Orbits of positive operators from a differentiable viewpoint

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Abstract

Let $\mathcal{A}$ be a unital C*-algebra and G the group of units of $\mathcal{A}$. A geometrical study of the action of G over the set $\mathcal{A}^+$ of all positive elements of $\mathcal{A}$ is presented. The orbits of elements with closed range by this action are provided with a structure of differentiable homogeneous space with a natural connection. The orbits are partitioned in "components" which also have a rich geometrical structure.

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1 Introduction

The differential geometry of the space of $n \times n$ positive definite complex matrices turns to be a relevant subject in problems coming from many different areas. To mention only a few, we refer the reader to the work of Ohara et al [29], [30], in linear systems, Amari [1], [2], Kass [21], Campbell [5], Murray and Rice [26] in statistics, Bougerol [4] in Kalman-Bucy filters, Liverani and Wojtkowski [25] in Lagrangian geometry, Hiai and Petz [19]-[20], Petz [31]-[32] in quantum systems. Moreover, Uhlmann [34], [35] proposed the extension to mixed states of the Berry phase using techniques from differential geometry on the set of density operators, i.e. trace class positive operators, on a Hilbert space. Uhlmann's work has been extended by Dabrowski and Jadczyk [12], Dabrowski and Grosse [11], Dittmann [13], [14], Dittmann and Rudolph [15], [16] and Uhlmann himself [36]. Uhlmann's approach is an invitation to study the space $L(\mathcal{H})^+$ of all positive bounded linear operators on the Hilbert space $\mathcal{H}$ from a differentiable viewpoint. A description of the differential geometry of $GL(\mathcal{H})^+$ (i.e. the set of invertible positive linear operators) has been done in [9], [6] and extended to the set of closed range positive operators in [7]. In [7], the set $L(\mathcal{H})^+$ is partitioned in certain "components", following ideas of Thompson [33], and each component is studied as a differential manifold. In [8] there is a comparison between these results and Uhlmann's.

The present paper continues this line of research. Instead of working on $L(\mathcal{H})$, we shall deal in a C*-algebras setting. So, in what follows, $\mathcal{A}$ denotes a unital C*-algebra represented on a Hilbert space $\mathcal{H}$, G is the group of invertible elements of $\mathcal{A}$, $\mathcal{U}$ the unitary group of $\mathcal{A}$, $\mathcal{A}^+$ the set of positive (semidefinite) operators, $\mathcal{A}^+_{cr}$ the subset of $\mathcal{A}^+$ of positive operators with closed range and $\mathcal{P}$ the set of orthogonal projections in $\mathcal{A}$. There is a natural action of G over $\mathcal{A}^+$ defined by $L_g a = gag^*$. In this paper we study the geometric structure of the orbits $\mathcal{O}_a = \{gag^* : g \in G\}$ corresponding to elements of $\mathcal{A}^+_{cr}$. In particular, when $a \in \mathcal{A}^+$ is invertible, the orbit of a is G^+, the set of positive invertible operators. G^+ is a homogeneous reductive space of G, with a canonical connection and a Finsler structure for which geodesics are short. A complete description of this structure and its properties can be found in [9], [6]. The cone $\mathcal{A}^+$ can also be described as a disjoint union of equivalence classes or "components", with a complete metric defined on each of them, the so called Thompson metric (for more details about this metric see [28]). Again each component is a homogeneous space of a certain group and has a Finsler metric which coincides with the Thompson metric, even if the elements of the component do not have closed range. In particular, the component of any $a \in G^+$ is G^+, so that in this case the orbit is also the Thompson component; if $a \in \mathcal{A}^+_{cr}$, the component of a can be identified with the invertible elements of $\mathcal{L}(R(a))$, and the acting group is the closed subgroup of G of the elements of G preserving $R(a)$ (see [6], [7]). It turns out that each orbit is the union of certain Thompson components: the component of b is included in the orbit of a if and only if the orthogonal projection onto the range of b belongs to the unitary orbit of the orthogonal projection onto the range of a, denoted by p_a. The aim of this paper is to provide the orbit $\mathcal{O}_a$ with a structure of differentiable homogeneous space such that each component belonging to $\mathcal{O}_a$ remains a submanifold. The case $\mathcal{A} = L(\mathcal{H})$ is studied in detail in order to show that some changes have to be done in the topology of $\mathcal{O}_a$ to obtain a structure of that type. Thus, we define a metric d on $\mathcal{O}_a$, which coincides with the one induced by the usual norm on each component. With this new topology, $(\mathcal{O}_a, d)$ can be given a structure of a differentiable manifold. Moreover, if $\mathcal{UO}_p = \{upu^* : u \in \mathcal{U}\}$ is the unitary orbit of $p \in \mathcal{P}$, then the metric d guarantees the smoothness of the map $\alpha : \mathcal{O}_a \to \mathcal{UO}_{p_a}$, $b \mapsto p_b$, whose fibres are the Thompson components. Also there exist local cross sections for the projection $\pi_a : G \to \mathcal{O}_a$, $\pi_a(g) = gag^*$ and the diagram

$$\begin{array}{ccc} G & \xrightarrow{\pi_a} & \mathcal{O}_a \\ \pi_{p_a} & \searrow & \downarrow \alpha \\ & & \mathcal{UO}_{p_a} \end{array}$$

is commutative, where $\pi_{p_a}(g) = p_{gp_ag^{-1}}$, $g \in G$, the natural action of G over $\mathcal{UO}_{p_a}$ (see [3]). Moreover $(\mathcal{O}_a, d)$ becomes a differentiable homogeneous space of G and the map α turns to be a locally trivial fibre bundle. A smooth connection is defined on $(G, \mathcal{O}_a)$ which provides a transport equation and a parallel lift for any smooth curve in $\mathcal{O}_a$. The parallel lift of a given curve is the solution of a certain variational problem. Finally, the existence of geodesics for this connection is studied and it is proved that two points on the same component are connected by a unique geodesic.

Section 2 contains a description of the differential geometry of G^+ and, in general, of all Thompson components of $\mathcal{A}^+$. It also contains certain known results on the geometry of the set $\mathcal{P}$. There is a natural action of G over the set $\mathcal{A}^+_{cr}$ of all closed range positive elements of A given by $(v, a) \to vav^*$ ($v \in G$, $a \in \mathcal{A}^+_{cr}$). Then $\mathcal{A}^+_{cr}$ is partitioned as the union of different orbits of this action and each orbit is the union of disjoint (Thompson) components. In Section 3, the orbit $\mathcal{O}_a$ of a closed range positive element $a \in \mathcal{A}$ is studied as a homogeneous space of G. Section 4 contains the description of the orbit $\mathcal{O}_a$ as a differentiable manifold and the Thompson component C_a as a submanifold of $\mathcal{O}_a$. In section 5 a connection is defined on the homogeneous space $G \to \mathcal{O}_a$, from which one gets a covariant derivative on the tangent bundle of $\mathcal{O}_a$ and the geodesics of this connection are determined.

