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Generalized Schur complements and oblique projections

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Abstract

Let $\mathscr{S}$ be a closed subspace of a Hilbert space $\mathscr{H}$ and A a bounded linear selfadjoint operator on $\mathscr{H}$. In this note, we show that the existence of A-selfadjoint projections with range $\mathscr{S}$ is related to some properties of shorted operators, Schur complements (in Ando's generalization of the classical concept) and compressions. © 2002 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let $\mathscr{S}$ be a closed subspace of a Hilbert space $\mathscr{H}$ and A a bounded linear selfadjoint operator on $\mathscr{H}$. In this note, we study the relationship between the properties

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of the shorted operator of A by $\mathscr{S}$ (if A is positive), the complementability of A by $\mathscr{S}$ and the compatibility of the pair $(A, \mathscr{S})$, which means that there exists an A-selfadjoint projection onto $\mathscr{S}$, i.e., a bounded operator Q on $\mathscr{H}$ such that $Q^2 = Q$, the range of Q is $\mathscr{S}$ and $AQ = Q^*A$.

Schur complements were introduced by Haynsworth [15], but they have been implicitly used since the beginning of the theory of matrices. Shorted operators, which are related to Schur complements, were defined by Anderson and Trapp [3,4] and applied it in electrical network theory, but it was Krein [17] who first studied a similar notion as part of the theory of extensions of Hermitian operators. Complementable matrices by a subspace were defined by Ando [2] as a useful generalization of Schur complements. Finally, compatible pairs have been recently studied by Hassi and Nordström [14] and Corach et al. [10], because they define classes of projections which have minimality properties that may be relevant in different areas, e.g., approximation theory, abstract splines and least square problems [11].

The relationship between shorted operators and compatible pairs is studied in the authors' work [10], where it is shown that, if A is positive and $A_{/\mathscr{S}}$ is the shorted operator of A by $\mathscr{S}$, then $(A, \mathscr{S})$ is compatible if and only if $A_{/\mathscr{S}} = \min\{Q^*AQ : Q^2 = Q,\ \ker Q = \mathscr{S}\}$. It is also shown in [10] that if A in injective, then $(A, \mathscr{S})$ is compatible if and only if $R(A_{/\mathscr{S}}) \subseteq R(A)$ and $\ker A_{/\mathscr{S}} = \mathscr{S}$.

This paper can be seen as a continuation of [10]. Several results of that work are collected as Theorem 2.6. Some other results proven in [10] appear in Section 3, some of them with a short proof. The main results of the present paper are the following:

1. The notions of compatibility and complementability for a selfadjoint operator on a Hilbert space are equivalent.
2. If A is positive, then $(A, \mathscr{S})$ is compatible if and only if $R(A_{/\mathscr{S}}) \subseteq R(A)$ and $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S}$ (see Theorem 3.8). This result generalizes the characterizations for compatibility for injective or closed range operators obtained in [10].
3. If A is positive, denote by $A_{\mathscr{S}} = A - A_{/\mathscr{S}}$ the *compression* of A by S, then $(A, \mathscr{S})$ is compatible if and only if $R(A_{\mathscr{S}}) = A(\mathscr{S})$ (see Proposition 3.9).
4. If A is positive, then $R(A_{/\mathscr{S}}) \subseteq R(A)$ if and only if the pair $(A, \ker A_{/\mathscr{S}})$ is compatible (see Proposition 3.6).

Section 2 contains the definitions and some properties of the three notions involved in the paper. In Section 3 we present several results which relate these concepts in a Hilbert space. Section 4 contains some examples.

2. Preliminaries

In this paper $\mathscr{H}$ is a Hilbert space, $L(\mathscr{H})$ is the algebra of all linear bounded operators on $\mathscr{H}$, $L(\mathscr{H})_{\mathrm{h}}$ is the real subspace of all Hermitian (or selfadjoint) operators in $L(\mathscr{H})$, $L(\mathscr{H})^+$ is the subset of $L(\mathscr{H})_{\mathrm{h}}$ of all positive (semi-definite) operators and $\mathrm{GL}(\mathscr{H})^+$ is the set of all positive invertible operators. We denote by $\mathcal{Q} = \{Q \in L(\mathcal{H}) : Q^2 = Q\}$ the set of projections and $\mathcal{P} = \{P \in \mathcal{Q} : P^* = P\}$ the set of orthogonal projections. For a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. For every $C \in L(\mathcal{H})$ its range is denoted by $R(C)$ and its nullspace by $\ker C$. Every closed subspace $\mathcal{S}$ of $\mathcal{H}$ induces a decomposition $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. In this descomposition every $C \in L(\mathcal{H})$ can be written as 2×2 operator matrix. Under this representation $P_{\mathcal{S}}$ can be identified with

$$\begin{pmatrix} I_{\mathcal{S}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and all idempotents Q with range $\mathcal{S}$ have the form

$$Q = \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

for some $x \in L(\mathcal{S}^\perp, \mathcal{S})$.

Along this note, the following result, due to Douglas [13], will be used several times. For $A, B \in L(\mathcal{H})$, the following conditions are equivalent:

(a) $R(B) \subseteq R(A)$;

(b) there exists a positive number λ such that $BB^* \leqslant \lambda AA^*$;

(c) there exists $D \in L(\mathcal{H})$ such that $B = AD$. In this case, there exists a unique $D \in L(\mathcal{H})$ such that

$$B = AD, \quad \ker D = \ker B, \quad \text{and} \quad R(D) \subseteq \overline{R(A^*)}.$$

D will be called the *reduced solution* of the equation $AX = B$.

An elementary result that we shall use sometimes is that, for $A \in L(\mathcal{H})^+$, it holds $R(A) \subseteq R(A^{1/2}) \subseteq \overline{R(A)}$; moreover, $R(A) = R(A^{1/2})$ if and only if $R(A)$ is closed.

