HOCHSCHILD (CO)HOMOLOGY OF HOPF CROSSED PRODUCTS

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Abstract. For a general crossed product $E = A\#_f H$, of an algebra A by a Hopf algebra H, we obtain complexes smaller than the canonical ones, giving the Hochschild homology and cohomology of E. These complexes are equipped with natural filtrations. The spectral sequences associated to them coincide with the ones obtained using a natural generalization of the of the direct method introduced in [H-S]. We also get that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex.

Introduction

Let G be a group, $S = \bigoplus S_g$ a strongly G-graded algebra and V an S-bimodule. In [L] was shown that there is a convergent spectral sequence

$$E^2_{rs} = \mathrm{H}_r(G, \mathrm{H}_s(S_e, V)) \Rightarrow \mathrm{H}_{r+s}(S, V),$$

where e denotes the identity of G. In [S] was shown that this result remains valid for H-Galois extensions (in his paper the author deals with both the homology and the cohomology of these algebras). An important particular type of H-Galois extensions are the crossed products with convolution invertible cocycle $E = A\#_f H$, of an algebra A by a Hopf algebra H (for the definition see Section 1). The purpose of our paper is to construct complexes smaller than the canonical ones, given the Hochschild (co)homology of E with coefficients in an arbitrary E-bimodule. These complexes are equipped with canonical filtrations. We show that the spectral sequences associated to them coincide with the ones obtained using a natural generalization of the direct method introduced in [H-S], and with the ones constructed in [S] (when these are specialize to crossed products). In the case of group extensions these results were proved in [E] and [B].

This paper is organized as follows: in Section 1 a resolution (X_*, d_*) of a crossed product $E = A\#_f H$ is given. To accomplish this construction we do not use the fact that the cocycle is convolution invertible. Moreover, we give a recursive construction of morphisms $\phi_*\colon (X_*, d_*) \to (\mathrm{B}_*(E), b'_*)$ and $\psi_*\colon (\mathrm{B}_*(E), b'_*) \to (X_*, d_*)$, where $(\mathrm{B}_*(E), b'_*)$ is the normalized Hochschild resolution of E, such that $\psi_*\,\phi_* = id$ and we show that $\phi_*\,\psi_*$ is homotopically equivalent to the identity map. Consequently

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our resolution is a direct sum of the normalized Hochschild resolution. We also recursively construct a homotopy $\phi_* \psi_* \xrightarrow{\omega_{*+1}} id_*$. Both the canonical normalized resolution and (X_*, d_*) are equipped with natural filtrations, which are preserved by the maps ϕ_*, ψ_* and ω_{*+1}.

In Section 2, for an E-bimodule M, we get complexes $(\widehat{X}_*, \widehat{d}_*)$ and $(\widehat{X}^*, \widehat{d}^*)$, giving the Hochschild homology and cohomology of E with coefficients in M, respectively. The filtration of (X_*, d_*) induces filtrations on $(\widehat{X}_*, \widehat{d}_*)$ and $(\widehat{X}^*, \widehat{d}^*)$. So, we obtain converging spectral sequences $E^1_{rs} = \mathrm{H}_r(A, M \otimes \overline{H}^{\otimes^s}) \Rightarrow \mathrm{H}_{r+s}(E, M)$ and $E_1^{rs} = \mathrm{H}^r(A, \mathrm{Hom}_k(\overline{H}^{\otimes^s}, M)) \Rightarrow \mathrm{H}^{r+s}(E, M)$. Using the results of Section 1, we get that these spectral sequences are the ones associated to suitable filtrations of the Hochschild normalized chain and cochain complexes $(M \otimes \overline{E}^{\otimes^*}, b_*)$ and $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*)$, respectively. This allows us to give very simple proofs of the main results of [H-S] and [G].

In Section 3, we show that, if the cocycle is convolution invertible, then the complexes $(\widehat{X}_*, \widehat{d}_*)$ and $(\widehat{X}^*, \widehat{d}^*)$ are isomorphic to simpler complexes $(\overline{X}_*, \overline{d}_*)$ and $(\overline{X}^*, \overline{d}^*)$, respectively. Then, we compute the term E^2_{rs} and E_2^{rs} of the spectral sequences obtained in Section 2. Moreover, using the above mentioned filtrations, we prove that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex.

In addition to the direct method developed in [H-S], there are another two classical methods to obtain spectral sequences converging to $\mathrm{H}_*(E, M)$ and with E^2-term $\mathrm{H}_*(H, \mathrm{H}_*(A, M))$. Namely the Cartan-Leray and the Grothendieck spectral sequences of a crossed product. In Section 4, we recall these constructions and we prove that these spectral sequences are isomorphic to the one obtained in Section 2. This generalizes the main results of [B]. Similar results are valid in the cohomological setting.

In a first appendix we give a method to construct (under suitable hypothesis) a projective resolution of the k-algebra E as $E^e = E \otimes E^{\mathrm{op}}$-bimodule, smaller than the canonical one of Hochschild. This method, which can be considered as a variant of the perturbation lemma, is used to prove the main result of Section 1. The boundary maps of the resolution (X_*, d_*) are recursively defined in Section 1. In a second appendix we give closed formulas for these maps.

Remark. Let A be an algebra and let H be a k-module provided of a base point $1_H \in H$. The results of Section 1 and 2 are valid for algebras $A \# H$, whose underlying k-module is $A \otimes H$ and whose multiplication map verifies $(a \otimes 1_H)(b \otimes h) = (ab \otimes h)$. We only must modify in an evident way the formulas in Theorem 1.1.3. These type of algebras were introduced in [Br].

Throughout this paper k denotes an arbitrary commutative base ring, all the algebras are over k, the unadorned tensor product is the tensor product over k and Υ denotes the family of all epimorphisms of E-bimodules which split as left E-module maps.

1. A resolution for a crossed product

Let A be an algebra and H a Hopf algebra. We will use the Sweedler notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$, with the summation understood and superindices instead of subindices. Recall some definitions of [B-C-M] and [D-T]. A *weak action* of H on A is a bilinear map $(h,a) \mapsto a^h$ from $H \times A$ to A such that, for $h \in H$, $a, b \in A$

1) $(ab)^h = a^{h^{(1)}} b^{h^{(2)}}$,
2) $1^h = \epsilon(h)1$,
3) $a^1 = a$.

Let A be an algebra and H a Hopf algebra with a weak action on A. Given a k-linear map $f\colon H \otimes H \to A$, let $A\#_f H$ be the algebra (in general non-associative and without 1) with underlying k-module $A \otimes H$ and multiplication map

$$(a \otimes h)(b \otimes l) = ab^{h^{(1)}} f(h^{(2)}, l^{(1)}) \otimes h^{(3)} l^{(2)},$$

for all $a, b \in A$, $h, l \in H$. The element $a \otimes h$ of $A\#_f H$ will usually be written $a\#h$ to remind us H is weakly acting on A. The algebra $A\#_f H$ is called a *crossed product* if it is associative with $1\#1$ as identity element. It is easy to check that this happens if and only if f and the weak action satisfy the following conditions:

i) (Normality of f) for all $h \in H$, we have $f(h,1) = f(1,h) = \epsilon(h)1_A$,
ii) (Cocycle condition) for all $h, l, m \in H$, we have

$$f\big(l^{(1)}, m^{(1)}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)} m^{(2)}\big) = f\big(h^{(1)}, l^{(1)}\big) f\big(h^{(2)} l^{(2)}, m\big),$$

iii) (Twisted module condition) for all $h, l \in H$, $a \in A$ we have

$$\big(a^{l^{(1)}}\big)^{h^{(1)}} f\big(h^{(2)}, l^{(2)}\big) = f\big(h^{(1)}, l^{(1)}\big) a^{h^{(2)} l^{(2)}}.$$

In this section we obtain a resolution (X_*, d_*) of a crossed product $E = A\#_f H$ as an E-bimodule, which is simpler than the canonical one of Hochschild. To begin, we fix some notations:

1) For each algebra B, we put $\overline{B} = B/k$. Moreover, given $b \in B$ we also let b denote the class of b in $\overline{B}$.
2) For each k-module V we write $V^{\otimes^l} = V \otimes \cdots \otimes V$ (l times), and for each algebra B we write $\mathrm{B}_l(B) = B \otimes \overline{B}^{\otimes^l} \otimes B$.
3) Given $a_0 \otimes \cdots \otimes a_r \in A^{\otimes^{r+1}}$ and $0 \le i < j \le r$, we write $\mathbf{a}_{ij} = a_i \otimes \cdots \otimes a_j$.
4) Given $h_0 \otimes \cdots \otimes h_s \in H^{\otimes^{s+1}}$ and $0 \le i < j \le s$, we write $\mathbf{h}_{ij} = h_i \otimes \cdots \otimes h_j$.

1.1. The resolution (X_*, d_*)

Let $Y_s = E \otimes_A (E/A)^{(\otimes_A)^s} \otimes_A E$ and $X_{rs} = E \otimes_A (E/A)^{(\otimes_A)^s} \otimes \overline{A}^{\otimes^r} \otimes E$, for all $r, s \ge 0$. Consider the diagram of E-bimodules and E-bimodule maps

$$\begin{array}{ccccccc}
\vdots & & & & & & \\
\big\downarrow{\scriptstyle \partial_2} & & & & & & \\
Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \dots \\
\big\downarrow{\scriptstyle \partial_1} & & & & & & \\
Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \dots,
\end{array}$$

where (Y_*, ∂_*) is the normalized Hochschild resolution of the algebra inclusion $A \subseteq E$, in the relative sense, introduced in [G-S]; for each $s \ge 0$, the complex (X_{*s}, d^0_{*s}) is the normalized bar resolution of A, tensored on the left over A with $E \otimes_A (E/A)^{(\otimes_A)^s}$, and on the right over A with E; and for each $s \ge 0$, the map μ_s is the canonical projection.

Note that $X_{rs} \simeq E \otimes \overline{H}^{\otimes^s} \otimes \overline{A}^{\otimes^r} \otimes E$. Moreover, each one of the rows of this diagram is contractible as a left E-module complex. A contracting homotopy $\sigma^0_{0s}\colon Y_s \to X_{0s}$ and $\sigma^0_{r+1,s}\colon X_{rs} \to X_{r+1,s}$ of the s-th row, is given by

$$\begin{aligned} &\sigma^0_{0s}(\mathbf{x} \otimes_A a\#h) = \mathbf{x}_{0,s-1} \otimes_A x_s a \otimes 1\#h, \\ &\sigma^0_{r+1,s}(\mathbf{x} \otimes \mathbf{a} \otimes a\#h) = (-1)^{r+1} \mathbf{x} \otimes \mathbf{a} \otimes a \otimes 1\#h, \end{aligned}$$

where $\mathbf{x} = x_0 \otimes_A \cdots \otimes_A x_s \in E \otimes_A (E/A)^{(\otimes_A)^s}$ and $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^{\otimes^r}$. So, we are in the situation considered in Appendix A. We define E-bimodule maps $d^l_{rs}\colon X_{rs} \to X_{r+l-1,s-l}$ ($r \ge 0$ and $1 \le l \le s$) recursively, by:

$$d^l_{rs}(\mathbf{x}) = \begin{cases} -\sigma^0_{0,s-1}\, \partial_s\, \mu_s(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1, \\ -\sum_{j=1}^{l-1} \sigma^0_{l-1,s-l}\, d^{l-j}_{j-1,s-j}\, d^j_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \le s, \\ -\sum_{j=0}^{l-1} \sigma^0_{r+l-1,s-l}\, d^{l-j}_{r+j-1,s-j}\, d^j_{rs}(\mathbf{x}) & \text{if } r > 0, \end{cases}$$

for $\mathbf{x} \in A \otimes_A (E/A)^{(\otimes_A)^s} \otimes \overline{A}^{\otimes^r} \otimes k$.

Theorem 1.1.1. *Let $\mu\colon X_{00} \to E$ be the multiplication map. There is a Υ-projective resolution of E*

$$E \xleftarrow{-\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} \dots, \tag{1}$$

where $X_n = \bigoplus_{r+s=n} X_{rs}$ *and* $d_n = \sum_{l=1}^{n} d^l_{0n} + \sum_{r=1}^{n}\sum_{l=0}^{n-r} d^l_{r,n-r}$.

Proof. Let $\widetilde{\mu}\colon Y_0 \to E$ be multiplication map. The complex of E-bimodules

$$E \xleftarrow{\widetilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_5 \xleftarrow{\partial_6} Y_6 \xleftarrow{\partial_7} \dots$$

is contractible as a complex of left E-modules. A chain contracting homotopy $\sigma^{-1}_0\colon E \to Y_0$ and $\sigma^{-1}_{s+1}\colon Y_s \to Y_{s+1}$, is given by $\sigma^{-1}_{s+1}(\mathbf{x}) = (-1)^s \mathbf{x} \otimes 1_E$. Hence, the theorem follows from Corollary A.2 of Appendix A. □

Remark 1.1.2. Let $\sigma^l_{l,s-l}\colon Y_s \to X_{l,s-l}$ and $\sigma^l_{r+l+1,s-l}\colon X_{rs} \to X_{r+l+1,s-l}$ be the maps recursively defined by

$$\sigma^l_{r+l+1,s-l} = -\sum_{i=0}^{l-1} \sigma^0_{r+l+1,s-l}\, d^{l-i}_{r+i+1,s-i}\, \sigma^i_{r+i+1,s-i} \qquad (0 < l \le s \text{ and } r \ge -1).$$

We will prove, in Corollary A.2, that the family $\overline{\sigma}_0\colon E \to X_0$, $\overline{\sigma}_{n+1}\colon X_n \to X_{n+1}$, defined by $\overline{\sigma}_0 = \sigma^0_{00}\, \sigma^{-1}_0$ and

$$\overline{\sigma}_{n+1} = -\sum_{l=0}^{n+1} \sigma^l_{l,n-l+1}\, \sigma^{-1}_{n+1}\, \mu_n + \sum_{r=0}^{n}\sum_{l=0}^{n-r} \sigma^l_{r+l+1,n-r-l} \qquad (n \ge 0),$$

is a contracting homotopy of the resolution (1) introduced in Theorem 1.1.1.

Theorem 1.1.3. *The following assertions hold:*

1) *Let* $\mathbf{x} = x_0 \otimes_A \cdots \otimes_A x_s \in E \otimes_A (E/A)^{(\otimes_A)^s}$ *with* $x_s = a_s \# h_s$, *and* $\mathbf{b} \in \overline{A}^{\otimes^r}$. *We have*

$$d^1_{rs}(\mathbf{x} \otimes \mathbf{b} \otimes 1_E) = \sum_{i=0}^{s-1} (-1)^{i+r} \mathbf{x}_{0,i-1} \otimes_A x_i x_{i+1} \otimes_A \mathbf{x}_{i+2,s} \otimes \mathbf{b} \otimes 1_E$$
$$+ (-1)^{r+s} \mathbf{x}_{0,s-2} \otimes_A x_{s-1} a_s \otimes \mathbf{b}^{h_s^{(1)}} \otimes 1 \# h_s^{(2)}.$$

2) *For each* $l \geq 2$, *the image of* d^l_{rs} *is contained in the* k*-submodule of* $X_{r+l-1,s-l}$ *generated by all the elementary tensors* $x_0 \otimes_A \cdots \otimes_A x_{s-l} \otimes a_1 \otimes \cdots \otimes a_{r+l-1}$ *with* $l-1$ *of the* a_i*'s in the image of the cocycle* f.

Proof. The computation of d^1_{rs} can be obtained easily by induction on r, using that $d^1_{0s} = -\sigma^0_{0,s-1}\, \partial_s\, \mu^0_s$ and $d^1_{rs} = -\sigma^0_{r,s-1}\, d^1_{r-1,s}\, d^0_{rs}$, for $r \geq 1$. The assertion for d^l_{rs}, with $l \geq 2$, follows by induction on l and r, using the recursive definition of d^l_{rs}. □

In Appendix B we will give precise formulas for the maps d^l_{rs}'s.

