\ -\sum_{j=0}^{l-1} \sigma_{r+l-1,s-l}^{0} d_{r+j-1,s-j}^{l-j} d_{rs}^{j}(\mathbf{x}) & \text{if } r > 0, \end{cases}$$

for $\mathbf{x} \in k \otimes \overline{H}^s \otimes \overline{A}^r \otimes k$.

Theorem 1.1.1. There is a relative projective resolution

(1)
$$E \xleftarrow{-\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} X_6 \xleftarrow{d_7} \dots,$$

where $X_n = \bigoplus_{r+s=n} X_{rs}, \mu$ is the multiplication map and $d_n = \sum_{\substack{r+s=n \ r+l>0}} \sum_{l=0}^s d_{rs}^l.$

Proof. Let $\tilde{\mu}: Y_0 \to E$ be the map $\tilde{\mu}(a \otimes (h_0 \otimes h_1)) = -af(h_0^{(1)}, h_1^{(1)}) \# h_0^{(2)} h_1^{(2)}$. The complex of *E*-bimodules

$$E \stackrel{\widetilde{\mu}}{\leftarrow} Y_0 \stackrel{\partial_1}{\leftarrow} Y_1 \stackrel{\partial_2}{\leftarrow} Y_2 \stackrel{\partial_3}{\leftarrow} Y_3 \stackrel{\partial_4}{\leftarrow} Y_4 \stackrel{\partial_5}{\leftarrow} Y_5 \stackrel{\partial_6}{\leftarrow} Y_6 \stackrel{\partial_7}{\leftarrow} \dots$$

is contractible as a complex of left *E*-modules. A chain contracting homotopy $\sigma_0^{-1}: E \to Y_0$ and $\sigma_{s+1}^{-1}: Y_s \to Y_{s+1}$ $(s \ge 0)$ is given by $\sigma_{s+1}^{-1}(x) = (-1)^s x \otimes 1_H$. Hence, the theorem follows from Corollary A.2 of Appendix A

Remark 1.1.2. Let $\sigma_{l,s-l}^l: Y_s \to X_{l,s-l}$ and $\sigma_{r+l+1,s-l}^l: X_{rs} \to X_{r+l+1,s-l}$ be the maps recursively defined by

$$\sigma_{r+l+1,s-l}^{l} = -\sum_{i=0}^{l-1} \sigma_{r+l+1,s-l}^{0} d_{r+i+1,s-i}^{l-i} \sigma_{r+i+1,s-i}^{i} \quad (0 < l \le s \text{ and } r \ge -1).$$

We will prove, in Corollary A.2, that the family $\overline{\sigma}_0 \colon E \to X_0, \overline{\sigma}_{n+1} \colon X_n \to X_{n+1}$, defined by $\overline{\sigma}_0 = \sigma_{00}^0 \sigma_0^{-1}$ and

$$\overline{\sigma}_{n+1} = -\sum_{l=0}^{n+1} \sigma_{l,n-l+1}^l \sigma_{n+1}^{-1} \mu_n + \sum_{r+s=n} \sum_{l=0}^s \sigma_{r+l+1,s-l}^l \quad (n \ge 0),$$

is a contracting homotopy of the resolution (1) introduced in Theorem 1.1.1.

Theorem 1.1.3. Let $\mathbf{x} = a_0 \otimes \mathbf{h} \otimes \mathbf{a} \otimes \mathbf{1}_E$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^r$ and $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H \otimes \overline{H}^s$. We have: 1) d_{rs}^1 is the map given by

$$d_{rs}^{1}(\mathbf{x}) = \sum_{i=0}^{s-1} (-1)^{i+r} a_{0} f(h_{i}^{(1)}, h_{i+1}^{(1)})^{\overline{\mathbf{h}_{0,i-1}^{(1)}}} \otimes \mathbf{h}_{0,i-1}^{(2)} \otimes \mathbf{h}_{i}^{(2)} h_{i+1}^{(2)} \otimes \mathbf{h}_{i+2,s} \otimes \mathbf{a} \otimes \mathbf{1}_{E}$$
$$+ (-1)^{r+s} a_{0} \otimes \mathbf{h}_{0,s-1} \otimes \mathbf{a}^{h_{s}^{(1)}} \otimes \mathbf{1} \# h_{s}^{(2)},$$

2) For each $l \geq 2$, there are maps $F_0^{(l)} : \overline{H}^l \to A^{l-1}$ and $F_r^{(l)} : \overline{H}^l \otimes \overline{A}^r \to A^{r+l-1}$ ($r \geq 1$), whose image is included in the k-submodule of A^{r+l-1} generated by all the elementary tensors $a_1 \otimes \cdots \otimes a_{r+l-1}$ with l-1 coordinates in the image of f, such that for $2 \leq l \leq s$,

$$d_{rs}^{l}(\mathbf{x}) = (-1)^{l(r+s)} a_0 \otimes \mathbf{h}_{0,s-l} \otimes F_r^{(l)}(\mathbf{h}_{s-l+1,s}^{(1)} \otimes \mathbf{a}) \otimes 1 \# \mathfrak{h}_{s-l+1,s}^{(2)},$$

where $F_r^{(l)}(\mathbf{h}_{s-l+1,s}^{(1)} \otimes \mathbf{a}) = F_0^{(l)}(\mathbf{h}_{s-l+1,s})$ if r = 0.

Proof. The computation of d_{rs}^1 can be obtained easily by induction on r, using that $d_{0s}^1 = -\sigma_{0,s-1}^0 \partial_s \mu_s^0$ and $d_{rs}^1 = -\sigma_{r,s-1}^0 d_{r-1,s}^1 d_{rs}^0$ for $r \ge 1$. The assertion for d_{rs}^l , with $l \ge 2$, follows easily by induction on l and r, using the recursive definition of $d_{rs}^l \square$

In Appendix B we will give more precise formulas for the maps $F_r^{(l)}$ completing the computation of the d_{rs}^{l} 's.

1.2. Comparison with the canonical resolution

Let $(B_*(E), b'_*)$ be the normalized Hochschild resolution of E. As it is well known, the complex

$$E \stackrel{\mu}{\leftarrow} E \otimes E \stackrel{b_1'}{\leftarrow} B_1(E) \stackrel{b_2'}{\leftarrow} B_2(E) \stackrel{b_3'}{\leftarrow} B_3(E) \stackrel{b_4'}{\leftarrow} \dots$$

is contractible as a complex of left *E*-modules, with contracting homotopy $\xi_n(\mathbf{x}) = (-1)^n \mathbf{x} \otimes 1$. Let $\overline{\sigma}_*$ be the contracting homotopy of (1) introduced in Remark 1.1.2. Let $\phi_* : (X_*, d_*) \to (B_*(E), b'_*)$ and $\psi_* : (B_*(E), b'_*) \to (X_*, d_*)$ be the morphisms of *E*-bimodule complexes, recursively defined by $\phi_0 = id$, $\psi_0 = id$, $\phi_{n+1}(\mathbf{x} \otimes 1) = \xi_{n+1} \phi_n d_{n+1}(\mathbf{x} \otimes 1)$ and $\psi_{n+1}(\mathbf{y} \otimes 1) = \overline{\sigma}_{n+1} \psi_n b'_{n+1}(\mathbf{y} \otimes 1)$.

Proposition 1.2.1. $\psi_* \phi_* = id_*$ and $\phi_* \psi_*$ is homotopically equivalent to the identity map. An homotopy $\phi_* \psi_* \xrightarrow{\omega_{*+1}} id_*$ is recursively defined by $\omega_1 = 0$ and $\omega_{n+1}(\mathbf{x}) = \xi_{n+1} (\phi_n \psi_n - id - \omega_n b'_n)(\mathbf{x})$, for $\mathbf{x} \in E \otimes \overline{E}^n \otimes k$.

Proof. We prove both assertions by induction. Let $U_n = \phi_n \psi_n - id_n$ and $T_n = U_n - \omega_n b'_n$. Assuming that $b'_n \omega_n + \omega_{n-1} b'_{n-1} = U_{n-1}$, we get that on $E \otimes \overline{E}^n \otimes k$,

$$b'_{n+1} \omega_{n+1} + \omega_n b'_n = b'_{n+1} \xi_{n+1} T_n + \omega_n b'_n$$

= $T_n - \xi_n b'_n T_n + \omega_n b'_n$
= $U_n - \xi_n U_{n-1} b'_n + \xi_n b'_n \omega_n b'_n$
= $U_n - \xi_n U_{n-1} b'_n + \xi_n T_{n-1} b'_n = U_n.$

Hence, $b'_{n+1}\omega_{n+1} + \omega_n b'_n = U_n$ on $B_n(E)$. Next, we prove that $\psi_* \phi_* = id_*$. It is clear that $\psi_0 \phi_0 = id_0$. Assume that $\psi_n \phi_n = id_n$. Since $\phi_{n+1}(E \otimes \overline{H}^s \otimes \overline{A}^r \otimes k) \subseteq E \otimes \overline{E}^{n+1} \otimes k$, we have that, on $k \otimes \overline{H}^s \otimes \overline{A}^{n+1-s} \otimes k$,

$$\begin{split} \psi_{n+1} \, \phi_{n+1} &= \overline{\sigma}_{n+1} \, \psi_n \, b'_{n+1} \, \phi_{n+1} \\ &= \overline{\sigma}_{n+1} \, \psi_n \, b'_{n+1} \, \xi_{n+1} \, \phi_n \, d_{n+1} \\ &= \overline{\sigma}_{n+1} \, \psi_n \, \phi_n \, d_{n+1} - \overline{\sigma}_{n+1} \, \psi_n \, \xi_n \, b'_n \, \phi_n \, d_{n+1} \\ &= \overline{\sigma}_{n+1} \, d_{n+1} = i d_{n+2} - d_{n+2} \, \overline{\sigma}_{n+2}. \end{split}$$

So, to finish the proof it suffices to check that $\overline{\sigma}_{n+2}(k \otimes \overline{H}^s \otimes \overline{A}^{n+1-s} \otimes k) = 0$, which follows easily from the definition of $\overline{\sigma}_*$ \Box

Let $F^i(X_n) = \bigoplus_{0 \le s \le i} E \otimes \overline{H}^s \otimes \overline{A}^{n-s} \otimes E$ and let $F^i(\mathcal{B}_n(E))$ be the sub-bimodule of $\mathcal{B}_n(E)$ generated by the tensors $1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1$ such that at least n-iof the x_j 's belong to A. The normalized Hochschild resolution $(\mathcal{B}_*(E), b'_*)$ and the resolution (X_*, d_*) are filtered by $F^0(\mathcal{B}_*(E)) \subseteq F^1(\mathcal{B}_*(E)) \subseteq F^2(\mathcal{B}_*(E)) \subseteq \cdots$ and $F^0(X_*) \subseteq F^1(X_*) \subseteq F^2(X_*) \subseteq \cdots$, respectively

Proposition 1.2.2. The maps ϕ_* , ψ_* and ω_{*+1} preserve filtrations.

Proof. Let
$$Q_j^i = E \otimes \overline{H}^i \otimes \overline{A}^{n-j} \otimes k$$
. We claim that
a) $\overline{\sigma}_{n+1}(F^i(X_n)) \subseteq F^i(X_{n+1})$ for all $0 \le i < n$,
b) $\overline{\sigma}_{n+1}(E \otimes \overline{H}^i \otimes \overline{A}^{n-i} \otimes A) \subseteq Q_{i-1}^i + F^{i-1}(X_{n+1})$ for all $0 \le i \le n$,
c) $\overline{\sigma}_{n+1}(E \otimes \overline{H}^n \otimes E) \subseteq E \otimes \overline{H}^{n+1} \otimes k + F^n(X_{n+1})$ for all $n \ge 0$,
d) $\psi_n(F^i(\mathbf{B}_n(E)) \cap E \otimes \overline{E}^n \otimes k) \subseteq Q_i^i + F^{i-1}(X_n)$.

In fact a), b) and c) follow immediately from the definition of $\overline{\sigma}_{n+1}$. Suppose d) is valid for n. Let $\mathbf{x} = x_0 \otimes \cdots \otimes x_{n+1} \otimes 1 \in F^i(\mathbf{B}_{n+1}(E)) \cap E \otimes \overline{E}^{n+1} \otimes k$. Using a) and b), we get that for $1 \leq j \leq n$,

$$\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,j-1}\otimes x_jx_{j+1}\otimes \mathbf{x}_{j+2,n+1}\otimes 1)) \subseteq \overline{\sigma}_{n+1}(Q_i^i + F^{i-1}(X_n)) \subseteq Q_{i-1}^i + F^{i-1}(X_n)$$

Since $\psi_{n+1}(\mathbf{x}) = \overline{\sigma}_{n+1} \psi_n b'_{n+1}(\mathbf{x})$, to prove d) for n+1 we only must check that $\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1})) \subseteq Q_{i-1}^i + F^{i-1}(X_n)$. If $x_{n+1} \in A$, then using a) and b), we get

$$\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1})) = \overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0n} \otimes \mathbf{1}_E)x_{n+1})$$
$$\subseteq \overline{\sigma}_{n+1}(E \otimes \overline{H}^i \otimes \overline{A}^{n-i} \otimes A + F^{i-1}(X_n))$$
$$\subseteq Q_{i-1}^i + F^{i-1}(X_n),$$

and if $x_{n+1} \notin A$, then $\mathbf{x}_{0,n+1} \in F^{i-1}(\mathbf{B}_n(E))$, which together a) and c), implies that

$$\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1})) \subseteq \overline{\sigma}_{n+1}(F^{i-1}(X_n)) \subseteq Q^i_{i-1} + F^{i-1}(X_{n+1}).$$