2 Preliminaries

Let $\mathcal{H}$ denote a Hilbert space, $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$ and $\mathcal{A} \subseteq L(\mathcal{H})$ a C*-algebra. Let G be the group of invertible elements of $\mathcal{A}$, $\mathcal{U}$ the unitary group of $\mathcal{A}$, $\mathcal{A}^+$ the set of positive semidefinite operators, $\mathcal{A}^+_{cr}$ the subset of $\mathcal{A}^+$ of positive operators with closed range and $\mathcal{P}$ the set of orthogonal projections in $\mathcal{A}$.

For every $c \in \mathcal{A}$, $R(c)$ denotes the range of c and $\ker(c)$ its kernel. For $c \in \mathcal{A}^+_{cr}$, $c^\dagger$ denotes the Moore-Penrose pseudoinverse of c and $p_c = cc^\dagger$, the orthogonal projection onto $R(c)$. For $\mathcal{M} \subseteq \mathcal{H}$ a closed subspace, $P_{\mathcal{M}} \in L(\mathcal{H})$ denotes the orthogonal projection onto $\mathcal{M}$. It is well known (see, for example, [18]) that, if $c \in \mathcal{A}^+_{cr}$, then $c^\dagger \in \mathcal{A}$ and $p_c \in \mathcal{P} \subseteq \mathcal{A}$. In this sense, $c \to c^\dagger$ is a well defined mapping from $\mathcal{A}^+_{cr}$ into itself. Its continuity properties have been determined by Labrousse and Mbekhta [23].

LEMMA 2.1 *Let $p \in \mathcal{P}$ and $q \in \mathcal{A}$ such that $q^2 = q$.*

1. *If $q = gpg^{-1}$, for some $g \in G$, then there exists $u \in \mathcal{U}$ such that $p_q = P_{gR(p)} = upu^*$.*

2. *If $q = q^*$ and $\|q - p\| < 1$, then there exists $u \in \mathcal{U}$ such that $q = upu^*$.*

Proof. The first item has been proven in [3]. Item 2 has been proven by B. Sz.-Nagy [27]. A nice proof of this fact, due to Kato [22], provides a positive invertible operator $a = 1 - (p - q)^2 \in G^+$ such that the unitary operator $u = qa^{-1/2}p + (1 - q)a^{-1/2}(1 - p)$ verifies $u \in \mathcal{A}$ and $upu^* = q$. ∎

Consider the following equivalence relation on the closed convex cone $\mathcal{A}^+$: for $a, b \in \mathcal{A}^+$, $a \sim b$ if there exist $r, s > 0$, such that $ra \le b \le sa$. C_a denotes the equivalence class or "Thompson component" of a. Each component is a closed convex cone and a complete metric can be naturally defined on it. As a consequence of Douglas majorization theorem [17], it follows that $a, b \in \mathcal{A}^+$ belong to the same component if and only if $R(a^{1/2}) = R(b^{1/2})$. For $a, b \in \mathcal{A}^+_{cr}$ it turns out that $a \sim b$ if and only if $R(a) = R(b)$ because $R(c) = R(c^{1/2})$ for all $c \in \mathcal{A}^+_{cr}$. In particular, for $a \in G^+$ it holds $C_a = G^+$; observe that the *orbit* of a under the action of G over $\mathcal{A}^+$, $\mathcal{O}_a = \{gag^* : g \in G\}$, also coincides with G^+. In [6] there is a study of G^+ as a homogeneous space of G with structural group $\mathcal{U}$. More generally, for every closed subspace $\mathcal{S}$ of $\mathcal{H}$ and $a \in \mathcal{A}^+_{cr}$ with range $\mathcal{S}$, the Thompson component C_a is the set of all $b \in \mathcal{A}^+$ such that $R(b) = \mathcal{S}$, so that C_a can be naturally identified with the space of all positive invertible operators in $L(\mathcal{S})$ and, thus, C_a is a homogeneous space of $GL(\mathcal{S})$ (see [7] for details).

For $p \in \mathcal{P}$, denote by $\mathcal{UO}_p$ the unitary orbit of p, $\mathcal{UO}_p = \{upu^* : u \in \mathcal{U}\}$. The following description of $\mathcal{UO}_p$ can be found in [3]: the action of G over $\mathcal{UO}_p$ given by the map $\pi_p : G \to \mathcal{UO}_p$, $\pi_p(g) = p_{gpg^{-1}} = P_{gR(p)}$, $g \in G$ defines an analytic homogeneous space. The structure group is the isotropy group

$$I_p = \{g \in G : P_{gR(p)} = p\} = \{g \in G : (1 - p)gp = 0 \text{ and } pgp \in G_{p\mathcal{A}p}\},$$

which is an union of connected components of the group of invertible elements of the subalgebra

$$\mathcal{T}_p(\mathcal{A}) = \{a \in \mathcal{A} : (1 - p)ap = 0\} \subseteq \mathcal{A}$$

of p-upper triangular elements of $\mathcal{A}$. This algebra is the tangent space at the identity of the group I_p. It is also the kernel of the differential $(T\pi_p)_1$ of π_p at 1, because $(T\pi_p)_1(a) = (1-p)ap$, for all $a \in \mathcal{A}$. The homogeneous space given by $\pi_p : G \to \mathcal{UO}_p$ admits a reductive structure given by the horizontal space $H_p = (1-p)\mathcal{A}p$ which can be homogeneously transported to all elements of G. Note that this horizontal space can be naturally identified with the tangent space $T(\mathcal{UO}_p)_p$ of $\mathcal{UO}_p$ at p, via the map $x \mapsto x + x^*$, $x \in H_p$. This connection gives rise to natural covariant derivative and parallel transport on $\mathcal{UO}_p$. Given $x \in (1-p)\mathcal{A}p$ and $X = x - x^*$, the geodesic γ_x starting at p with $\dot{\gamma_x}(0) = x + x^*$ is

$$\gamma_x(t) = e^{tX} p e^{-tX} \quad , \quad t \in \mathbb{R}.$$

The Finsler structure of $\mathcal{UO}_p$ is obtained by taking the usual norm of the tangent vectors at any point. It allows to compute the length of smooth curves with the usual formulae. In [10], it has been proved that if $p, r \in \mathcal{UO}_p$ and $\|p - r\| < 1$, then there exists a unique geodesic of $\mathcal{UO}_p$ joining them and it has minimal length.

3 The orbit of a positive operator with closed range

Let $\mathcal{A} \subseteq L(\mathcal{H})$ be a C*-algebra and consider the action $L : G \times \mathcal{A}^+ \to \mathcal{A}^+$

$$L(g, b) = L_g(b) = gbg^* \quad g \in G, \; b \in \mathcal{A}^+.$$

For every $a \in \mathcal{A}^+_{cr}$ let p_a be the orthogonal projection onto $R(a)$. The *orbit* of a is

$$\mathcal{O}_a = \{L_g a : g \in G\} = \{gag^* : g \in G\}.$$

Observe that $\mathcal{O}_a \subseteq \mathcal{A}^+_{cr}$, i.e. $L_g(\mathcal{A}^+_{cr}) \subset \mathcal{A}^+_{cr}$, for all $g \in G$. Denote $\mathcal{UO}_p$ the unitary orbit of p, i.e. $\mathcal{UO}_p = \{upu^* \; : \; u \in \mathcal{U}\}$.