1.2. Comparison with the canonical resolution

Let $(\mathrm{B}_*(E), b'_*)$ be the normalized Hochschild resolution of E. As is well known, the complex

$$E \xleftarrow{\mu} \mathrm{B}_0(E) \xleftarrow{b'_1} \mathrm{B}_1(E) \xleftarrow{b'_2} \mathrm{B}_2(E) \xleftarrow{b'_3} \mathrm{B}_3(E) \xleftarrow{b'_4} \dots$$

is contractible as a complex of left E-modules, with contracting homotopy $\xi_n(\mathbf{x}) = (-1)^n \mathbf{x} \otimes 1$. Let $\overline{\sigma}_*$ be the contracting homotopy of (1) introduced in Remark 1.1.2. Let $\phi_*\colon (X_*, d_*) \to (\mathrm{B}_*(E), b'_*)$ and $\psi_*\colon (\mathrm{B}_*(E), b'_*) \to (X_*, d_*)$ be the morphisms of E-bimodule complexes, recursively defined by $\phi_0 = id$, $\psi_0 = id$, $\phi_{n+1}(\mathbf{x} \otimes 1) = \xi_{n+1}\, \phi_n\, d_{n+1}(\mathbf{x} \otimes 1)$ and $\psi_{n+1}(\mathbf{y} \otimes 1) = \overline{\sigma}_{n+1}\, \psi_n\, b'_{n+1}(\mathbf{y} \otimes 1)$.

Proposition 1.2.1. $\psi_*\, \phi_* = id_*$ *and* $\phi_*\, \psi_*$ *is homotopically equivalent to the identity map. A homotopy* $\phi_*\, \psi_* \xrightarrow{\omega_{*+1}} id_*$ *is recursively defined by* $\omega_1 = 0$ *and* $\omega_{n+1}(\mathbf{x}) = \xi_{n+1}\, (\phi_n\, \psi_n - id - \omega_n\, b'_n)(\mathbf{x})$, *for* $\mathbf{x} \in E \otimes \overline{E}^{\otimes^n} \otimes k$.

Proof. We prove both assertions by induction. Let $U_n = \phi_n\, \psi_n - id_n$ and $T_n = U_n - \omega_n\, b'_n$. Assuming that $b'_n\, \omega_n + \omega_{n-1}\, b'_{n-1} = U_{n-1}$, we get that, on $E \otimes \overline{E}^{\otimes^n} \otimes k$,

$$\begin{aligned} b'_{n+1}\, \omega_{n+1} + \omega_n\, b'_n &= b'_{n+1}\, \xi_{n+1}\, T_n + \omega_n\, b'_n \\ &= T_n - \xi_n\, b'_n\, T_n + \omega_n\, b'_n \\ &= U_n - \xi_n\, U_{n-1}\, b'_n + \xi_n\, b'_n\, \omega_n\, b'_n \\ &= U_n - \xi_n\, U_{n-1}\, b'_n + \xi_n\, T_{n-1}\, b'_n = U_n. \end{aligned}$$

Hence, $b'_{n+1}\, \omega_{n+1} + \omega_n\, b'_n = U_n$ on $\mathrm{B}_n(E)$. Next, we prove that $\psi_*\, \phi_* = id_*$. It is clear that $\psi_0\, \phi_0 = id_0$. Assume that $\psi_n\, \phi_n = id_n$. Since

$$\phi_{n+1}(E \otimes_A (E/A)^{(\otimes_A)^s} \otimes \overline{A}^{\otimes^{n+1-s}} \otimes k) \subseteq E \otimes \overline{E}^{\otimes^{n+1}} \otimes k,$$

we have that, on $A\otimes_A (E/A)^{(\otimes_A)^s}\otimes \overline{A}^{\otimes^{n+1-s}}\otimes k$,

$$\begin{aligned}\psi_{n+1}\,\phi_{n+1} &= \overline{\sigma}_{n+1}\,\psi_n\, b'_{n+1}\,\phi_{n+1}\\ &= \overline{\sigma}_{n+1}\,\psi_n\, b'_{n+1}\,\xi_{n+1}\,\phi_n\, d_{n+1}\\ &= \overline{\sigma}_{n+1}\,\psi_n\,\phi_n\, d_{n+1} - \overline{\sigma}_{n+1}\,\psi_n\,\xi_n\, b'_n\,\phi_n\, d_{n+1}\\ &= \overline{\sigma}_{n+1}\, d_{n+1} = id_{n+2} - d_{n+2}\,\overline{\sigma}_{n+2}.\end{aligned}$$

So, to end the proof it suffices to see that $\overline{\sigma}_{n+2}(A\otimes_A (E/A)^{(\otimes_A)^s}\otimes \overline{A}^{\otimes^{n+1-s}}\otimes k) = 0$, which follows easily from the definition of $\overline{\sigma}_*$. □

Let $F^i(X_n) = \bigoplus_{0\le s\le i} E\otimes_A (E/A)^{(\otimes_A)^s}\otimes \overline{A}^{\otimes^{n-s}}\otimes k$ and let $F^i(\mathrm{B}_n(E))$ be the subbimodule of $\mathrm{B}_n(E)$ generated by the tensors $1\otimes x_1\otimes\cdots\otimes x_n\otimes 1$ such that at least $n-i$ of the x_j's belong to A. The normalized Hochschild resolution $(\mathrm{B}_*(E), b'_*)$ and the resolution (X_*, d_*) are filtered by $F^0(\mathrm{B}_*(E))\subseteq F^1(\mathrm{B}_*(E))\subseteq F^2(\mathrm{B}_*(E))\subseteq\dots$ and $F^0(X_*)\subseteq F^1(X_*)\subseteq F^2(X_*)\subseteq\dots$, respectively

Proposition 1.2.2. *The maps ϕ_*, ψ_* and ω_{*+1} preserve filtrations.*

Proof. Let $Q^i_j = E\otimes_A (E/A)^{(\otimes_A)^i}\otimes \overline{A}^{\otimes^{n-j}}\otimes k$. We claim that

a) $\overline{\sigma}_{n+1}(F^i(X_n))\subseteq F^i(X_{n+1})$ for all $0\le i<n$,

b) $\overline{\sigma}_{n+1}(E\otimes_A (E/A)^{(\otimes_A)^i}\otimes \overline{A}^{\otimes^{n-i}}\otimes A)\subseteq Q^i_{i-1} + F^{i-1}(X_{n+1})$ for all $0\le i\le n$,

c) $\overline{\sigma}_{n+1}(E\otimes_A (E/A)^{(\otimes_A)^n}\otimes E)\subseteq E\otimes_A (E/A)^{(\otimes_A)^{n+1}}\otimes k + F^n(X_{n+1})$,

d) $\psi_n(F^i(\mathrm{B}_n(E))\cap E\otimes \overline{E}^{\otimes^n}\otimes k)\subseteq Q^i_i + F^{i-1}(X_n)$ for all $0\le i\le n$.

In fact a), b) and c) follow immediately from the definition of $\overline{\sigma}_{n+1}$. Suppose d) is valid for n. Let $\mathbf{x} = x_0\otimes\cdots\otimes x_{n+1}\otimes 1\in F^i(\mathrm{B}_{n+1}(E))\cap E\otimes \overline{E}^{\otimes^{n+1}}\otimes k$, with $0\le i\le n+1$. Using a) and b), we get that for $1\le j\le n$,

$$\begin{aligned}\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,j-1}\otimes x_jx_{j+1}\otimes \mathbf{x}_{j+2,n+1}\otimes 1)) &\subseteq \overline{\sigma}_{n+1}(Q^i_i + F^{i-1}(X_n))\\ &\subseteq Q^i_{i-1} + F^{i-1}(X_{n+1}).\end{aligned}$$

Since $\psi_{n+1}(\mathbf{x}) = \overline{\sigma}_{n+1}\,\psi_n\, b'_{n+1}(\mathbf{x})$, to prove d) for $n+1$ we only must check that $\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1}))\subseteq Q^i_{i-1} + F^{i-1}(X_{n+1})$. If $x_{n+1}\in A$, then using a), b) and the inductive hypothesis, we get

$$\begin{aligned}\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1})) &= \overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0n}\otimes 1_E)x_{n+1})\\ &\subseteq \overline{\sigma}_{n+1}(E\otimes_A (E/A)^{(\otimes_A)^i}\otimes \overline{A}^{\otimes^{n-i}}\otimes A + F^{i-1}(X_n))\\ &\subseteq Q^i_{i-1} + F^{i-1}(X_{n+1}),\end{aligned}$$

and if $x_{n+1}\notin A$, then $\mathbf{x}_{0,n+1}\in F^{i-1}(\mathrm{B}_n(E))$, which together with a), c) and the inductive hypothesis, implies that

$$\overline{\sigma}_{n+1}(\psi_n(\mathbf{x}_{0,n+1}))\subseteq \overline{\sigma}_{n+1}(F^{i-1}(X_n))\subseteq Q^i_{i-1} + F^{i-1}(X_{n+1}).$$

From d) it follows immediately that ψ_* preserves filtrations. Next, assuming that ϕ_n preserves filtrations, we prove that ϕ_{n+1} does. Let $\mathbf{y} = \mathbf{x}\otimes \mathbf{a}\otimes 1$, where

$\mathbf{x} = x_0 \otimes_A \cdots \otimes_A x_i \in E \otimes_A (E/A)^{(\otimes_A)^i}$ and $\mathbf{a} \otimes 1 = a_1 \otimes \cdots \otimes a_{n+1-i} \otimes 1 \in \overline{A}^{\otimes^{n+1-i}} \otimes k$. Since $\phi_{n+1}(\mathbf{y}) = \xi_{n+1}\, \phi_n\, d_n(\mathbf{y})$ and

$$\begin{aligned} \xi_{n+1}(\phi_n(d^l_{rs}(\mathbf{y}))) &\subseteq \xi_{n+1}(\phi_n(F^{i-l}(X_n))) \\ &\subseteq \xi_{n+1}(F^{i-l}(\mathrm{B}_n(E))) \subseteq F^{i-l+1}(\mathrm{B}_{n+1}(E)), \end{aligned}$$

it suffices to see that $\xi_{n+1}(\phi_n(d^0_{rs}(\mathbf{y}))) \subseteq F^i(\mathrm{B}_{n+1}(E))$. But

$$\begin{aligned} \xi_{n+1}\, \phi_n\, d^0_{rs}(\mathbf{y}) &= (-1)^r \xi_{n+1}\, \phi_n(\mathbf{x} \otimes \mathbf{a}) \\ &= (-1)^r \xi_{n+1}(\phi_n(\mathbf{x}_{0i} \otimes \mathbf{a}_{1,n-i} \otimes 1) a_{n+1-i}) \\ &\subseteq \xi_{n+1}(F^i(\mathrm{B}_n(E)) \cap E \otimes \overline{E}^{\otimes^n} \otimes A) \\ &\subseteq F^i(\mathrm{B}_{n+1}(E)), \end{aligned}$$

since $\phi_n(Q^i_i) \subseteq E \otimes \overline{E}^{\otimes^n} \otimes k$. Next, we prove that ω_* preserves filtrations. Assume that ω_n does. Let $\mathbf{x} = x_0 \otimes \cdots \otimes x_n \otimes 1 \in F^i(\mathrm{B}_n(E)) \cap E \otimes \overline{E}^{\otimes^n} \otimes k$. It is evident that $\omega_{n+1}(\mathbf{x}) = \xi_{n+1}\, \phi_n\, \psi_n(\mathbf{x}) - \xi_{n+1}\, \omega_n\, b'_n(\mathbf{x})$. From d) we get

$$\xi_{n+1}\, \phi_n\, \psi_n(\mathbf{x}) \in \xi_{n+1}\, \phi_n\big(Q^i_i + F^{i-1}(X_n)\big) \subseteq \xi_{n+1}(F^{i-1}(\mathrm{B}_n(E))) \subseteq F^i(\mathrm{B}_n(E)),$$

since $\xi_{n+1}(\phi_n(Q^i_i)) \subseteq \xi_{n+1}(E \otimes \overline{E}^{\otimes^n} \otimes k) = 0$. To finish, it remains to check that $\xi_{n+1}\, \omega_n\, b'_n(\mathbf{x}) \subseteq F^i(\mathrm{B}_n(E))$. Since $\omega_n(E \otimes \overline{E}^{\otimes^{n-1}} \otimes k) \subseteq E \otimes \overline{E}^{\otimes^n} \otimes k$, we have $\xi_{n+1}\, \omega_n\, b'_n(\mathbf{x}) = (-1)^{n-1} \xi_{n+1}\, \omega_n(\mathbf{x}_{0n})$. Hence, if $x_n \in A$, then

$$\begin{aligned} \xi_{n+1}\, \omega_n\, b'_n(\mathbf{x}) &= (-1)^{n-1} \xi_{n+1}(\omega_n(\mathbf{x}_{0,n-1} \otimes 1) x_n) \\ &\subseteq \xi_{n+1}(F^i(\mathrm{B}_n(E)) \cap E \otimes \overline{E}^{\otimes^n} \otimes A) \\ &\subseteq F^i(\mathrm{B}_{n+1}(E)), \end{aligned}$$

and if $x_n \notin A$, then $\mathbf{x}_{0n} \in F^{i-1}(\mathrm{B}_{n-1}(E))$, and so

$$\xi_{n+1}\, \omega_n\, b'_n(\mathbf{x}) = (-1)^{n-1} \xi_{n+1}\, \omega_n(\mathbf{x}_{0n}) \subseteq \xi_{n+1}(F^{i-1}(\mathrm{B}_n(E))) \subseteq F^i(\mathrm{B}_{n+1}\, E). \quad \square$$

2. The Hochschild (co)homology of a crossed product

Let $E = A\#_f H$ and let M be an E-bimodule. In this section we use Theorem 1.1.1 in order to construct complexes, smaller than the canonical ones, giving the Hochschild homology and cohomology of A with coefficients in M, respectively. These complexes have natural filtrations that allow us to obtain spectral sequences converging to $\mathrm{H}_*(E, M)$ and $\mathrm{H}^*(E, M)$, respectively. We compare these spectral sequences with the ones obtained from a generalization of the Hochschild-Serre direct method.

Theorem 2.1. *The Hochschild homology of E with coefficients in M is the homology of $M \otimes_{E^e} (X_*, d_*)$.*

Proof. It is immediate. $\square$

Let $\widehat{X}_{rs} = M \otimes_A (E/A)^{(\otimes_A)^s} \otimes \overline{A}^{\otimes^r}$. It is well known that $\widehat{X}_{rs} \simeq M \otimes_{E^e} X_{rs}$. Let $\widehat{d}^l_{rs}\colon \widehat{X}_{rs} \to \widehat{X}_{r+l-1,s-l}$ be the map induced by $id_M \otimes_{E^e} d^l_{rs}$. It is easy to see that $\widehat{d}^0_{rs}$ is the boundary map of the normalized chain Hochschild complex of A with coefficients in $M \otimes_A (E/A)^{(\otimes_A)^s}$. With the above identifications the complex $M \otimes_{E^e} (X_*, d_*)$ becomes $(\widehat{X}_*, \widehat{d}_*)$, where

$$\widehat{X}_n = \bigoplus_{r+s=n} \widehat{X}_{rs} \qquad \text{and} \qquad \widehat{d}_n := \sum_{l=1}^{n} \widehat{d}^l_{0n} + \sum_{r=1}^{n}\sum_{l=0}^{n-r} \widehat{d}^l_{r,n-r}.$$

Let $F^i(\widehat{X}_n) = \bigoplus_{0\le s\le i} \widehat{X}_{n-s,s}$. Clearly $F^0(\widehat{X}_*) \subseteq F^1(\widehat{X}_*) \subseteq \dots$ is a filtration of $(\widehat{X}_*, \widehat{d}_*)$. Using this fact we obtain:

Corollary 2.2. *There is a convergent spectral sequence*

$$E^1_{rs} = \mathrm{H}_r(A, M \otimes_A (E/A)^{(\otimes_A)^s}) \Rightarrow \mathrm{H}_{r+s}(E, M).$$

The normalized Hochschild complex $(M \otimes \overline{E}^{\otimes^*}, b_*)$ is filtered by

$$F^0(M \otimes \overline{E}^{\otimes^*}) \subseteq F^1(M \otimes \overline{E}^{\otimes^*}) \subseteq F^2(M \otimes \overline{E}^{\otimes^*}) \subseteq \dots,$$

where $F^i(M \otimes \overline{E}^{\otimes^n})$ is the k-submodule of $M \otimes \overline{E}^{\otimes^n}$ generated by the tensors $m \otimes x_1 \otimes \dots \otimes x_n$ such that at least $n-i$ of the x_j's belong to A. The spectral sequence associated to this filtration is called the homological Hochschild-Serre spectral sequence. Since, for each extension of groups $N \subseteq G$, with N a normal subgroup, $k[G]$ is a crossed product of $k[G/N]$ on $k[N]$, the following theorem (joint with Corollary 3.1.3 below) gives, as a particular case, the homological version of the main results of [H-S].