2.1. Shorted operator and compressions

Anderson [3] showed that if $A = \begin{pmatrix} B & C \\ C^* & D \end{pmatrix}$ is an $n \times n$ positive (semi-definite) matrix and B is a square $k \times k$ submatrix, then the operator

$$A_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & D - C^* B^\dagger C \end{pmatrix},$$

where $B^\dagger$ is the Moore–Penrose pseudoinverse of B and $\mathcal{S}$ the subspace of $\mathbb{C}^n$ generated by the first k canonical vectors, has the following interpretation in electrical network theory: if A is the impedance matrix of a resistive n-port network, then $A_{/\mathcal{S}}$ is the impedance matrix of the network obtained by shorting the first k ports. He proved that

$$A_{/\mathcal{S}} = \max \left\{ X \in \mathbb{C}^{n \times n} : 0 \leqslant X \leqslant A \text{ and } R(X) \subseteq \mathcal{S}^\perp \right\}$$

and used this property to extend the notion to Hilbert space positive operators.

Definition 2.1. Let $A \in L(\mathcal{H})^+$ and $\mathcal{S}$ a closed subspace of $\mathcal{H}$. Then:

1. The *shorted operator* of A by $\mathcal{S}$ is defined by

$$A_{/\mathscr{S}} = \max\left\{X \in L(\mathscr{H})^+ : X \leqslant A \text{ and } R(X) \subseteq \mathscr{S}^\perp\right\},$$

where the maximum is taken for the natural order relation in $L(\mathscr{H})^+$ (see [4]).
2. The $\mathscr{S}$*-compression* $A_{\mathscr{S}}$ of A is defined as $A_{\mathscr{S}} = A - A_{/\mathscr{S}}$.

The existence of this maximum has been proven by Krein [17] but the result has been widely used only after its rediscovering by Anderson and Trapp [4]. The reader is also referred to [1,6,18,21,22]. In the following theorem we collect some results on shorted operators proved by Anderson and Trapp [4] and Krein [17] (see also [16,21]) which are relevant in this paper.

Theorem 2.2. *Let $\mathscr{S}$ be a closed subspace of a Hilbert space $\mathscr{H}$ and let $A \in L(\mathscr{H})^+$ with matrix representation $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of $\mathscr{S}$. Then*:

1. *$R(b) \subseteq R(a^{1/2})$ and if $d \in L(\mathscr{S}^\perp, \mathscr{S})$ is the reduced solution of the equation $a^{1/2}\, x = b$, then*
$$A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.$$
2. *If $\mathscr{M} = \overline{A^{1/2}(\mathscr{S})}$ and $P_{\mathscr{M}}$ is the orthogonal projection onto $\mathscr{M}$, then*
$$A_{/\mathscr{S}} = A^{1/2}(1 - P_{\mathscr{M}})A^{1/2}.$$
3. *$A_{/\mathscr{S}}$ is the infimum of the set $\{R^*AR : R \in \mathscr{Q}, \ker R = \mathscr{S}\}$; in general, the infimum is not attained.*
4. *$R(A) \cap \mathscr{S}^\perp \subseteq R(A_{/\mathscr{S}}) \subseteq R((A_{/\mathscr{S}})^{1/2}) = R(A^{1/2}) \cap \mathscr{S}^\perp$; in general, the inclusions are strict.*

The reader is referred to [4,17] for proofs of these facts. One of the goals of this paper is to find conditions such that the infimum of item 3 becomes a minimum and the first inclusion of item 4 becomes an equality.

Corollary 2.3. *Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Then*:
1. $\overline{\ker A + \mathscr{S}} \subseteq \ker(A_{/\mathscr{S}}) = A^{-1/2}(\overline{A^{1/2}(\mathscr{S})})$.
2. $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S}$ *if and only if $A^{1/2}(\mathscr{S})$ is closed in $R(A^{1/2})$.*

Proof.
1. By Theorem 2.2, if $\mathscr{M} = \overline{A^{1/2}(\mathscr{S})}$, then $A_{/\mathscr{S}} = A^{1/2}(1 - P_{\mathscr{M}})A^{1/2}$. Hence both $\ker A$ and $\mathscr{S}$ are contained in $\ker A_{/\mathscr{S}}$. On the other hand,
$$\ker A_{/\mathscr{S}} = \ker A^{1/2}(1 - P_{\mathscr{M}})A^{1/2} = \ker(1 - P_{\mathscr{M}})A^{1/2} = A^{-1/2}(\mathscr{M})$$
2. It is clear that $A^{1/2}(\mathscr{S})$ is closed in $R(A^{1/2})$ if and only if $\mathscr{M} \cap R(A^{1/2}) = A^{1/2}(\mathscr{S})$, which is equivalent to $A^{-1/2}(\mathscr{M}) = A^{-1/2}(A^{1/2}(\mathscr{S})) = \ker A + \mathscr{S}$. □

Proposition 2.4. *Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Then the following properties hold*:
1. $(A_{\mathscr{S}})_{/\mathscr{S}} = 0$.
2. *If* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *and d is the reduced solution of the equation $a^{1/2}x = b$, then*
$$A_{\mathscr{S}} = \begin{pmatrix} a & b \\ b^* & d^*d \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ d^* & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & d \\ 0 & 0 \end{pmatrix}.$$
3. $A_{\mathscr{S}} = A^{1/2} P_{\mathscr{M}} A^{1/2}$, *where* $\mathscr{M} = \overline{A^{1/2}(\mathscr{S})}$.
4. $\ker A_{\mathscr{S}} = A^{-1}(\mathscr{S}^{\perp})$.
5. $A(\mathscr{S}) \subseteq R(A_{\mathscr{S}}) \subseteq \overline{A(\mathscr{S})}$.