From d) follows immediately that ψ_* preserves filtrations. Next, assuming that ϕ_n preserve filtrations, we prove that ϕ_{n+1} does it. Let $\mathbf{x} \in F^i(X_{n+1}) \cap Q_{i-1}^i$. Since $\phi_{n+1}(\mathbf{x}) = \xi_{n+1} \phi_n d_n(\mathbf{x})$ and

$$\xi_{n+1}(\phi_n(d_{rs}^l(\mathbf{x}))) \subseteq \xi_{n+1}(\phi_n(F^{i-l}(X_n))) \subseteq \xi_{n+1}(F^{i-l}(\mathbf{B}_n(E))) \subseteq F^{i-l+1}(\mathbf{B}_{n+1}(E)),$$

it suffices to see that $\xi_{n+1}(\phi_n(d_{rs}^0(\mathbf{x}))) \subseteq F^i(\mathbf{B}_{n+1}(E))$ for $\mathbf{x} = 1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1$, with $\mathbf{h} = h_1 \otimes \cdots \otimes h_i$ and $\mathbf{a} = a_1 \otimes \cdots \otimes a_{n+1-i}$. Since $\phi_n(Q_i^i) \subseteq E \otimes \overline{E}^n \otimes k$, we have

$$\xi_{n+1} \phi_n d_{rs}^0(\mathbf{x}) = (-1)^r \xi_{n+1} \phi_n (1 \otimes \mathbf{h} \otimes \mathbf{a}) = (-1)^r \xi_{n+1} (\phi_n (1 \otimes \mathbf{h} \otimes \mathbf{a}_{1,n-i} \otimes 1) a_{n+1-i}) \subseteq \xi_{n+1} (F^i(\mathbf{B}_n(E)) \cap E \otimes \overline{E}^n \otimes A) \subseteq F^i(\mathbf{B}_{n+1}(E)).$$

Next, we prove that ω_* preserves filtrations. Assume that ω_n does it. Let $\mathbf{x} = x_0 \otimes \cdots \otimes x_n \otimes 1 \in F^i(\mathbf{B}_n(E)) \cap E \otimes \overline{E}^n \otimes k$. It is evident that $\omega_{n+1}(\mathbf{x}) = \xi_{n+1} \phi_n \psi_n(\mathbf{x}) - \xi_{n+1} \omega_n b'_n(\mathbf{x})$. Since $\xi_{n+1}(\phi_n(Q_i^i)) \subseteq \xi_{n+1}(E \otimes \overline{E}^n \otimes k) = 0$, from d) we get

$$\xi_{n+1}\phi_n\psi_n(\mathbf{x}) \in \xi_{n+1}\phi_n(Q_i^i + F^{i-1}(X_n)) \subseteq \xi_{n+1}(F^{i-1}(B_n(E))) \subseteq F^i(B_n(E)).$$

It remains to check that $\xi_{n+1} \omega_n b'_n(\mathbf{x}) \subseteq F^i(\mathbf{B}_n(E))$. Since $\omega_n(E \otimes \overline{E}^{n-1} \otimes k) \subseteq E \otimes \overline{E}^n \otimes k$, we have $\xi_{n+1} \omega_n b'_n(\mathbf{x}) = (-1)^{n-1} \xi_{n+1} \omega_n(\mathbf{x}_{0n})$. Hence, if $x_n \in A$, then

$$\xi_{n+1} \,\omega_n \, b'_n(\mathbf{x}) = (-1)^{n-1} \xi_{n+1}(\omega_n(\mathbf{x}_{0,n-1} \otimes 1) x_n)$$
$$\subseteq \xi_{n+1}(F^i(\mathbf{B}_n(E)) \cap E \otimes \overline{E}^n \otimes A)$$
$$\subseteq F^i(\mathbf{B}_{n+1}(E)),$$

and if $x_n \notin A$, then $\mathbf{x} \in F^{i-1}(\mathbf{B}_{n-1}(E))$, and so

$$\xi_{n+1}\,\omega_n\,b'_n(\mathbf{x}) = (-1)^{n-1}\xi_{n+1}\,\omega_n(\mathbf{x}_{0n}) \subseteq \xi_{n+1}(F^{i-1}(\mathbf{B}_n(E))) \subseteq F^i(\mathbf{B}_{n+1}\,E) \quad \Box$$

2. The Hochschild (co)homology of a crossed product

Let $E = A \#_f H$ and M an E-bimodule. In this section we use Theorem 1.1.1 in order to construct complexes $\widehat{X}_*(E, M)$ and $\widehat{X}_*(E, M)$, simpler than the canonical ones, giving the Hochschild homology and cohomology of A with coefficients in M respectively. These complexes have natural filtrations that allow us to obtain spectral sequences converging to $H_*(E, M)$ and $H^*(E, M)$ respectively.

2.1. Hochschild homology

Let $\widehat{d}_{rs}^l \colon M \otimes \overline{H}^s \otimes \overline{A}^r \to M \otimes \overline{H}^{s-l} \otimes \overline{A}^{r+l-1}$ $(r, s \ge 0, 0 \le l \le s \text{ and } r+l > 0)$ be the morphisms defined by:

$$\begin{split} \widehat{d}_{rs}^{0}(\mathbf{x}) &= ma_{1}^{\overline{\mathbf{h}^{(1)}}} \otimes \mathbf{h}^{(2)} \otimes \mathbf{a}_{2r} + (-1)^{r}a_{r}m \otimes \mathbf{h} \otimes \mathbf{a}_{1,r-1} \\ &+ \sum_{i=1}^{r-1} (-1)^{i}m \otimes \mathbf{h} \otimes \mathbf{a}_{1,i-1} \otimes a_{i}a_{i+1} \otimes \mathbf{a}_{i+2,r}, \\ \widehat{d}_{rs}^{1}(\mathbf{x}) &= (-1)^{r}m(1\#h_{1}) \otimes \mathbf{h}_{2s} \otimes \mathbf{a} + (-1)^{r+s}(1\#h_{s}^{(2)})m \otimes \mathbf{h}_{1,s-1}^{(1)} \otimes \mathbf{a}_{s}^{h_{s}^{(1)}} \\ &+ \sum_{i=1}^{s-1} (-1)^{r+i}mf(h_{i}^{(1)}, h_{i+1}^{(1)})^{\mathbf{h}_{1,i-1}^{(1)}} \otimes \mathbf{h}_{1,i-1}^{(2)} \otimes h_{i}^{(2)}h_{i+1}^{(2)} \otimes \mathbf{h}_{i+2,s} \otimes \mathbf{a} \\ \widehat{d}_{rs}^{l}(\mathbf{x}) &= (-1)^{l(r+s)}(1\#\mathfrak{h}_{s-l+1,s}^{(2)})m \otimes \mathbf{h}_{1,s-l} \otimes F_{r}^{(l)}(\mathbf{h}_{s-l+1,s}^{(1)} \otimes \mathbf{a}), \\ \end{split}$$
where $\mathbf{x} = m \otimes \mathbf{h} \otimes \mathbf{a}$, with $\mathbf{a} = a_{1} \otimes \cdots \otimes a_{r}$ and $\mathbf{h} = h_{1} \otimes \cdots \otimes h_{s}. \end{split}$

Theorem 2.1.1. The Hochschild homology of E with coefficients in M is the homology of the chain complex

$$\widehat{X}_{*}(E,M) = \widehat{X}_{0} \xleftarrow{\widehat{d}_{1}} \widehat{X}_{1} \xleftarrow{\widehat{d}_{2}} \widehat{X}_{2} \xleftarrow{\widehat{d}_{3}} \widehat{X}_{3} \xleftarrow{\widehat{d}_{4}} \widehat{X}_{4} \xleftarrow{\widehat{d}_{5}} \widehat{X}_{5} \xleftarrow{\widehat{d}_{6}} \widehat{X}_{6} \xleftarrow{\widehat{d}_{7}} \dots$$
where $\widehat{X}_{n} = \bigoplus_{r+s=n} M \otimes \overline{H}^{s} \otimes \overline{A}^{r}$ and $\widehat{d}_{n} = \sum_{r+s=n \atop l \ge 0} \sum_{l=0}^{s} \widehat{d}_{rs}^{l}$.

Proof. It follows from the fact that $\widehat{X}_*(E, M) \simeq M \otimes_{E^e} (X_*, d_*)$. An isomorphism is provided by the maps $\widehat{\theta}_{rs} \colon M \otimes \overline{H}^s \otimes \overline{A}^r \to M \otimes_{E^e} X_{rs}$, defined by $\widehat{\theta}_{rs}(m \otimes \mathbf{h} \otimes \mathbf{a}) = m \otimes (\mathbf{1}_E \otimes \mathbf{h} \otimes \mathbf{a} \otimes \mathbf{1}_E)$

2.1.2. A spectral sequence. Let $F^i(\widehat{X}_n) = \bigoplus_{0 \le s \le i} M \otimes \overline{H}^s \otimes \overline{A}^{n-s}$. Clearly $F^0(\widehat{X}_*) \subseteq F^1(\widehat{X}_*) \subseteq \ldots$ is a filtration of $\widehat{X}_*(E, M)$. Using this fact we obtain:

Corollary 2.1.2.1. There is a convergent spectral sequence

$$E_{rs}^1 = \mathrm{H}_r(A, M \otimes \overline{H}^s) \Rightarrow \mathrm{H}_{r+s}(E, M),$$

where $M \otimes \overline{H}^s$ is considered as an A-bimodule via $a_1(m \otimes \mathbf{h}_{1s})a_2 = a_1 m a_2^{\overline{\mathbf{h}_{1s}^{(1)}}} \otimes \mathbf{h}_{1s}^{(2)}$.

The normalized Hochschild complex $(M \otimes \overline{E}^*, b_*)$ has a filtration $F^0(M \otimes \overline{E}^*) \subseteq F^1(M \otimes \overline{E}^*) \subseteq F^2(M \otimes \overline{E}^*) \subseteq \ldots$, where $F^i(M \otimes \overline{E}^n)$ is the k-submodule of $M \otimes \overline{E}^n$ generated by the tensors $m \otimes x_1 \otimes \cdots \otimes x_n$ such that at least n - i of the x_j 's belong to A. The spectral sequence associate to this filtration is called the homological Hochschild-Serre spectral sequence. Since, for each extension of groups $N \subseteq G$ with N a normal subgroup, it is hold that k[G] is a crossed product of k[G/N] on k[N], the following theorem (joint with Corollary 3.1.3 below) gives, as a particular case, the homological version of the main results of [H-S].

Theorem 2.1.2.2. The homological Hochschild-Serre spectral sequence is isomorphic to the one obtained in Corollary 2.1.2.1.

Proof. It is an easy consequence of Propositions 1.2.1 and 1.2.2.

2.1.3. A decomposition of $\widehat{X}_*(E, M)$. Let [H, H] be the k-submodule of H spanned by the set of all elements ab - ba $(a, b \in H)$. It is easy to see that [H, H] is a coideal in H. Let \check{H} be the quotient coalgebra H/[H, H]. Given $h \in H$, we let [h] denote the class of h in \check{H} . Given a subcoalgebra C of \check{H} and a right \check{H} -comodule (N, ρ) , we put $N^C = \{n \in N : \rho(n) \in N \otimes C\}$. It is well known that if \check{H} decomposes as a direct sum of subcoalgebras C_i $(i \in I)$, then $N = \bigoplus_{i \in I} N^{C_i}$.

Now, let us assume that M is a Hopf bimodule. That is, M is an E-bimodule and a right H-comodule, and the coaction $m \mapsto m^{(0)} \otimes m^{(1)}$ verifies:

$$((a\#h)m(b\#l))^{(0)} \otimes ((a\#h)m(b\#l))^{(1)} = (a\#h^{(1)})m^{(0)}(b\#l^{(1)}) \otimes (a\#h^{(2)})m^{(1)}(b\#l^{(2)}).$$

For each $n \ge 0$, \widehat{X}_n is an \check{H} -comodule via

$$\rho_n(m \otimes \mathbf{h}_{1s} \otimes \mathbf{a}_{1r}) = m^{(0)} \otimes \mathbf{h}_{1s}^{(1)} \otimes \mathbf{a}_{1r} \otimes [m^{(1)}\mathfrak{h}_{1s}^{(2)}] \qquad (r+s=n).$$

Moreover, the map $\rho_*: \widehat{X}_*(E, M) \to \widehat{X}_*(E, M) \otimes \check{H}$ is a map of complexes. This fact implies that if C is a subcoalgebra of \check{H} , then $\widehat{d}_n(\widehat{X}_n^C) \subseteq \widehat{X}_{n-1}^C$. We consider the subcomplex $\widehat{X}_*^C(E, M)$ of $\widehat{X}_*(E, M)$, with modules \widehat{X}_n^C , and we let $\mathrm{H}_*^C(E, M)$ denote its homology. By the above discussion, if \check{H} decomposes as a direct sum of subcoalgebras C_i $(i \in I)$, then $\widehat{X}_*(E, M) = \bigoplus_{i \in I} \widehat{X}_*^{C_i}(E, M)$. Consequently $\mathrm{H}_*(E, M) = \bigoplus_{i \in I} \mathrm{H}_*^{C_i}(E, M)$. Finally, the filtration of $\widehat{X}_*(E, M)$ induces a filtration on $\widehat{X}_*^C(E, M)$. Hence we have a convergent spectral sequence

$$E_{rs}^{1} = \mathrm{H}_{r}(A, (M \otimes \overline{H}^{s})^{C}) \Rightarrow \mathrm{H}_{r+s}^{C}(E, M),$$

where $(M \otimes \overline{H}^s)^C$ is an A-bimodule via $a_1(m \otimes \mathbf{h}_{1s})a_2 = a_1ma_2^{\overline{\mathbf{h}_{1s}^{(1)}}} \otimes \mathbf{h}_{1s}^{(2)}$.

2.1.4. Compatibility with the canonical decomposition. Let us assume that $k \supseteq \mathbb{Q}$, H is cocommutative, A is commutative, M is symmetric as an A-bimodule and the cocycle f takes its values in k. In [G-S1] was obtained a decomposition of the canonical Hochschild complex $(M \otimes \overline{A}^*, b_*)$. It is easy to check that the maps \widehat{d}_0 and \widehat{d}_1 are compatible with this decomposition. Since $\widehat{d}_l = 0$ for all $l \ge 2$, we obtain a decomposition of $\widehat{X}_*(E, M)$, and then a decomposition of $H_*(E, M)$.