Theorem 2.3. *The homological Hochschild-Serre spectral sequence is isomorphic to the one obtained in Corollary 2.2.*

Proof. It is immediate that the filtrations of $(\widehat{X}_*, \widehat{d}_*)$ and $(M \otimes \overline{E}^{\otimes^*}, b_*)$ are the ones induced by the filtrations of (X_*, d_*) and $(\mathrm{B}_*(E), b'_*)$, introduced above Proposition 1.1.2, respectively. Hence, the result follows from Propositions 1.2.1 and 1.2.2. □

Theorem 2.4. *The Hochschild cohomology of E with coefficients in M is the cohomology of* $\mathrm{Hom}_{E^e}((X_*, d_*), M)$.

Proof. It is immediate. □

Let $\widehat{X}^{rs} = \mathrm{Hom}_A((E/A)^{(\otimes_A)^s} \otimes \overline{A}^{\otimes^r}, M) \simeq \mathrm{Hom}_k(\overline{A}^{\otimes^r}, \mathrm{Hom}_A((E/A)^{(\otimes_A)^s}, M))$. Clearly $\widehat{X}^{rs} \simeq \mathrm{Hom}_{E^e}(X_{rs}, M)$. Let $\widehat{d}^{rs}_l\colon \widehat{X}^{r+l-1,s-l} \to \widehat{X}^{rs}$ be the map induced by $\mathrm{Hom}_{E^e}(d^l_{rs}, M)$. It is easy to see that $\widehat{d}^{rs}_0$ is the boundary map of the normalized cochain Hochschild complex of A with coefficients in $\mathrm{Hom}_A((E/A)^{(\otimes_A)^s}, M)$. With the above identifications $\mathrm{Hom}_{E^e}((X_*, d_*), M)$ becomes $(\widehat{X}^*, \widehat{d}^*)$, where

$$\widehat{X}^n = \bigoplus_{r+s=n} \widehat{X}^{rs} \qquad \text{and} \qquad \widehat{d}^n := \sum_{l=1}^{n} \widehat{d}^{0n}_l + \sum_{r=1}^{n}\sum_{l=0}^{n-r} \widehat{d}^{r,n-r}_l.$$

Let $F_i(\widehat{X}^n) = \bigoplus_{s\ge i} \widehat{X}^{n-s,s}$. Clearly $F_0(\widehat{X}^*) \supseteq F_1(\widehat{X}^*) \supseteq F_2(\widehat{X}^*) \supseteq \dots$ is a filtration of $(\widehat{X}^*, \widehat{d}^*)$. Using this fact we obtain:

Corollary 2.5. *There is a convergent spectral sequence*

$$E_1^{rs} = \mathrm{H}^r(A, \mathrm{Hom}_A((E/A)^{(\otimes_A)^s}, M)) \Rightarrow \mathrm{H}^{r+s}(E, M).$$

Let $F_i(\mathrm{Hom}_k(\overline{E}^{\otimes^n}, M))$ be the k-submodule of $\mathrm{Hom}_k(\overline{E}^{\otimes^n}, M)$ consisting of maps $\varphi \in \mathrm{Hom}_k(\overline{E}^{\otimes^n}, M)$, for which $\varphi(x_1 \otimes \cdots \otimes x_n) = 0$ whenever $n - i$ of the x_j's belong to A. The normalized Hochschild complex $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*)$ is filtered by

$$F_0(\mathrm{Hom}_k(\overline{E}^*, M)) \supseteq F_1(\mathrm{Hom}_k(\overline{E}^*, M)) \supseteq F_2(\mathrm{Hom}_k(\overline{E}^*, M)) \supseteq \dots.$$

The spectral sequence associated to this filtration is called the cohomological Hochschild-Serre spectral sequence. The following theorem, joint with Corollary 3.2.3 below, gives, as a particular case, the main results of [H-S].

Theorem 2.6. *The cohomological Hochschild-Serre spectral sequence is isomorphic to the one obtained in Corollary 2.5.*

Proof. It is immediate that the filtrations of $(\widehat{X}^*, \widehat{d}^*)$ and $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*)$ are the ones induced by the filtrations of (X_*, d_*) and $(\mathrm{B}_*(E), b'_*)$, introduced above Proposition 1.1.2, respectively. Hence, the result follows from Propositions 1.2.1 and 1.2.2. □

3. The Hochschild (co)homology of a crossed product with invertible cocycle

Let $E = A\#_f H$ and let M be an E-bimodule. Assume that the cocycle f is invertible. Then, the map $h \mapsto 1\#h$ is convolution invertible and its inverse is the map $h \mapsto (1\#h)^{-1} = f^{-1}(S(h^{(2)}), h^{(3)})\#S(h^{(1)})$. Under this hypothesis, we prove that the complexes $\widehat{X}_*(E,M)$ and $\widehat{X}^*(E,M)$ of Section 2 are isomorphic to simpler complexes. We use these new complexes in order to compute the term E^2 of the spectral sequences obtained in Section 2. Moreover, using a theorem of Gerstenhaber and Schack, we prove that if the 2-cocycle f takes its values in a separable subalgebra of A, then the Hochschild (co)homology of E with coefficients in M is the (co)homology of H with coefficients in a (co)chain complex.

3.1. Hochschild homology

Let $\overline{X}_{rs} = M \otimes \overline{A}^{\otimes^r} \otimes \overline{H}^{\otimes^s}$. The map $\theta_{rs} \colon \widehat{X}_{rs} \to \overline{X}_{rs}$, defined by

$$\theta_{rs}(m\otimes_A(1\#h_1)\otimes_A\cdots\otimes_A(1\#h_s)\otimes\mathbf{a}) = m(1\#h_1^{(1)})\cdots(1\#h_s^{(1)})\otimes\mathbf{a}\otimes h_1^{(2)}\otimes\cdots\otimes h_s^{(2)},$$

where $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$, is an isomorphism. The inverse map of θ_{rs} is the map

$$m\otimes\mathbf{a}\otimes h_1\otimes\cdots\otimes h_s \mapsto m(1\#h_s^{(1)})^{-1}\cdots(1\#h_1^{(1)})^{-1}\otimes_A(1\#h_1^{(2)})\otimes_A\cdots\otimes_A(1\#h_s^{(2)})\otimes\mathbf{a}.$$

Let $\overline{d}^l_{rs} \colon \overline{X}_{rs} \to \overline{X}_{r+l-1,s-l}$ be the map $\overline{d}^l_{rs} := \theta_{r+l-1,s-l}\, \widehat{d}^l_{rs}\, \theta^{-1}_{rs}$.

Theorem 3.1.1. *The Hochschild homology of E with coefficients in M is the homology of $(\overline{X}_*, \overline{d}_*)$, where*

$$\overline{X}_n = \bigoplus_{r+s=n} \overline{X}_{rs} \qquad \textit{and} \qquad \overline{d}_n := \sum_{l=1}^{n} \overline{d}^l_{0n} + \sum_{r=1}^{n}\sum_{l=0}^{n-r} \overline{d}^l_{r,n-r}.$$

Proof. This is an immediate consequence of Theorem 2.1 and the fact that the map $\theta_*\colon (\widehat{X}_*, \widehat{d}_*) \to (\overline{X}_*, \overline{d}_*)$, given by $\theta_n = \sum_{r+s=n} \theta_{rs}$, is an isomorphism. □

Note that when f takes its values in k, then $(\overline{X}_*, \overline{d}_*)$ is the total complex of the double complex $\bigl(M \otimes \overline{A}^{\otimes^*} \otimes \overline{H}^{\otimes^*}, \overline{d}^0_{**}, \overline{d}^1_{**}\bigr)$.

It is easy to see that $\overline{d}^0_{rs}$ is the the boundary map of the normalized chain Hochschild complex of A with coefficients in M, tensored on the right over k with $id_{\overline{H}^{\otimes s}}$, and

$$\begin{aligned}\overline{d}^1_{rs}(\mathbf{x}) &= (-1)^{r+s}(1\#h_s^{(3)})m(1\#h_s^{(1)})^{-1} \otimes \mathbf{a}^{h_s^{(2)}} \otimes \mathbf{h}_{1,s-1}\\ &+ \sum_{i=1}^{s-1}(-1)^{r+i} m \otimes \mathbf{a} \otimes \mathbf{h}_{1,i-1} \otimes h_i h_{i+1} \otimes \mathbf{h}_{i+2,s} + (-1)^r m\epsilon(h_1) \otimes \mathbf{a} \otimes \mathbf{h}_{2s},\end{aligned}$$

where $\mathbf{x} = m \otimes \mathbf{a} \otimes \mathbf{h}$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s$.

For each $h \in H$, we have the morphism $\vartheta^h_*\colon (M \otimes \overline{A}^{\otimes^*}, b_*) \to (M \otimes \overline{A}^{\otimes^*}, b_*)$, defined by $\vartheta^h_r(m \otimes \mathbf{a}) = (1\#h_s^{(3)})m(1\#h_s^{(1)})^{-1} \otimes \mathbf{a}^{h_s^{(2)}}$.

Proposition 3.1.2. *For each $h, l \in H$, the endomorphisms of* $\mathrm{H}_*(A, M)$ *induced by $\vartheta^h_* \vartheta^l_*$ and by ϑ^{hl}_* coincide. Consequently* $\mathrm{H}_*(A, M)$ *is a left H-module.*

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}_0(A, M)$, and in this case the result is immediate. □

Let $F^0(\overline{X}_*) \subseteq F^1(\overline{X}_*) \subseteq \dots$ be the filtration of $(\overline{X}_*, \overline{d}_*)$ obtained transporting the one of $(\widehat{X}_*, \widehat{d}_*)$, given above Corollary 2.2, through θ_*. It is immediate that $F^i(\overline{X}_n) = \bigoplus_{0\le s\le i} \overline{X}_{n-s,s}$. Then, we have the following:

Corollary 3.1.3. *if H is flat over k, then the spectral sequence of Corollary 2.2 satisfies $E^1_{rs} = \mathrm{H}_r(A, M) \otimes \overline{H}^{\otimes^s}$ and $E^2_{rs} = \mathrm{H}_s(H, \mathrm{H}_r(A, M))$.*

Let S be a separable subalgebra of A. Next we prove that if the 2-cocycle f takes its values in S, then the Hochschild homology of E with coefficients in M is the homology of H with coefficients in a chain complex. Assume that $f(h, l) \in S$ for all $h, l \in H$. Let $\widetilde{A} = A/S$, $\widetilde{A}^{(\otimes_S)^0} = S$ and $\widetilde{A}^{(\otimes_S)^r} = \widetilde{A} \otimes_S \cdots \otimes_S \widetilde{A}$ (r-times) for $r > 0$, and let $M \otimes_S \widetilde{A}^{(\otimes_S)^r} \otimes_S = M \otimes_{A^e} (A \otimes_s \widetilde{A}^{(\otimes_S)^r} \otimes_s A) = M \otimes_S \widetilde{A}^{(\otimes_S)^r} \otimes_{S^e} S$ be the cyclic tensor product over S of M and $\widetilde{A}^{(\otimes_S)^r}$ (see [G-S] or [Q]). Using the fact that f takes its values in S, it is easy to see that H acts on $(M \otimes_S \widetilde{A}^{(\otimes_S)^*} \otimes_S, b_*)$ via

$$h \cdot \bigl(m \otimes_S \widetilde{\mathbf{a}}\bigr) = (1\#h^{(3)})m(1\#h^{(1)})^{-1} \otimes_S \widetilde{\mathbf{a}}^{h^{(2)}},$$

where $m \otimes_S \widetilde{\mathbf{a}} = m \otimes_S a_1 \otimes_S \cdots \otimes_S a_r \otimes_S$ and $\widetilde{\mathbf{a}}^{h^{(2)}} = a_1^{h^{(2)}} \otimes_S \cdots \otimes_S a_r^{h^{(r+1)}} \otimes_S$.

Theorem 3.1.4. *The Hochschild homology* $\mathrm{H}_*(E,M)$*, of* E *with coefficients in* M*, is the homology of* H *with coefficients in* $(M\otimes_S \widetilde{A}^{(\otimes_S)^*}\otimes_S, b_*)$.

Proof. Let $((M\otimes_S \widetilde{A}^{(\otimes_S)^*}\otimes_S)\otimes \overline{H}^{\otimes^*}, \widetilde{d}^0_{**}, \widetilde{d}^1_{**})$ be the double complex with horizontal differentials

$$\widetilde{d}^0_{rs}(\mathbf{x}) = ma_1\otimes_S \widetilde{\mathbf{a}}_{2r}\otimes \mathbf{h} + (-1)^r a_r m\otimes_S \widetilde{\mathbf{a}}_{1,r-1}\otimes \mathbf{h} + \sum_{i=1}^{r-1}(-1)^i m\otimes_S \widetilde{\mathbf{a}}_{1,i-1}\otimes_S a_ia_{i+1}\otimes_S \widetilde{\mathbf{a}}_{i+2,r}\otimes \mathbf{h},$$

and vertical differentials

$$\widetilde{d}^1_{rs}(\mathbf{x}) = (-1)^r m\otimes_s \widetilde{\mathbf{a}}\otimes \mathbf{h}_{2s} + (-1)^{r+s}(1\# h_s^{(3)})m(1\# h_s^{(1)})^{-1}\otimes_S \widetilde{\mathbf{a}}^{h_s^{(2)}}\otimes \mathbf{h}_{1,s-1} + \sum_{i=1}^{s-1}(-1)^{r+i} m\otimes_S \widetilde{\mathbf{a}}\otimes h_1^{(2)}\otimes\cdots\otimes h_{i-1}^{(2)}\otimes h_i^{(2)}h_{i+1}^{(2)}\otimes \mathbf{h}_{i+2,s},$$

where $\mathbf{x} = m\otimes \widetilde{\mathbf{a}}\otimes \mathbf{h}$, with $\widetilde{\mathbf{a}} = a_1\otimes_S\cdots\otimes_S a_r\otimes_S$ and $\mathbf{h} = h_1\otimes\cdots\otimes h_s$. Let $(\overline{X}^S_*, \overline{d}^S_*)$ be the total complex of $((M\otimes_S \widetilde{A}^{(\otimes_S^*)}\otimes_S)\otimes \overline{H}^{\otimes^*}, \widetilde{d}^0_{**}, \widetilde{d}^1_{**})$. We must prove that $\mathrm{H}_*(E,M)$ is the homology of $(\overline{X}^S_*, \overline{d}^S_*)$. Let $\pi_*\colon (\overline{X}_*, \overline{d}_*)\to (\overline{X}^S_*, \overline{d}^S_*)$ be the map $m\otimes \mathbf{a}\otimes \mathbf{h}\mapsto m\otimes_S \widetilde{\mathbf{a}}\otimes \mathbf{h}$. Using item b) of Theorem 1.1.3 it is easy to check that π_* is a map of complexes. Consider the filtration $F^{0S}_*\subseteq F^{1S}_*\subseteq F^{2S}_*\subseteq\dots$ of $(\overline{X}^S_*, \overline{d}^S_*)$, where $F^{iS}_n = \bigoplus_{0\le s\le i}(M\otimes_S \widetilde{A}^{(\otimes_S)^{n-s}}\otimes_S)\otimes \overline{H}^{\otimes^s}$. From Theorem 1.2 of [G-S], it follows that π_* is a morphism of filtered complexes inducing an quasi-isomorphism between the graded complexes associated to the filtrations of $(\overline{X}_*, \overline{d}_*)$ and $(\overline{X}^S_*, \overline{d}^S_*)$. Consequently π_* is a quasi-isomorphism. The proof can be finished by applying Theorem 3.1.1. □

3.2. Hochschild cohomology

Let $\overline{X}^{rs} = \mathrm{Hom}_k(\overline{A}^{\otimes^r}\otimes \overline{H}^{\otimes^s}, M)$. The map $\theta^{rs}\colon \overline{X}^{rs}\to \widehat{X}^{rs}$, defined by

$$\theta^{rs}(\varphi)((1\# h_1)\otimes_A\cdots\otimes_A(1\# h_s)\otimes \mathbf{a}) = (1\# h_1^{(1)})\cdots(1\# h_s^{(1)})\varphi(\mathbf{a}\otimes h_1^{(2)}\otimes\cdots\otimes h_s^{(2)}),$$

is an isomorphism. Let $\overline{d}^{rs}_l\colon \overline{X}^{r+l-1,s-l}\to \overline{X}^{rs}$ be the map defined by $\overline{d}^{rs}_l := (\theta^{rs})^{-1}\,\widehat{d}^{rs}_l\,\theta^{r+l-1,s-l}$.