Proof. Items 1–3 are direct consequences of Theorem 2.2.
4. Note that $\mathscr{M}^{\perp} = A^{-1/2}(\mathscr{S}^{\perp})$. Therefore
$$\ker A_{\mathscr{S}} = \ker P_{\mathscr{M}} A^{1/2} = A^{-1/2}(\mathscr{M}^{\perp}) = A^{-1/2}(A^{-1/2}(\mathscr{S}^{\perp})) = A^{-1}(\mathscr{S}^{\perp}).$$
5. Since $\mathscr{S} \subseteq \ker A_{/\mathscr{S}}$, then $A(\mathscr{S}) = A_{\mathscr{S}}(\mathscr{S}) \subseteq R(A_{\mathscr{S}}) \subseteq (\ker A_{\mathscr{S}})^{\perp} = \overline{A(\mathscr{S})}$. This completes the proof. □

2.2. *Complementable matrices*

We start with Haynsworth's definition of Schur complements of square matrices [15]: given a block matrix $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$, with B invertible, then $A/_B = E - DB^{-1}C$ is the Schur complement of B in A. The reader is referred to the nice surveys by Cottle [9], Carlson [7] and Ouellette [19] for many properties and applications. The notion was extended in several directions. In particular, Ando [2] introduced the following generalization.

Definition 2.5. Given a closed subspace $\mathscr{S}$ of $\mathscr{H}$ and $A \in L(\mathscr{H})$, A is said to be *$\mathscr{S}$-complementable* if there exist operators $M_r, M_l \in L(\mathscr{H})$ such that $PM_r = M_r$, $M_l P = M_l$, $PAM_r = PA$ and $M_l AP = AP$, where $P = P_{\mathscr{S}}$. In this case it can be shown that AM_r only depends on A and $\mathscr{S}$, not on M_r. The operator $A_{\mathscr{S}} = AM_r$ is called the *$\mathscr{S}$-compression* of A to $\mathscr{S}$ and $A_{/\mathscr{S}} = A - A_{\mathscr{S}}$ is the *Schur complement* of A to $\mathscr{S}$.

The use of the same notation as for shorted operators can be justified as follows: if $\mathscr{H} = \mathbb{C}^n$, $\mathscr{S} = \mathbb{C}^k \times \{0\}$ and $\mathscr{S}^{\perp} = \{0\} \times \mathbb{C}^{n-k}$, then every $A \in L(\mathscr{H})$ can be written as $A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$ and, if B is invertible, then $M_r = \begin{pmatrix} 1 & B^{-1}C \\ 0 & 0 \end{pmatrix}$ and $M_l = \begin{pmatrix} 1 & 0 \\ DB^{-1} & 0 \end{pmatrix}$ satisfy the equations above and

$$A_{\mathscr{S}} = \begin{pmatrix} B & C \\ D & DB^{-1}C \end{pmatrix}, \quad A_{/\mathscr{S}} = \begin{pmatrix} 0 & 0 \\ 0 & E - DB^{-1}C \end{pmatrix}.$$

Thus, $A_{/\mathscr{S}}$ is determined by the classical Schur complement. On the other hand, if $A \in L(\mathscr{H})^+$, we shall see in Corollary 3.5 below that, if A is $\mathscr{S}$-complementable, then the Schur complement $A_{/\mathscr{S}}$ coincides with the shorted operator of Anderson and Trapp. The reader will find more information on these matters in [2,7,8].

2.3. *A-selfadjoint projections and compatibility*

Any selfadjoint operator $A \in L(\mathscr{H})$ defines a bounded Hermitian sesquilinear form $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$, $\xi,\ \eta \in \mathscr{H}$. The A-orthogonal subspace of a subset $\mathscr{S}$ of $\mathscr{H}$ is

$$\mathscr{S}^{\perp_A} := \left\{ \xi : \langle A\xi, \eta \rangle = 0\ \forall \eta \in \mathscr{S} \right\} = A^{-1}(\mathscr{S}^{\perp}) = A(\mathscr{S})^{\perp}.$$

Given $T \in L(\mathscr{H})$, an operator $W \in L(\mathscr{H})$ is called an *A-adjoint* of T if $\langle T\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$ for every $\xi, \eta \in \mathscr{H}$, or, which is the same, if $T^*A = AW$. Therefore, the existence of an A-adjoint W of T is equivalent to $R(T^*A) \subseteq R(A)$. In particular, if $Q \in L(\mathscr{H})$ and $Q^2 = Q$, then the existence of an A-adjoint of Q is also equivalent to the decomposition

$$R(A) = R(A) \cap \ker Q^* \oplus R(A) \cap R(Q^*). \tag{1}$$

We say that $C \in L(\mathscr{H})$ is *A-selfadjoint* if $AC = C^*A$. We consider the existence and uniqueness of A-selfadjoint projections whose range is exactly $\mathscr{S}$, namely in the elements of

$$\mathscr{P}(A, \mathscr{S}) = \left\{ Q \in \mathscr{Q} : R(Q) = \mathscr{S},\ AQ = Q^*A \right\}.$$

A pair $(A, \mathscr{S})$ is called *compatible* if $\mathscr{P}(A, \mathscr{S})$ is not empty. Sometimes we say that A is $\mathscr{S}$-compatible or that $\mathscr{S}$ is A-compatible. In the following theorem we present several results about compatibility, extracted from the authors' paper [10] (see also [14]).