LEMMA 3.1 *Let $a \in \mathcal{A}^+_{cr}$ and $p = p_a$. Then*

1. $\mathcal{O}_a = \mathcal{O}_p$.
2. *If $b \in \mathcal{A}^+_{cr}$, then $b \in \mathcal{O}_a$ if and only if $p_b \in \mathcal{UO}_p$.*

Proof. Consider $g = (a^{1/2})^\dagger + (1-p)$. Then $g \in G$ and $gag^* = p$, so that p and a belong to the same orbit. Then $\mathcal{O}_a = \mathcal{O}_p$, because two orbits which are not disjoint must be equal.

If $b \in \mathcal{O}_a$ there exists $g \in G$ such that $gag^* = b$ and then $R(b) = g(R(a))$ or, equivalently, $R(p_b) = g(R(p))$. Then $q = gqg^{-1}$ is an oblique projection with $R(q) = R(p_b)$. By Lemma 2.1, there exists $u \in \mathcal{U}$ such that $p_b = upu^*$ and then $p_b \in \mathcal{O}_a$.

Conversely if $p_b \in \mathcal{UO}_p$ there exists a $u \in \mathcal{U}$ such that $upu^* = p_b$. If $v = b^{1/2} + 1 - p_b$ then $v \in G$ and $b = vp_bv^* = vupu^*v^*$, so that $b \in \mathcal{O}_a$. ∎

COROLLARY 3.2 *Fix $a \in \mathcal{A}^+_{cr}$ and $p = p_a$. Then $\mathcal{UO}_p = \mathcal{O}_a \cap \mathcal{P}$.*

For $a \in \mathcal{A}^+_{cr}$ let $\pi_a : G \to \mathcal{O}_a$, $\pi_a(g) = L_g a$ and define the retraction

$$\alpha : \mathcal{O}_a \to \mathcal{UO}_{p_a} \qquad \alpha(b) = bb^\dagger = p_b \ , \quad b \in \mathcal{O}_a.$$

Consider also the mapping $\pi_{p_a} : G \to \mathcal{UO}_{p_a}$ $\pi_{p_a}(g) = p_{gp_a g^{-1}}$, $g \in G$; π_{p_a} is a natural action of G over $\mathcal{UO}_{p_a}$ (see [3]). Observe that $\alpha \circ \pi_a : G \to \mathcal{UO}_{p_a}$, $\alpha \circ \pi_a(g) = \alpha(gag^*) = P_{gR(a)} = \pi_{p_a}(g)$. Therefore the following diagram is commutative:

$$\begin{tikzcd} G \arrow[r, "\pi_a"] \arrow[dr, "\pi_{p_a}"'] & \mathcal{O}_a \arrow[d, "\alpha"] \\ & \mathcal{UO}_{p_a}. \end{tikzcd} \tag{1}$$

With this diagram it is easy to relate the orbit of a with the Thompson components of its elements: in fact by Lemma 3.1 it follows that, if $\mathcal{C}_a$ is the Thompson component of a, then $b \in \mathcal{O}_a$ if and only if there exists $u \in \mathcal{U}$ such that $ubu^* \in \mathcal{C}_a$. Then

$$\mathcal{O}_a = \bigcup_{u \in \mathcal{U}} u\mathcal{C}_a u^* = \bigcup_{u \in \mathcal{U}} \mathcal{C}_{upu^*}.$$

PROPOSITION 3.3 *Let $b \in \mathcal{O}_a$. Then $\alpha^{-1}(\{p_b\}) = C_a$.*

Proof. From the characterization of the Thompson components stated in the Preliminaries, $c \in \mathcal{C}_b$ if and only if $R(c^{1/2}) = R(b^{1/2})$, or equivalently $R(c) = R(b)$ because $R(c)$ and $R(b)$ are closed; but $\alpha(c) = cc^\dagger = p_c$, so that $\alpha(c) = p_b$ if and only if $p_c = p_b$ if and only if $R(c) = R(b)$, or equivalently $c \in \mathcal{C}_b$. ∎

Diagram (1) has at least two disadvantages: first, the map α may not be continuous ; second, π_a may not have local cross sections. see [18].

In the following theorem we state necessary and sufficient conditions for α to be continuous and to obtain continuous local cross sections for π_a, when $\mathcal{A} = L(\mathcal{H})$. We also give another characterization of the orbits.

THEOREM 3.4 *Let $\mathcal{H}$ be an infinite dimensional Hilbert space and $\mathcal{A} = L(\mathcal{H})$. For $n, m \in \mathbb{N} \cup \{0, \infty\}$ such that $n + m = \infty$, define*

$$\mathcal{A}^+_{n,m} = \{a \in \mathcal{A}^+_{cr} : \dim \ker a = n, \dim R(a) = m\}.$$

Let $a \in \mathcal{A}^+_{n,m}$. Then:

1. *$\mathcal{A}^+_{n,m} = \mathcal{O}_a$.*

2. *The mapping $\alpha : \mathcal{O}_a \to \mathcal{UO}_p$, $\alpha(b) = p_b$ is continuous if and only if $n < \infty$ or $m < \infty$.*

3. *The map $\pi_a : G \to \mathcal{O}_a$, $\pi_a(g) = gag^*$ has continuous local cross sections if and only if $n < \infty$ or $m < \infty$.*

Proof. The identity $\mathcal{O}_a = \mathcal{A}^+_{n,m}$ if $a \in \mathcal{A}^+_{n,m}$ can be easily deduced from Lemma 3.1.

Suppose that $a \in \mathcal{A}^+_{n,m}$ with $n < \infty$ (the case $m < \infty$ is similar). We recall the definition of the (Apostol's) *reduced minimum modulus* of $a \in \mathcal{A}$:

$$\gamma(a) = \max\{c \geq 0 : \|a\xi\| \geq c\|\xi\|,\ \xi \in (\ker a)^\perp\} = \inf\{\|a\xi\| : \xi \in (\ker a)^\perp,\ \|\xi\| = 1\},$$

Observe that $\gamma(a) > 0$ if and only if $R(a)$ is closed. In particular, if $a \geq 0$, then $a \in \mathcal{A}^+_{cr}$ if and only if $\gamma(a) > 0$ if and only if 0 is an isolated point of the spectrum of a. Denote by $r = \gamma(a) > 0$. We shall see that, if $b \in \mathcal{O}_a$ and $\|a - b\| < r/2$, then $\gamma(b) \geq r/2$. Indeed, for $\xi \in R(a) = \ker a^\perp$, $\|b\xi\| \geq \|a\xi\| - \|(a-b)\xi\| \geq r/2$. Hence $\ker b \cap R(a) = \{0\}$. Since $\dim \ker b = \dim \ker a = n < \infty$ (resp. $\dim R(a) = \dim R(b) < \infty$), then $\ker b \oplus R(a) = \mathcal{H}$. Let q be the bounded projection onto $R(a)$ given by this decomposition. If $\eta \in (\ker b)^\perp$ then $\|q(\eta)\| = (\|\eta\|^2 + \|(1-q)\eta\|^2)^{1/2} \geq \|\eta\|$ so that

$$\|b\eta\| = \|b(q\eta)\| \geq \frac{r}{2}\|q\eta\| \geq \frac{r}{2}\|\eta\|.$$

Hence $\gamma(b) \geq r/2$ near a. Using the continuous functional calculus (the map $f(t) = t^{-1}\aleph_{[r/2,\infty)}$), the map $b \mapsto b^\dagger$ and therefore also the map $b \mapsto \alpha(b) = bb^\dagger = p_b$ are continuous when restricted to $\mathcal{O}_a$.