Theorem 3.2.1. *The Hochschild cohomology of* E *with coefficients in* M *is the cohomology of* $(\overline{X}^*, \overline{d}^*)$*, where*

$$\overline{X}^n = \bigoplus_{r+s=n}\overline{X}^{rs} \qquad \text{and} \qquad \overline{d}^n := \sum_{l=1}^{n}\overline{d}^{0n}_l + \sum_{r=1}^{n}\sum_{l=0}^{n-r}\overline{d}^{r,n-r}_l.$$

Proof. This is an immediate consequence of Theorem 2.4 and the fact that the map $\theta^*\colon (\overline{X}^*, \overline{d}^*)\to (\widehat{X}^*, \widehat{d}^*)$, given by $\theta^n = \sum_{r+s=n}\theta^{rs}$, is an isomorphism. □

Note that when f takes its values in k, then $(\overline{X}^*, \overline{d}^*)$ is the total complex of the double complex $\big(\mathrm{Hom}_k(\overline{A}^*\otimes \overline{H}^*, M), \overline{d}^{**}_0, \overline{d}^{**}_1\big)$.

It is easy to see that

$$\overline{d}_0^{rs}(\varphi)(\mathbf{x}) = a_1\varphi(\mathbf{a}_{2r}\otimes\mathbf{h}) + (-1)^r\varphi(\mathbf{a}_{1,r-1}\otimes\mathbf{h})a_r + \sum_{i=1}^{r-1}(-1)^i\varphi(\mathbf{a}_{1,i-1}\otimes a_ia_{i+1}\otimes\mathbf{a}_{i+2,r}\otimes\mathbf{h}),$$

$$\overline{d}_1^{rs}(\varphi)(\mathbf{x}) = (-1)^r\epsilon(h_1)\varphi(\mathbf{a}\otimes\mathbf{h}_{2s}) + (-1)^{r+s}(1\#h_s^{(1)})^{-1}\varphi(\mathbf{a}^{h_s^{(2)}}\otimes\mathbf{h}_{1,s-1})(1\#h_s^{(3)}) + \sum_{i=1}^{s-1}(-1)^{r+i}\varphi(\mathbf{a}\otimes\mathbf{h}_{1,i-1}\otimes h_ih_{i+1}\otimes\mathbf{h}_{i+2,s}),$$

where $\mathbf{x} = \mathbf{a}\otimes\mathbf{h}$, with $\mathbf{a} = a_1\otimes\cdots\otimes a_r$ and $\mathbf{h} = h_1\otimes\cdots\otimes h_s$.

For each $h\in H$ we have the map $\vartheta_h^*\colon (\operatorname{Hom}_k(\overline{A}^*, M), b^*) \to (\operatorname{Hom}_k(\overline{A}^*, M), b^*)$, defined by $\vartheta_h^r(\varphi)(\mathbf{a}) = (1\#h^{(1)})^{-1}\varphi(\mathbf{a}^{h^{(2)}})(1\#h^{(3)})$.

Proposition 3.2.2. *For each $h,l\in H$, the endomorphisms of $\mathrm{H}^*(A,M)$ induced by $\vartheta_l^*\,\vartheta_h^*$ and by ϑ_{hl}^* coincide. Consequently $\mathrm{H}^*(A,M)$ is a right H-module.*

Proof. By a standard argument it is sufficient to prove it for $\mathrm{H}^0(A,M)$, and in this case the result is immediate. □

Let $F_0(\overline{X}^*)\supseteq F_1(\overline{X}^*)\subseteq\dots$ be the filtration of $(\overline{X}^*,\overline{d}^*)$ obtained transporting the one of $(\widehat{X}_*,\widehat{d}_*)$, given above Corollary 2.5, through θ^*. It is immediate that $F_i(\overline{X}_n) = \bigoplus_{s\ge i}\overline{X}^{n-s,s}$. Then, we have the following:

Corollary 3.2.3. *if H is flat over k, then the spectral sequence of Corollary 2.6 satisfies $E_1^{rs} = \operatorname{Hom}_k(\overline{H}^s, \mathrm{H}^r(A,M))$ and $E_2^{rs} = \mathrm{H}^s(H,\mathrm{H}^r(A,M))$.*

Let S be a separable subalgebra of A and let $\widetilde{A}^{(\otimes_S)^r}$ $(r\ge 0)$ be as in 3.1.4. Suppose $f(h,l)\in S$ for all $h,l\in H$. Using the fact that f takes its values in S it is easy to see that H acts on $\big(\operatorname{Hom}_{A^e}(A\otimes_s\widetilde{A}^{(\otimes_S)^r}\otimes_s A, M), b^*\big) = \big(\operatorname{Hom}_{S^e}(\widetilde{A}^{(\otimes_S)^r}, M), b^*\big)$ via $\big(\varphi\cdot h\big)(\widetilde{\mathbf{a}}) = (1\#h^{(1)})^{-1}\varphi\big(\widetilde{\mathbf{a}}^{h^{(2)}}\big)(1\#h^{(3)})$.

Theorem 3.2.4. *The Hochschild cohomology $\mathrm{H}^*(E,M)$, of E with coefficients in M, is the cohomology of H with coefficients in $\big(\operatorname{Hom}_{S^e}(\widetilde{A}^{(\otimes_S)^r}, M), b^*\big)$.*

Proof. It is similar to the proof of Theorem 3.1.4. □

4. The Cartan-Leray and Grothendieck spectral sequences

Assume that E is a crossed product with invertible cocycle. In this case another two spectral sequences converging to $\mathrm{H}_*(E,M)$ and with E^2-term $\mathrm{H}_*(H,\mathrm{H}_*(A,M))$ can be considered. They are the Cartan-Leray and the Grothendieck spectral sequences. The last one was introduced for the more general setting of Galois extension in [S]. In this Section we recall these constructions and we prove that both coincide with the Hochschild-Serre spectral sequence. Similar results are valid in the cohomological setting. Recall that $(\mathrm{B}_*(E), b'_*)$ is the normalized Hochschild resolution of E.

Let $(\overline{H}^{\otimes^*}\otimes H, d_*)$ be the canonical resolution of k as a right H-module and $(Z_*,\partial_*) = (\mathrm{B}_*(E),b'_*)\otimes(\overline{H}^{\otimes^*}\otimes H, d_*)$. Consider $E\otimes\overline{E}^{\otimes^r}\otimes E\otimes\overline{H}^{\otimes^s}\otimes H$ as an E-bimodule via

$$(a\#l)(\mathbf{x}\otimes\mathbf{h})(b\#q) = ((a\#l)x_0\otimes\mathbf{x}_{1r}\otimes x_{r+1}(b\#q^{(1)}))\otimes(\mathbf{h}_{1s}\otimes h_{s+1}q^{(2)}),$$

where $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_{s+1}$. It is clear that

$$E \xleftarrow{\mu} Z_0 \xleftarrow{\partial_1} Z_1 \xleftarrow{\partial_2} Z_2 \xleftarrow{\partial_3} Z_3 \xleftarrow{\partial_4} Z_4 \xleftarrow{\partial_5} Z_5 \xleftarrow{\partial_6} \dots, \tag{3}$$

where $\mu((a_0\#h_0 \otimes a_1\#h_1) \otimes l) = \epsilon(l)a_0a_1f(h_0^{(1)}, h_1^{(1)})\#h_0^{(2)}h_1^{(2)}$, is a complex of E-bimodules. Moreover (3) is contractible as a complex of left E-modules, with contracting homotopy ζ_n $(n \geq 0)$ given by $\zeta_0(1_E) = 1_E \otimes 1_E \otimes 1_H$ and

$$\zeta_{n+1}(\mathbf{y}) = \begin{cases} -\mathbf{x} \otimes 1_E \otimes \mathbf{h} + (-1)^{n+1}x_0x_1 \otimes 1_E \otimes \mathbf{h} \otimes 1_H & \text{if } r = 0 \\ (-1)^{r+1}\mathbf{x} \otimes 1_E \otimes \mathbf{h} & \text{if } r > 0 \end{cases},$$

where $\mathbf{y} = \mathbf{x} \otimes \mathbf{h}$, with $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_{n-r+1}$. Since the map

$$\tau\colon E \otimes \overline{E}^{\otimes^r} \otimes \overline{H}^{\otimes^s} \otimes H \otimes E \to E \otimes \overline{E}^{\otimes^r} \otimes E \otimes \overline{H}^{\otimes^s} \otimes H,$$

given by $\tau(\mathbf{x}_{0r} \otimes \mathbf{h} \otimes x_{r+1}) = (\mathbf{x}_{0r} \otimes 1_E \otimes \mathbf{h})x_{r+1}$, is an isomorphism of E-bimodules (the inverse of τ is the map $\mathbf{x}_{0r} \otimes a\#h \otimes \mathbf{h} \mapsto \mathbf{x}_{0r} \otimes \mathbf{h}_{1s} \otimes h_{s+1}S^{-1}(h^{(2)}) \otimes a\#h^{(1)}$), (Z_*, ∂_*) is an Υ-projective resolution of E.

We consider (Z_*, ∂_*) filtered by $F^0(Z_*) \subseteq F^1(Z_*) \subseteq F^2(Z_*) \subseteq \dots$, where $F^i(Z_n) = \bigoplus_{j=0}^{i} E \otimes \overline{E}^{\otimes^{n-j}} \otimes E) \otimes (\overline{H}^{\otimes^j} \otimes H)$.

Let M be an E-bimodule. The groups $M \otimes_{E\otimes A^{\mathrm{op}}} (E \otimes \overline{E}^{\otimes^r} \otimes E)$ are left H-modules via

$$h(m \otimes \mathbf{x}) = (1\#h^{(2)})m \otimes \mathbf{x}_{0r} \otimes x_{r+1}(1\#h^{(1)})^{-1},$$

where $\mathbf{x} = x_0 \otimes \cdots \otimes x_{r+1}$. There is an isomorphism

$$M \otimes_{E^e} (Z_*, \partial_*) \simeq (\overline{H}^{\otimes^*} \otimes H, d_*) \otimes_H (M \otimes_{E\otimes A^{\mathrm{op}}} (\mathrm{B}_*(E), b'_*)).$$

Let $F^0_* \subseteq F^1_* \subseteq F^2_* \subseteq F^3_* \subseteq \dots$ be the filtration of $M \otimes_{E^e} (Z_*, \partial_*)$ induced by the filtration of (Z_*, ∂_*). It is easy to see that $F^i_n \simeq \bigoplus_{j=0}^{i} (\overline{H}^{\otimes^j} \otimes H) \otimes_H (M \otimes_{A\otimes E^{\mathrm{op}}} E \otimes \overline{E}^{\otimes^{n-j}} \otimes E)$. The spectral sequence associated to this filtration converges to $\mathrm{H}_*(E, M)$ and has E^2-term $\mathrm{H}_*(H, \mathrm{H}_*(A, M))$. This spectral sequence is called the homological Cartan-Leray spectral sequence. Similarly the groups $\mathrm{Hom}_{E\otimes A^{\mathrm{op}}}(E \otimes \overline{E}^{\otimes^r} \otimes E, M)$ are right H modules via $f.h(\mathbf{x}) = f(\mathbf{x}_{0r} \otimes x_{r+1}(1\#h^{(1)})^{-1})(1\#h^{(2)})$ and there is an isomorphism

$$\mathrm{Hom}_{E^e}((Z_*, \partial_*), M) \simeq \mathrm{Hom}_H\big((\overline{H}^{\otimes^*} \otimes H, d_*), \mathrm{Hom}_{E\otimes A^{\mathrm{op}}}((\mathrm{B}_*(E), b'_*), M)\big).$$

Let $F_0^* \supseteq F_1^* \supseteq F_2^* \supseteq F_3^* \supseteq F_4^* \supseteq \dots$ be the filtration of $\mathrm{Hom}_{E^e}((Z_*, \partial_*), M)$ induced by the filtration of (Z_*, ∂_*). It is easy to see that $F_i^n = \bigoplus_{j\geq i} \mathrm{Hom}_H\big(\overline{H}^{\otimes^j} \otimes H, \mathrm{Hom}_{E\otimes A^{\mathrm{op}}}(E \otimes \overline{E}^{\otimes^{n-j}} \otimes E, M)\big)$. The spectral sequence associated to this filtration converges to $\mathrm{H}^*(E, M)$ and has E^2-term $\mathrm{H}^*(H, \mathrm{H}^*(A, M))$. This spectral sequence is called the cohomological Cartan-Leray spectral sequence.

Let $\Phi_*\colon (\mathrm{B}_*(E), b'_*) \to (Z_*, \partial_*)$ and $\Psi_*\colon (Z_*, \partial_*) \to (\mathrm{B}_*(E), b'_*)$ be the morphisms of E-bimodule complexes, recursively defined by

$$\Phi_0(x \otimes 1_E) = x \otimes 1_E \otimes 1_H, \quad \Psi_0(x \otimes 1_E \otimes h) = \epsilon(h)x \otimes 1_E,$$
$$\Phi_{n+1}(\mathbf{x} \otimes 1_E) = \zeta_{n+1}\,\Phi_n\, b'_{n+1}(\mathbf{x} \otimes 1_E),$$
$$\Psi_{n+1}(\mathbf{x} \otimes 1_E \otimes \mathbf{h}) = \xi_{n+1}\,\psi_n\,\partial_{n+1}(\mathbf{x} \otimes 1_E \otimes \mathbf{h}).$$

Proposition 4.1. *It is true that* $\Psi_* \Phi_* = id_*$ *and that* $\Phi_* \Psi_*$ *is homotopically equivalent to the identity map. The homotopy* $\Phi_* \Psi_* \xrightarrow{\Omega_{*+1}} id_*$ *is recursively defined by* $\Omega_1(x \otimes 1_E \otimes h) = x \otimes 1_E \otimes h \otimes 1_H$ *and*

$$\Omega_{n+1}(\mathbf{x} \otimes 1_E \otimes \mathbf{h}) = \zeta_{n+1} (\Phi_n \Psi_n - id - \Omega_n \partial_n)(\mathbf{x} \otimes 1_E \otimes \mathbf{h}),$$

for $\mathbf{x} = x_0 \otimes \cdots \otimes x_r$ *and* $\mathbf{h} = h_1 \otimes \cdots \otimes h_{n+1-r}$.

Proof. It is easy to see that Φ_* and Ψ_* are morphisms of complexes. Arguing as in Proposition 1.2.1 we get that Ω_{*+1} is a homotopy from $\Phi_* \Psi_*$ to the identity map. It remains to prove that $\Psi_* \Phi_* = id_*$. It is clear that $\Psi_0 \Phi_0 = id_0$. Assume that $\Psi_n \Phi_n = id_n$. Since $\Phi_{n+1}(E \otimes \overline{E}^{\otimes^n} \otimes k) \subseteq \sum_{r=0}^{n+1} E \otimes \overline{E}^{\otimes^r} \otimes k \otimes \overline{H}^{\otimes^{n+1-r}} \otimes H$, we have that on $E \otimes \overline{E}^{\otimes^n} \otimes k$

$$\begin{aligned} \Psi_{n+1} \Phi_{n+1} &= \xi_{n+1} \Psi_n \partial_{n+1} \Phi_{n+1} = \xi_{n+1} \Psi_n \partial_{n+1} \zeta_{n+1} \Phi_n b'_{n+1} \\ &= \xi_{n+1} \Psi_n \Phi_n b'_{n+1} - \xi_{n+1} \Psi_n \zeta_n \partial_n \Phi_n b'_{n+1} = \xi_{n+1} b'_{n+1} = id_{n+1}. \quad \square \end{aligned}$$

Next, we consider the normalized Hochschild resolution $(\mathrm{B}_*(E), b'_*)$ filtered as in Proposition 1.2.2 and the resolution (Z_*, ∂_*) filtered by $F^0(Z_*) \subseteq F^1(Z_*) \subseteq \dots$, where $F^i(Z_n) = \bigoplus_{j=0}^{i} (E \otimes \overline{E}^{\otimes^{n-j}} \otimes E) \otimes (\overline{H}^{\otimes^j} \otimes H)$.