Theorem 2.6. *Let $A \in L(\mathscr{H})_h$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Write* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$.
Then:
1. *$(A, \mathscr{S})$ is compatible if and only if $R(b) \subseteq R(a)$ if and only if $\mathscr{S} + A^{-1}(\mathscr{S}^{\perp}) = \mathscr{H}$.*
2. *In this case, if $d \in L(\mathscr{S}^{\perp}, \mathscr{S})$ is the reduced solution of the equation $ax = b$, then*

$$P_{A,\mathscr{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix} \in \mathscr{P}(A, \mathscr{S})$$

   and, if $\mathscr{N} = A^{-1}(S^{\perp}) \cap \mathscr{S}$, then $\ker P_{A,\mathscr{S}} = A^{-1}(\mathscr{S}^{\perp}) \ominus \mathscr{N}$.
3. *$\mathscr{P}(A, \mathscr{S})$ is an affine manifold that can be parametrized as*

$$\mathscr{P}(A, \mathscr{S}) = P_{A,\mathscr{S}} + L(\mathscr{S}^{\perp}, \mathscr{N}),$$

where $L(\mathscr{S}^{\perp}, \mathscr{N})$ *is viewed as a subspace of* $L(\mathscr{H})$. *Observe that* $\mathscr{P}(A, \mathscr{S})$ *has a unique element (namely,* $P_{A,\mathscr{S}}$*) if and only if* $\mathscr{N} = \{0\}$, *i.e., if* $\mathscr{S} \oplus A^{-1}(\mathscr{S}^{\perp}) = \mathscr{H}$.

4. $P_{A,\mathscr{S}}$ *has minimal norm in* $\mathscr{P}(A, \mathscr{S})$. *Nevertheless,* $P_{A,\mathscr{S}}$ *is not in general the unique element of* $\mathscr{P}(A, \mathscr{S})$ *that realizes the minimum.*
5. *If* $A \in L(\mathscr{H})^+$, *then* $\mathscr{N} = \ker A \cap \mathscr{S}$.
6. *If* $A \in L(\mathscr{H})^+$ *has closed range, then the pair* $(A, \mathscr{S})$ *is compatible if and only if any of the following subspaces:* $R(P_{\mathscr{S}} A P_{\mathscr{S}})$, $\mathscr{S} + \ker A$, $R(A) + \mathscr{S}^{\perp}$, $A^{1/2}(\mathscr{S})$ *or* $A(\mathscr{S})$ *is closed.*
7. *If* $A \in L(\mathscr{H})^+$ *is injective, then the following conditions are equivalent:*
   (a) *The pair* $(A, \mathscr{S})$ *is compatible.*
   (b) $\mathscr{S}^{\perp} \oplus \overline{A(\mathscr{S})}$ *is closed.*
   (c) $\ker A_{/\mathscr{S}} = \mathscr{S}$ *and* $R(A_{/\mathscr{S}}) \subseteq R(A)$.
   (d) $\mathscr{S}^{\perp} \subseteq R(A + \lambda(1 - P_{\mathscr{S}}))$ *for some (and then for any)* $\lambda > 0$.

For a proof of these facts see [10]. In [11] estimations for the norm of $P_{A,\mathscr{S}}$ are given, in terms of its applications to abstract spline interpolation.

Remark 2.7.
1. Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Denote by $P = P_{\mathscr{S}}$. Then, if $R(PAP)$ is closed, the pair $(A, \mathscr{S})$ is compatible. Indeed, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, then, by Theorem 2.2, $R(b) \subseteq R(a^{1/2})$. But if $R(PAP)$ is closed, $R(a^{1/2}) = R(a)$, then, by Theorem 2.6, the pair $(A, \mathscr{S})$ is compatible. In particular, if $\dim \mathscr{S} < \infty$, then $(A, \mathscr{S})$ is compatible. As a consequence, if $\dim \mathscr{H} < \infty$, then every pair $(A, \mathscr{S})$ is compatible.
2. If $A \in \mathrm{GL}(\mathscr{H})^+$, then $R(PAP) = \mathscr{S}$ so that $(A, \mathscr{S})$ is compatible. In this case, $P_{A,\mathscr{S}}$ is determined (see [5] or [20]) by the formulae
$$P_{A,\mathscr{S}} = P(1 + P - A^{-1}PA)^{-1} = (PAP + (1 - P)A(1 - P))^{-1} PA = \begin{pmatrix} 1 & a^{-1}b \\ 0 & 0 \end{pmatrix}.$$
3. Given two subspaces $\mathscr{S}, \mathscr{T}$ the cosine of the Friedrichs angle between them is defined by $c(\mathscr{S}, \mathscr{T}) = \sup\{|\langle \xi, \eta\rangle| : \xi \in \mathscr{S} \cap (\mathscr{S} \cap \mathscr{T})^{\perp},\ \|\xi\| \leqslant 1,\ \eta \in \mathscr{T} \cap (\mathscr{S} \cap \mathscr{T})^{\perp},\ \|\eta\| \leqslant 1\}$. It is well known (see [12]) that $c(\mathscr{S}, \mathscr{T}) < 1$ if and only if $\mathscr{S} + \mathscr{T}$ is closed. Then compatibility, if A is injective or $R(A)$ is closed, is related to an angle condition between two subspaces (see Theorem 2.6):
   (a) If $A \in L(\mathscr{H})^+$ has closed range, then $(A, \mathscr{S})$ is compatible if and only if $c(\mathscr{S}, \ker A) < 1$.
   (b) If $A \in L(\mathscr{H})^+$ is injective, then $(A, \mathscr{S})$ is compatible if and only if $c(\mathscr{S}^{\perp}, \overline{A(\mathscr{S})}\,) < 1$.

Example 2.8. Let $A \in L(\mathscr{H})^+$ be injective such that $R(A)$ is not closed. Consider

$$M = \begin{pmatrix} A & A^{1/2} \\ A^{1/2} & I \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ I & 0 \end{pmatrix} \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} \in L(\mathscr{H} \oplus \mathscr{H})^+.$$

If $\mathscr{S} = \mathscr{H} \oplus \{0\}$, then $(M, \mathscr{S})$ is not compatible, because $R(A)$ is properly contained in $R(A^{1/2})$. Also $M/_{\mathscr{S}} = 0$, because the reduced solution (actually, the unique solution) of the equation $A^{1/2}X = A^{1/2}$ is $D = I$. Therefore, $M = M_{\mathscr{S}}$ and $M(\mathscr{S}) \neq R(M_{\mathscr{S}})$. Moreover, $R(M_{\mathscr{S}})$ is closed, because

$$R \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} = (R(A^{1/2}) + R(I)) \oplus \{0\} = \mathscr{S}$$

is closed. Hence, in this example, $R(M) = R(M_{\mathscr{S}}) = \overline{M(\mathscr{S})}$ while $M(\mathscr{S})$ is not closed (see Propositions 2.4 and 3.9). Observe that

$$\ker M = \ker M_{\mathscr{S}} = \ker \begin{pmatrix} A^{1/2} & I \\ 0 & 0 \end{pmatrix} = \left\{ \xi \oplus -A^{1/2}\xi : \xi \in \mathscr{H} \right\},$$

which is the graph of $-A^{1/2}$. Note that $c(\mathscr{S}, \ker M) = 1$.