Since π_a is surjective, in order to show that π_a has local cross sections near every $b \in \mathcal{A}^+_{n,m}$, it suffices to show this fact just near a. By Lemma 2.1 if $p, q \in \mathcal{P}$ and $\|p - q\| < 1$ then they are unitarily equivalent in $\mathcal{A}$. Moreover, there exists a continuous map

$$s : \{q \in \mathcal{P} :\ \|p - q\| < 1\} \to \mathcal{U}$$

such that $s(p) = 1$ and $s(q)ps(q)^* = q$ for every q. See the proof of Lemma 2.1 or [10].

For $b \in \mathcal{A}^+_{n,m}$ close to a, using that $p \mapsto p_b$ is continuous, we can suppose $\|p - q\| < 1$. Then by considering $s(p_b)^* b s(p_b)$, instead of b we can assume that b is still close to a and $p_a = p_b$, i.e. $\ker b = \ker a$. In this case define $g(b) = b^{1/2}(a^\dagger)^{1/2} + (1 - p_a)$. It is clear that $g(b)ag(b)^* = b$. Also $g(b) \in G$, because

$$\|1 - g(b)\| = \|a^{1/2}(a^\dagger)^{1/2} + (1 - p_a) - g(b)\| = \|(a^{1/2} - b^{1/2})(a^\dagger)^{1/2}\| < 1$$

for b close enough to a.

Let $a \in \mathcal{A}^+_{\infty,\infty}$ and consider $q \in \mathcal{P}$ such that $q \neq 0$, $R(q) \subset \ker(a)$ and $\dim(\ker(a) \ominus R(q)) = \infty$. Define $a_n = a + \frac{1}{n}q$, $n \in \mathbb{N}$. Observe that $\ker(a_n) = \ker(a) \ominus R(q)$ and $R(a_n)) = R(a) \oplus R(q)$ so that $a_n \in \mathcal{A}^+_{\infty,\infty} = \mathcal{O}_a$ for every n. The fact that $\ker(a_n) = \ker(a) \ominus R(q)$ is properly included in $\ker(a)$ clearly implies that the map $b \to p_b$ cannot be continuous.

Also, observe that it is not possible that $a_n = g_n a g_n^*$, with $g_n \in G$, $g_n \to 1$. For, in this case it would be $\ker(a_n) = g_n^{*-1}(\ker(a)) = R(g_n^{*-1} P_{\ker(a)} g_n^*)$ for every n.

But $g_n^{*-1} P_{\ker(a)} g_n^* \to P_{\ker(a)}$ and this implies that $P_{\ker(a_n)} \to P_{\ker(a)}$ (see the proof of Proposition 3.7 below) which is false. Thus, in this case there exists no local cross section for the action of G on $\mathcal{O}_a$. ∎

REMARK 3.5 Observe that in the proof above it was shown that the map $b \to b^\dagger$ is continuous when restricted to $\mathcal{O}_a = \mathcal{A}^+_{n,m}$, with $n < \infty$ or $m < \infty$.

In many infinite dimensional problems, the lack of continuity is the only obstruction for differentiability. Therefore we shall define a new metric on the orbits $\mathcal{O}_a$, in order to make α a continuous map. Doing this, $\mathcal{O}_a$ turns to be a homogeneous space of G. The differentiable structure will be studied in the next section.

Consider the following metric on $\mathcal{O}_a$:

$$d(b, c) = \|b - c\| + \|p_b - p_c\| = \|b - c\| + \|\alpha(b) - \alpha(c)\|, \; b, c \in \mathcal{O}_a.$$

PROPOSITION 3.6 *1. The map $\alpha : (\mathcal{O}_a,\, d\,) \to \mathcal{U}\mathcal{O}_p$ is continuous.*

2. For $b, c \in \mathcal{O}_a$, $d(b, c) = \|b-c\|$ if and only if b and c belong to the same Thompson component; in particular d coincides with the usual metric in each component.

Proof. The continuity of α follows from the definition of d. If $b\,, c \in \mathcal{C}_b$ then $p_c = \alpha(b) = \alpha(c) = p_b$ so that $d(b, c) = \|b - c\|$. Conversely, if $d(b, c) = \|b - c\|$ then $p_b = p_c$ so that b and c belong to the same component. ∎

Fix $a \in \mathcal{A}^+_{cr}$. For each $b \in \mathcal{O}_a$ consider the map $\pi_b : G \to \mathcal{O}_a$, $\pi_b(g) = L(g, b) = gbg^*$. Denote $\pi = \pi_a$ and $p = p_a$.

PROPOSITION 3.7 *The map $\pi : (G, \|\cdot\|) \to (\mathcal{O}_a, d)$ is continuous and it admits continuous local cross sections.*

Proof. Note that π is continuous if the norm metric is considered on $\mathcal{O}_a$. Also, by diagram (1), $\alpha \circ \pi = \pi_p$, which is known to be continuous (see [3]). Therefore $\pi : (G, \|\cdot\|) \to (\mathcal{O}_a, d)$ is continuous. Let $b \in \mathcal{O}_a$ such that $d(b, a) < 1$; then $\|p_b - p\| < 1$, $(1 - (p_b - p)^2) \in G^+$ and it is easy to see that if

$$s(b) = b^{1/2}(1 - (p_b - p)^2)^{-1/2}(a^\dagger)^{1/2} + (1 - p_b)(1 - (p_b - p)^2)^{-1/2}(1 - p) \in G$$

then $\pi \circ s(b) = b$ on $\{b \in \mathcal{O}_a : d(b, a) < 1\}$; thus s is a continuous local section of π in a neighbourhood of a. If $c = gag^*$, $g \in G$, consider $s' = l_g \circ s \circ L_{g^{-1}}$, where $l_g : G \to G$ is the left multiplication by g, then s' is a local section of π in a neighbourhood of c. This completes the proof. ∎

COROLLARY 3.8 *Let $a \in \mathcal{A}_{cr}^+$. Denote by $I_a = \{g \in G : gag^* = a\}$, the isotropy group of a by the action of G. Then the metric space $(\mathcal{O}_a, d)$ is homeomorphic to the quotient space G/I_a, where the quotient topology is considered.*

REMARK 3.9 Observe that the continuity of $b \to p_b$ on $\mathcal{O}_a$ is equivalent to the continuity of the Moore-Penrose pseudoinverse $b \to b^\dagger$ on $\mathcal{O}_a$ (see [23]. Both are provided by the metric d defined on the orbit.