Proposition 4.2. *We have that*

$$\begin{aligned} \Phi_n(a_0 \# h_0 \otimes \cdots \otimes a_{n+1} \# h_{n+1}) = \sum_{j=0}^{n} (-1)^{j(n+1)} (a_0 \# h_0)(a_1 \# h_1^{(1)}) \dots (a_j \# h_j^{(1)}) \\ \otimes (a_{j+1} \# h_{j+1}^{(1)}) \otimes \cdots \otimes (a_{n+1} \# h_{n+1}^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_j^{(2)} \otimes h_{j+1}^{(2)} \cdots h_{n+1}^{(2)}. \end{aligned}$$

Consequently the map Φ_* *preserves filtrations.*

Proof. It follows by induction on n, using the recursive definition of Φ_*. $\square$

Proposition 4.3. *The map* Φ_* *induces a homotopy equivalence of* E*-bimodule complexes between the graded complexes associated to the filtrations of* $(\mathrm{B}_*(E), b'_*)$ *and* (Z_*, ∂_*).

Proof. Note that

$$\begin{aligned} \frac{F^s(X_*)}{F^{s-1}(X_*)} &= (X_{*s}, d^0_{*s}) = (E \otimes \overline{H}^{\otimes^s} \otimes \overline{A}^{\otimes^*} \otimes E, d^0_{*s}), \\ \frac{F^s(Z_*)}{F^{s-1}(Z_*)} &= (\mathrm{B}_*(E), b'_*) \otimes \overline{H}^{\otimes^s} \otimes H, \end{aligned}$$

where $d^0_{*,s}$ is the boundary map introduced in Subsection 1.1. By Proposition 1.2.2 it suffices to check that $\mathit{\Phi}_* = \Phi_* \phi_*$ induces a homotopy equivalence $\widetilde{\mathit{\Phi}}^s_*$ of E-bimodules complexes, from $(E \otimes \overline{H}^{\otimes^s} \otimes \overline{A}^{\otimes^*} \otimes E, d^0_{*s})$ to $(\mathrm{B}_*(E), b'_*) \otimes \overline{H}^{\otimes^s} \otimes H$. Let Y_s and μ_s be as in Subsection 1.1 and $\widetilde{Y}_s = E \otimes \overline{H}^{\otimes^s} \otimes H$ endowed with the structure of E-bimodule given by $x_0(x_1\otimes \mathbf{h})x_2 = x_0x_1x_2\otimes \mathbf{h}$, where $\mathbf{h} = h_0\otimes\cdots\otimes h_{s+1}$. Consider the diagram

$$
(4)\qquad
\begin{array}{ccccccc}
Y_s & \xleftarrow{\mu_s} & E\otimes \overline{H}^{\otimes^s}\otimes E & \xleftarrow{d^0_{1s}} & E\otimes \overline{H}^{\otimes^s}\otimes \overline{A}\otimes E & \xleftarrow{d^0_{2s}} & \dots\\
\big\downarrow{\scriptstyle \widetilde{\Phi}^s} & & \big\downarrow{\scriptstyle \widetilde{\Phi}^s_0} & & \big\downarrow{\scriptstyle \widetilde{\Phi}^s_1} & & \\
\widetilde{Y}_s & \xleftarrow{\widetilde{\mu}_s} & E\otimes E\otimes \overline{H}^{\otimes^s}\otimes H & \xleftarrow{b'_1} & E\otimes \overline{E}\otimes E\otimes \overline{H}^{\otimes^s}\otimes H & \xleftarrow{b'_2} & \dots,
\end{array}
$$

where

$$\widetilde{\mu}_s((x_0\otimes x_1)\otimes \mathbf{h}) = x_0x_1\otimes \mathbf{h}\qquad\text{and}$$
$$\widetilde{\Phi}^s(x\otimes \mathbf{h}) = x(1\#h_1^{(1)})\cdots(1\#h_{s+1}^{(1)})\otimes h_1^{(2)}\otimes\cdots\otimes h_{s+1}^{(2)}.$$

We assert that $\widetilde{\Phi}^s_0(\mathbf{x}) = 1_E\otimes(1\#h_1^{(1)})\cdots(1\#h_s^{(1)})\otimes h_1^{(2)}\otimes\cdots\otimes h_s^{(2)}\otimes 1_H$, where $\mathbf{x} = 1_E\otimes\mathbf{h}\otimes 1_E$, with $\mathbf{h} = h_1\otimes\cdots\otimes h_s$. To prove this it suffices to check that

$$\Phi_s\,\phi_s(\mathbf{x}) \in 1_E\otimes(1\#h_1^{(1)})\cdots(1\#h_s^{(1)})\otimes h_1^{(2)}\otimes\cdots\otimes h_s^{(2)}\otimes 1_H + F_{s-1},$$

which follows by induction on s, using that $\Phi_s\,\phi_s(\mathbf{x}) = \zeta_s\,\Phi_{s-1}\,\phi_{s-1}\,d_s(\mathbf{x})$. Now, it is immediate that $\widetilde{\mu}_s\,\widetilde{\Phi}^s_0 = \widetilde{\Phi}^s\,\mu_s$. Since $\widetilde{\Phi}^s$ is an isomorphism and the rows of (4) are Υ-projective resolutions of Y_s and $\widetilde{Y}_s$ respectively, it follows that $\widetilde{\Phi}^s_*$ is a homotopy equivalence. □

Corollary 4.4. *The (co)homological Cartan-Leray spectral sequence is isomorphic to the (co)homological Hochschild-Serre spectral sequence.*

Proof. We prove the assertion for the homology. The proof in the cohomological setting is similar. It is immediate that the filtration of $(M\otimes\overline{E}^{\otimes^*}, b_*)$ is the one induced by the filtration of $(\mathrm{B}_*(E), b'_*)$, defined above Proposition 1.1.2. Hence, it is clear that the map $id_M\otimes_{E^e}\Phi_*$ preserve filtrations. Let $\mathrm{gr}(id_M\otimes_{E^e}\Phi_*)$ be the map induced by $id_M\otimes_{E^e}\Phi_*$, between the graded complexes associated to the filtrations of $(M\otimes\overline{E}^{\otimes^*}, b_*)$ and $M\otimes_{E^e}(Z_*,\partial_*)$, respectively. Since, for all $i,n\ge 0$ the E-bimodules $F^i(\mathrm{B}_n(E))$ and $F^{i+1}(\mathrm{B}_n(E))$ are direct summands of $F^i(Z_n)$ and $F^{i+1}(Z_n)$ respectively, $\mathrm{gr}(id_M\otimes_{E^e}\Phi_*) = id_M\otimes_{E^e}\mathrm{gr}(\Phi_*)$, where $\mathrm{gr}(\Phi_*)$ is the map induced by Φ_*, between the graded complexes associated to the filtrations of $(\mathrm{B}_*(E), b'_*)$ and (Z_*,∂_*), respectively. By Proposition 4.3, $\mathrm{gr}(id_M\otimes_{E^e}\Phi_*)$ is a quasi-isomorphism. The assertion follows immediately from this fact. □

4.5 The Grothendieck spectral sequence. If M is an E-bimodule, then the group $\mathrm{H}_0(A,M) = M/[A,M]$ is a left H-module via $h\cdot\overline{m} = \overline{(1\#h^{(2)})m(1\#h^{(1)})^{-1}}$, where the $\overline{m}$ denotes the class of m in $M/[A,M]$. Let us consider the functors $M\mapsto \mathrm{H}_0(E,M)$ from the category of E-bimodules to the category of k-modules, $M\mapsto \mathrm{H}_0(A,M)$ from the category of E-bimodules to the category of left H-modules and $M\mapsto \mathrm{H}_0(H,M)$ from the category of left H-modules to the category of k-modules. It is easy to see that $\mathrm{H}_0(E,M) = \mathrm{H}_0(H,\mathrm{H}_0(A,M))$ and that if M is a Υ-projective module, then $\mathrm{H}_0(A,M)$ is a projective H-module, relative to the family of all epimorphisms of H-modules which split as k-linear maps. In fact, if $M = E\otimes N$, then the map $h\otimes n\mapsto \overline{1\#h^{(2)}\otimes n(1\#h^{(1)})^{-1}}$ is an isomorphism of left H-modules from $H \otimes N$ to $\mathrm{H}_0(A, M)$. Thus we have a Grothendieck spectral sequence

$$E^2_{rs} = \mathrm{H}_s(H, \mathrm{H}_r(A, M)) \to \mathrm{H}_{r+s}(E, M)).$$

We assert that the Grothendieck spectral sequence and the Cartan-Leray spectral sequence coincide. To prove this we use a concrete construction of the Grothendieck spectral sequence. Let $(P_*, \partial_*) = (M \otimes \overline{E}^{\otimes^*} \otimes E, b'_*)$ be the normalized canonical resolution of M as a right E-module. Let us write $(\overline{P}_*, \overline{\partial}_*) = (P_*, \partial_*) \otimes_{A^e} A$. Consider the double complex

$$C_{**} := \begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \downarrow & & \downarrow & & \downarrow & & \\ H \otimes_H \overline{P}_1 & \longleftarrow & \overline{H} \otimes H \otimes_H \overline{P}_1 & \longleftarrow & \overline{H}^{\otimes^2} \otimes H \otimes_H \overline{P}_1 & \longleftarrow & \dots \\ \downarrow & & \downarrow & & \downarrow & & \\ H \otimes_H \overline{P}_0 & \longleftarrow & \overline{H} \otimes H \otimes_H \overline{P}_0 & \longleftarrow & \overline{H}^{\otimes^2} \otimes H \otimes_H \overline{P}_0 & \longleftarrow & \dots, \end{array}$$

whose r-th column is $(-1)^r$ times $\overline{H}^{\otimes^r} \otimes H \otimes_H (\overline{P}_*, \overline{\partial}_*)$ and whose s-th row is the canonical complex $(\overline{H}^{\otimes^*} \otimes H \otimes_H \overline{P}_s, d_*)$ giving the homology $\mathrm{H}_*(H, \overline{P}_s)$ of k as a trivial right H-module with coefficients in $\overline{P}_s$. By definition, the Grothendieck spectral sequence is the spectral sequence associated to the filtrations by columns of C_{**}. Since $C_{**} \simeq (\overline{H}^{\otimes^*} \otimes H, d_*) \otimes_H (M \otimes_{E \otimes A^{\mathrm{op}}} (\mathrm{B}_*(E), b'_*))$ as filtered complexes, the homological Cartan-Leray and the Grothendieck spectral sequence coincide. The same is valid in the cohomological setting.

Appendix A

Let $R \to S$ be an unitary ring map and let N be a left S-module. In this section, under suitable conditions, we construct a projective resolution of N, relative to the family of all epimorphisms of S-modules, which split as R-linear maps. We need this result (with $R = E$, $S = E^e$ and $N = E$) to complete the proof of Theorem 1.1.1. The general case considered here simplifies the notation and enables us to consider other cases, for instance algebras of groups having particular resolutions.

Let us consider a diagram of left S-modules and S-module maps

$$\begin{array}{ccccccc} \vdots & & & & & & \\ \downarrow{\scriptstyle \partial_2} & & & & & & \\ Y_1 & \xleftarrow{\mu_1} & X_{01} & \xleftarrow{d^0_{11}} & X_{11} & \xleftarrow{d^0_{21}} & \dots \\ \downarrow{\scriptstyle \partial_1} & & & & & & \\ Y_0 & \xleftarrow{\mu_0} & X_{00} & \xleftarrow{d^0_{10}} & X_{10} & \xleftarrow{d^0_{20}} & \dots, \end{array}$$

such that:

a) The column and the rows are chain complexes.

b) For each $r,s\ge 0$ we have a left R-module $\overline{X}_{rs}$ and S-module maps

$$s_{rs}\colon X_{rs}\to S\otimes \overline{X}_{rs} \qquad \text{and} \qquad \pi_{rs}\colon S\otimes \overline{X}_{rs}\to X_{rs}$$

verifying $\pi_{rs}\, s_{rs} = id$.

c) Each row is contractible as a complex of R-modules, with a chain contracting homotopy $\sigma^0_{0s}\colon Y_s\to X_{0s}$ and $\sigma^0_{r+1,s}\colon X_{rs}\to X_{r+1,s}$ $(r\ge 0)$.

We are going to modify this diagram by adding S-module maps

$$d^l_{rs}\colon X_{rs}\longrightarrow X_{r+l-1,s-l} \qquad (r,s\ge 0 \text{ and } 1\le l\le s).$$

For each $n\ge 0$, let

$$X_n = \bigoplus_{r+s=n} X_{rs} \qquad \text{and} \qquad d_n=\sum_{l=1}^{n} d^l_{0n} + \sum_{r=1}^{n}\sum_{l=0}^{n-r} d^l_{r,n-r}.$$

Consider the maps $\mu'_n\colon X_n\to Y_n$ $(n\ge 0)$, given by:

$$\mu'_n(x)=\begin{cases}\mu_n(x) & \text{for } x\in X_{0n}\\ 0 & \text{for } x\in X_{r,n-r}, \text{ with } r>0.\end{cases}$$

We define the arrows d^l_{rs} in such a way that (X_*,d_*) becomes a chain complex of S-modules and $\mu'_*\colon (X_*,d_*)\to (Y_*,-\partial_*)$ becomes a chain homotopy equivalence of complexes of R-modules. In fact, we are going to build R-module morphisms

$$\sigma^l_{l,s-l}\colon Y_s\to X_{l,s-l} \;\text{ and }\; \sigma^l_{r+l+1,s-l}\colon X_{rs}\to X_{r+l+1,s-l} \;\; (r,s\ge 0 \text{ and } 1\le l\le s),$$

satisfying the following:

Theorem A.1. *Let* $\mathrm{C}_*(\mu'_*)$ *be the mapping cone of* μ'_**, that is,* $\mathrm{C}_*(\mu'_*)=(C_*,\delta_*)$*, where* $C_n=Y_n\oplus X_{n-1}$ *and* $\delta_n(y_n,x_{n-1})=\big(-\partial(y_n)-\mu'_{n-1}(x_{n-1}),-d_{n-1}(x_{n-1})\big)$*. The family of* R*-module maps* $\sigma_{n+1}\colon \mathrm{C}_n(\mu'_*)\to \mathrm{C}_{n+1}(\mu'_*)$ $(n\ge 0)$*, defined by:*

$$\sigma_{n+1}=-\sum_{r=-1}^{n-1}\sum_{l=0}^{n-r-1}\sigma^l_{r+l+1,n-r-l-1},$$

is a chain contracting homotopy of $\mathrm{C}_*(\mu'_*)$.