3. Compatibility and shorted operators

In this section we compare the notions introduced in the previous one. As a first result, we prove that $\mathscr{S}$-compatibility and $\mathscr{S}$-complementability are equivalent properties for selfadjoint operators.

Proposition 3.1. *Let $A \in L(\mathscr{H})_h$. Then the pair $(A, \mathscr{S})$ is compatible if and only if A is $\mathscr{S}$-complementable.*

Proof. Suppose that $(A, \mathscr{S})$ is compatible and let $Q \in \mathscr{P}(A, \mathscr{S})$. Then $M_r = Q$ and $M_l = Q^*$ satisfy the equations of Definition 2.5. Conversely, if $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ is $\mathscr{S}$-complementable, then the equation $P_{\mathscr{S}} M_r = M_r$ implies that M_r has matrix form $\begin{pmatrix} u & v \\ 0 & 0 \end{pmatrix}$ and the equation $P_{\mathscr{S}} A M_r = P_{\mathscr{S}} A$ implies that $av = b$. Then $R(b) \subseteq R(a)$ and $(A, \mathscr{S})$ is compatible by Theorem 2.6 (1). □

Proposition 3.2. *Let $A \in L(\mathscr{H})^+$. Then*:
1. $(A^2_{/\mathscr{S}})^{1/2} \leqslant A_{/\mathscr{S}}$.
2. *If $(A, \mathscr{S})$ is compatible, then $A(\mathscr{S})$ is closed in $R(A)$.*
3. *If $A(\mathscr{S})$ is closed in $R(A)$, then $A^{1/2}(\mathscr{S})$ is closed in $R(A^{1/2})$.*

Proof.
1. Observe that, by the monotonicity of the square root, the inequality $A^2_{\mathscr{S}} \leqslant A^2$ implies that $(A^2_{/\mathscr{S}})^{1/2} \leqslant A$. On the other hand, $R((A^2_{/\mathscr{S}})^{1/2}) \subseteq \mathscr{S}^\perp$ so that $(A^2_{/\mathscr{S}})^{1/2} \leqslant A_{/\mathscr{S}}$ by Definition 2.1.

2. Assume that $(A, \mathscr{S})$ is compatible and let $Q \in \mathscr{P}(A, \mathscr{S})$. Then, by Eq. (1), $R(A) = R(A) \cap R(Q^*) \oplus R(A) \cap \ker Q^*$. Therefore $A(\mathscr{S}) = R(AQ) = R(Q^*A) = R(Q^*) \cap R(A)$ is closed in $R(A)$.
3. Using Corollary 2.3, the fact that $A(\mathscr{S})$ is closed in $R(A)$ implies that $\ker A^2_{/\mathscr{S}} = \ker A^2 + \mathscr{S} = \ker A + \mathscr{S}$. Then, by item 1, $\ker A_{/\mathscr{S}} \subseteq \ker A^2_{/\mathscr{S}} \subseteq \ker A + \mathscr{S}$ so that $A^{1/2}(\mathscr{S})$ is closed in $R(A^{1/2})$, again by Corollary 2.3. □

The following lemma gives equivalent conditions for a projection to be A-selfadjoint.

Lemma 3.3. *Let $A \in L(\mathscr{H})^+$ and $Q \in \mathscr{Q}$. Then the following conditions are equivalent*:
1. *Q is A-selfadjoint.*
2. $\ker Q \subseteq R(Q)^{\perp_A} = A^{-1}(R(Q)^{\perp}) = \ker Q^*A$.
3. *Q is an A-contraction, i.e., $Q^*AQ \leqslant A$.*

Proof. See 3.2 of [10]. □

The following proposition relates, when $(A, \mathscr{S})$ is compatible, the shorted operator $A_{/\mathscr{S}}$ with the elements of $\mathscr{P}(A, \mathscr{S})$. Items 1, 2 and the first part of item 3 were previously proven in [10]. We include a short proof of them because they appear as the key point of the relationship between shorted operators and A-selfadjoint projections.

Proposition 3.4. *Let $A \in L(\mathscr{H})^+$ such that the pair $(A, \mathscr{S})$ is compatible. Then*:
1. *$A_{/\mathscr{S}} = AQ = Q^*AQ$ for every $Q \in \mathscr{Q}$ such that $1 - Q \in \mathscr{P}(A, \mathscr{S})$.*
2. *$A_{/\mathscr{S}} = \min\{R^*AR : R \in \mathscr{Q}, \ker R = \mathscr{S}\}$. Actually, this property is equivalent to the compatibility of the pair $(A, \mathscr{S})$.*
3. *$R(A_{/\mathscr{S}}) = R(A) \cap S^{\perp}$ and $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S}$.*

Proof.
1. Let $Q \in \mathscr{Q}$ such that $1 - Q \in \mathscr{P}(A, \mathscr{S})$. Note that $0 \leqslant AQ = Q^*AQ \leqslant A$, by Lemma 3.3. Also $R(AQ) = R(Q^*A) \subseteq R(Q^*) = \mathscr{S}^{\perp}$. Therefore, $AQ \leqslant A_{/\mathscr{S}}$. On the other hand, if $0 \leqslant X \leqslant A$ and $R(X) \subseteq \mathscr{S}^{\perp}$, then $X = Q^*XQ \leqslant Q^*AQ = AQ$, where the first equality holds because $\ker Q = \mathscr{S}$ and X has the form $\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$.
2. By item 1, $Q^*AQ = A_{/\mathscr{S}}$ and $\ker Q = \mathscr{S}$. So the minimum is attained at Q by Theorem 2.2. On the other hand, if the minimum is attained at some projection Y, then $Y^*AY = A_{/\mathscr{S}} \leqslant A$ implies that Y is A-selfadjoint, by Lemma 3.3. Therefore $1 - Y \in \mathscr{P}(A, \mathscr{S})$.