2.2. Hochschild cohomology

Let \widehat{d}_l^{rs} : $\operatorname{Hom}_k(\overline{H}^{s-l} \otimes \overline{A}^{s+l-1}, M) \to \operatorname{Hom}_k(\overline{H}^s \otimes \overline{A}^r, M) \ (0 \le l \le s, r+l > 0)$ be the morphisms defined by:

$$\begin{split} \widehat{d}_{0}^{rs}(\varphi)(\mathbf{x}) &= a_{1}^{\mathbf{h}^{(1)}}\varphi(\mathbf{h}^{(2)}\otimes\mathbf{a}_{2r}) + (-1)^{r}\varphi(\mathbf{h}\otimes\mathbf{a}_{1,r-1})a_{r} \\ &+ \sum_{i=1}^{r-1} (-1)^{i}\varphi(\mathbf{h}\otimes\mathbf{a}_{1,i-1}\otimes a_{i}a_{i+1}\otimes\mathbf{a}_{i+2,r}), \\ \widehat{d}_{1}^{rs}(\varphi)(\mathbf{x}) &= (-1)^{r}(1\#h_{1})\varphi(\mathbf{h}_{2s}\otimes\mathbf{a}) + (-1)^{r+s}\varphi(\mathbf{h}_{1,s-1}\otimes\mathbf{a}^{h_{s}^{(1)}})(1\#h_{s}^{(2)}) \\ &+ \sum_{i=1}^{s-1} (-1)^{r+i}f(h_{i}^{(1)},h_{i+1}^{(1)})^{\mathbf{h}_{1,i-1}^{(1)}}\varphi(\mathbf{h}_{1,i-1}^{(2)}\otimes h_{i}^{(2)}h_{i+1}^{(2)}\otimes\mathbf{h}_{i+2,s}\otimes\mathbf{a}), \\ \widehat{d}_{l}^{rs}(\varphi)(\mathbf{x}) &= (-1)^{l(r+s)}\varphi(\mathbf{h}_{1,s-l}\otimes F_{r}^{(l)}(\mathbf{h}_{s-l+1,s}^{(1)}\otimes\mathbf{a}))(1\#\mathfrak{h}_{s-l+1,s}^{(2)}), \end{split}$$

where $\mathbf{x} = \mathbf{h} \otimes \mathbf{a}$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$.

Theorem 2.2.1. The Hochschild cohomology of E with coefficients in M is the homology of

$$\widehat{X}^*(E,M) = \quad \widehat{X}^0 \xrightarrow{\widehat{d}^1} \widehat{X}^1 \xrightarrow{\widehat{d}^2} \widehat{X}^2 \xrightarrow{\widehat{d}^3} \widehat{X}^3 \xrightarrow{\widehat{d}^4} \widehat{X}^4 \xrightarrow{\widehat{d}_5} \widehat{X}^5 \xrightarrow{\widehat{d}^6} \widehat{X}^6 \xrightarrow{\widehat{d}^7} \dots$$

$$where \ \widehat{X}^n = \bigoplus_{r+s=n} \operatorname{Hom}_k(\overline{H}^s \otimes \overline{A}^r, M) \ and \ \widehat{d}^n = \sum_{r+s=n \atop r+l>0} \sum_{l=0}^s \widehat{d}_l^{rs}.$$

Proof. It follows from the fact that $\widehat{X}^*(E, M) \simeq \operatorname{Hom}_{E^e}((X_*, d_*), M)$. An isomorphism is provided by the maps $\widehat{\theta}^{rs} \colon \operatorname{Hom}_k(\overline{H}^s \otimes \overline{A}^r, M) \to \operatorname{Hom}_{E^e}(X_{rs}, M)$, defined by $\widehat{\theta}^{rs}(\varphi)(1_E \otimes \mathbf{x} \otimes 1_E) = \varphi(\mathbf{x})$

2.2.2. A spectral sequence. Let $F^i(\widehat{X}^n) = \bigoplus_{s \ge i} \operatorname{Hom}_k(\overline{H}^s \otimes \overline{A}^{n-s}, M)$. Clearly $F_0 \supseteq F_1 \supseteq F_2 \supseteq \ldots$ is a filtration of $\widehat{X}^*(E, M)$. Using this fact we obtain:

Corollary 2.2.2.1. There is a convergent spectral sequence

$$E_1^{rs} = \mathrm{H}^r(A, \mathrm{Hom}_k(\overline{H}^s, M)) \Rightarrow \mathrm{H}^{r+s}(E, M),$$

where $\operatorname{Hom}_{k}(\overline{H}^{s}, M)$ is considered as an A-bimodule via $a_{1}\varphi a_{2}(\mathbf{h}) = a_{1}^{\mathbf{h}^{(1)}}\varphi(\mathbf{h}^{(2)})a_{2}$.

Let $F_i(\operatorname{Hom}_k(\overline{E}^n, M))$ be the k-submodule of $(\operatorname{Hom}_k(\overline{E}^*, M), b^*)$ consisting of maps $f \in \operatorname{Hom}_k(\overline{E}^n, M)$, for which $f(x_1 \otimes \cdots \otimes x_n) = 0$ whenever n - i of the x_j 's belong to A. The normalized Hochschild complex $(\operatorname{Hom}_k(\overline{E}^*, M), b^*)$ is filtered by $F_0(\operatorname{Hom}_k(\overline{E}^*, M)) \supseteq F_1(\operatorname{Hom}_k(\overline{E}^*, M)) \supseteq F_2(\operatorname{Hom}_k(\overline{E}^*, M)) \supseteq \cdots$ The spectral sequence associated to this filtration is called the cohomological Hochschild-Serre spectral sequence. The following theorem (joint with Corollary 3.2.3 below) gives, as a particular case, of the main results of [H-S].

Theorem 2.2.2.2. The cohomological Hochschild-Serre spectral sequence is isomorphic to the one obtained in Corollary 2.2.2.1.

Proof. It is an easy consequence of Propositions 1.2.1 and 1.2.2.

2.2.3. Compatibility with the canonical decomposition. Assume that $k \supseteq \mathbb{Q}$, H is cocommutative, A is commutative, M is symmetric as an A-bimodule and the cocycle f takes its values in k. Then, the Hochschild cohomology $H^*(E, M)$ has a decomposition similar to the one obtained in 2.1.4 for the Hochschild homology.

3. The Hochschild (co)homology of a crossed product with invertible cocycle

Let $E = A \#_f H$ and M an E-bimodule. Assume that the cocycle f is invertible. Then, the map $h \mapsto 1 \# h$ is convolution invertible and its inverse is the map $h \mapsto (1 \# h)^{-1} = f^{-1}(S(h^{(2)}), h^{(3)}) \# S(h^{(1)})$. Under this hypothesis, we prove that the complexes $\widehat{X}_*(E, M)$ and $\widehat{X}^*(E, M)$ of Section 2 are isomorphic to simpler complexes. These complexes have natural filtrations, which give the spectral sequences obtained in [S]. Using these facts and a theorem of Gerstenhaber and Schack, we prove that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex. Finally, as an application we obtain some results about the Tor^E_* and Ext^*_E functors and an upper bound for the global dimension of E.

3.1. Hochschild homology

Let $\overline{d}_{rs}^{l} \colon M \otimes \overline{A}^{r} \otimes \overline{H}^{s} \to M \otimes \overline{A}^{r+l-1} \otimes \overline{H}^{s-l}$ $(r, s \ge 0, 0 \le l \le s \text{ and } r+l > 0)$ be the morphisms defined by:

$$\overline{d}_{rs}^{0}(\mathbf{x}) = ma_1 \otimes \mathbf{a}_{2r} \otimes \mathbf{h} + (-1)^r a_r m \otimes \mathbf{a}_{1,r-1} \otimes \mathbf{h} + \sum_{i=1}^{r-1} (-1)^i m \otimes \mathbf{a}_{1,i-1} \otimes a_i a_{i+1} \otimes \mathbf{a}_{i+2,r} \otimes \mathbf{h},$$

$$\begin{aligned} \overline{d}_{rs}^{l}(\mathbf{x}) &= (-1)^{r} m \epsilon(h_{1}) \otimes \mathbf{a} \otimes \mathbf{h}_{2s} + (-1)^{r+s} (1\#h_{s}^{(3)}) m (1\#h_{s}^{(1)})^{-1} \otimes \mathbf{a}^{h_{s}^{(2)}} \otimes \mathbf{h}_{1,s-1} \\ &+ \sum_{i=1}^{s-1} (-1)^{r+i} m \otimes \mathbf{a} \otimes \mathbf{h}_{1,i-1} \otimes h_{i} h_{i+1} \otimes \mathbf{h}_{i+2,s}, \\ \overline{d}_{rs}^{l}(\mathbf{x}) &= (-1)^{l(r+s)} (1\#\mathfrak{h}_{s-l+1,s}^{(3)}) m (1\#\mathfrak{h}_{s-l+1,s}^{(1)})^{-1} \otimes F_{r}^{(l)} (\mathbf{h}_{s-l+1,s}^{(2)} \otimes \mathbf{a}), \end{aligned}$$

where $\mathbf{x} = m \otimes \mathbf{a} \otimes \mathbf{h}$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$. Let $\overline{X}_*(E, M)$ be the complex

$$\overline{X}_*(E,M) = \overline{X}_0 \xleftarrow{\overline{d}_1} \overline{X}_1 \xleftarrow{\overline{d}_2} \overline{X}_2 \xleftarrow{\overline{d}_3} \overline{X}_3 \xleftarrow{\overline{d}_4} \overline{X}_4 \xleftarrow{\overline{d}_5} \overline{X}_5 \xleftarrow{\overline{d}_6} \overline{X}_6 \xleftarrow{\overline{d}_7} \dots,$$

where $\overline{X}_n = \bigoplus_{r+s=n} M \otimes \overline{A}^r \otimes \overline{H}^s$ and $\overline{d}_n = \sum_{\substack{r+s=n\\r+l>0}} \sum_{l=0}^s \overline{d}_{rs}^l$.

Theorem 3.1.1. The map $\theta_* \colon \widehat{X}_*(E, M) \to \overline{X}_*(E, M)$, given by

$$\theta_n(m \otimes \mathbf{h} \otimes \mathbf{a}) = m(1 \# h_1^{(1)}) \cdots (1 \# h_s^{(1)}) \otimes \mathbf{a} \otimes \mathbf{h}^{(2)} \qquad (r+s=n),$$

is an isomorphism of complexes. Consequently, the Hochschild homology of E with coefficients in M is the homology of $\overline{X}_*(E, M)$.

Proof. A direct computation shows that θ_* is a morphism of complexes. The inverse map of θ_n is the map $m \otimes \mathbf{a} \otimes \mathbf{h} \mapsto m(1\#h_s^{(1)})^{-1} \cdots (1\#h_1^{(1)})^{-1} \otimes \mathbf{h}^{(2)} \otimes \mathbf{a}$ \square

Note that when f takes its values in k, then $\overline{X}_*(E, M)$ is the total complex of the double complex $(M \otimes \overline{A}^* \otimes \overline{H}^*, \overline{d}^0_{**}, \overline{d}^1_{**})$.

For each $h \in H$, we have the morphism $\theta_*^h : (M \otimes \overline{A}^*, b_*) \to (M \otimes \overline{A}^*, b_*)$, defined by $\theta_r^h(m \otimes \mathbf{a}) = (1 \# h_s^{(3)}) m (1 \# h_s^{(1)})^{-1} \otimes \mathbf{a}^{h_s^{(2)}}$.

Proposition 3.1.2. For each $h, l \in H$ the endomorphisms of $H_*(A, M)$ induced by $\theta_*^h \theta_*^l$ and by θ_*^{hl} coincide. Consequently $H_*(A, M)$ is a left *H*-module.

Proof. By a standard argument it is sufficient to prove it for $H_0(A, M)$, and in this case the result is immediate \Box

Corollary 3.1.3. The chain complex $\overline{X}_*(E, M)$ has a filtration $F^0 \subseteq F^1 \subseteq \ldots$, where $F^i(\overline{X}_n) = \bigoplus_{0 \leq s \leq i} M \otimes \overline{A}^{n-s} \otimes \overline{H}^s$. The spectral sequence of this filtration is isomorphic to the one obtained in Corollary 2.1.2. From Proposition 3.1.2 it follows that if H is a flat k-module, then $E_{rs}^1 = \operatorname{H}_r(A, M) \otimes \overline{H}^s$ and $E_{rs}^2 = \operatorname{H}_s(H, \operatorname{H}_r(A, M))$.

Given an A-bimodule M we let [A, M] denote the k-submodule of M generated by the commutators am - ma ($a \in A$ and $m \in M$).

Remark 3.1.4. From Corollary 3.1.3 it follows immediately that if A is separable, then $H_*(E, M) = H_*(H, M/[A, M])$, and if A is quasi-free, then there is a long exact sequence

$$\dots \to \mathrm{H}_{n+1}(H, \mathrm{H}_0(A, M)) \to \mathrm{H}_{n-1}(H, \mathrm{H}_1(A, M)) \to \mathrm{H}_n(E, M) \to \mathrm{H}_n(H, \mathrm{H}_0(A, M)) \to \mathrm{H}_{n-2}(H, \mathrm{H}_1(A, M)) \to \mathrm{H}_{n-1}(E, M) \to \dots$$

3.1.5. Separable subalgebras. Let S be a separable subalgebra of A. Next we prove that if the 2-cocycle f takes its values in S, then the Hochschild homology of E with coefficients in M is the homology of H with coefficients in a chain complex. When S equals A we recover the first part of Remark 3.1.4. Assume that $f(h, l) \in S$ for all $h, l \in H$. Let $\tilde{A} = A/S$, $\tilde{A}^0 = S$ and $\tilde{A}^r = \tilde{A} \otimes_S \cdots \otimes_S \tilde{A}$ (r-times) for r > 0, and let $M \otimes_S \tilde{A}^r \otimes_S = M \otimes_{A^e} (A \otimes_s \tilde{A}^r \otimes_s A) = M \otimes_S \tilde{A}^r \otimes_{S^e} S$ be the cyclic tensor product over S of M and \tilde{A}^r (see [G-S2] or [Q]). Using the fact that f takes its values in S, it is easy to see that H acts on $(M \otimes_S \tilde{A}^r \otimes_S b_*)$ via

$$h \cdot \left(m \otimes_S \widetilde{\mathbf{a}} \right) = (1 \# h^{(3)}) m (1 \# h^{(1)})^{-1} \otimes_S \widetilde{\mathbf{a}}^{\overline{h^{(2)}}},$$

where $m \otimes_S \widetilde{\mathbf{a}} = m \otimes_S a_1 \otimes_S \cdots \otimes_S a_r \otimes_S$ and $\widetilde{\mathbf{a}}^{\overline{h^{(2)}}} = a_1^{\overline{h^{(2)}}} \otimes_S \cdots \otimes_S a_r^{\overline{h^{(r+1)}}} \otimes_S$.