Corollary A.2. *Let* N *be a left* S*-module. If there is an* S*-module map* $\widetilde{\mu}\colon Y_0\to N$*, such that*

$$(*)\qquad N \xleftarrow{\widetilde{\mu}} Y_0 \xleftarrow{\partial_1} Y_1 \xleftarrow{\partial_2} Y_2 \xleftarrow{\partial_3} Y_3 \xleftarrow{\partial_4} Y_4 \xleftarrow{\partial_5} Y_6 \xleftarrow{\partial_7} \dots$$

is contractible as a complex of left R*-modules, then*

$$(**)\qquad N \xleftarrow{\mu} X_0 \xleftarrow{d_1} X_1 \xleftarrow{d_2} X_2 \xleftarrow{d_3} X_3 \xleftarrow{d_4} X_4 \xleftarrow{d_5} X_5 \xleftarrow{d_6} X_6 \xleftarrow{d_7} \dots,$$

where $\mu=\widetilde{\mu}\,\mu_0$*, is a relative projective resolution of* N*, relative to the family of all epimorphisms of* S*-modules, which split as* R*-linear maps. Moreover, if* $\sigma_0^{-1}\colon N\to$

Y_0, $\sigma_{n+1}^{-1}\colon Y_n \to Y_{n+1}$ ($n \ge 0$) is a chain contracting homotopy of (*), then we obtain a chain contracting homotopy $\overline{\sigma}_0\colon N \to X_0$, $\overline{\sigma}_{n+1}\colon X_n \to X_{n+1}$ ($n \ge 0$) of (**), defining $\overline{\sigma}_0 = \sigma_{00}^0\,\sigma_0^{-1}$ and

$$\overline{\sigma}_{n+1} = -\sum_{l=0}^{n+1} \sigma_{l,n-l+1}^l\,\sigma_{n+1}^{-1}\,\mu_n + \sum_{r=0}^{n}\sum_{l=0}^{n-r} \sigma_{r+l+1,n-r-l}^l.$$

Proof. Write

$$\widetilde{\sigma}_n = \sum_{r=0}^{n-1}\sum_{l=0}^{n-r-1} \sigma_{r+l+1,n-r-l-1}^l \quad (n \ge 1) \qquad \text{and} \qquad \widehat{\sigma}_n = \sum_{l=0}^{n} \sigma_{l,n-l}^l \quad (n \ge 0).$$

From Theorem A.1, we have

$$\text{(*n)} \qquad \widehat{\sigma}_n\,\partial_{n+1} = \sum_{l=0}^{n} \sigma_{l,n-l}^l\,\partial_{n+1} = -\sum_{l=0}^{n}\sum_{i=0}^{l+1} d_{i,n+1-i}^{l+1-i}\,\sigma_{i,n+1-i}^i = -d_{n+1}\,\widehat{\sigma}_{n+1}.$$

It is clear that $\mu\,\overline{\sigma}_0 = id$. Moreover

$$\begin{aligned}\overline{\sigma}_0\,\mu &= \sigma_{00}^0\,\sigma_0^{-1}\,\widetilde{\mu}\,\mu_0 = \sigma_{00}^0\,\mu_0 - \sigma_{00}^0\,\partial_1\,\sigma_1^{-1}\,\mu_0\\ &= id - d_{10}^0\,\sigma_{10}^0 + d_{01}^1\,\sigma_{01}^0\,\sigma_1^{-1}\,\mu_0 + d_{10}^0\,\sigma_{10}^1\,\sigma_1^{-1}\,\mu_0,\end{aligned}$$

where the last equality follows from (*0). Now, let $n \ge 1$. Take $x \in X_{r,n-r}$. If $r \ge 1$, then the equality $(0,x) = \delta_{n+2}\,\sigma_{n+2}(0,x) + \sigma_{n+1}\,\delta_{n+1}(0,x)$ implies that $x = d_{n+1}\,\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n\,d_n(x)$. Hence, we can suppose $r = 0$. Then, from $(0,x) = \delta_{n+2}\,\sigma_{n+2}(0,x) + \sigma_{n+1}\,\delta_{n+1}(0,x)$, we get

$$\begin{aligned}x &= d_{n+1}\,\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n\,d_n(x) + \widehat{\sigma}_n\,\mu_n(x)\\ &= d_{n+1}\,\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n\,d_n(x) + \widehat{\sigma}_n\,\sigma_n^{-1}\,\partial_n\,\mu_n(x) + \widehat{\sigma}_n\,\partial_{n+1}\,\sigma_{n+1}^{-1}\,\mu_n(x)\\ &= d_{n+1}\,\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n\,d_n(x) - \widehat{\sigma}_n\,\sigma_n^{-1}\,\mu_{n-1}\,d_n(x) + \widehat{\sigma}_n\,\partial_{n+1}\,\sigma_{n+1}^{-1}\,\mu_n(x)\\ &= d_{n+1}\,\widetilde{\sigma}_{n+1}(x) + \widetilde{\sigma}_n\,d_n(x) - \widehat{\sigma}_n\,\sigma_n^{-1}\,\mu_{n-1}\,d_n(x) - d_{n+1}\,\widehat{\sigma}_{n+1}\,\sigma_{n+1}^{-1}\,\mu_n(x),\end{aligned}$$

where the last equality follows from (*n). □

Next we define the morphisms d_{rs}^l and we prove that (X_*, d_*) is a chain complex.

Definition A.3. *We define the S-module maps $d_{rs}^l\colon X_{rs} \to X_{r+l-1,s-l}$ ($r \ge 0$ and $1 \le l \le s$), recursively by $d_{rs}^l = \overline{d}_{rs}^l\,s_{rs}$, where $\overline{d}_{rs}^l\colon S \otimes \overline{X}_{rs} \to X_{r+l-1,s-l}$ ($r \ge 0$ and $1 \le l \le s$) is the S-module map defined by*

$$\overline{d}_{rs}^l(\mathbf{x}) = \begin{cases} -\sigma_{0,s-1}^0\,\partial_s\,\mu_s\,\pi_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } l = 1,\\ -\sum_{j=1}^{l-1} \sigma_{l-1,s-l}^0\,d_{j-1,s-j}^{l-j}\,d_{0s}^j\,\pi_{0s}(\mathbf{x}) & \text{if } r = 0 \text{ and } 1 < l \le s,\\ -\sum_{j=0}^{l-1} \sigma_{r+l-1,s-l}^0\,d_{r+j-1,s-j}^{l-j}\,d_{rs}^j\,\pi_{rs}(\mathbf{x}) & \text{if } r > 0,\end{cases}$$

for each $\mathbf{x} = 1 \otimes \overline{\mathbf{x}} \in S \otimes \overline{X}_{rs}$.

Proposition A.4. *We have* $\mu_{s-1}\, d^1_{0s} = -\partial_s\, \mu_s$ *and*

$$d^0_{r+l-1,s-l}\, d^l_{rs} = \begin{cases} -\sum_{j=1}^{l-1} d^{l-j}_{j-1,s-j}\, d^j_{0s} & \textit{if } r=0 \textit{ and } 1<l\le s\\ -\sum_{j=0}^{l-1} d^{l-j}_{r+j-1,s-j}\, d^j_{rs} & \textit{if } r>0 \textit{ and } 1\le l\le s.\end{cases}$$

Consequently (X_*, d_*) *is a chain complex.*

Proof. We prove the proposition by induction on l and r. To simplify the expressions we put $d^0_{0s} := \mu_s$, $d^1_{-1,s} := \partial_s$ and $d^l_{-1,s} := 0$ for all $l>1$. Moreover to abbreviate we do not write the subindices. Let $\mathbf{x} = 1\otimes \overline{\mathbf{x}}$ with $\overline{\mathbf{x}} \in \overline{X}_{0s}$. Since $\overline{d}^1_0(\mathbf{x}) = -\sigma^0\, d^1\, d^0\, \pi(\mathbf{x})$, we have $d^0\, \overline{d}^1(\mathbf{x}) = -d^0\,\sigma^0\, d^1\, d^0\,\pi(\mathbf{x}) = -d^1\, d^0\,\pi(\mathbf{x})$, which implies $d^0\, d^1 = -d^1\, d^0$. Let $l+r>1$ and suppose the result is valid for d^j_{p*} with $j<l$ or $j=l$ and $p<r$. Let $\mathbf{x} = 1\otimes\overline{\mathbf{x}}$ with $\overline{\mathbf{x}}\in \overline{X}_{rs}$. Since $\overline{d}^l(\mathbf{x}) = -\sum_{j=0}^{l-1}\sigma^0\, d^{l-j}\, d^j\,\pi(\mathbf{x})$, then

$$d^0\,\overline{d}^l(\mathbf{x}) = -\sum_{j=0}^{l-1} d^0\,\sigma^0\, d^{l-j}\, d^j\,\pi(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{l-j}\, d^j\,\pi(\mathbf{x}) + \sum_{j=0}^{l-1}\sigma^0\, d^0\, d^{l-j}\, d^j\,\pi(\mathbf{x}).$$

Applying first the inductive hypothesis to $d^0\, d^{l-j}$ with $(0\le j<l)$ and then to $d^0\, d^j$ with $(0<j<l)$, we obtain:

$$\begin{aligned} d^0\,\overline{d}^l(\mathbf{x}) &= -\sum_{j=0}^{l-1} d^{l-j}\, d^j\,\pi(\mathbf{x}) - \sum_{j=0}^{l-1}\sum_{i=0}^{l-j-1}\sigma^0\, d^{l-j-i}\, d^i\, d^j\,\pi(\mathbf{x})\\ &= -\sum_{j=0}^{l-1} d^{l-j}\, d^j\,\pi(\mathbf{x}) - \sum_{j=0}^{l-2}\sum_{i=1}^{l-j-1}\sigma^0\, d^{l-j-i}\, d^i\, d^j\,\pi(\mathbf{x})\\ &\quad+\sum_{j=1}^{l-1}\sum_{h=0}^{j-1}\sigma^0\, d^{l-j}\, d^{j-h}\, d^h\,\pi(\mathbf{x}) = -\sum_{j=0}^{l-1} d^{l-j}\, d^j\,\pi(\mathbf{x}).\end{aligned}$$

The desired equality follows immediately from this fact. ☐

It is immediate that $\mu'_*\colon (X_*, d_*) \to (Y_*, -\partial_*)$ is a morphism of S-module chain complexes. Next, we construct the chain contracting homotopy of $\mathrm{C}_*(\mu'_*)$.

Definition A.5. *We define* $\sigma^l_{l,s-l}\colon Y_s \to X_{l,s-l}$ *and* $\sigma^l_{r+l+1,s-l}\colon X_{rs}\to X_{r+l+1,s-l}$ $(0<l\le s,\ r\ge 0)$*, recursively by:*

$$\sigma^l_{r+l+1,s-l} = -\sum_{i=0}^{l-1}\sigma^0_{r+l+1,s-l}\, d^{l-i}_{r+i+1,s-i}\,\sigma^i_{r+i+1,s-i}\qquad (0<l\le s \textit{ and } r\ge -1).$$

Proof of Theorem A.1. To simplify the expressions we put $d^0_{-1,s} := 0$, $d^0_{0s} := \mu_s$, $d^1_{-1,s} := \partial_s$ and $d^l_{-1,s} := 0$ for all $l>1$. Because of the definitions of d_* and σ_*, it suffices to check that $\sigma^0_{rs}\, d^0_{rs} + d^0_{r+1,s}\,\sigma^0_{r+1,s} = id$ and

$$\sum_{i=0}^{l}\sigma^{l-i}_{r+l,s-l}\, d^i_{rs} + \sum_{i=0}^{l} d^{l-i}_{r+i+1,s-i}\,\sigma^i_{r+i+1,s-i} = 0\qquad \text{for } l>0,$$

where we put $d^0_{-1,s} = 0$. The first formula simply says that σ^0_* is a chain contracting homotopy of d^0_*. Let us see the second one. To abbreviate we do not write the subindices. From the definition of σ^l we have:

$$d^0\,\sigma^l = -\sum_{i=0}^{l-1} d^0\,\sigma^0\,d^{l-i}\,\sigma^i = \sum_{i=0}^{l-1}\sigma^0\,d^0\,d^{l-i}\,\sigma^i - \sum_{i=0}^{l-1} d^{l-i}\,\sigma^i.$$

Consequently

$$\sum_{i=0}^{l}\sigma^{l-i}\,d^i + \sum_{i=0}^{l} d^{l-i}\,\sigma^i = \sum_{i=0}^{l}\sigma^{l-i}\,d^i + \sum_{i=0}^{l-1}\sigma^0\,d^0\,d^{l-i}\,\sigma^i.$$

Then, it suffices to prove that the term appearing on the right side of the equality is zero. We prove this by induction on l. For $l = 1$ we have:

$$\sigma^0\,d^0\,d^1\,\sigma^0 = -\sigma^0\,d^1\,d^0\,\sigma^0 = \sigma^0\,d^1\,\sigma^0\,d^0 - \sigma^0\,d^1 = -\sigma^1\,d^0 - \sigma^0\,d^1.$$

Suppose $l > 1$. From Proposition A.5,

$$\sum_{i=0}^{l-1}\sigma^0\,d^0\,d^{l-i}\,\sigma^i = -\sum_{i=0}^{l-1}\sum_{j=0}^{l-i-1}\sigma^0\,d^{l-i-j}\,d^j\,\sigma^i = -\sum_{h=0}^{l-1}\sum_{i=0}^{h}\sigma^0\,d^{l-h}\,d^{h-i}\,\sigma^i.$$

So, applying the inductive hypothesis to $\sum_{i=0}^{h} d^{h-i}\,\sigma^i$ $(h \ge 0)$, we obtain

$$\begin{aligned}\sum_{i=0}^{l-1}\sigma^0\,d^0\,d^{l-i}\,\sigma^i &= \sum_{h=0}^{l-1}\sum_{i=0}^{h}\sigma^0\,d^{l-h}\,\sigma^{h-i}\,d^i - \sigma^0\,d^l\\ &= \sum_{i=0}^{l-1}\sum_{j=0}^{l-i-1}\sigma^0\,d^{l-i-j}\,\sigma^j\,d^i - \sigma^0\,d^l\\ &= -\sum_{i=0}^{l}\sigma^{l-i}\,d^i. \quad \square\end{aligned}$$

Appendix B

In this appendix we compute explicitly the maps d^l_{rs} introduced in Section 1, completing the results of Theorem 1.1.3. We use the following notations:

1) Given $h_0 \otimes \cdots \otimes h_s \in H^{\otimes^{s+1}}$ and $0 \le i < j \le s$, we write $\mathfrak{h}_{ij} = h_i \cdots h_j \in H$.
2) Given $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H^{\otimes^{s+1}}$, we let $\mathbf{h}^{(1)} \otimes \mathbf{h}^{(2)}$ denote the comultiplication of $\mathbf{h}$ in $H^{\otimes^{s+1}}$. So, $\mathbf{h}^{(1)} \otimes \mathbf{h}^{(2)} = (h_0^{(1)} \otimes \cdots \otimes h_s^{(1)}) \otimes (h_0^{(2)} \otimes \cdots \otimes h_s^{(2)})$.
3) Given $a \in A$, $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in A^{\otimes^r}$ and $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H^{\otimes^{s+1}}$, we write $a^{\mathbf{h}} = (\ldots(((a^{h_s})^{h_{s-1}})^{h_{s-2}})^{h_{s-3}}\ldots)^{h_0}$ and $\mathbf{a}^{\mathbf{h}} = a_1^{\mathbf{h}^{(1)}} \otimes \cdots \otimes a_r^{\mathbf{h}^{(r)}}$.

Definition B.1. Given $\mathbf{h} = h_1 \otimes \cdots \otimes h_l \in \overline{H}^{\otimes^l}$, we define $F_0^{(l)}(\mathbf{h}) \in \overline{A}^{\otimes^{l-1}}$, recursively by:

$$\begin{aligned}F_0^{(2)}(\mathbf{h}) &= -f(h_1, h_2),\\ F_0^{(l+1)}(\mathbf{h}) &= \sum_{j=1}^{l}(-1)^j f(h_j^{(1)}, h_{j+1}^{(1)})^{\mathbf{h}^{(1)}_{1,j-1}} \otimes F^{(l)}(\mathbf{h}^{j(2)}),\end{aligned}$$

where $\mathbf{h}^{j(2)} = \mathbf{h}^{(2)}_{1,j-1} \otimes \mathrm{h}^{(2)}_j\, h^{(2)}_{j+1} \otimes \mathbf{h}_{j+2,l+1}$. For instance, we have

$$F^{(3)}_0(\mathbf{h}) = f(h^{(1)}_1, h^{(1)}_2) \otimes f(\mathfrak{h}^{(2)}_{12}, h_3) - f(h^{(1)}_2, h^{(1)}_3)^{h^{(1)}_1} \otimes f(h^{(2)}_1, \mathfrak{h}^{(2)}_{23})$$

and

$$\begin{aligned} F^{(4)}_0(\mathbf{h}) = &- f(h^{(1)}_1, h^{(1)}_2) \otimes f(\mathfrak{h}^{(2)}_{12}, h^{(1)}_3) \otimes f(\mathfrak{h}^{(3)}_{12} h^{(2)}_3, h_4) \\ &+ f(h^{(1)}_1, h^{(1)}_2) \otimes f(h^{(1)}_3, h^{(1)}_4)^{\mathfrak{h}^{(2)}_{12}} \otimes f(\mathfrak{h}^{(2)}_{12}, \mathfrak{h}^{(2)}_{34}) \\ &+ f(h^{(1)}_2, h^{(1)}_3)^{h^{(1)}_1} \otimes f(h^{(2)}_1, \mathfrak{h}^{(2)}_{23}) \otimes f(\mathfrak{h}^{(3)}_{13}, h_4) \\ &- f(h^{(1)}_2, h^{(1)}_3)^{h^{(1)}_1} \otimes f(\mathfrak{h}^{(2)}_{23}, h^{(1)}_4)^{h^{(2)}_1} \otimes f(h^{(3)}_1, \mathfrak{h}^{(3)}_{23} h^{(2)}_4) \\ &- f(h^{(1)}_3, h^{(1)}_4)^{\mathbf{h}^{(1)}_{12}} \otimes f(h^{(2)}_1, h^{(2)}_2) \otimes f(\mathfrak{h}^{(3)}_{12}, \mathfrak{h}^{(2)}_{34}) \\ &+ f(h^{(1)}_3, h^{(1)}_4)^{\mathbf{h}^{(1)}_{12}} \otimes f(h^{(2)}_2, \mathfrak{h}^{(2)}_{34})^{h^{(2)}_1} \otimes f(h^{(3)}_1, \mathfrak{h}^{(3)}_{24}). \end{aligned}$$

For the following definition we adopt the convention that $\mathbf{a}_{10} = \mathbf{a}_{r+1,r} = 1_k \in k$.