3. Clearly the equation $A_{/\mathscr{S}} = AQ$ implies that $R(A_{/\mathscr{S}}) \subseteq R(A) \cap S^{\perp}$. The other inclusion always holds by Theorem 2.2. The equality $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S}$ follows from Proposition 3.2 and Corollary 2.3. □

The next result justifies the use of the same symbol $A_{/\mathscr{S}}$ (resp. $A_{\mathscr{S}}$) denoting shorted operators and Schur complements (resp. compressions).

Corollary 3.5. *Let $A \in L(\mathscr{H})^{+}$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$ such that A is S-complementable. Then the Schur complement* (*as in Definition* 2.5) *and the shorted operator of A by $\mathscr{S}$ are the same operator $A_{/\mathscr{S}}$.*

Proof. If A is S-complementable, then $(A, \mathscr{S})$ is compatible and, if $Q \in \mathscr{P}(A, \mathscr{S})$, Proposition 3.4 says that the shorted operator $A_{/\mathscr{S}} = A(1 - Q)$. On the other hand, by the proof of Proposition 3.1, the Schur complement also equals $A - A_{\mathscr{S}} = A - AQ = A(1 - Q)$. □

The condition $R(A_{/\mathscr{S}}) \subseteq R(A)$, which is necessary for compatibility, implies that some subspace bigger than $\mathscr{S}$ (actually, $\ker A_{/\mathscr{S}}$) is A-compatible:

Proposition 3.6. *Let $A \in L(\mathscr{H})^{+}$. Denote* $\ker A_{/\mathscr{S}} = \mathscr{T}$. *Then*:
1. $A_{/\mathscr{T}} = A_{/\mathscr{S}}$.
2. *The pair $(A, \mathscr{T})$ is compatible if and only if $R(A_{/\mathscr{S}}) \subseteq R(A)$.*

Proof. Item 1 follows directly from the definition of shorted operator, because $\mathscr{S} \subseteq \mathscr{T}$ and $R(A_{/\mathscr{S}}) \subseteq \mathscr{T}^{\perp}$. Suppose that $R(A_{/\mathscr{S}}) \subseteq R(A)$. By Douglas theorem, there exists $Q \in L(\mathscr{H})$ such that $AQ = A_{/\mathscr{S}}$ and $\ker Q = \mathscr{T}$. Then $Q^{*}A = AQ$, because $A_{/\mathscr{S}}$ is selfadjoint. In order to prove that $1 - Q \in \mathscr{P}(A, \mathscr{T})$, it just remains to show that $Q^2 = Q$. Let us first prove that, if $\mathscr{Z} = A^{-1/2}(\mathscr{S}^{\perp}) = A^{1/2}(\mathscr{S})^{\perp}$, then Q is a solution of the equation $A^{1/2}X = P_{\mathscr{Z}}A^{1/2}$. Recall that, by Theorem 2.2, $A_{/\mathscr{S}} = A^{1/2}P_{\mathscr{Z}}A^{1/2}$, so $A^{1/2}(A^{1/2}Q - P_{\mathscr{Z}}A^{1/2}) = 0$. Then, for $\xi \in \mathscr{H}$, it holds $P_{\mathscr{Z}}A^{1/2}\xi - A^{1/2}Q\xi = \eta \in \ker A^{1/2} = R(A^{1/2})^{\perp}$. Therefore, $\|\eta\|^2 = \langle P_{\mathscr{Z}}A^{1/2}\xi, \eta\rangle = \langle \xi, A^{1/2}\eta\rangle = 0$, because $\eta \in \ker A^{1/2} \subseteq \mathscr{Z}$. Thus, $A^{1/2}Q = P_{\mathscr{Z}}A^{1/2}$ and, as a consequence, $A^{1/2}Q^2 = (P_{\mathscr{Z}})^2A^{1/2} = A^{1/2}Q$, so $A^{1/2}(Q^2 - Q) = 0$. Let $\rho \in R(Q)$. Then $Q\rho - \rho \in \ker A \cap R(Q)$. If $Q\rho - \rho = Q\omega$ for some $\omega \in \mathscr{H}$, then $0 = AQ\omega = A_{/\mathscr{S}}\omega$. So $\omega \in \ker A_{/\mathscr{S}} = \mathscr{T} = \ker Q$. Therefore $Q\rho = \rho$ for every $\rho \in R(Q)$. This clearly implies that $Q^2 = Q$ and $1 - Q \in \mathscr{P}(A, \mathscr{T})$. Conversely, if $(A, \mathscr{T})$ is compatible, then $R(A_{/\mathscr{S}}) = R(A_{/\mathscr{T}}) \subseteq R(A)$ by Proposition 3.4. □

Remark 3.7. In this remark we present a characterization of compatibility in terms of the range and nullspace of the shorted operator, which holds if A has closed range or if it is injective. They are deduced from Theorem 2.6, i.e., from authors' paper [10].