4 Differentiable structure

Recall some definitions and results on Banach-Lie groups. As a general reference about this subject, see, for example, [24].

DEFINITION 4.1 Given a Banach-Lie group G, a subgroup H of G is **regular** if it is a Banach-Lie group and if $(TH)_1$ is a closed and complemented subspace of $(TG)_1$.

THEOREM 4.2 *Let G be a Banch-Lie group, $H \subseteq G$ a regular subgroup. Then*

1. *G/H has a unique structure of differentiable manifold such that $G \to G/H$ is a submersion*
2. *$G \to G/H$ is a principal bundle with structure group H.*
3. *The action $G \times G/H \to G/H$ is smooth.*

In order to provide $\mathcal{O}_a \simeq G/I_a$ with a differential structure using Theorem 4.2, we need to prove that I_a is a regular subgroup of G. First observe that if $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ then $(\mathcal{S}, \langle\, , \,\rangle)$ is a Hilbert space and every positive invertible operator $c \in L(\mathcal{S})$ determines an inner product on $\mathcal{S}$ by

$$\langle \xi, \eta \rangle_c = \langle c\xi, \eta \rangle \ , \quad \xi, \ \eta \in \mathcal{S} \, ;$$

$\langle\, , \,\rangle_c$ is equivalent to the original inner product $\langle\, , \,\rangle$. Observe that, given $v \in L(\mathcal{S})$, $w \in L(\mathcal{S})$ is the adjoint of v respect to $\langle\, , \,\rangle_c$ if

$$\langle v\xi, \eta \rangle_c = \langle \xi, w\eta \rangle_c, \quad \xi, \ \eta \in \mathcal{S},$$

or, equivalently, if $v^*c = cw$.

Consider $c \in \mathcal{A}_{cr}^+$ and $\mathcal{A}_c = p_c \mathcal{A} p_c$; as $R(x) \subset (\ker c)^\perp$ for every $x \in \mathcal{A}_c$ we can naturally identify $\mathcal{A}_c$ with a subalgebra of $L(R(c))$, by restricting the elements of $\mathcal{A}_c$ to $(\ker c)^\perp$. Then $c \in G^+(\mathcal{A}_c)$. Also $\mathcal{A}_c$ is a C*-algebra when we consider the inner product $\langle\, , \,\rangle_c$

in $R(c)$. Denote $\mathcal{U}_c$ the unitary group of $(\mathcal{A}_c, \langle\ ,\ \rangle_c)$, i.e. the operators v in $G(\mathcal{A}_c)$ such that

$$v^* c = c v^{-1}. \tag{2}$$

The *isotropy group* I_b of $b \in \mathcal{O}_a$ is the group of all $g \in G$ such that $L_g b = b$, i.e.

$$I_b = \{g \in G : gbg^* = b\}.$$

The following proposition gives a characterization of I_a. First, let us denote by $a_0 \in L(R(a))$ the compression of a to $R(a)$, so that $a = \begin{pmatrix} a_0 & 0 \\ 0 & 0 \end{pmatrix}$. Observe that $a^\dagger = \begin{pmatrix} a_0^{-1} & 0 \\ 0 & 0 \end{pmatrix}$, hence $a_0^{-1} = (a^\dagger)_0$.

PROPOSITION 4.3 *Let $g \in \mathcal{A}$. Then $g \in I_a$ if and only if $g \in G$ and the matrix representation of g induced by p is given by $g = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}$ where $g_{11} \in \mathcal{U}_{a_0{}^{-1}}$.*

Proof. Let $g \in I_a$. Then $g(R(a)) = R(a)$ so that $pgp = gp$. As $g \in I_a$ if and only if $g^{-1} \in I_a$, also $pg^{-1}p = g^{-1}p$. Then if $g_{11} = pgp$ and $w_{11} = pg^{-1}p$, $g_{11}w_{11} = w_{11}g_{11} = p$ so that $g_{11} \in G(\mathcal{A}_a)$. Also $g_{11}a_0g_{11}^* = a_0$ or $a_0g_{11}^* = g_{11}^{-1}a_0$. Then, by (2), $g_{11} \in \mathcal{U}_{a_0^{-1}}$. Easy matrix computations show that the invertibility of g_{11} implies that $g_{21} = 0$. ∎

COROLLARY 4.4 *Let $b \in \mathcal{A}^+_{cr}$. Then I_b is a Banach-Lie group, and*

$$(TI_b)_1 = \{X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix} \in \mathcal{A} :\ X_{11}b = -bX_{11}^*\},$$

where we use matrix representations in terms of p_b. Therefore I_b is a regular subgroup of G.

THEOREM 4.5 $(\mathcal{O}_a,\, d\,)$ *is a differentiable manifold and, for every $b \in \mathcal{O}_a$, $\mathcal{C}_b$ is a submanifold of $\mathcal{O}_a$. Moreover, the d-open subsets $U_b = \{c \in \mathcal{O}_a : \|p_c - p_b\| < 1\}$ are diffeomorphic to the product spaces $\mathcal{C}_b \times V_{p_b}$, where $V_{p_b} = \{q \in \mathcal{P} : \|q - p_b\| < 1\}$.*

Proof. By Propositions 3.7 and 4.3, the space $(\mathcal{O}_a,\, d\,)$ is homeomorphic to the quotient space G/I_a. On the other hand, by Proposition 4.3, I_a is a regular subgroup of the Lie-Banach group G. Therefore there exists a unique smooth manifold structure on G/I_a which makes the quotient map (or, modulo the mentioned homeomorphism, the map π) a submersion.

For every $b \in \mathcal{O}_a$ the Thompson component $\mathcal{C}_b$ of b is a differentiable manifold, [7]; also the unitary orbit of p_b, $\mathcal{UO}_{p_b}$, is a differentiable manifold, [10]. Then $\mathcal{C}_b \times \mathcal{UO}_{p_b}$ admits a structure of differentiable manifold. Fix $b \in \mathcal{O}_a$ and consider the d-open set $U_b = \{c \in \mathcal{O}_a : \|p_c - p_b\| < 1\}$. Now consider $V_{p_b} = \{q \in \mathcal{P} : \|q - p_b\| < 1\}$. V_{p_b} is a neighbourhood of p_b in $\mathcal{P}$ and $V_{p_b} \subset \mathcal{UO}_{p_b}$, (see Preliminaries). For $q \in V_{p_b}$, set

$e = 1-(q-p_b)^2$, then $e \in G^+$ so that we can define $\phi(q) = qe^{-1/2}p_b + (1-q)e^{-1/2}(1-p_b)$. It is easy to see that $\phi(q) \in \mathcal{U}$ and $\phi(q)p_b\phi(q)^* = q$. Observe that $\phi(q)(R(b)) = R(q)$.