Definition B.2. Given $\mathbf{h} = h_1 \otimes \cdots \otimes h_l \in \overline{H}^{\otimes^l}$ and $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^{\otimes^r}$, we define $F^{(l)}_r(\mathbf{h} \otimes \mathbf{a})$, recursively by:

$$F^{(2)}_r(\mathbf{h} \otimes \mathbf{a}) = \sum_{i=0}^{r} (-1)^{i+1} \mathbf{a}^{\mathbf{h}^{(1)}_{12}}_{1i} \otimes f(h^{(2)}_1, h^{(2)}_2) \otimes \mathbf{a}^{\mathfrak{h}^{(3)}_{12}}_{i+1,r},$$

$$F^{(l+1)}_r(\mathbf{h} \otimes \mathbf{a}) = \sum_{j=1}^{l} \sum_{i=0}^{r} (-1)^{il+j} \mathbf{a}^{\mathbf{h}^{(1)}_{1,l+1}}_{1i} \otimes f(h^{(2)}_j, h^{(2)}_{j+1})^{\mathbf{h}^{(2)}_{1,j-1}} \otimes F^{(l)}_{r-i}(\mathbf{h}^{j(3)} \otimes \mathbf{a}_{i+1,r}),$$

where $\mathbf{h}^{j(3)} = \mathbf{h}^{(3)}_{1,j-1} \otimes \mathrm{h}^{(3)}_j\, h^{(3)}_{j+1} \otimes \mathbf{h}^{(2)}_{j+2,l+1}$ and $F^{(l)}_0(\mathbf{h}^{j(3)} \otimes \mathbf{a}_{r+1,r}) = F^{(l)}_0(\mathbf{h}^{j(3)})$. For instance, we have

$$\begin{aligned} F^{(3)}_r(\mathbf{h} \otimes \mathbf{a}) = &\sum_{0 \le i \le j \le r} (-1)^{i+j} \mathbf{a}^{\mathbf{h}^{(1)}_{13}}_{1i} \otimes f(h^{(2)}_1, h^{(2)}_2) \otimes \mathbf{a}^{\mathbf{h}^{1(3)}_{13}}_{i+1,j} \otimes f(\mathfrak{h}^{(4)}_{12}, h^{(3)}_3) \otimes \mathbf{a}^{\mathfrak{h}^{(5)}_{12} h^{(4)}_3}_{j+1,r} \\ &+ \sum_{0 \le i \le j \le r} (-1)^{i+j+1} \mathbf{a}^{\mathbf{h}^{(1)}_{13}}_{1i} \otimes f(h^{(2)}_2, h^{(2)}_3)^{h^{(2)}_1} \otimes \mathbf{a}^{\mathbf{h}^{2(3)}_{13}}_{i+1,j} \otimes f(h^{(4)}_1, \mathfrak{h}^{(4)}_{23}) \otimes \mathbf{a}^{\mathfrak{h}^{(5)}_{13}}_{j+1,r}. \end{aligned}$$

We set $F^{(1)}_0(h_s) = 1_k \in k$, $F^{(1)}_r(h_s \otimes \mathbf{a}) = \mathbf{a}^{h_s}$ and $F^{(l)}_0(\mathbf{h}_{s-l-1,s} \otimes 1_k) = F^{(l)}_0(\mathbf{h}_{s-l-1,s})$. Moreover, to abbreviate we write $F^{(l)}(\mathbf{h}) = F^{(l)}_0(\mathbf{h})$ and $F^{(l)}\binom{\mathbf{a}}{\mathbf{h}} = F^{(l)}_r(\mathbf{h} \otimes \mathbf{a})$.

Lemma B.3. *Let $\mathbf{a} = a_1 \otimes \cdots \otimes a_r$ and $\mathbf{h}_{s-l,s} = h_{s-l} \otimes \cdots \otimes h_l$. We have:*

$$F^{(l+1)}(\mathbf{h}_{s-l,s}) = \sum_{i=1}^{l} (-1)^i F^{(l-i+1)} \binom{F^{(i)}(\mathbf{h}^{(1)}_{s-i+1,s})}{\mathbf{h}^{(1)}_{s-l,s-i}} \otimes f(\mathfrak{h}^{(2)}_{s-l,s-i}, \mathfrak{h}^{(2)}_{s-i+1,s})$$

and

$$
\begin{aligned}
F^{(l+1)}\binom{\mathbf{a}}{\mathbf{h}_{s-l,s}} &= F^{(l+1)}\binom{\mathbf{a}_{1,r-1}}{\mathbf{h}^{(1)}_{s-l,s}} \otimes a_r^{\mathfrak{h}^{(2)}_{s-l,s}} \\
&+ \sum_{i=1}^{l}(-1)^{r+i} F^{(l-i+1)}\begin{pmatrix} F^{(i)}\binom{\mathbf{a}}{\mathbf{h}^{(1)}_{s-i+1,s}} \\ \mathbf{h}^{(1)}_{s-l,s-i} \end{pmatrix} \otimes f(\mathfrak{h}^{(2)}_{s-l,s-i}, \mathfrak{h}^{(2)}_{s-i+1,s}).
\end{aligned}
$$

where $F^{(l+1)}\binom{\mathbf{a}_{1,r-1}}{\mathbf{h}_{s-l,s}} = F^{(l+1)}(\mathbf{h}_{s-l,s})$ *if* $r = 1$.

Proof. We prove the second formula. The proof of the first one is similar. It is clear that the lemma is valid for $l = 1$. Let $l > 1$ and suppose the result is valid for $l - 1$. To abbreviate we put

$$
\begin{aligned}
&\xi = u(l-1) + j + s \\
&\mathfrak{h}^{j(4)}_{s-l,s} = \mathfrak{h}^{(4)}_{s-l,j+1}\mathfrak{h}^{(3)}_{j+2,s}, \\
&\mathbf{h}^{j(3)}_{s-l,s} = \mathbf{h}^{(3)}_{s-l,j-1} \otimes h^{(3)}_j h^{(3)}_{j+1} \otimes \mathbf{h}^{(2)}_{j+2,s}, \\
&\mathbf{f}^{(2)}_j = f(h^{(2)}_j, h^{(2)}_{j+1})^{\mathbf{h}^{(2)}_{s-l,j-1}}, \\
&\mathbf{f}^{j(4)}_{s-l,s-i,s} = f(\mathfrak{h}^{(4)}_{s-l,j+1}\mathfrak{h}^{(3)}_{j+2,s-i}, \mathfrak{h}^{(3)}_{s-i+1,s}), \\
&\mathbf{f}^{(4)j}_{s-l,s-i,s} = f(\mathfrak{h}^{(4)}_{s-l,s-i}, \mathfrak{h}^{(4)}_{s-i+1,j+1}\mathfrak{h}^{(3)}_{j+2,s}) \\
&\mathbf{f}^{(2)}_{s-l,s-i,s} = f(\mathfrak{h}^{(2)}_{s-l,s-i}, \mathfrak{h}^{(2)}_{s-i+1,s}).
\end{aligned}
$$

We have:

$$
\begin{aligned}
F^{(l+1)}\binom{\mathbf{a}}{\mathbf{h}_{s-l,s}} &= \sum_{j=s-l}^{s-1}\sum_{u=0}^{r}(-1)^{\xi-1}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l)}\binom{\mathbf{a}_{u+1,r}}{\mathbf{h}^{j(3)}_{s-l,s}} \\
&= \sum_{j=s-l}^{s-1}\sum_{u=0}^{r-1}(-1)^{\xi-1}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l)}\binom{\mathbf{a}_{u+1,r-1}}{\mathbf{h}^{j(3)}_{s-l,s}} \otimes a_r^{\mathfrak{h}^{j(4)}_{s-l,s}} \\
&+ \sum_{j=s-l}^{s-2}\sum_{u=0}^{r}\sum_{i=1}^{s-j-1}(-1)^{\xi-1+r-u+i}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l-i)}\begin{pmatrix} F^{(i)}\binom{\mathbf{a}_{u+1,r}}{\mathbf{h}^{(2)}_{s-i+1,s}} \\ \mathbf{h}^{j(3)}_{s-l,s-i}\end{pmatrix} \otimes \mathbf{f}^{j(4)}_{s-l,s-i,s} \\
&+ \sum_{j=s-l+1}^{s-1}\sum_{u=0}^{r}\sum_{i=s-j}^{l-1}(-1)^{\xi-1+r-u+i}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l-i)}\begin{pmatrix} F^{(i)}\binom{\mathbf{a}_{u+1,r}}{\mathbf{h}^{j(3)}_{s-i,s}} \\ \mathbf{h}^{(3)}_{s-l,s-i-1}\end{pmatrix} \otimes \mathbf{f}^{(4)j}_{s-l,s-i-1,s}.
\end{aligned}
$$

Permuting the order of the summands, we obtain

$$
\begin{aligned}
F^{(l+1)}\binom{\mathbf{a}}{\mathbf{h}_{s-l,s}} &= \sum_{j=s-l}^{s-1}\sum_{u=0}^{r-1}(-1)^{\xi-1}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l)}\binom{\mathbf{a}_{u+1,r-1}}{\mathbf{h}^{j(3)}_{s-l,s}} \otimes a_r^{\mathfrak{h}^{j(4)}_{s-l,s}} \\
&+ \sum_{i=1}^{l-1}\sum_{u=0}^{r}\sum_{j=s-l}^{s-i-1}(-1)^{\xi-1+r-u+i}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l-i)}\begin{pmatrix} F^{(i)}\binom{\mathbf{a}_{u+1,r}}{\mathbf{h}^{(2)}_{s-i+1,s}} \\ \mathbf{h}^{j(3)}_{s-l,s-i}\end{pmatrix} \otimes \mathbf{f}^{j(4)}_{s-l,s-i,s} \\
&+ \sum_{i=2}^{l}\sum_{u=0}^{r}\sum_{j=s-i+1}^{s-1}(-1)^{\xi+r-u+i}\mathbf{a}_{1u}^{\mathbf{h}^{(1)}_{s-l,s}} \otimes \mathbf{f}^{(2)}_j \otimes F^{(l-i+1)}\begin{pmatrix} F^{(i-1)}\binom{\mathbf{a}_{u+1,r}}{\mathbf{h}^{j(3)}_{s-i+1,s}} \\ \mathbf{h}^{(3)}_{s-l,s-i}\end{pmatrix} \otimes \mathbf{f}^{(4)j}_{s-l,s-i,s}
\end{aligned}
$$

$$= F^{(l+1)}\binom{\mathbf{a}_{1,r-1}}{\mathbf{h}^{(1)}_{s-l,s}} \otimes a_r^{\mathfrak{h}^{(2)}_{s-l,s}} + \sum_{i=1}^{l}(-1)^{r+i}F^{(l-i+1)}\begin{pmatrix} F^{(i)} & \mathbf{a}_{1r} \\ & \mathbf{h}^{(1)}_{s-i+1,s} \\ \mathbf{h}^{(1)}_{s-l,s-i} & \end{pmatrix} \otimes \mathbf{f}^{(2)}_{s-l,s-i,s},$$

which ends the proof. □

In the following theorem we identify X_{rs} with $E \otimes \overline{H}^{\otimes^s} \otimes \overline{A}^{\otimes^r} \otimes E$.

Theorem B.4. *Let $\mathbf{x} = a_0 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1_E$, with $\mathbf{a} = a_1 \otimes \cdots \otimes a_r \in \overline{A}^{\otimes^r}$ and $\mathbf{h} = h_0 \otimes \cdots \otimes h_s \in H \otimes \overline{H}^{\otimes^s}$. For $2 \le l \le s$, we have*

$$d^l_{rs}(\mathbf{x}) = (-1)^{l(r+s)} a_0 \otimes \mathbf{h}_{0,s-l} \otimes F^{(l)}_r(\mathbf{h}^{(1)}_{s-l+1,s} \otimes \mathbf{a}) \otimes 1\#\mathfrak{h}^{(2)}_{s-l+1,s},$$

where $F^{(l)}_r(\mathbf{h}^{(1)}_{s-l+1,s} \otimes \mathbf{a}) = F^{(l)}_0(\mathbf{h}_{s-l+1,s})$ if $r = 0$.

Proof. Let us compute d^{l+1}_{rs} for $l \ge 1$. First we suppose the formula is valid for d^j_{rs} with $j \le l$ and we see that it is valid for d^{l+1}_{0s}. To abbreviate we write $\zeta_i = is+(l-i+1)(s-1)+1$. Using the inductive hypothesis and the fact that $\sigma^0\, d^l\big(a_0 \otimes \mathbf{h}_{0s} \otimes 1\#1\big) = 0$, we obtain:

$$d^{l+1}\big(1 \otimes \mathbf{h} \otimes 1_E\big) = -\sum_{i=1}^{l} \sigma^0\, d^{l+1-i}\, d^i\big(1 \otimes \mathbf{h} \otimes 1_E\big)$$

$$= \sum_{i=1}^{l}(-1)^{is+1}\sigma^0\, d^{l+1-i}\Big(1 \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)}(\mathbf{h}^{(1)}_{s-i+1,s}) \otimes 1\#\mathfrak{h}^{(2)}_{s-i+1,s}\Big)$$

$$= \sum_{i=1}^{l}\sigma^0\left((-1)^{\zeta_i} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1-i)}\begin{pmatrix} F^{(i)}(\mathbf{h}^{(1)}_{s-i+1,s}) \\ \mathbf{h}^{(1)}_{s-l,s-i}\end{pmatrix} \otimes f(\mathfrak{h}^{(2)}_{s-l,s-i}, \mathfrak{h}^{(2)}_{s-i+1,s})\#\mathfrak{h}^{(3)}_{s-l,s}\right)$$

$$= (-1)^{(l+1)s} 1 \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)}(\mathbf{h}^{(1)}_{s-l,s}) \otimes 1\#\mathfrak{h}^{(2)}_{s-l,s},$$

where the last equality follows from the definition of σ^0 and Lemma B.3. Now, we suppose the result is valid for $d^{l+1}_{r's}$ with $r' < r$ and we show that it is valid for d^{l+1}_{rs}. To abbreviate we write $\zeta_i = i(r+s) + (l-i+1)(r+s-1) + 1$.