Theorem 3.1.5.1. The Hochschild homology $H_*(E, M)$, of E with coefficients in M, is the homology of H with coefficients in $(M \otimes_S \widetilde{A}^r \otimes_S, b_*)$.

Proof. Let $((M \otimes_S \widetilde{A}^* \otimes_S) \otimes \overline{H}^*, \widetilde{d}^0_{**}, \widetilde{d}^1_{**})$ be the double complex with horizontal differentials

$$d_{rs}^{0}(\mathbf{x}) = ma_{1} \otimes_{S} \widetilde{\mathbf{a}}_{2r} \otimes \mathbf{h} + (-1)^{r} a_{r} m \otimes_{S} \widetilde{\mathbf{a}}_{1,r-1} \otimes \mathbf{h} + \sum_{i=1}^{r-1} (-1)^{i} m \otimes_{S} \widetilde{\mathbf{a}}_{1,i-1} \otimes_{S} a_{i} a_{i+1} \otimes_{S} \widetilde{\mathbf{a}}_{i+2,r} \otimes \mathbf{h},$$

and vertical differentials

$$\widetilde{d}_{rs}^{1}(\mathbf{x}) = (-1)^{r} m \otimes_{s} \widetilde{\mathbf{a}} \otimes \mathbf{h}_{2s} + (-1)^{r+s} (1\#h_{s}^{(3)}) m (1\#h_{s}^{(1)})^{-1} \otimes_{s} \widetilde{\mathbf{a}}^{h_{s}^{(2)}} \otimes \mathbf{h}_{1,s-1} + \sum_{i=1}^{s-1} (-1)^{r+i} m \otimes_{s} \widetilde{\mathbf{a}} \otimes \mathbf{h}_{1,i-1}^{(2)} \otimes h_{i}^{(2)} h_{i+1}^{(2)} \otimes \mathbf{h}_{i+2,s},$$

where $\mathbf{x} = m \otimes \tilde{\mathbf{a}} \otimes \mathbf{h}$, with $\tilde{\mathbf{a}} = a_1 \otimes_S \cdots \otimes_S a_r \otimes_S$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$. Let $\overline{X}^S_*(E, M)$ be the total complex of $((M \otimes_S \widetilde{A}^* \otimes_S) \otimes \overline{H}^*, \widetilde{d}^0_{**}, \widetilde{d}^1_{**})$. We must prove that $\mathbf{H}_*(E, M)$ is the homology of $\overline{X}^S_*(E, M)$. Let $\pi_* \colon \overline{X}_*(E, M) \to \overline{X}^S_*(E, M)$ be the map $m \otimes \mathbf{a}_{1r} \otimes \mathbf{h}_{1s} \mapsto m \otimes_S \widetilde{\mathbf{a}}_{1r} \otimes \mathbf{h}_{1s}$. Consider the filtration $F^{0S}_* \subseteq F^{1S}_* \subseteq F^{2S}_* \subseteq \ldots$ of $\overline{X}^S_*(E, M)$, where $F^{iS}_n = \bigoplus_{0 \le s \le i} (M \otimes_S \widetilde{A}^{n-s} \otimes_S) \otimes \overline{H}^s$. From Theorem 1.2 of [G-S2], it follows that π_* is a morphism of filtered complexes inducing an quasi-isomorphism between the graded complexes associated to the filtrations of $\overline{X}_*(E, M)$ and $\overline{X}^S_*(E, M)$. Consequently π_* is a quasi-isomorphism. The proof can be finished by applying Theorem 3.1.1 \Box

3.1.6. A decomposition of $\overline{X}_*(E, M)$. Here we freely use the notations of Subsection 2.1.3. Suppose M is a Hopf bimodule. A direct computation shows that the H-coaction of $\overline{X}_*(E, M)$, obtained transporting the one of $\widehat{X}_*(E, M)$ through $\theta_*: \widehat{X}_*(E, M) \to \overline{X}_*(E, M)$, is given by

(2)
$$m \otimes \mathbf{a} \otimes \mathbf{h} \mapsto m^{(0)} \otimes \mathbf{a} \otimes \mathbf{h}^{(2)} \otimes m^{(1)} S(h_s^{(1)}) \cdots S(h_1^{(1)}) h_1^{(3)} \cdots h_s^{(3)}$$
.

For each subcoalgebra C of \check{H} , we consider the subcomplex $\overline{X}_*^C(E, M)$ of $\overline{X}_*(E, M)$ with modules \overline{X}_n^C , and we let $\mathrm{H}_*^C(E, M)$ denote its homology. If \check{H} decomposes as a direct sum of subcoalgebras C_i $(i \in I)$, then $\overline{X}_*(E, M) = \bigoplus_{i \in I} \overline{X}_*^{C_i}(E, M)$. Consequently $\mathrm{H}_*(E, M) = \bigoplus_{i \in I} \mathrm{H}_*^{C_i}(E, M)$. From (2) it follows that if \check{H} is cocommutative, then $\overline{X}_n^C = \bigoplus_{r+s=n} M^C \otimes \overline{A}^r \otimes \overline{H}^s$. Finally, the filtration of $\overline{X}_*(E, M)$ induces a filtration on $\overline{X}_*^C(E, M)$. Hence, when \check{H} is cocommutative and H is a flat k-module, we have a convergent spectral sequence

$$E_{rs}^2 = \mathrm{H}_r(H, \mathrm{H}_s(A, M^C)) \Rightarrow \mathrm{H}_{r+s}^C(E, M),$$

where $H_r(A, M^C)$ is a left *H*-module via the action introduced in Proposition 3.1.2.

3.1.7. An application to Tor_*^E . Let k be a field, B an arbitrary k-algebra, M a right B-module and N a left B-module. It is well known that $\operatorname{Tor}_*^B(M, N) \simeq \operatorname{H}_*(B, N \otimes M)$ (here $N \otimes M$ is an B-bimodule via $a(n \otimes m)b = an \otimes mb$). This fact and Corollary 3.1.3 show that if k is a field, M is a right E-module and N is a left E-module, then there is a convergent spectral sequence

$$E_{rs}^2 = \operatorname{H}_r(H, \operatorname{Tor}_s^A(M, N)) \Rightarrow \operatorname{Tor}_{r+s}^E(M, N).$$

3.2. Hochschild cohomology

Let \overline{d}_l^{rs} : Hom_k $(\overline{A}^{r+l-1} \otimes \overline{H}^{s-l}, M) \to \text{Hom}_k(\overline{A}^r \otimes \overline{H}^s, M) \ (0 \le l \le s, r+l > 0)$ be the morphisms defined by:

$$\begin{split} \overline{d}_{0}^{rs}(\varphi)(\mathbf{x}) &= a_{1}\varphi(\mathbf{a}_{2r}\otimes\mathbf{h}) + (-1)^{r}\varphi(\mathbf{a}_{1,r-1}\otimes\mathbf{h})a_{r} \\ &+ \sum_{i=1}^{r-1} (-1)^{i}\varphi(\mathbf{a}_{1,i-1}\otimes a_{i}a_{i+1}\otimes\mathbf{a}_{i+2,r}\otimes\mathbf{h}), \\ \overline{d}_{1}^{rs}(\varphi)(\mathbf{x}) &= (-1)^{r}\epsilon(h_{1})\varphi(\mathbf{a}\otimes\mathbf{h}_{2s}) + (-1)^{r+s}(1\#h_{s}^{(1)})^{-1}\varphi(\mathbf{a}^{h_{s}^{(2)}}\otimes\mathbf{h}_{1,s-1})(1\#h_{s}^{(3)}) \\ &+ \sum_{i=1}^{s-1} (-1)^{r+i}\varphi(\mathbf{a}\otimes\mathbf{h}_{1,i-1}\otimes h_{i}h_{i+1}\otimes\mathbf{h}_{i+2,s}), \\ \overline{d}_{l}^{rs}(\varphi)(\mathbf{x}) &= ((-1)^{l(r+s)}\#\mathfrak{h}_{s-l+1,s}^{(1)})^{-1}\varphi(F_{r}^{(l)}(\mathbf{h}_{s-l+1,s}^{(2)}\otimes\mathbf{a})\otimes\mathbf{h}_{1,s-l})(1\#\mathfrak{h}_{s-l+1,s}^{(3)}), \end{split}$$

where $\mathbf{x} = \mathbf{a} \otimes \mathbf{h}$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$. Let $\overline{X}^*(E, M)$ be the complex

$$\overline{X}^*(E,M) = \overline{X}^0 \xrightarrow{\overline{d}^1} \overline{X}^1 \xrightarrow{\overline{d}^2} \overline{X}^2 \xrightarrow{\overline{d}^3} \overline{X}^3 \xrightarrow{\overline{d}^4} \overline{X}^4 \xrightarrow{\overline{d}_5} \overline{X}^5 \xrightarrow{\overline{d}^6} \overline{X}^6 \xrightarrow{\overline{d}^7} \dots$$

where $\overline{X}^n = \bigoplus_{r+s=n} \operatorname{Hom}_k(\overline{A}^r \otimes \overline{H}^s, M)$ and $\overline{d}^n = \sum_{\substack{r+s=n\\r+l>0}} \sum_{l=0}^s \overline{d}_l^{rs}$.

Theorem 3.2.1. The map $\theta^* : \overline{X}^*(E, M) \to \widehat{X}^*(E, M)$, given by

$$\theta^n(\varphi)(\mathbf{h}\otimes\mathbf{a}) = (1\#h_1^{(1)})\cdots(1\#h_s^{(1)})\varphi(\mathbf{a}\otimes\mathbf{h}^{(2)}) \qquad (r+s=n),$$

is an isomorphism of complexes. Consequently, the Hochschild cohomology of E with coefficients in M is the homology of $\overline{X}^*(E, M)$.

Proof. It is similar to the proof of Theorem 3.1.1 \Box

Note that when f takes its values in k, then $\overline{X}^*(E, M)$ is the total complex of the double complex $(\operatorname{Hom}_k(\overline{A}^* \otimes \overline{H}^*, M), \overline{d}_0^{**}, \overline{d}_1^{**})$.

For each $h \in H$ we have the map θ_h^* : $(\operatorname{Hom}_k(\overline{A}^*, M), b^*) \to (\operatorname{Hom}_k(\overline{A}^*, M), b^*)$, defined by $\theta_h^r(\varphi)(\mathbf{a}) = (1 \# h^{(1)})^{-1} \varphi(\mathbf{a}^{h^{(2)}})(1 \# h^{(3)}).$

Proposition 3.2.2. For each $h, l \in H$ the endomorphisms of $H^*(A, M)$ induced by $\theta_l^* \theta_h^*$ and by θ_{hl}^* coincide. Consequently $H^*(A, M)$ is a right *H*-module.

Proof. By a standard argument it is sufficient to prove it for $H^0(A, M)$, and in this case the result is immediate \Box

Corollary 3.2.3. The cochain complex $\overline{X}^*(E, M)$ has a filtration $F_0 \supseteq F_1 \supseteq \ldots$, where $F_i(\overline{X}^n) = \bigoplus_{0 \le r \le n-i} \operatorname{Hom}_k(\overline{A}^r \otimes \overline{H}^{n-r}, M)$. The spectral sequence of this filtration is isomorphic to the one obtained in Corollary 2.2.2. From Proposition 3.2.2 it follows that $E_1^{rs} = \operatorname{Hom}_k(\overline{H}^s, \operatorname{H}^r(A, M))$ and $E_2^{rs} = \operatorname{H}^s(H, \operatorname{H}^r(A, M))$.

Given an A-bimodule M, we let M^A denote the k-submodule of M consisting of the elements m verifying am = ma for all $a \in A$.

Remark 3.2.4. From Corollary 3.2.3, it follows immediately that if A is separable, then $H^*(E, M) = H^*(H, M^A)$ and if A is quasi-free, then there is a long exact sequence

$$\dots \to \mathrm{H}^{n-2}(H, \mathrm{H}^{1}(A, M)) \to \mathrm{H}^{n}(H, \mathrm{H}^{0}(A, M)) \to \mathrm{H}^{n}(E, M) \to \\ \mathrm{H}^{n-1}(H, \mathrm{H}^{1}(A, M)) \to \mathrm{H}^{n+1}(H, \mathrm{H}^{0}(A, M)) \to \mathrm{H}^{n+1}(E, M) \to \dots$$

3.2.5. Separable subalgebras. Let *S* be a separable subalgebra of *A* and let \widetilde{A}^r $(r \geq 0)$ be as in 3.1.5. Suppose $f(h,l) \in S$ for all $h, l \in H$. Using the fact that *f* takes its values in *S* it is easy to see that *H* acts on $(\operatorname{Hom}_{A^e}(A \otimes_s \widetilde{A}^r \otimes_s A, M), b^*) = (\operatorname{Hom}_{S^e}(\widetilde{A}^r, M), b^*)$ via $(\varphi \cdot h)(\widetilde{\mathbf{a}}) = (1 \# h^{(1)})^{-1} \varphi(\widetilde{\mathbf{a}}^{h^{(2)}})(1 \# h^{(3)}).$

Theorem 3.2.5.1. The Hochschild cohomology $H^*(E, M)$, of E with coefficients in M, is the cohomology of H with coefficients in $(\text{Hom}_{S^e}(\tilde{A}^r, M), b^*)$.