Define $f_b : \mathcal{C}_b \times V_{p_b} \to U_b \subseteq \mathcal{A}$, by $f_b(c,q) = \phi(q)c\phi(q)^*$. As $R(f_b(c,q)) = \phi(q)(R(c)) = \phi(q)(R(b)) = R(q)$, it follows that $f_b(c,q) \in \mathcal{C}_q$. Clearly f_b is a C^∞ map. In order to see that f_b is a diffeomorphism, observe that the map $h : U_b \to \mathcal{C}_b \times V_{p_b}$ defined by $h(x) = (\phi(p_x)^* x \phi(p_x), p_x)$ is the inverse of f_b and h is a C^∞ map, because the map $x \mapsto p_x$ is C^∞: this can be easily verified by composing with π_b. ∎

REMARK 4.6 The homogeneous space $(GL(\mathcal{S}), C_a, I_a)$ mentioned in the Preliminaries is a subbundle of $(G, \mathcal{O}_a, I_a)$ with the same structure group I_a, see [7].

REMARK 4.7 Observe that, in general, $\mathcal{O}_a$ is not a submanifold of $\mathcal{A}$ (because the topologies induced by d and the usual norm may be different, see Theorem 3.4). So that, in principle, we can not identify its "abstract" tangent space with the image of the tangent map of the map $\pi : G \to \mathcal{O}_a \subseteq \mathcal{A}$. Each tangent space $(T\mathcal{O}_a)_b$ should be identify with the image of tangent map of $\pi_b : G \to G/I_b$, because this is the way the manifold structure of $\mathcal{O}_a$ is constructed. However, both processes give the same result: in fact, because the kernel of $(T\pi_b)_1$ is the tangent space at 1 of the isotropy group, we can identify $(T\mathcal{O}_a)_b$ with $TG_1/(TI_b)_1$. Recall that $(TG)_1 = \mathcal{A}$ and, by Proposition 4.3,

$$(TI_b)_1 = \{X = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix} \in \mathcal{A} : \; X_{11}b = -bX_{11}^*\},$$

where we use matrix representations in terms of p_b. On the other hand, if we consider $(T\pi_b)_1 : \mathcal{A} \to \mathcal{A}$, it is easy to see that $(T\pi_b)_1(Y) = Yb + bY^*$, and it is clear that $\ker(T\pi_b)_1 = (TI_b)_1$. So we shall identify any abstract tangent vector $(T\pi_b)_1(Y)$, $Y \in \mathcal{A}$, in $T(\mathcal{O}_a)_b$ with the concrete vector $Yb + bY^* \in \mathcal{A}$.

PROPOSITION 4.8 *For $b \in \mathcal{O}_a$ the tangent space $(T\mathcal{O}_a)_b$ identifies with the set*

$$\{X = \begin{pmatrix} X_1 & W^* \\ W & 0 \end{pmatrix} \in \mathcal{A} : X_1 = X_1^*\} = \{X \in \mathcal{A} : X = X^*, (1-p_b)X(1-p_b) = 0\},$$

where the matrix representation of X is given by p_b.

Proof. First consider $b = a$. Consider $\gamma(t) \subset \mathcal{O}_a$ a smooth curve such that $\gamma(0) = a$ and $\dot\gamma(0) = X$. Using the existence of a local cross section in a neighbourhood of a, there exists a curve $g(t) \subset G$ such that $\gamma(t) = g(t)ag(t)^*$, $g(0) = 1$, $\dot g(0) = Y$. Computing the derivative of γ at $t = 0$ we get that $\dot\gamma(0) = \dot g(0)a + a\dot g(0)^*$ or $X = Ya + aY^*$.

If $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$ is the matrix representation given by p, then

$$X = \begin{pmatrix} Y_{11}\,a & 0 \\ Y_{21}\,a & 0 \end{pmatrix} + \begin{pmatrix} aY_{11}^* & aY_{21}^* \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} Y_{11}\,a + aY_{11}^* & aY_{21}^* \\ Y_{21}\,a & 0 \end{pmatrix}.$$

Hence $X = \begin{pmatrix} X_1 & W^* \\ W & 0 \end{pmatrix}$ with $X_1 = X_1^*$. Conversely if X has this form consider $Y_{11} = \frac{1}{2}X_1 a^\dagger$, $Y_{21} = Wa^\dagger$ and $Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}$, then $Ya + aY^* = X$ and if $g(t) = e^{Xt}$ we get that $\gamma = gag^*$, $\gamma(0) = a$ and $\dot\gamma(0) = X$. The proof is similar for any $b \in \mathcal{O}_a$. ∎

5 The transport equation

In this section we define a natural connection on the homogeneous space $(G, \mathcal{O}_a, I_a)$ by giving a smooth distribution of horizontal spaces. The existence of a connection allows to obtain the horizontal lift for any curve γ in $\mathcal{O}_a$, as the solution to a linear differential equation, the so called transport equation. Then, it is possible to define a covariant derivative and an associated notion of geodesic. Some results of existence of geodesics are given.

First observe that the tangent map of the projection π at $1 \in \mathcal{A}$, $(T\pi)_1 : \mathcal{A} \to (T\mathcal{O}_a)_a$ identifies with

$$(T\pi)_1(X) = Xa + aX^*$$

and

$$\ker(T\pi)_1 = \{\begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix} \in \mathcal{A} : \ X_{11}a = -aX_{11}^*\} = (TI_a)_1.$$

Observe that X_{11} is $a^\dagger$-antisymmetric, if it is considered as an operator in $L(R(a))$. Define the *horizontal space* at 1 as

$$H_a = \{\begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \in \mathcal{A} : \ X_{11}a = aX_{11}^*\}$$

and the *vertical space* at 1 as $V_a = (TI_a)_1$. It is easy to see that $\mathcal{A} = V_a \oplus H_a$ and that the restriction of $(T\pi)_1$ to H_a, $(T\pi)_1\big|_{H_a} : H_a \to (T\mathcal{O}_a)_a$ is an isomorphism. Explicitely,

$$(T\pi)_1\big|_{H_a}(X) = 2X_{11}a + X_{21}a + aX_{21}^* \quad , \qquad \text{where} \qquad X = \begin{pmatrix} X_{11} & 0 \\ X_{21} & 0 \end{pmatrix} \in H_a,$$

and the inverse map $((T\pi)_1\big|_{H_a})^{-1} = K_a : (T\mathcal{O}_a)_a \to H_a$ is given by

$$K_a(X) = \frac{1}{2}pXpa^\dagger + (1-p)Xpa^\dagger = \frac{1}{2}pXa^\dagger + (1-p)Xa^\dagger = (1 - \frac{p}{2})Xa^\dagger;$$

equivalently, if $X = \begin{pmatrix} X_1 & W^* \\ W & 0 \end{pmatrix}$ then $K_a(X) = \frac{1}{2}X_1 a^\dagger + Wa^\dagger$.

For every $g \in G$ set $H_{a,g} = gH_a$ the *horizontal space* at g and $V_{a,g} = gV_a$ the *vertical space* at g. Then $\mathcal{A} = H_{a,g} \oplus V_{a,g}$ and this distribution of horizontal subspaces defines a smooth connection.

Given a smooth curve $\gamma \subset \mathcal{O}_a$, a smooth curve $\Gamma \subset G$ is a *lift* of γ if $\gamma = \pi(\Gamma) = \Gamma a \Gamma^*$ and Γ is a *horizontal lift* of γ if Γ is a lift of γ and $\dot{\Gamma} \in H_{a,\Gamma}$.