$$d^{l+1}\big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1_E\big) = -\sum_{i=0}^{l} \sigma^0\, d^{l+1-i}\, d^i\big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1_E\big)$$

$$= (-1)^{r+1}\sigma_0\, d^{l+1}\big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1\big) - (-1)^{r+s}\sigma_0\, d^l\big(1 \otimes \mathbf{h}_{0,s-1} \otimes \mathbf{a}^{h^{(1)}_s} \otimes 1\#h^{(2)}_s\big)$$

$$- \sum_{i=2}^{l}\sigma^0\, d^{l+1-i}\left((-1)^{i(r+s)} \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)}\binom{\mathbf{a}}{\mathbf{h}^{(1)}_{s-i+1,s}} \otimes 1\#\mathbf{h}^{(2)}_{s-i+1,s}\right)$$

$$= (-1)^{r+1}\sigma_0\, d^{l+1}\big(1 \otimes \mathbf{h} \otimes \mathbf{a} \otimes 1\big)$$

$$- \sum_{i=1}^{l}\sigma^0\, d^{l+1-i}\left((-1)^{i(r+s)} \otimes \mathbf{h}_{0,s-i} \otimes F^{(i)}\binom{\mathbf{a}}{\mathbf{h}^{(1)}_{s-i+1,s}} \otimes 1\#\mathbf{h}^{(2)}_{s-i+1,s}\right)$$

$$= \sigma^0\Bigg((-1)^{(l+1)(r+s-1)+r+1} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)}\binom{\mathbf{a}_{1,r-1}}{\mathbf{h}^{(1)}_{s-l,s}} \otimes a_r^{\mathfrak{h}^{(2)}_{s-l,s}}\#\mathfrak{h}^{(3)}_{s-l,s}$$

$$+ \sum_{i=1}^{l}(-1)^{\zeta_i} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1-i)}\begin{pmatrix} F^{(i)} & \mathbf{a} \\ & \mathbf{h}^{(1)}_{s-i+1,s} \\ \mathbf{h}^{(1)}_{s-l,s-i} & \end{pmatrix} \otimes f(\mathbf{h}^{(2)}_{s-l,s-i}, \mathbf{h}^{(2)}_{s-i+1,s})\#\mathbf{h}^{(3)}_{s-l,s}\Bigg)$$

$$= (-1)^{(l+1)(r+s)} \otimes \mathbf{h}_{0,s-l-1} \otimes F^{(l+1)}\binom{\mathbf{a}}{\mathbf{h}^{(1)}_{s-l,s}} \otimes 1\#\mathfrak{h}^{(2)}_{s-l,s},$$

where the last equality follows from the definition of σ^0 and Lemma B.3. □

Remark B.5. When H is a group algebra $k[G]$ and the 2-cocycle f takes its values in the center of A, then

$$d^l_{rs}\big(a_0\otimes \mathbf{g}_{0s}\otimes \mathbf{a}_{1r}\otimes 1_E\big) = (-1)^{l(r+s)} a_0\otimes \mathbf{g}_{0,s-l}\otimes F^{(l)}_0(\mathbf{g}_{s-l+1,s}) * \mathbf{a}_{1r}\otimes 1\#\mathfrak{g}_{s-l+1,s},$$

where $*$ denotes the shuffle product:

$$\mathbf{a}_{1r} * \mathbf{b}_{1l} = \sum_{0\le i_1\le\cdots\le i_r\le l} (-1)^{i_1+\cdots+i_l} b_1\otimes\cdots\otimes b_{i_1}\otimes a_1\otimes b_{i_1+1}\otimes\cdots\otimes b_{i_r}\otimes a_r\otimes b_{i_r+1}\otimes\cdots .$$

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Decompositions

5.1. A decomposition of $(\widehat{X}_*, \widehat{d}_*)$. Let $[H,H]$ be the k-submodule of H spanned by the set of all elements $ab-ba$ $(a,b\in H)$. It is easy to see that $[H,H]$ is a coideal in H. Let $\breve{H}$ be the quotient coalgebra $H/[H,H]$. Given $h\in H$, we let $[h]$ denote the class of h in $\breve{H}$. Given a subcoalgebra C of $\breve{H}$ and a right $\breve{H}$-comodule (N,ρ), we put $N^C=\{n\in N\colon \rho(n)\in N\otimes C\}$. It is well known that if $\breve{H}$ decomposes as a direct sum of subcoalgebras C_i $(i\in I)$, then $N=\bigoplus_{i\in I} N^{C_i}$.

Now, let us assume that M is a Hopf bimodule. That is, M is an E-bimodule and a right H-comodule, and the coaction $m\mapsto m^{(0)}\otimes m^{(1)}$ verifies:

$$((a\#h)m(b\#l))^{(0)}\otimes((a\#h)m(b\#l))^{(1)} = (a\#h^{(1)})m^{(0)}(b\#l^{(1)})\otimes(a\#h^{(2)})m^{(1)}(b\#l^{(2)}).$$

For each $n\ge 0$, $\widehat{X}_n$ is an $\breve{H}$-comodule via

$$\begin{aligned}&\rho_n(m\otimes_A(1\#h_1)\otimes_A\cdots(1\#h_s)\otimes \mathbf{a}_{1r})\\ &= m^{(0)}\otimes_A(1\#h_1^{(1)})\otimes_A\cdots(1\#h_s^{(1)})\otimes\mathbf{a}_{1r}\otimes[m^{(1)}h_1^{(2)}\dots h_s^{(2)}]\end{aligned}$$

where $r+s=n$. Moreover, the map $\rho_*\colon \widehat{X}_*\to\widehat{X}_*\otimes\breve{H}$ is a morphism of complexes. This fact implies that if C is a subcoalgebra of $\breve{H}$, then $\widehat{d}_n(\widehat{X}_n^C)\subseteq\widehat{X}_{n-1}^C$. We consider the subcomplex $(\widehat{X}_*^C,\widehat{d}^C)$ of $(\widehat{X}_*,\widehat{d}_*)$, with modules $\widehat{X}_n^C$, and we let $\mathrm{H}_*^C(E,M)$ denote its homology. The filtration of $(\widehat{X}_*,\widehat{d}_*)$ introduced above Corollary 2.2 induces a filtration $F^0(\widehat{X}_*^C)\subseteq F^1(\widehat{X}_*^C)\subseteq\dots$ on $(\widehat{X}_*^C,\widehat{d}_*^C)$. It is clear that $F^i(\widehat{X}_n^C)=\bigoplus_{0\le s\le i}\widehat{X}_{n-s,s}^C$. Hence, we have a convergent spectral sequence

$$E^1_{rs}=\mathrm{H}_r(A,(M\otimes_A(E/A)^{(\otimes_A)^s})^C)\Rightarrow \mathrm{H}^C_{r+s}(E,M). \tag{1}$$

By the previous discussion, if $\breve{H}$ decomposes as a direct sum of subcoalgebras C_i $(i\in I)$, then $(\widehat{X}_*,\widehat{d}_*)=\bigoplus_{i\in I}(\widehat{X}_*^{C_i},\widehat{d}_*^{C_i})$. Consequently $\mathrm{H}_*(E,M)=\bigoplus_{i\in I}\mathrm{H}_*^{C_i}(E,M)$ and the spectral sequence of Corollary 2.2 decomposes as direct sum of the above mentioned spectral sequences. Finally, the canonical normalized Hochschild complex $(M\otimes\overline{E}^{\otimes^*},b_*)$ is a differential $\breve{H}$-comodule via

$$\rho_n(m\otimes a_1\#h_1\otimes\cdots\otimes a_n\#h_n)=m^{(0)}\otimes a_1\#h_1^{(0)}\otimes\cdots\otimes a_n\#h_n^{(0)}\otimes[m^{(1)}h_1^{(1)}\cdots h_n^{(0)}]$$

and the morphism from $(\widehat{X}_*,\widehat{d}_*)$ to $(M\otimes\overline{E}^{\otimes^*},b_*)$, induced by the map ϕ_* of Section 1.2, is an $\breve{H}$-comodule morphism. Hence, $\mathrm{H}_*^C(E,M)=\mathrm{H}_*(((M\otimes\overline{E}^{\otimes^*})^C,b_*))$, for each subcoalgebra C of $\breve{H}$. Note that the filtration of the normalized Hochschild complex given below Corollary 2.2 induces a filtration of $((M\otimes\overline{E}^{\otimes^*})^C,b_*)$. The spectral sequence of this filtration is isomorphic to the spectral sequence (1).

5.2. A decomposition of $(\overline{X}_*,\overline{d}_*)$. Now, suppose the cocycle f is invertible. A direct computation shows that the $\breve{H}$-coaction of $(\overline{X}_*,\overline{d}_*)$, obtained transporting the one of $(\widehat{X}_*,\widehat{d}_*)$ through $\theta_*\colon(\widehat{X}_*,\widehat{d}_*)\to(\overline{X}_*,\overline{d}_*)$, is given by

$$m\otimes\mathbf{a}\otimes\mathbf{h}\mapsto m^{(0)}\otimes\mathbf{a}\otimes h_1^{(2)}\otimes\cdots\otimes h_s^{(2)}\otimes m^{(1)}S(h_s^{(1)})\cdots S(h_1^{(1)})h_1^{(3)}\cdots h_s^{(3)}, \tag{2}$$

where $\mathbf{a} \in \overline{A}^{\otimes^r}$ and $\mathbf{h} = h_1 \otimes \cdots \otimes h_s \in \overline{H}^{\otimes^s}$. For each subcoalgebra C of $\breve{H}$, we consider the subcomplex $(\overline{X}^C_*, \overline{d}^C_*)$ of $(\overline{X}_*, \overline{d}_*)$ with modules $\overline{X}^C_n$. It is clear that θ_* induces an isomorphism $\theta^C_*\colon (\widehat{X}^C_*, \widehat{d}^C_*) \to (\overline{X}^C_*, \overline{d}^C_*)$. Let $F^0(\overline{X}^C_*) \subseteq F^1(\overline{X}^C_*) \subseteq \dots$ be the filtration of $(\overline{X}^C_*, \overline{d}^C_*)$ obtained transporting the one of $(\widehat{X}^C_*, \widehat{d}^C_*)$, through θ^C_*. It is clear that $F^i(\overline{X}^C_n) = \bigoplus_{0\le s\le i} \overline{X}^C_{n-s,s}$. Moreover, from (2) it follows that if $\breve{H}$ is cocommutative, then $\overline{X}^C_{rs} = M^C \otimes \overline{A}^{\otimes^r} \otimes \overline{H}^{\otimes^s}$. Hence, when $\breve{H}$ is cocommutative and H is a flat k-module, the spectral sequence of (1) verifies $E^1_{rs} = \mathrm{H}_r(A, M^C) \otimes \overline{H}^{\otimes^s}$ and $E^2_{rs} = \mathrm{H}_r(H, \mathrm{H}_s(A, M^C))$, where $\mathrm{H}_r(A, M^C)$ is a left H-module via the action introduced in Proposition 3.1.2.

5.3. An application to group crossed products. Let G be a finite group, $E = A\#_f G$ a crossed product and $M = \bigoplus_{g\in G} M_g$ a G-graded E-bimodule. Let $\langle G\rangle$ be the set of conjugation classes of G. For each $g \in G$ we let $\langle g\rangle$ denote the conjugation class of g and we write $M_{[g]} = \bigoplus_{h\in\langle g\rangle} M_h$. The complex $(\overline{X}^*, \overline{d}^*)$ decomposes as a direct sum of chain complexes $(\overline{X}^*, \overline{d}^*) = \bigoplus_{\langle g\rangle\in\langle G\rangle} (\overline{X}^*, \overline{d}^*)^{\langle g\rangle}$, where

$$(\overline{X}^*, \overline{d}^*)^{\langle g\rangle} = \bigoplus_{r+s=n} \mathrm{Hom}_k(\overline{A}^{\otimes^r} \otimes \overline{k[G]}^{\otimes^s}, M_{[g]})$$

Hence $\mathrm{H}^n(E, M) = \bigoplus_{\langle g\rangle\in\langle G\rangle} \mathrm{H}^n((\overline{X}^*, \overline{d}^*)^{\langle g\rangle})$. The filtration introduced in Corollary 3.2.3 induces a filtration in each one of the terms of the decomposition of $(\overline{X}^*, \overline{d}^*)$. Taking the spectral sequences of these filtrations we get converging spectral sequences

$$E^{rs}_2 = \mathrm{H}^s(G, \mathrm{H}^r(A, M_{[g]})) \Rightarrow \mathrm{H}^{r+s}((\overline{X}^*, \overline{d}^*)^{\langle g\rangle}) \qquad \langle g\rangle \in \langle G\rangle.$$

Now, it is immediate that the spectral sequence of Corollary 3.2.3 is the direct sum of these spectral sequences. Finally the canonical complex $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*)$ decomposes as a direct sum $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*) = \bigoplus_{\langle g\rangle\in\langle G\rangle} (\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M)^{\langle g\rangle}, b^*)$, where

$$\mathrm{Hom}_k(\overline{E}^{\otimes^n}, M)^{\langle g\rangle} = \bigoplus_{\substack{g_1,\dots,g_{n+1}\in G\\ g_n^{-1}\cdots g_1^{-1} g_{n+1}\in\langle g\rangle}} \mathrm{Hom}_k(\overline{A\#g_1} \otimes \cdots \otimes \overline{A\#g_n}, M_{g_{n+1}})$$

We assert that $\mathrm{H}^n((\overline{X}^*, \overline{d}^*)^{\langle g\rangle}) = \mathrm{H}^n((\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M)^{\langle g\rangle}, b^*))$. In fact, this follows from the fact that the complex $(\widehat{X}^*, \widehat{d}^*)$ introduced in Section 2 decomposes as direct sums of chain complexes $(\widehat{X}^*, \widehat{d}^*) = \bigoplus_{\langle g\rangle\in\langle G\rangle} (\widehat{X}^*, \widehat{d}^*)^{\langle g\rangle}$, where

$$\widehat{X}^n = \bigoplus_{r+s=n} \bigoplus_{\substack{g_1,\dots,g_{s+1}\in G\\ g_s^{-1}\cdots g_1^{-1} g_{s+1}\in\langle g\rangle}} \mathrm{Hom}_k(\overline{A}^{\otimes^r} \otimes kg_1 \otimes \cdots \otimes kg_s, M_{g_{s+1}})$$

and from the fact that the morphism $\theta^*\colon (\overline{X}^*, \overline{d}^*) \to (\widehat{X}^*, \widehat{d}^*)$ introduced in Theorem 3.2.1 and the morphism from $(\mathrm{Hom}_k(\overline{E}^{\otimes^*}, M), b^*)$ to $(\widehat{X}^*, \widehat{d}^*)$, induced by the map ϕ_* of Section 1.2, preserve the decompositions.

Miscelaneas

2.1.4. Compatibility with the canonical decomposition. Let us assume that $k \supseteq \mathbb{Q}$, H is cocommutative, A is commutative, M is symmetric as an A-bimodule and the cocycle f takes its values in k. In [G-S1] was obtained a decomposition of the canonical Hochschild complex $(M \otimes \overline{A}^{\otimes^*}, b_*)$. It is easy to check that the maps $\widehat{d}_0$ and $\widehat{d}_1$ are compatible with this decomposition. Since $\widehat{d}_l = 0$ for all $l \geq 2$, we obtain a decomposition of $(\widehat{X}_*, \widehat{d}_*)$, and then a decomposition of $\mathrm{H}_*(E, M)$.

2.2.3. Compatibility with the canonical decomposition. Assume that $k \supseteq \mathbb{Q}$, H is cocommutative, A is commutative, M is symmetric as an A-bimodule and the cocycle f takes its values in k. Then, the Hochschild cohomology $\mathrm{H}^*(E, M)$ has a decomposition similar to the one obtained in 2.1.4 for the Hochschild homology.

3.1.7. An application to Tor_*^E. Let k be a field, B an arbitrary k-algebra, M a right B-module and N a left B-module. It is well known that $\mathrm{Tor}_*^B(M, N) \simeq \mathrm{H}_*(B, N \otimes M)$ (here $N \otimes M$ is an B-bimodule via $a(n \otimes m)b = an \otimes mb$). This fact and Corollary 3.1.3 show that if k is a field, M is a right E-module and N is a left E-module, then there is a convergent spectral sequence

$$E_{rs}^2 = \mathrm{H}_r(H, \mathrm{Tor}_s^A(M, N)) \Rightarrow \mathrm{Tor}_{r+s}^E(M, N).$$

3.2.6. An application to Ext_E^*. Let k be a field, B an arbitrary k-algebra and M, N two left B-modules. It is well known that $\mathrm{Ext}_B^*(M, N) \simeq \mathrm{H}^*(B, \mathrm{Hom}_k(M, N))$ (here $\mathrm{Hom}_k(M, N)$ is an B-bimodule via $(a\varphi b)(m) = a\varphi(bm)$). This fact and Corollary 3.2.3 show that if k is a field and M and N are left E-modules, then there is a convergent spectral sequence

$$E_2^{rs} = \mathrm{H}^r(H, \mathrm{Ext}_A^s(M, N)) \Rightarrow \mathrm{Ext}_E^{r+s}(M, N).$$

As a corollary we obtain that $\mathrm{gl.\,dim}(E) \leq \mathrm{gl.\,dim}(A) + \mathrm{gl.\,dim}(H)$, where gl. dim denotes the left global dimension. Note that this result implies Maschke's Theorem for crossed product, as it was established in [B-M].

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