Proof. It is similar to the proof of Theorem 3.1.5.1 \Box

3.2.6. An application to $\operatorname{Ext}_{E}^{*}$. Let k be a field, B an arbitrary k-algebra and M, N two left B-modules. It is well known that $\operatorname{Ext}_{B}^{*}(M, N) \simeq \operatorname{H}^{*}(B, \operatorname{Hom}_{k}(M, N))$ (here $\operatorname{Hom}_{k}(M, N)$ is an B-bimodule via $(a\varphi b)(m) = a\varphi(bm)$). This fact and Corollary 3.2.3 show that if k is a field and M and N are left E-modules, then there is a convergent spectral sequence

$$E_2^{rs} = \mathrm{H}^r(H, \mathrm{Ext}^s_A(M, N)) \Rightarrow \mathrm{Ext}^{r+s}_E(M, N).$$

As a corollary we obtain that gl. $\dim(E) \leq \text{gl.} \dim(A) + \text{gl.} \dim(H)$, where gl. \dim denotes the left global dimension. Note that this result implies Maschke's Theorem for crossed product, as it was established in [B-M].

4. The Cartan-Leray and Grothendieck spectral sequences

Assume that E is a crossed product with invertible cocycle. In this case another two spectral sequences converging to $H_*(E, M)$ and with E^2 -term $H_*(H, H_*(A, M))$ can be considered. They are the Cartan-Leray and the Grothendieck spectral sequences. The last one was introduced for the more general setting of Galois extension in [S]. In this Section we recall these constructions and we prove that both coincide with the Hochschild-Serre spectral sequence. Similar results are valid in the cohomological setting.

Let $(\overline{H}^* \otimes H, d_*)$ be the canonical resolution of k as a right H-module and $(Z_*, \partial_*) = (E \otimes \overline{E}^* \otimes E, b'_*) \otimes (\overline{H}^* \otimes H, d_*)$. Consider $E \otimes \overline{E}^r \otimes E \otimes \overline{H}^s \otimes H$ as an E-bimodule via

$$(a\#l)(\mathbf{x} \otimes \mathbf{h})(b\#q) = ((a\#l)x_0 \otimes \mathbf{x}_{1r} \otimes x_{r+1}(b\#q^{(1)})) \otimes (\mathbf{h}_{1s} \otimes h_{s+1}q^{(2)}),$$

where $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_{s+1}$. It is clear that

$$(3) \qquad E \stackrel{\mu}{\leftarrow} Z_0 \stackrel{\partial_1}{\leftarrow} Z_1 \stackrel{\partial_2}{\leftarrow} Z_2 \stackrel{\partial_3}{\leftarrow} Z_3 \stackrel{\partial_4}{\leftarrow} Z_4 \stackrel{\partial_5}{\leftarrow} Z_5 \stackrel{\partial_6}{\leftarrow} Z_6 \stackrel{\partial_7}{\leftarrow} Z_7 \stackrel{\partial_8}{\leftarrow} \dots,$$

where $\mu((a_0 \# h_0 \otimes a_1 \# h_1) \otimes l) = \epsilon(l)a_0a_1f(h_0^{(1)}h_1^{(1)})\# h_0^{(2)}h_1^{(2)}$, is a complex of *E*-bimodules. Moreover (3) is contractible as a complex of left *E*-modules, with contracting homotopy ζ_n $(n \ge 0)$ given by $\zeta_0(1_E) = 1_E \otimes 1_E \otimes 1_H$ and

$$\zeta_{n+1}(\mathbf{y}) = \begin{cases} -\mathbf{x} \otimes \mathbf{1}_E \otimes \mathbf{h} + (-1)^{n+1} x_0 x_1 \otimes \mathbf{1}_E \otimes \mathbf{h} \otimes \mathbf{1}_H & \text{if } r = 0\\ (-1)^{r+1} \mathbf{x} \otimes \mathbf{1}_E \otimes \mathbf{h} & \text{if } r > 0 \end{cases},$$

where $\mathbf{y} = \mathbf{x} \otimes \mathbf{h}$, with $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_{n-r+1}$. Since the map

$$\tau \colon E \otimes \overline{E}^r \otimes \overline{H}^s \otimes H \otimes E \to E \otimes \overline{E}^r \otimes E \otimes \overline{H}^s \otimes H,$$

given by $\tau(\mathbf{x}_{0r} \otimes \mathbf{h} \otimes x_{r+1}) = (\mathbf{x}_{0r} \otimes \mathbf{1}_E \otimes \mathbf{h}) x_{r+1}$, is an isomorphism of *E*-bimodules (the inverse of τ is the map $\mathbf{x}_{0r} \otimes a \# h \otimes \mathbf{h} \mapsto \mathbf{x}_{0r} \otimes \mathbf{h}_{1s} \otimes h_{s+1} S^{-1}(h^{(2)}) \otimes a \# h^{(1)})$, (Z_*, ∂_*) is a relative projective resolution of *E*.

Let M be an E-bimodule. The groups $M \otimes_{E \otimes A^{\text{op}}} (E \otimes \overline{E}^r \otimes E)$ are left H-modules via $h(m \otimes \mathbf{x}) = (1 \# h^{(2)})m \otimes \mathbf{x}_{0r} \otimes x_{r+1}(1 \# h^{(1)})^{-1}$, where $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$. There is an isomorphism

$$M \otimes_{E^e} (Z_*, \partial_*) \simeq (\overline{H}^* \otimes H, d_*) \otimes_H (M \otimes_{E \otimes A^{\mathrm{op}}} (E \otimes \overline{E}^* \otimes E, b'_*)).$$

Let $F^i = \bigoplus_{j=0}^i (\overline{H}^j \otimes H) \otimes_H (M \otimes_{A \otimes E^{\mathrm{op}}} E \otimes \overline{E}^* \otimes E)$. It is immediate that $F^0 \subseteq F^1 \subseteq F^2 \subseteq F^3 \subseteq \ldots$, is a filtration of the last complex. The spectral sequence associate to this filtration converges to $H_*(E, M)$ and has E^2 -term $H_*(H, H_*(A, M))$. This spectral sequence is called the homological Cartan-Leray spectral sequence. Similarly the groups $\operatorname{Hom}_{E \otimes A^{\mathrm{op}}}(E \otimes \overline{E}^r \otimes E, M)$ are right H modules via $f.h(\mathbf{x}_{0,r+1}) = f(\mathbf{x}_{0r} \otimes x_{r+1}(1\#h^{(1)})^{-1})(1\#h^{(2)})$ and there is an isomorphism

$$\operatorname{Hom}_{E^{e}}((Z_{*},\partial_{*}),M) \simeq \operatorname{Hom}_{H}\left((\overline{H}^{*} \otimes H, d_{*}), \operatorname{Hom}_{E \otimes A^{\operatorname{op}}}((E \otimes \overline{E}^{*} \otimes E, b_{*}'), M)\right)$$

This complex has a filtration $F_0 \supseteq F_1 \supseteq F_2 \supseteq F_3 \supseteq F_4 \supseteq \ldots$, defined by $F_i^n = \bigoplus_{j \ge i} \operatorname{Hom}_H(\overline{H}^j \otimes H, \operatorname{Hom}_{E \otimes A^{\operatorname{op}}}(E \otimes \overline{E}^{n-j} \otimes E, M))$. The spectral sequence associate to this filtration converges to $\operatorname{H}^*(E, M)$ and has E^2 -term $\operatorname{H}^*(H, \operatorname{H}^*(A, M))$. This spectral sequence is called the cohomological Cartan-Leray spectral sequence.

Let $\Phi_* : (E \otimes \overline{E}^* \otimes E, b'_*) \to (Z_*, \partial_*)$ and $\Psi_* : (Z_*, \partial_*) \to (E \otimes \overline{E}^* \otimes E, b'_*)$ be the morphisms of *E*-bimodule complexes, recursively defined by

$$\begin{split} \Phi_0(x \otimes 1_E) &= x \otimes 1_E \otimes 1_H, \quad \Psi_0(x \otimes 1_E \otimes h) = \epsilon(h)x \otimes 1_E, \\ \Phi_{n+1}(\mathbf{x} \otimes 1_E) &= \zeta_{n+1} \Phi_n \, b'_{n+1}(\mathbf{x} \otimes 1_E) \text{ for } \mathbf{x} \in E \otimes \overline{E}^{n+1}, \\ \Psi_{n+1}(\mathbf{x} \otimes 1_E \otimes \mathbf{h}) &= \xi_{n+1} \, \psi_n \, \partial_{n+1}(\mathbf{x} \otimes 1_E \otimes \mathbf{h}) \text{ for } \mathbf{x} \in E \otimes \overline{E}^r, \, \mathbf{h} \in \overline{H}^{n+1-r} \otimes H. \end{split}$$

Proposition 4.1. It is hold that $\Psi_* \Phi_* = id_*$ and that $\Phi_* \Psi_*$ is homotopically equivalent to the identity map. The homotopy $\Phi_* \Psi_* \xrightarrow{\Omega_{*+1}} id_*$ is recursively defined by $\Omega_1(x \otimes 1_E \otimes h) = x \otimes 1_E \otimes h \otimes 1_H$ and

$$\Omega_{n+1}(\mathbf{x} \otimes \mathbf{1}_E \otimes \mathbf{h}) = \zeta_{n+1} \left(\Phi_n \, \Psi_n - id - \Omega_n \, \partial_n \right) (\mathbf{x} \otimes \mathbf{1}_E \otimes \mathbf{h}),$$

for $\mathbf{x} = x_0 \otimes \cdots \otimes x_r$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_{n+1-r}$.

Proof. It is easy to see that Φ_* and Ψ_* are morphisms of complexes. Arguing as in Proposition 1.2.1 we get that Ω_{*+1} is an homotopy from $\Phi_*\Psi_*$ to the identity map. It remains to prove that $\Psi_*\Phi_* = id_*$. It is clear that $\Psi_0\Phi_0 = id_0$. Assume that $\Psi_n\Phi_n = id_n$. Since $\Phi_{n+1}(E \otimes \overline{E}^n \otimes k) \subseteq \sum_{r=0}^{n+1} E \otimes \overline{E}^r \otimes k \otimes \overline{H}^{n+1-r} \otimes H$, we have that on $E \otimes \overline{E}^n \otimes k$

$$\begin{split} \Psi_{n+1} \, \Phi_{n+1} &= \xi_{n+1} \, \Psi_n \, \partial_{n+1} \, \Phi_{n+1} = \xi_{n+1} \, \Psi_n \, \partial_{n+1} \, \zeta_{n+1} \, \Phi_n \, b'_{n+1} \\ &= \xi_{n+1} \, \Psi_n \, \Phi_n \, b'_{n+1} - \xi_{n+1} \, \Psi_n \, \zeta_n \, \partial_n \, \Phi_n \, b'_{n+1} = \xi_{n+1} \, b'_{n+1} = i d_{n+1} \quad \Box \end{split}$$

Next, we consider the normalized Hochschild resolution $(E \otimes \overline{E}^* \otimes E, b'_*)$ filtered as in Proposition 1.2.2 and the resolution (Z_*, ∂_*) filtered by $F^0_* \subseteq F^1_* \subseteq F^2_* \subseteq \ldots$, where $F^i_* = \bigoplus_{j=0}^i (E \otimes \overline{E}^{n-j} \otimes E) \otimes (\overline{H}^j \otimes H)$.

Proposition 4.2. We have that

$$\Phi_n(a_0 \# h_0 \otimes \dots \otimes a_{n+1} \# h_{n+1}) = \sum_{j=0}^n (-1)^{j(n+1)} (a_0 \# h_0) (a_1 \# h_1^{(1)}) \dots (a_j \# h_j^{(1)})$$
$$\otimes (a_{j+1} \# h_{j+1}^{(1)}) \otimes \dots \otimes (a_{n+1} \# h_{n+1}^{(1)}) \otimes h_1^{(2)} \otimes \dots \otimes h_j^{(2)} \otimes h_{j+1}^{(2)} \dots h_{n+1}^{(2)}$$

Consequently the map Φ_* preserve filtrations.

Proof. It follows by induction on n, using the recursive definition of Φ_*

Proposition 4.3. The map Φ_* induces an homotopy equivalence of *E*-bimodule complexes between the graded complexes associated to the filtrations of $(B_*(E), b'_*)$ and $(B_*(E), b'_*) \otimes (\overline{H}^* \otimes H, d_*)$.