1. If A has closed range, then $(A, \mathscr{S})$ is compatible if and only if $\ker(A_{/\mathscr{S}}) = \mathscr{S} + \ker A$. Indeed, $(A, \mathscr{S})$ is compatible if and only if $A^{1/2}(\mathscr{S})$ is closed (see Theorem 2.6) if and only if $A^{1/2}(\mathscr{S})$ is closed in $R(A^{1/2})$ (because $R(A^{1/2}) = R(A)$ is closed) if and only if $\ker A_{/\mathscr{S}} = \mathscr{S} + \ker A$ (see Corollary 2.3). Note that $R(A_{/\mathscr{S}}) = R(A) \cap \mathscr{S}^{\perp}$ if $R(A)$ closed.
2. If A is injective, then, by Theorem 2.6, $(A, \mathscr{S})$ is compatible if and only if $R(A_{/\mathscr{S}}) = R(A) \cap \mathscr{S}^{\perp}$ and $\ker(A_{/\mathscr{S}}) = \mathscr{S}$ (observe that this is also a direct consequence of Propositions 3.4 and 3.6).

The following theorem generalizes these results to the general case:

Theorem 3.8. *Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Then $(A, \mathscr{S})$ is compatible if and only if $R(A_{/\mathscr{S}}) = R(A) \cap \mathscr{S}^{\perp}$ and* $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S}$.

Proof. The necessity has been proved in Proposition 3.4. Conversely, if $R(A_{/\mathscr{S}}) = R(A) \cap \mathscr{S}^{\perp}$ and $\ker A_{/\mathscr{S}} = \ker A + \mathscr{S} = \mathscr{T}$, then, by Proposition 3.6, $(A, \mathscr{T})$ is compatible, or equivalently $\mathscr{T} + A^{-1}(\mathscr{T}^{\perp}) = \mathscr{H}$. But

$$\ker A \subseteq A^{-1}(\mathscr{S}^{\perp}) = A(\mathscr{S})^{\perp} = A(\mathscr{T})^{\perp} = A^{-1}(\mathscr{T}^{\perp})$$

so that $\mathscr{S} + A^{-1}(\mathscr{S}^{\perp}) = \mathscr{H}$. Then $(A, \mathscr{S})$ is compatible. □

3.1. Compressions

Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Recall that the *compression* of A by $\mathscr{S}$ is defined as $A_{\mathscr{S}} = A - A_{/\mathscr{S}}$. Using Proposition 3.4, if $(A, \mathscr{S})$ is compatible, then $A_{\mathscr{S}} = AP_{A,\mathscr{S}}$. Therefore $R(A_{\mathscr{S}}) = A(\mathscr{S})$. The next result shows that this equality actually characterizes compatibility:

Proposition 3.9. *Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. Then*:
1. *The pair $(A, \mathscr{S})$ is compatible if and only if $R(A_{\mathscr{S}}) = A(\mathscr{S})$.*
2. *If $(A, \mathscr{S})$ is compatible, Y is the reduced solution of the equation $(AP)X = A_{\mathscr{S}}$ and $\mathscr{N} = \ker A \cap \mathscr{S}$, then $Y = P_{A,\mathscr{S}\ominus\mathscr{N}}$ and $P_{A,\mathscr{S}} = Y + P_{\mathscr{N}}$.*

Proof. If $(A, \mathscr{S})$ is compatible, then from the properties of $A_{\mathscr{S}}$ above, $R(A_{\mathscr{S}}) = A(\mathscr{S})$. Conversely, $R(A_{\mathscr{S}}) = A(\mathscr{S})$ implies that

$$\begin{aligned}\mathscr{H} &= A_{\mathscr{S}}^{-1}(R(A_{\mathscr{S}})) = A_{\mathscr{S}}^{-1}(A(\mathscr{S})) = A_{\mathscr{S}}^{-1}(A_{\mathscr{S}}(\mathscr{S})) = \mathscr{S} + \ker A_{\mathscr{S}} \\ &= \mathscr{S} + A^{-1}(\mathscr{S}^{\perp}),\end{aligned}$$

by Proposition 2.4. Then, by using Theorem 2.6, the pair $(A, \mathscr{S})$ is compatible.

Suppose that $R(A_{\mathscr{S}}) = A(\mathscr{S})$ and denote by Y the reduced solution of the equation $AP_{\mathscr{S}}X = A_{\mathscr{S}}$. Then

$$\begin{aligned} &\ker Y = \ker A_{\mathscr{S}} = A^{-1}(\mathscr{S}^{\perp}), \\ &R(Y) \subseteq (\ker AP_{\mathscr{S}})^{\perp} = (\mathscr{S}^{\perp} + \mathscr{N})^{\perp} = \mathscr{S} \ominus \mathscr{N}. \end{aligned} \tag{2}$$

So $P_{\mathscr{S}}Y = Y$ and $A_{\mathscr{S}} = AY = Y^*A$, which means that Y is A-selfadjoint. On the other hand, because $A|_{\mathscr{S}} = A_{\mathscr{S}}|_{\mathscr{S}}$ and the fact that $A|_{\mathscr{S}\ominus\mathscr{N}}$ is injective, we can deduce that $Y\xi = \xi$ for every $\xi \in \mathscr{S} \ominus \mathscr{N}$, which means that $Y^2 = Y$ and $R(Y) = \mathscr{S} \ominus \mathscr{N}$. The equality $P_{A,\mathscr{S}} = Y + P_{\mathscr{N}}$ follows from Theorem 2.6, because $\mathscr{N} = \mathscr{S} \cap A^{-1}(\mathscr{S}^{\perp})$ so that $YP_{\mathscr{N}} = P_{\mathscr{N}}Y = 0$. □

Corollary 3.10. *Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. If $(A, \mathscr{S})$ is compatible and $\mathscr{N} = \ker A \cap \mathscr{S}$, then*:

1. *The pairs $(A, \mathscr{S} \ominus \mathscr{N})$, $(A, A(\mathscr{S})^{\perp})$ and $(A, A(\mathscr{S})^{\perp} \ominus \mathscr{N})$ are compatible.*
2. *$A_{/\mathscr{S}} = A_{A(\mathscr{S})^{\perp}\ominus\mathscr{N}}$ and $A_{\mathscr{S}} = A_{/A(\mathscr{S})^{\perp}\ominus\mathscr{N}}$.*
3. *$A_{/A(\mathscr{S})^{\perp}} = A_{\mathscr{S}\ominus\mathscr{N}}$ and $A_{/\mathscr{S}\ominus\mathscr{N}} = A_{A(\mathscr{S})^{\perp}}$.*

Proof. Recall that $A(\mathscr{S})^{\perp} = A^{-1}(\mathscr{S}^{\perp})$.