LEMMA 5.1 *Γ is a horizontal lift of γ if and only if Γ is a solution to $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$.*

Proof. To prove this assertion suppose first that Γ is a horizontal lift of γ. Differentiating $\pi(\Gamma) = \gamma$ we get $(T\pi)_\Gamma(\dot{\Gamma}) = \dot{\gamma}$. Also, from $\pi = L_g \circ \pi \circ l_{g^{-1}}$, $g \in G$, where l_g is the left multiplication by g, we obtain, at $g = \Gamma$, that $(T\pi)_\Gamma = (TL_\Gamma)_a (T\pi)_1 l_{\Gamma^{-1}}$, then

$$(TL_\gamma)_a (T\pi)_1 (\Gamma^{-1}\dot{\Gamma}) = \dot{\gamma}, \text{ or } (T\pi)_1(\Gamma^{-1}\dot{\Gamma}) = (TL_\gamma)_a^{-1}\dot{\gamma}.$$

$\dot{\Gamma} \in H_{a,\Gamma}$ because Γ is horizontal; then $\Gamma^{-1}\dot{\Gamma} \in H_a$ so that $\Gamma^{-1}\dot{\Gamma} = K_a (TL_\gamma)_a^{-1}\dot{\gamma}$. Now observe that $K_\gamma = Aut_\Gamma K_a (TL_\gamma)_a^{-1}$, where $Aut_g(h) = ghg^{-1}$, $g, h \in G$. Then $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$.

Conversely, if Γ is a solution to $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$ then $\dot{\Gamma}\Gamma^{-1} = K_\gamma(\dot{\gamma}) \in H_\Gamma = \Gamma H_a \Gamma^{-1}$; therefore $\dot{\Gamma} \in H_a = \Gamma H_{a,\Gamma}$ and Γ is horizontal.

PROPOSITION 5.2 *Consider $\gamma : [0,1] \to \mathcal{O}_a$ a smooth curve such that $\gamma(0) = a$ and $\Gamma : [0,1] \to G$ a lift of γ. Then Γ is horizontal if and only if Γ is the solution to the differential equations*

$$\begin{cases} p_\gamma(\dot{\Gamma}\Gamma^{-1} - \frac{1}{2}\dot{\gamma}\gamma^\dagger)p_\gamma & = 0 \\ \dot{\Gamma}\Gamma^{-1}(1 - p_\gamma) & = 0. \end{cases}$$

Proof. By Lemma 5.1 Γ is a horizontal lift of γ if and only if Γ is a solution to $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$, where $K_\gamma(\dot{\gamma}) = \frac{1}{2}p_\gamma\dot{\gamma}\gamma^\dagger + (1 - p_\gamma)\dot{\gamma}\gamma^\dagger$.

It is well known that the unique solution of a linear differential problem $\dot{\Gamma} = K\Gamma$, $\Gamma(0) = 1$ satisfies $\Gamma(t) \in G$, for all $t \in [0,1]$. Then, $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$ is equivalent to

$$\dot{\Gamma}\Gamma^{-1} = \frac{1}{2}p_\gamma\dot{\gamma}\gamma^\dagger + (1 - p_\gamma)\dot{\gamma}\gamma^\dagger,$$

or, what is the same, to

$$\begin{cases} \dot{\Gamma}\Gamma^{-1}p_\gamma & = \frac{1}{2}p_\gamma\dot{\gamma}\gamma^\dagger + (1 - p_\gamma)\dot{\gamma}\gamma^\dagger \\ \dot{\Gamma}\Gamma^{-1}(1 - p_\gamma) & = 0. \end{cases}$$

Multiplying conveniently by p_γ and by $1 - p_\gamma$ we obtain the equivalent equations

$$\begin{cases} p_\gamma(\dot{\Gamma}\Gamma^{-1} - \frac{1}{2}\dot{\gamma}\gamma^\dagger)p_\gamma & = 0 \\ (1 - p_\gamma)(\dot{\Gamma}\Gamma^{-1} - \dot{\gamma}\gamma^\dagger)p_\gamma & = 0 \\ \dot{\Gamma}\Gamma^{-1}(1 - p_\gamma) & = 0. \end{cases}$$

But the second equation is satisfied by every lift of γ, not necessarily horizontal. In fact differentiating $a = \Gamma^{-1}\gamma\Gamma^{*-1}$ we get $\dot{\gamma} = \dot{\Gamma}\Gamma^{-1}\gamma + \gamma\Gamma^{*-1}\dot{\Gamma}^*$. Then $(\dot{\Gamma}\Gamma^{-1} - \dot{\gamma}\gamma^\dagger)p_\gamma = -\gamma\Gamma^{*-1}\dot{\Gamma}^*\gamma^\dagger$ and the equation follows.

Then Γ is horizontal if and only if

$$\begin{cases} p_\gamma(\dot{\Gamma}\Gamma^{-1} - \frac{1}{2}\dot{\gamma}\gamma^\dagger)p_\gamma & = 0 \\ \dot{\Gamma}\Gamma^{-1}(1 - p_\gamma) & = 0. \end{cases}$$

∎

REMARK 5.3 i) The equations in the proposition above can be rewritten as

$$\begin{cases} (p_\gamma \dot{\Gamma} a \Gamma^* p_\gamma)^* & = p_\gamma \dot{\Gamma} a \Gamma^* p_\gamma \\ \dot{\Gamma}\Gamma^{-1}(1-p_\gamma) & = 0. \end{cases}$$

ii) As we have already seen Γ is a horizontal lift if and only if $\dot{\Gamma} = K_\gamma(\dot{\gamma})\Gamma$.

On the other side $K_\gamma = Aut_\Gamma K_a (TL_\Gamma)_a^{-1}$, where $Aut_\Gamma(X) = \Gamma X \Gamma^{-1}$ and $(TL_\Gamma)_a^{-1}$ is the inverse of the tangent map of L_Γ at a at each $t \in [0,1]$ and it is given by $(TL_{\Gamma(t)})_a^{-1} = L_{\Gamma(t)^{-1}}$. Then

$$\Gamma^{-1}\dot{\Gamma} = K_a(L_{\Gamma^{-1}}\dot{\gamma}) = K_a(\Gamma^{-1}\dot{\gamma}\Gamma^{-1*})$$

so that

$$\Gamma^{-1}\dot{\Gamma} = \frac{1}{2}p\,\Gamma^{-1}\dot{\gamma}\Gamma^{-1*}p + (1-p)\Gamma^{-1}\dot{\gamma}\Gamma^{-1*}p.$$

Or, equivalently,

$$\begin{cases} p\Gamma^{-1}\dot{\Gamma}p & = \frac{1}{2}p\Gamma^{-1}\dot{\gamma}\Gamma^{-1*}p \\ (1-p)\Gamma^{-1}\dot{\Gamma}p & = (1-p)\Gamma^{-1}\dot{\gamma}\Gamma^{-1*}p \\ \Gamma^{-1}\dot{\Gamma}(1-p) & = 0\ . \end{cases}$$

Again, the second equation is verified by every lift of γ and the first one is equivalent to $p\Gamma^{-1}\dot{\Gamma}p = (p\Gamma^{-1}\dot{\Gamma}p)^*$. Then a lift Γ of γ is horizontal if and only if it satisfies

$$\begin{cases} p\Gamma^{-1}\dot{\Gamma}p & = (p\Gamma^{-1}\dot{\Gamma}p)^* \\ \Gamma^{-1}\dot{\Gamma}(1-p) & = 0\ . \end{cases}$$

If $a \in G^+$ then $C_a = \mathcal{O}_a = G^+$. The geometric structure of this set has already been described, [9] and [6]. If $\alpha : [0,1] \to G^+$ is a smooth curve the *length* of α is defined as $L(\alpha) = \int_0^1 \|\alpha^{-1/2}\dot{\alpha}\alpha^{-1/2}\|dt$; this is the natural definition considering the Finsler metric defined in G^+ (see the references above).