Proof. Note that

$$\frac{F^s(X_{*}, d_{*})}{F^{s-1}(X_{*}, d_{*})} = (X_{*s}, d^0_{*s}) = (E \otimes \overline{H}^s \otimes \overline{A}^* \otimes E, d^0_{*s}),$$
$$\frac{F^s(Z_{*}, \partial_{*})}{F^{s-1}(Z_{*}, \partial_{*})} = (B_*(E), b'_*) \otimes \overline{H}^s \otimes H,$$

where $d^0_{*,s}$ is the boundary map introduced in Subsection 1.1. By Proposition 1.2.2 it suffices to check that $\Phi_* = \Phi_* \phi_*$ induces an homotopy equivalence $\widetilde{\Phi}^s_*$ of *E*bimodules complexes, from $(E \otimes \overline{H}^s \otimes \overline{A}^* \otimes E, d^0_{*,s})$ to $(B_*(E), b'_*) \otimes \overline{H}^s \otimes H$. Let Y_s and μ_s be as in Subsection 1.1 and $\widetilde{Y}_s = E \otimes \overline{H}^s \otimes H$ endowed with the structure of *E*-bimodule given by $x_0(x_1 \otimes \mathbf{h})x_2 = x_0x_1x_2 \otimes \mathbf{h}$, where $\mathbf{h} = h_0 \otimes \cdots \otimes h_{s+1}$. Consider the diagram

where $\tilde{\mu}_s((x_0 \otimes x_1) \otimes \mathbf{h}) = x_0 x_1 \otimes \mathbf{h}$ and $\tilde{\Phi}^s(x \otimes \mathbf{h}) = x(1 \# h_1^{(1)}) \cdots (1 \# h_{s+1}^{(1)}) \otimes \mathbf{h}^{(2)}$. We assert that $\tilde{\Phi}_0^s(\mathbf{x}) = \mathbf{1}_E \otimes (1 \# h_1^{(1)}) \cdots (1 \# h_s^{(1)}) \otimes \mathbf{h}^{(2)} \otimes \mathbf{1}_H$, where $\mathbf{x} = \mathbf{1}_E \otimes \mathbf{h} \otimes \mathbf{1}_E$, with $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$. To prove this it suffices to check that

$$\Phi_s \phi_s(\mathbf{x}) \in \mathbb{1}_E \otimes (\mathbb{1} \# h_1^{(1)}) \cdots (\mathbb{1} \# h_s^{(1)}) \otimes \mathbf{h}^{(2)} \otimes \mathbb{1}_H + F_{s-1}$$

which follows by induction on s, using that $\Phi_s \phi_s(\mathbf{x}) = \zeta_s \Phi_{s-1} \phi_{s-1} d_s(\mathbf{x})$. Now, it is immediate that $\tilde{\mu}_s \tilde{\Phi}_0^s = \tilde{\Phi}^s \mu_s$. Since $\tilde{\Phi}^s$ is an isomorphism and the rows of (4) are relative projective resolutions of Y_s and \tilde{Y}_s respectively, it follows that $\tilde{\Phi}_s^s$ is an homotopy equivalence \Box

Corollary 4.4. The (co)homological Cartan-Leray spectral sequence is isomorphic to the (co)homological Hochschild-Serre spectral sequence.

4.5 The Grothendieck spectral sequence. If M is an E-bimodule, then the group $H_0(A, M) = M/[A, M]$ is a left H-module via $h \cdot \overline{m} = (1\#h^{(2)})m(1\#h^{(1)})^{-1}$, where the \overline{m} denotes the class of m in M/[A, M]. Let us consider the functors $M \mapsto H_0(E, M)$ from the category of E-bimodules to the category of k-modules, $M \mapsto H_0(A, M)$ from the category of E-bimodules to the category of left H-modules and $M \mapsto H_0(H, M)$ from the category of left H-modules to the category of k-modules. It is easy to see that $H_0(E, M) = H_0(H, H_0(A, M))$ and that if M is a relatively projective E^e/E^{op} -module, then $H_0(A, M)$ is a relatively projective H/k-module. In fact, if $M = E \otimes N$, then the map $h \otimes n \mapsto \overline{1\#h^{(2)} \otimes n(1\#h^{(1)})^{-1}}$ is an isomorphism of left H-modules from $H \otimes N$ to $H_0(A, M)$. Thus we have a Grothendieck spectral sequence

$$E_{rs}^2 = \mathrm{H}_s(H, \mathrm{H}_r(A, M)) \to \mathrm{H}_{r+s}(E, M)).$$

We assert that the Grothendieck spectral sequence and the Cartan-Leray spectral sequence coincide. To prove this we use a concrete construction of the Grothendieck spectral sequence. Let $(P_*, \partial_*) = (M \otimes \overline{E}^* \otimes E, b'_*)$ be the normalized canonical resolution of M as a right E^e -module. Let us write $(\overline{P}_*, \overline{\partial}_*) = (P_*, \partial_*) \otimes_{A^e} A$. Consider the double complex

whose r-th column is $(-1)^r$ times $\overline{H}^r \otimes H \otimes_H (\overline{P}_*, \overline{\partial}_*)$ and whose s-th row is the canonical complex $(\overline{H}^* \otimes H \otimes_H \overline{P}_s, d_*)$ giving the homology $H_*(H, \overline{P}_s)$ of k as a trivial right H-module with coefficients in \overline{P}_s . By definition, the Grothendieck spectral sequence is the spectral sequence associate to the filtrations by columns of C_{**} . Since $C_{**} \simeq (\overline{H}^* \otimes H, d_*) \otimes_H (M \otimes_{E \otimes A^{\text{op}}} (E \otimes \overline{E}^* \otimes E, b'_*))$ as filtered complexes, the homological Cartan-Leray and the Grothendieck spectral sequence coincide. The same is valid in the cohomological setting.

APPENDIX A

Let $R \to S$ be an unitary ring map and let N be a left S-module. In this section, under suitable conditions, we construct a projective relative resolution of N. We need this result (with R = E, $S = E^e$ and N = E) to complete the proof of Theorem 1.1.1. The general case considered here simplifies the notation and enables us to consider other cases, for instance algebras of groups having particular resolutions.

Let us consider a diagram of left S-modules and S-module maps

$$\begin{array}{c} \vdots \\ \downarrow \partial_2 \\ Y_1 \xleftarrow{\mu_1} X_{01} \xleftarrow{d_{11}^0} X_{11} \xleftarrow{d_{21}^0} \dots \\ \downarrow \partial_1 \\ Y_0 \xleftarrow{\mu_0} X_{00} \xleftarrow{d_{10}^0} X_{10} \xleftarrow{d_{20}^0} \dots, \end{array}$$

such that:

a) The column and the rows are chain complexes.

b) For each $r, s \ge 0$ we have a left *R*-module \overline{X}_{rs} and *S*-module maps

$$s_{rs} \colon X_{rs} \to S \otimes \overline{X}_{rs}$$
 and $\pi_{rs} \colon S \otimes \overline{X}_{rs} \to X_{rs}$

verifying $\pi_{rs} s_{rs} = id$.

c) Each row is contractible as a complex of *R*-modules, with a chain contracting homotopy $\sigma_{0s}^0: Y_s \to X_{0s}$ and $\sigma_{r+1,s}^0: X_{rs} \to X_{r+1,s}$ $(r \ge 0)$.

We are going to modify this diagram by adding S-module maps

$$d_{rs}^l \colon X_{rs} \to X_{r+l-1,s-l} \qquad (r,s \ge 0 \text{ and } 1 \le l \le s).$$

Let $X_n = \bigoplus_{r+s=n} X_{rs}$ and $d_n = \sum_{r+s=n} \sum_{l=0}^{s} d_{rs}^l$ $(n \ge 1)$. Consider the maps $\mu'_n: X_n \to Y_n \ (n \ge 0)$, given by:

$$\mu'_n(x) = \begin{cases} \mu_n(x) & \text{for } x \in X_{0n} \\ 0 & \text{for } x \in X_{r,n-r} \text{ with } r > 0. \end{cases}$$

We define the arrows d_{rs}^l in such a way that (X_*, d_*) becomes a chain complex of S-modules and $\mu'_*: (X_*, d_*) \to (Y_*, -\partial_*)$ becomes a chain homotopy equivalence of complexes of *R*-modules. In fact, we are going to build *R*-module morphisms

$$\sigma_{l,s-l}^{l} \colon Y_s \to X_{l,s-l} \text{ and } \sigma_{r+l+1,s-l}^{l} \colon X_{rs} \to X_{r+l+1,s-l} \quad (r,s \ge 0 \text{ and } 1 \le l \le s),$$

satisfying the following:

Theorem A.1. Let $C_*(\mu'_*)$ be the mapping cone of μ'_* , that is, $C_*(\mu'_*) = (C_*, \delta_*)$, where $C_n = Y_n \oplus X_{n-1}$ and $\delta_n(y_n, x_{n-1}) = (-\partial(y_n) - \mu'_{n-1}(x_{n-1}), -d_{n-1}(x_{n-1}))$. The family of *R*-module maps $\sigma_{n+1} \colon C_n(\mu'_*) \to C_{n+1}(\mu'_*)$ $(n \ge 0)$, defined by:

$$\sigma_{n+1} = -\sum_{\substack{r+s=n-1\\r\geq -1}} \sum_{l=0}^{s} \sigma_{r+l+1,s-l}^{l},$$

is a chain contracting homotopy of $C_*(\mu'_*)$.

Corollary A.2. Let N be a left S-module. If there is a S-module map $\tilde{\mu}: Y_0 \to N$, such that

$$(*) N \stackrel{\widetilde{\mu}}{\leftarrow} Y_0 \stackrel{\partial_1}{\leftarrow} Y_1 \stackrel{\partial_2}{\leftarrow} Y_2 \stackrel{\partial_3}{\leftarrow} Y_3 \stackrel{\partial_4}{\leftarrow} Y_4 \stackrel{\partial_5}{\leftarrow} Y_6 \stackrel{\partial_7}{\leftarrow} \dots$$

is contractible as a complex of left R-modules, then

$$(**) \qquad N \stackrel{\mu}{\leftarrow} X_0 \stackrel{d_1}{\leftarrow} X_1 \stackrel{d_2}{\leftarrow} X_2 \stackrel{d_3}{\leftarrow} X_3 \stackrel{d_4}{\leftarrow} X_4 \stackrel{d_5}{\leftarrow} X_5 \stackrel{d_6}{\leftarrow} X_6 \stackrel{d_7}{\leftarrow} \dots,$$

where $\mu = \tilde{\mu} \mu_0$, is a relative projective resolution. Moreover, if $\sigma_0^{-1} \colon N \to Y_0$, $\sigma_{n+1}^{-1} \colon Y_n \to Y_{n+1} \ (n \ge 0)$ is a chain contracting homotopy of (*), then we obtain a chain contracting homotopy $\overline{\sigma}_0 \colon N \to X_0$, $\overline{\sigma}_{n+1} \colon X_n \to X_{n+1} \ (n \ge 0)$ of (**), defining $\overline{\sigma}_0 = \sigma_{00}^0 \sigma_0^{-1}$ and

$$\overline{\sigma}_{n+1} = -\sum_{l=0}^{n+1} \sigma_{l,n-l+1}^l \sigma_{n+1}^{-1} \mu_n + \sum_{r+s=n} \sum_{l=0}^s \sigma_{r+l+1,s-l}^l.$$

Proof. Write

$$\widetilde{\sigma}_n = \sum_{\substack{r+s=n-1\\r\geq 0}} \sum_{l=0}^s \sigma_{r+l+1,s-l}^l \quad (n\geq 1) \quad \text{and} \quad \widehat{\sigma}_n = \sum_{l=0}^n \sigma_{l,n-l}^l \quad (n\geq 0).$$

From Theorem A.1, we have

(*n)
$$\widehat{\sigma}_n \,\partial_{n+1} = \sum_{l=0}^n \sigma_{l,n-l}^l \,\partial_{n+1} = -\sum_{l=0}^n \sum_{i=0}^{l+1} d_{i,n+1-i}^{l+1-i} \,\sigma_{i,n+1-i}^i = -d_{n+1} \,\widehat{\sigma}_{n+1}$$

It is clear that $\mu \overline{\sigma}_0 = id$. Moreover

$$\overline{\sigma}_{0} \mu = \sigma_{00}^{0} \sigma_{0}^{-1} \widetilde{\mu} \mu_{0} = \sigma_{00}^{0} \mu_{0} - \sigma_{00}^{0} \partial_{1} \sigma_{1}^{-1} \mu_{0}$$
$$= id - d_{10}^{0} \sigma_{10}^{0} + d_{01}^{1} \sigma_{01}^{0} \sigma_{1}^{-1} \mu_{0} + d_{10}^{0} \sigma_{10}^{1} \sigma_{1}^{-1} \mu_{0},$$

where the last equality follows from (*0). Now, let $n \ge 1$. Take $x \in X_{r,n-r}$. If $r \ge 1$, then the equality $(0,x) = \delta_{n+2} \sigma_{n+2}(0,x) + \sigma_{n+1} \delta_{n+1}(0,x)$ implies that $x = d_{n+1} \tilde{\sigma}_{n+1}(x) + \tilde{\sigma}_n d_n(x)$. Hence, we can suppose r = 0. Then, from $(0,x) = \delta_{n+2} \sigma_{n+2}(0,x) + \sigma_{n+1} \delta_{n+1}(0,x)$, we get

$$\begin{aligned} x &= d_{n+1}\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n d_n(x) + \widehat{\sigma}_n \mu_n(x) \\ &= d_{n+1}\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n d_n(x) + \widehat{\sigma}_n \sigma_n^{-1} \partial_n \mu_n(x) + \widehat{\sigma}_n \partial_{n+1} \sigma_{n+1}^{-1} \mu_n(x) \\ &= d_{n+1}\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n d_n(x) - \widehat{\sigma}_n \sigma_n^{-1} \mu_{n-1} d_n(x) + \widehat{\sigma}_n \partial_{n+1} \sigma_{n+1}^{-1} \mu_n(x) \\ &= d_{n+1}\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n d_n(x) - \widehat{\sigma}_n \sigma_n^{-1} \mu_{n-1} d_n(x) - d_{n+1} \widehat{\sigma}_{n+1} \sigma_{n+1}^{-1} \mu_n(x). \end{aligned}$$

where the last equality follows from (*n)

Next we define the morphisms d_{rs}^l and we prove that (X_*, d_*) is a chain complex. **Definition A.3.** We define the S-module maps $d_{rs}^l \colon X_{rs} \to X_{r+l-1,s-l}$ $(r \ge 0$ and $1 \le l \le s)$, recursively by $d_{rs}^l = \overline{d}_{rs}^l s_{rs}$, where $\overline{d}_{rs}^l \colon S \otimes \overline{X}_{rs} \to X_{r+l-1,s-l}$ $(r \ge 0$ and $1 \le l \le s)$ is the S-module map defined by

$$\overline{d}_{rs}^{l}(\mathbf{x}) = \begin{cases} -\sigma_{0,s-1}^{0} \partial_{s} \mu_{s} \pi_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1, \\ -\sum_{j=1}^{l-1} \sigma_{l-1,s-l}^{0} d_{j-1,s-j}^{l-j} d_{0s}^{j} \pi_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \le s, \\ -\sum_{j=0}^{l-1} \sigma_{r+l-1,s-l}^{0} d_{r+j-1,s-j}^{l-j} d_{rs}^{j} \pi_{rs}(\mathbf{x}) & \text{if } r > 0, \end{cases}$$

for each $\mathbf{x} = 1 \otimes \overline{\mathbf{x}} \in S \otimes \overline{X}_{rs}$.