Consider the map $\pi_p : G \to \mathcal{O}_a$ defined by $\pi_p(g) = gpg^*$, $g \in G$.

PROPOSITION 5.4 *If $\Gamma_0 : [0,1] \to G$ is a horizontal lift (with respect to π_p) of $\gamma : [0,1] \to \mathcal{O}_a$ then*

$$\|\Gamma_0^{-1}\dot{\gamma}\Gamma_0^{*-1}\| = \|\Gamma_0^{-1}\dot{\Gamma}_0 + (\Gamma_0^{-1}\dot{\Gamma}_0)^*\| = \|\alpha^{-1/2}\dot{\alpha}\alpha^{-1/2}\| = \|\dot{\alpha}\|_\alpha$$

where $\alpha \subset G^+$ is the curve $\alpha = \Gamma\Gamma^*$ so that $L(\alpha) = \int_0^1 \|\dot{\alpha}\|_\alpha dt = \int_0^1 \|\Gamma_0^{-1}\dot{\gamma}\Gamma_0^{*-1}\|dt$.

Moreover, Γ_0 is a solution of the variational problem

$$\min \int_0^1 \|p\Gamma^{-1}\dot{\Gamma}\|dt,$$

where the minimum is taken over all lifts of γ.

Proof. Γ is a lift of γ if and only if $\gamma = \Gamma p \Gamma^*$. Then computing the derivative of $p = \Gamma^{-1}\gamma\Gamma^{*-1}$ we get that $\Gamma^{-1}\dot\gamma\Gamma^{*-1} = \Gamma^{-1}\dot\Gamma p + p\dot\Gamma^*\Gamma^{*-1}$. If Γ_0 is horizontal then Γ_0 satisfies $\Gamma_0^{-1}\dot\Gamma_0 p = \Gamma_0^{-1}\dot\Gamma_0$ so that $\Gamma_0^{-1}\dot\gamma\Gamma_0^{*-1} = \Gamma_0^{-1}\dot\Gamma_0 + ({\Gamma_0}^{-1}\dot\Gamma_0)^*$.

Now consider $\alpha = \Gamma_0\Gamma_0^* \subset G^+$, then $\Gamma_0(t) = \alpha^{1/2}(t)u(t)$, with $u(t) \subset \mathcal{U}$. Then $\dot\alpha = \dot\Gamma_0\Gamma_0^* + \Gamma_0\dot\Gamma_0^*$ or $\Gamma_0^{-1}\dot\alpha\Gamma_0^{*-1} = \Gamma_0^{-1}\dot\Gamma_0 + ({\Gamma_0}^{-1}\dot\Gamma_0)^*$, so that $\|\Gamma_0^{-1}\dot\Gamma_0 + ({\Gamma_0}^{-1}\dot\Gamma_0)^*\| = \|\Gamma_0^{-1}\dot\alpha\Gamma_0^{*-1}\| = \|\alpha^{-1/2}\dot\alpha\alpha^{-1/2}\|$.

Also $p\Gamma_0^{-1}\dot\Gamma_0 = p\Gamma_0^{-1}\dot\Gamma_0 p = \frac{1}{2}p\Gamma_0^{-1}\dot\gamma\Gamma_0^{*-1}p$ so that $\|p\Gamma_0^{-1}\dot\Gamma_0\| = \frac{1}{2}\|p\Gamma_0^{-1}\dot\gamma\Gamma_0^{*-1}p\|$.

If Γ is any lift of γ then $\gamma = \Gamma p\Gamma^*$ and $\|p\Gamma^{-1}\dot\gamma\Gamma^{*-1}p\| = \|p\Gamma^{-1}\dot\Gamma p + p(\Gamma^{-1}\dot\Gamma)^*p\| \leq 2\|p\Gamma^{-1}\dot\Gamma p\| \leq 2\|p\Gamma^{-1}\dot\Gamma\|$. ∎

Given $X \in (T\mathcal{O}_a)_a$, then $X = \begin{pmatrix} X_1 & W^* \\ W & 0 \end{pmatrix}$ with $X_1 = X_1^*$. We look for a geodesic $\gamma \subset \mathcal{O}_a$ such that $\gamma(0) = a$ and $\dot\gamma(0) = X$. If $\gamma = \Gamma a\Gamma^*$ then Γ verifies

$$\dot\Gamma = K_a(X)\Gamma$$

with $K_a(X) = (\frac{1}{2}X_1 + W)a^\dagger = (\frac{1}{2}X_1 + W)\widetilde{a}^{-1}$ where $\widetilde{a} = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \in G$. Then

$$\Gamma(t) = e^{tK_a(X)} = e^{t(\frac{1}{2}X_1 + W)a^\dagger}$$

and

$$\gamma(t) = e^{tYa^\dagger} a e^{ta^\dagger Y^*}$$

where $Y = \frac{1}{2}X_1 + W$. Observe that $Y + Y^* = X$ and $Y = Yp$. Easy computations show that

$$\gamma(t) = \widetilde{a}e^{t\widetilde{a}^{-1}Y}e^{t\widetilde{a}^{-1}Y^*} - (1-p).$$

and that $\gamma(0) = a$, $\dot\gamma(0) = X$.

REMARK 5.5 If $a \in G$, $1 - p = 0$, $\widetilde{a} = a$ and $Y = X_1 = X$ then $\gamma(t) = ae^{ta^{-1}X}$ is the usual geodesic as in the invertible case. Observe that γ can also be expressed as

$$\gamma(t) = ae^{t\widetilde{a}^{-1}Y}e^{t\widetilde{a}^{-1}Y^*} + (1-p)(e^{t\widetilde{a}^{-1}Y}e^{t\widetilde{a}^{-1}Y^*} - 1).$$

In particular, if $a = p$ then $\widetilde{a} = 1$ and $\gamma(t) = e^{tY}e^{tY^*} - (1-p)$. Finally, if b and c belong to the same component there exists a unique geodesic with endpoints b and c and it coincides with the geodesic provided by the structure studied in [7].

Some questions: Some of the results about the orbit of a closed range positive element remain valid if the closed range condition is dropped. However in this case it is not obvious how to provide $\mathcal{O}_a$ with a differentiable structure compatible with the structure of homogeneous reductive space of its Thompson components, as in the closed range case.

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