Proposition A.4. We have $\mu_{s-1} d_{0s}^1 = -\partial_s \mu_s$ and

$$d_{r+l-1,s-l}^{0} d_{rs}^{l} = \begin{cases} -\sum_{j=1}^{l-1} d_{j-1,s-j}^{l-j} d_{0s}^{j} & \text{if } r = 0 \text{ and } 1 < l \le s \\ -\sum_{j=0}^{l-1} d_{r+j-1,s-j}^{l-j} d_{rs}^{j} & \text{if } r > 0 \text{ and } 1 \le l \le s. \end{cases}$$

Consequently (X_*, d_*) is a chain complex.

Proof. We prove the proposition by induction on l and r. To simplify the expressions we put $d_{0s}^0 := \mu_s$, $d_{-1,s}^1 := \partial_s$ and $d_{-1,s}^l := 0$ for all l > 1. Moreover to abbreviate we do not write the subindices. Let $\mathbf{x} = 1 \otimes \overline{\mathbf{x}}$ with $\overline{\mathbf{x}} \in \overline{X}_{0s}$. Since $\overline{d}_0^1(\mathbf{x}) = -\sigma^0 d^1 d^0 \pi(\mathbf{x})$, we have $d^0 \overline{d}^1(\mathbf{x}) = -d^0 \sigma^0 d^1 d^0 \pi(\mathbf{x}) = -d^1 d^0 \pi(\mathbf{x})$, which implies $d^0 d^1 = -d^1 d^0$. Let l + r > 1 and suppose the result is valid for

 d_{p*}^{j} with j < l or j = l and p < r. Let $\mathbf{x} = 1 \otimes \overline{\mathbf{x}}$ with $\overline{\mathbf{x}} \in \overline{X}_{rs}$. Since $\overline{d}^{l}(\mathbf{x}) = -\sum_{j=0}^{l-1} \sigma^{0} d^{l-j} d^{j} \pi(\mathbf{x})$, then

$$d^{0} \overline{d}^{l}(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{0} \sigma^{0} d^{l-j} d^{j} \pi(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{l-j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{0} d^{l-j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{0} d^{l-j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{0} d^{j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{j} d^{j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{j} d^{j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{j} d^{j} d^{j} d^{j} \pi(\mathbf{x}) + \sum_{j=0}^{l-1} \sigma^{0} d^{j} d^{j}$$

Applying first the inductive hypothesis to $d^0 d^{l-j}$ with $(0 \le j < l)$ and then to $d^0 d^j$ with (0 < j < l), we obtain:

$$d^{0} \overline{d}^{l}(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{l-j} d^{j} \pi(\mathbf{x}) - \sum_{j=0}^{l-1} \sum_{i=0}^{l-j-1} \sigma^{0} d^{l-j-i} d^{i} d^{j} \pi(\mathbf{x})$$
$$= -\sum_{j=0}^{l-1} d^{l-j} d^{j} \pi(\mathbf{x}) - \sum_{j=0}^{l-2} \sum_{i=1}^{l-j-1} \sigma^{0} d^{l-j-i} d^{i} d^{j} \pi(\mathbf{x})$$
$$+ \sum_{j=1}^{l-1} \sum_{h=0}^{j-1} \sigma^{0} d^{l-j} d^{j-h} d^{h} \pi(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{l-j} d^{j} \pi(\mathbf{x}).$$

The desired equality follows immediately from this fact \Box

It is immediate that $\mu'_*: (X_*, d_*) \to (Y_*, -\partial_*)$ is a morphism of S-module chain complexes. Next, we construct the chain contracting homotopy of $C_*(\mu'_*)$.

Definition A.5. We define $\sigma_{l,s-l}^l: Y_s \to X_{l,s-l}$ and $\sigma_{r+l+1,s-l}^l: X_{rs} \to X_{r+l+1,s-l}$ $(0 < l \le s, r \ge 0)$, recursively by:

$$\sigma_{r+l+1,s-l}^{l} = -\sum_{i=0}^{l-1} \sigma_{r+l+1,s-l}^{0} \, d_{r+i+1,s-i}^{l-i} \, \sigma_{r+i+1,s-i}^{i} \quad (0 < l \le s \text{ and } r \ge -1).$$

Proof of Theorem A.1. To simplify the expressions we put $d_{-1,s}^0 := 0$, $d_{0s}^0 := \mu_s$, $d_{-1,s}^1 := \partial_s$ and $d_{-1,s}^l := 0$ for all l > 1. Because of the definitions of d_* and σ_* , it suffices to check that $\sigma_{rs}^0 d_{rs}^0 + d_{r+1,s}^0 \sigma_{r+1,s}^0 = id$ and

$$\sum_{i=0}^{l} \sigma_{r+l,s-l}^{l-i} d_{rs}^{i} + \sum_{i=0}^{l} d_{r+i+1,s-i}^{l-i} \sigma_{r+i+1,s-i}^{i} = 0 \quad \text{for } l > 0,$$

where we put $d^0_{-1,s} = 0$. The first formula simply says that σ^0_* is a chain contracting homotopy of d^0_* . Let us see the second one. To abbreviate we do not write the subindices. From the definition of σ^l we have:

$$d^{0} \sigma^{l} = -\sum_{i=0}^{l-1} d^{0} \sigma^{0} d^{l-i} \sigma^{i} = \sum_{i=0}^{l-1} \sigma^{0} d^{0} d^{l-i} \sigma^{i} - \sum_{i=0}^{l-1} d^{l-i} \sigma^{i}.$$

Consequently

$$\sum_{i=0}^{l} \sigma^{l-i} d^{i} + \sum_{i=0}^{l} d^{l-i} \sigma^{i} = \sum_{i=0}^{l} \sigma^{l-i} d^{i} + \sum_{i=0}^{l-1} \sigma^{0} d^{0} d^{l-i} \sigma^{i}.$$

Then, it suffices to prove that the term appearing on the right side of the equality is zero. We prove this by induction on l. For l = 1 we have:

$$\sigma^0 d^0 d^1 \sigma^0 = -\sigma^0 d^1 d^0 \sigma^0 = \sigma^0 d^1 \sigma^0 d^0 - \sigma^0 d^1 = -\sigma^1 d^0 - \sigma^0 d^1.$$

Suppose l > 1. From Proposition A.5,

$$\sum_{i=0}^{l-1} \sigma^0 \, d^0 \, d^{l-i} \, \sigma^i = -\sum_{i=0}^{l-1} \sum_{j=0}^{l-i-1} \sigma^0 \, d^{l-i-j} \, d^j \, \sigma^i = -\sum_{h=0}^{l-1} \sum_{i=0}^{h} \sigma^0 \, d^{l-h} \, d^{h-i} \, \sigma^i.$$

So, applying the inductive hypothesis to $\sum_{i=0}^{h} d^{h-i}\,\sigma^i \ (h\geq 0),$ we obtain

$$\sum_{i=0}^{l-1} \sigma^0 d^0 d^{l-i} \sigma^i = \sum_{h=0}^{l-1} \sum_{i=0}^{h} \sigma^0 d^{l-h} \sigma^{h-i} d^i - \sigma^0 d^l$$
$$= \sum_{i=0}^{l-1} \sum_{j=0}^{l-i-1} \sigma^0 d^{l-i-j} \sigma^j d^i - \sigma^0 d^l$$
$$= -\sum_{i=0}^{l} \sigma^{l-i} d^i \quad \Box$$

In this appendix we compute explicitly the maps d_{rs}^l introduced in Section 1, completing the results of Theorem 1.1.3.

Definition B.1. Given $\mathbf{h} = h_1 \otimes \cdots \otimes h_l \in \overline{H}^l$, we define $F_0^{(l)}(\mathbf{h})$, recursively by:

$$F_0^{(2)}(\mathbf{h}) = -f(h_1, h_2),$$

$$F_0^{(l+1)}(\mathbf{h}) = \sum_{j=1}^l (-1)^j f(h_j^{(1)}, h_{j+1}^{(1)})^{\overline{\mathbf{h}_{1,j-1}^{(1)}}} \otimes F^{(l)}(\mathbf{h}^{j(2)}),$$

where $\mathbf{h}^{j(2)} = \mathbf{h}_{1,j-1}^{(2)} \otimes \mathbf{h}_{j}^{(2)} h_{j+1}^{(2)} \otimes \mathbf{h}_{j+2,l+1}$. For instance, we have

$$F_0^{(3)}(\mathbf{h}) = f(h_1^{(1)}, h_2^{(1)}) \otimes f(\mathfrak{h}_{12}^{(2)}, h_3) - f(h_2^{(1)}, h_3^{(1)})^{h_1^{(1)}} \otimes f(h_1^{(2)}, \mathfrak{h}_{23}^{(2)})$$

and

$$\begin{split} F_{0}^{(4)}(\mathbf{h}) &= -f(h_{1}^{(1)},h_{2}^{(1)}) \otimes f(\mathfrak{h}_{12}^{(2)},h_{3}^{(1)}) \otimes f(\mathfrak{h}_{12}^{(3)}h_{3}^{(2)},h_{4}) \\ &+ f(h_{1}^{(1)},h_{2}^{(1)}) \otimes f(h_{3}^{(1)},h_{4}^{(1)})^{\mathfrak{h}_{12}^{(2)}} \otimes f(\mathfrak{h}_{12}^{(2)},\mathfrak{h}_{34}^{(2)}) \\ &+ f(h_{2}^{(1)},h_{3}^{(1)})^{h_{1}^{(1)}} \otimes f(h_{1}^{(2)},\mathfrak{h}_{23}^{(2)}) \otimes f(\mathfrak{h}_{13}^{(3)},h_{4}) \\ &- f(h_{2}^{(1)},h_{3}^{(1)})^{h_{1}^{(1)}} \otimes f(\mathfrak{h}_{23}^{(2)},h_{4}^{(1)})^{h_{1}^{(2)}} \otimes f(h_{1}^{(3)},\mathfrak{h}_{23}^{(3)}h_{4}^{(2)}) \\ &- f(h_{3}^{(1)},h_{4}^{(1)})^{\overline{\mathfrak{h}_{12}^{(1)}}} \otimes f(h_{1}^{(2)},h_{2}^{(2)}) \otimes f(\mathfrak{h}_{12}^{(3)},\mathfrak{h}_{34}^{(3)}) \\ &+ f(h_{3}^{(1)},h_{4}^{(1)})^{\overline{\mathfrak{h}_{12}^{(1)}}} \otimes f(h_{2}^{(2)},\mathfrak{h}_{34}^{(2)})^{h_{1}^{(2)}} \otimes f(h_{1}^{(3)},\mathfrak{h}_{24}^{(3)}). \end{split}$$

For the following definition we adopt the convention that $\mathbf{a}_{10} = \mathbf{a}_{r+1,r} = \mathbf{1}_k \in k$.

Definition B.2. Given $\mathbf{h} = h_1 \otimes \cdots \otimes h_l \in \overline{H}^l$ and $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^r$, we define $F_r^{(l)}(\mathbf{h} \otimes \mathbf{a})$, recursively by:

$$F_{r}^{(2)}(\mathbf{h}\otimes\mathbf{a}) = \sum_{i=0}^{r} (-1)^{i+1} \mathbf{a}_{1i}^{\overline{\mathbf{h}_{12}^{(1)}}} \otimes f(h_{1}^{(2)}, h_{2}^{(2)}) \otimes \mathbf{a}_{i+1,r}^{\mathfrak{h}_{12}^{(3)}},$$

$$F_{r}^{(l+1)}(\mathbf{h}\otimes\mathbf{a}) = \sum_{j=1}^{l} \sum_{i=0}^{r} (-1)^{il+j} \mathbf{a}_{1i}^{\overline{\mathbf{h}_{1,l+1}^{(1)}}} \otimes f(h_{j}^{(2)}, h_{j+1}^{(2)})^{\overline{\mathbf{h}_{1,j-1}^{(2)}}} \otimes F_{r-i}^{(l)}(\mathbf{h}^{j(3)} \otimes \mathbf{a}_{i+1,r})$$

where $\mathbf{h}^{j(3)} = \mathbf{h}_{1,j-1}^{(3)} \otimes \mathbf{h}_{j}^{(3)} h_{j+1}^{(3)} \otimes \mathbf{h}_{j+2,l+1}^{(2)}$ and $F_{0}^{(l)}(\mathbf{h}^{j(3)} \otimes \mathbf{a}_{r+1,r}) = F_{0}^{(l)}(\mathbf{h}^{j(3)})$. For instance, we have

$$\begin{split} F_r^{(3)}(\mathbf{h}\otimes\mathbf{a}) &= \sum_{0\leq i\leq j\leq r} (-1)^{i+j} \mathbf{a}_{1i}^{\overline{\mathbf{h}_{13}^{(1)}}} \otimes f(h_1^{(2)}, h_2^{(2)}) \otimes \mathbf{a}_{i+1,j}^{\overline{\mathbf{h}_{13}^{(3)}}} \otimes f(\mathfrak{h}_{12}^{(4)}, h_3^{(3)}) \otimes \mathbf{a}_{j+1,r}^{\mathfrak{h}_{13}^{(5)}, h_3^{(4)}} \\ &+ \sum_{0\leq i\leq j\leq r} (-1)^{i+j+1} \mathbf{a}_{1i}^{\overline{\mathbf{h}_{13}^{(1)}}} \otimes f(h_2^{(2)}, h_3^{(2)})^{h_1^{(2)}} \otimes \mathbf{a}_{i+1,j}^{\overline{\mathbf{h}_{13}^{(3)}}} \otimes f(h_1^{(4)}, \mathfrak{h}_{23}^{(4)}) \otimes \mathbf{a}_{j+1,r}^{\mathfrak{h}_{13}^{(5)}}. \end{split}$$

We set $F_0^{(1)}(h_s) = 1_k \in k$, $F_r^{(1)}(h_s \otimes \mathbf{a}) = \mathbf{a}^{h_s}$ and $F_0^{(l)}(\mathbf{h}_{s-l-1,s} \otimes 1_k) = F_0^{(l)}(\mathbf{h}_{s-l-1,s})$. Moreover, to abbreviate we write $F^{(l)}(\mathbf{h}) = F_0^{(l)}(\mathbf{h})$ and $F^{(l)}(\mathbf{a}) = F_r^{(l)}(\mathbf{h} \otimes \mathbf{a})$.

Lemma B.3. Let $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h}_{s-l,s} = h_{s-l} \otimes \cdots \otimes h_l$. We have:

$$F^{(l+1)}(\mathbf{h}_{s-l,s}) = \sum_{i=1}^{l} (-1)^{i} F^{(l-i+1)} \binom{F^{(i)}(\mathbf{h}_{s-i+1,s}^{(1)})}{\mathbf{h}_{s-l,s-i}^{(1)}} \otimes f(\mathfrak{h}_{s-l,s-i}^{(2)},\mathfrak{h}_{s-i+1,s}^{(2)})$$

and

$$\begin{split} F^{(l+1)} \binom{\mathbf{a}}{\mathbf{h}_{s-l,s}} &= F^{(l+1)} \binom{\mathbf{a}_{1,r-1}}{\mathbf{h}_{s-l,s}^{(1)}} \otimes a_{r}^{\mathfrak{h}_{s-l,s}^{(2)}} \\ &+ \sum_{i=1}^{l} (-1)^{r+i} F^{(l-i+1)} \binom{F^{(i)} \binom{\mathbf{a}_{1}}{\mathbf{h}_{s-i+1,s}^{(1)}}}{\mathbf{h}_{s-l,s-i}^{(1)}} \otimes f(\mathfrak{h}_{s-l,s-i}^{(2)},\mathfrak{h}_{s-i+1,s}^{(2)}). \end{split}$$
where $F^{(l+1)} \binom{\mathbf{a}_{1,r-1}}{\mathbf{h}_{s-l,s}} = F^{(l+1)}(\mathbf{h}_{s-l,s}) \text{ if } r = 1.$

Proof. We prove the second formula. The proof of the first one is similar. It is clear that the lemma is valid for l = 1. Let l > 1 and suppose the result is valid for l - 1. To abbreviate we put

$$\begin{split} &\xi = u(l-1) + j + s \\ &\mathfrak{h}_{s-l,s}^{j(4)} = \mathfrak{h}_{s-l,j+1}^{(4)} \mathfrak{h}_{j+2,s}^{(3)}, \\ &\mathbf{h}_{s-l,s}^{j(3)} = \mathbf{h}_{s-l,j-1}^{(3)} \otimes h_{j}^{(3)} h_{j+1}^{(3)} \otimes \mathbf{h}_{j+2,s}^{(2)}, \\ &\mathbf{f}_{j}^{(2)} = f(h_{j}^{(2)}, h_{j+1}^{(2)})^{\overline{\mathbf{h}_{s-l,j-1}^{(2)}}}, \\ &\mathbf{f}_{s-l,s-i,s}^{j(4)} = f(\mathfrak{h}_{s-l,j+1}^{(4)} \mathfrak{h}_{j+2,s-i}^{(3)}, \mathfrak{h}_{s-i+1,s}^{(3)}), \\ &\mathbf{f}_{s-l,s-i,s}^{(4)j} = f(\mathfrak{h}_{s-l,s-i}^{(4)}, \mathfrak{h}_{s-i+1,j+1}^{(4)} \mathfrak{h}_{j+2,s}^{(3)}) \\ &\mathbf{f}_{s-l,s-i,s}^{(2)} = f(\mathfrak{h}_{s-l,s-i}^{(2)}, \mathfrak{h}_{s-i+1,s}^{(2)}). \end{split}$$

We have:

$$\begin{split} F^{(l+1)} \begin{pmatrix} \mathbf{a} \\ \mathbf{h}_{s-l,s} \end{pmatrix} &= \sum_{j=s-l}^{s-1} \sum_{u=0}^{r} (-1)^{\xi-1} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l)} \begin{pmatrix} \mathbf{a}_{u+1,r} \\ \mathbf{h}_{s-l,s}^{(3)} \end{pmatrix} \\ &= \sum_{j=s-l}^{s-1} \sum_{u=0}^{r-1} (-1)^{\xi-1} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l)} \begin{pmatrix} \mathbf{a}_{u+1,r-1} \\ \mathbf{h}_{s-l,s}^{(3)} \end{pmatrix} \otimes a_{r}^{\mathbf{h}_{s-l,s}^{(4)}} \\ &+ \sum_{j=s-l}^{s-2} \sum_{u=0}^{r} \sum_{i=1}^{s-j-1} (-1)^{\xi-1+r-u+i} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l-i)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{u+1,r} \\ \mathbf{h}_{s-l,s}^{(2)} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{(3)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i,s}^{j(4)} \\ &+ \sum_{j=s-l+1}^{s-1} \sum_{u=0}^{r} \sum_{i=s-j}^{l-1} (-1)^{\xi-1+r-u+i} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l-i)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{u+1,r} \\ \mathbf{h}_{s-i,s-i}^{(2)} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i-1}^{(3)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i-1,s}^{j(4)}. \end{split}$$

Permuting the order of the summands, we obtain

$$\begin{split} F^{(l+1)} \begin{pmatrix} \mathbf{a} \\ \mathbf{h}_{s-l,s} \end{pmatrix} &= \sum_{j=s-l}^{s-1} \sum_{u=0}^{r-1} (-1)^{\xi-1} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l)} \begin{pmatrix} \mathbf{a}_{u+1,r-1} \\ \mathbf{h}_{s-l,s}^{(3)} \end{pmatrix} \otimes a_{r}^{\mathfrak{h}_{s-l,s}^{j(4)}} \\ &+ \sum_{i=1}^{l-1} \sum_{u=0}^{r} \sum_{j=s-l}^{s-i-1} (-1)^{\xi-1+r-u+i} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l-i)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{u+1,r} \\ \mathbf{h}_{s-i+1,s} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{j(3)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i,s}^{j(4)} \\ &+ \sum_{i=2}^{l} \sum_{u=0}^{r} \sum_{j=s-i+1}^{s-1} (-1)^{\xi+r-u+i} \mathbf{a}_{1u}^{\overline{\mathbf{h}_{s-l,s}^{(1)}}} \otimes \mathbf{f}_{j}^{(2)} \otimes F^{(l-i+1)} \begin{pmatrix} F^{(i-1)} \begin{pmatrix} \mathbf{a}_{u+1,r} \\ \mathbf{h}_{s-i+1,s} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{(3)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i,s}^{(4)j} \\ &= F^{(l+1)} \begin{pmatrix} \mathbf{a}_{1,r-1} \\ \mathbf{h}_{s-l,s}^{(1)} \end{pmatrix} \otimes a_{r}^{\mathfrak{h}_{s-l,s}^{(2)}} + \sum_{i=1}^{l} (-1)^{r+i} F^{(l-i+1)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{1,r} \\ \mathbf{h}_{s-i,s-i} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{(2)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i,s}^{(2)} \\ &+ \sum_{i=1}^{l} (-1)^{r+i} F^{(l-i+1)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{1,r} \\ \mathbf{h}_{s-i,s-i} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{(2)} \end{pmatrix} \otimes \mathbf{f}_{s-l,s-i,s}^{(2)} \\ &+ \sum_{i=1}^{l} (-1)^{r+i} F^{(l-i+1)} \begin{pmatrix} F^{(i)} \begin{pmatrix} \mathbf{a}_{1,r} \\ \mathbf{h}_{s-i,s-i} \end{pmatrix} \\ \mathbf{h}_{s-l,s-i}^{(2)} \end{pmatrix} \\ &= F^{(l+1)} \begin{pmatrix} \mathbf{a}_{1,r-1} \\ \mathbf{h}_{s-l,s} \end{pmatrix} \\ &= F^{(l+1)} \begin{pmatrix} \mathbf{a}_{1,r-1} \\ \mathbf{h}_{s-l,s} \end{pmatrix} \\ &= F^{(l+1)} \begin{pmatrix} \mathbf{a}_{1,r-1} \\ \mathbf{h}_{s-l,s-i} \end{pmatrix} \\ &= F^{(l+1)} \begin{pmatrix} \mathbf{a}_{1,r-1} \\ \mathbf{h}_{1,r-1} \end{pmatrix} \\ &= F^{($$

which ends the proof \Box

Computation of d_{rs}^l . Let us compute d_{rs}^{l+1} for $l \ge 1$. First we suppose the formula is valid for d_{rs}^j with $j \le l$ and we see that it is valid for d_{0s}^{l+1} . To abbreviate we write $\zeta_i = is + (l - i + 1)(s - 1) + 1$. Using the inductive hypothesis and the fact that $\sigma^0 d^l (a_0 \otimes \mathbf{h}_{0s} \otimes 1 \# 1) = 0$, we obtain:

$$d^{l+1}(1 \otimes \mathbf{h} \otimes 1_{E}) = -\sum_{i=1}^{l} \sigma^{0} d^{l+1-i} d^{i}(1 \otimes \mathbf{h} \otimes 1_{E})$$

$$= \sum_{i=1}^{l} (-1)^{is+1} \sigma^{0} d^{l+1-i} \Big(1 \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)}(\mathbf{h}_{s-i+1,s}^{(1)}) \otimes 1 \# \mathfrak{h}_{s-i+1,s}^{(2)} \Big)$$

$$= \sum_{i=1}^{l} \sigma^{0} \Big((-1)^{\zeta_{i}} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1-i)} {F^{(i)}(\mathbf{h}_{s-i+1,s}^{(1)}) \otimes f(\mathfrak{h}_{s-l,s-i}^{(2)}, \mathfrak{h}_{s-i+1,s}^{(2)})} \# \mathfrak{h}_{s-l,s}^{(3)} \Big)$$

$$= (-1)^{(l+1)s} 1 \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)}(\mathbf{h}_{s-l,s}^{(1)}) \otimes 1 \# \mathfrak{h}_{s-l,s}^{(2)},$$

where the last equality follows from the definition of σ^0 and Lemma B.3. Now, we suppose the result is valid for $d_{r's}^{l+1}$ with r' < r and we show that it is valid for d_{rs}^{l+1} .

To abbreviate we write $\zeta_i = i(r+s) + (l-i+1)(r+s-1) + 1$.

$$\begin{split} d^{l+1} \left(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1_E \right) &= -\sum_{i=0}^l \sigma^0 \, d^{l+1-i} \, d^i \big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1_E \big) \\ &= (-1)^{r+1} \sigma_0 \, d^{l+1} \big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1 \big) - (-1)^{r+s} \sigma_0 \, d^l \big(1 \otimes \mathbf{h}_{0,s-1} \otimes \mathbf{a}^{\mathbf{h}_s^{(1)}} \otimes 1 \# \mathbf{h}_s^{(2)} \big) \\ &- \sum_{i=2}^l \sigma^0 \, d^{l+1-i} \left((-1)^{i(r+s)} \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)} \Big(\mathbf{h}_{s-i+1,s}^{(1)} \Big) \otimes 1 \# \mathbf{h}_{s-i+1,s}^{(2)} \right) \\ &= (-1)^{r+1} \sigma_0 \, d^{l+1} \big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1 \big) \\ &- \sum_{i=1}^l \sigma^0 \, d^{l+1-i} \left((-1)^{i(r+s)} \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)} \Big(\mathbf{h}_{s-i+1,s}^{(1)} \Big) \otimes 1 \# \mathbf{h}_{s-i+1,s}^{(2)} \right) \\ &= \sigma^0 \bigg((-1)^{(l+1)(r+s-1)+r+1} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)} \Big(\mathbf{h}_{s-l,s}^{(1)} \Big) \otimes a_r^{\mathfrak{h}_{s-l,s}^{(2)}} \# \mathfrak{h}_{s-l,s}^{(3)} \\ &+ \sum_{i=1}^l (-1)^{\zeta_i} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1-i)} \Big(\begin{array}{c} F^{(i)} \Big(\mathbf{h}_{s-i+1,s}^{(1)} \Big) \\ \mathbf{h}_{s-l,s-i}^{(1)} \Big) \otimes f(\mathbf{h}_{s-l,s-i}^{(2)}, \mathbf{h}_{s-i+1,s}^{(2)}) \# \mathbf{h}_{s-l,s}^{(3)} \bigg) \\ &= (-1)^{(l+1)(r+s)} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)} \Big(\mathbf{h}_{s-l,s-i}^{(1)} \Big) \otimes 1 \# \mathfrak{h}_{s-l,s}^{(2)}, \end{split}$$

where the last equality follows from the definition of σ^0 and Lemma B.3

Remark B.4. When H is a group algebra k[G] and the 2-cocycle f takes its values in the center of A, then

$$d_{rs}^{l}\left(a_{0}\otimes\mathbf{g}_{0s}\otimes\mathbf{a}_{1r}\otimes\mathbf{1}_{E}\right)=(-1)^{l(r+s)}a_{0}\otimes\mathbf{g}_{0,s-l}\otimes F_{0}^{(l)}(\mathbf{g}_{s-l+1,s})\ast\mathbf{a}_{1r}\otimes\mathbf{1}\#\mathfrak{g}_{s-l+1,s},$$

where * denotes the shuffle product:

$$\mathbf{a}_{1r} * \mathbf{b}_{1l} = \sum_{0 \le i_1 \le \dots \le i_r \le l} (-1)^{i_1 + \dots + i_l} b_1 \otimes \dots \otimes b_{i_1} \otimes a_1 \otimes b_{i_1+1} \otimes \dots \otimes b_{i_r} \otimes a_r \otimes b_{i_r+1} \otimes \dots$$

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