1. Let Y be the projection of Proposition 3.6. Then, by Eq. (2), $Y \in \mathscr{P}(A, \mathscr{S} \ominus \mathscr{N})$ and $1 - Y \in \mathscr{P}(A, A(\mathscr{S})^{\perp})$. On the other hand, by Theorem 2.6, $\ker P_{A,\mathscr{S}} = A(\mathscr{S})^{\perp} \ominus \mathscr{N}$ so that $1 - P_{A,\mathscr{S}} \in \mathscr{P}(A, A(\mathscr{S})^{\perp} \ominus \mathscr{N})$.
2. Since $1 - P_{A,\mathscr{S}} \in \mathscr{P}(A, A(\mathscr{S})^{\perp} \ominus \mathscr{N})$, by Proposition 3.4, $A_{/\mathscr{S}} = A(1 - P_{A,\mathscr{S}}) = A_{A(\mathscr{S})^{\perp}\ominus\mathscr{N}}$ and $A_{\mathscr{S}} = AP_{A,\mathscr{S}} = A_{/A(\mathscr{S})^{\perp}\ominus\mathscr{N}}$.
3. As before $A_{/A(\mathscr{S})^{\perp}} = A(1 - Y) = A_{\mathscr{S}\ominus\mathscr{N}}$ and $A_{/\mathscr{S}\ominus\mathscr{N}} = AY = A_{A(\mathscr{S})^{\perp}}$. □

4. Some examples

Example 4.1. Given a positive injective operator $A \in L(\mathscr{H})$ with non-closed range. Let $\xi \in R(A^{1/2})$ and let P_ξ be the orthogonal projection onto the subspace $\langle\xi\rangle$ generated by ξ. Then $R(P_\xi) \subseteq R(A^{1/2})$ so that, by Douglas' theorem, $P_\xi \leqslant \lambda A$ for some positive number λ which we can suppose equal to 1, by changing A by λA. It is well known that this implies that the operator $B \in L(\mathscr{H} \oplus \mathscr{H})$ defined by

$$B = \begin{pmatrix} A & P_\xi \\ P_\xi & A \end{pmatrix}$$

is positive. Suppose that $\xi \in R(A^{1/2})\backslash R(A)$. Let $\mathscr{S} = \mathscr{H} \oplus \{0\}$. Then the pair $(B, \mathscr{S})$ is incompatible, because $R(P_\xi) \not\subseteq R(A)$. We shall see that B is injective, that $\ker B_{/\mathscr{S}} = \mathscr{S}$ and, moreover, that $B(\mathscr{S})$ is closed in $R(B)$. Indeed, because $\xi \notin R(A)$ and A is injective,

$$\begin{aligned} B(\omega \oplus \eta) = 0 \oplus 0 &\Rightarrow A\omega + P_\xi\eta = 0 = A\eta + P_\xi\omega \\ &\Rightarrow A\omega = A\eta = 0 \\ &\Rightarrow \omega = \eta = 0 \end{aligned}$$

so that B is injective. Finally, $(\mathscr{H} \oplus \langle\xi\rangle) \cap R(B) = B(\mathscr{H} \oplus \{0\})$, because if $\omega \neq 0$, then $A\omega \notin \langle\xi\rangle$ and $B(\eta \oplus \omega) \notin \mathscr{H} \oplus \langle\xi\rangle$ for every $\eta \in \mathscr{H}$. Therefore $B(\mathscr{S})$ is closed in $R(B)$. This implies, by Proposition 3.2, that $\ker B_{/\mathscr{S}} = \mathscr{S} + \ker B = \mathscr{S}$. This example shows that the condition $R(B_{/\mathscr{S}}) \subseteq R(B)$ (which does not hold in this example) is indeed necessary in Theorem 3.8.

Remark 4.2. Let $A \in L(\mathscr{H})^+$ and $\mathscr{S}$ a closed subspace of $\mathscr{H}$. If $\dim \mathscr{S} < \infty$, then $(A, \mathscr{S})$ is compatible by Remark 2.7. If $\dim \mathscr{S}^\perp < \infty$ and $R(A)$ is closed, then $\mathscr{S}^\perp + R(A)$ is closed and $(A, \mathscr{S})$ is compatible, by Theorem 2.6. If $R(A)$ is not closed, the result may fail. This is shown in the next example.

Example 4.3. Let $A \in L(\mathscr{H})^+$ be injective non-invertible. Let $\xi \in \mathscr{H} \setminus R(A)$ a unit vector. Denote by $\mathscr{S} = \langle\xi\rangle^\perp$, $P = P_{\mathscr{S}}$ and $P_\xi = 1 - P$. If $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ in terms of P and $A\xi = \lambda\xi + \eta$ with $\eta \in \mathscr{S}$, then $\lambda = \langle A\xi, \xi\rangle \neq 0$ and $\eta \neq 0$ (otherwise $\xi \in R(A)$). Therefore $c = \lambda P_\xi$ and $b(\mu\xi) = \mu\eta$, $\mu \in \mathbb{C}$. Suppose that $\eta \in R(a)$, i.e., there exists $\nu \in \mathscr{S}$ which verifies $a\nu = b\xi$. Then $PA(\nu - \xi) = a\nu - b\xi = 0$, so $A(\nu - \xi)$ is a multiple of ξ, which must be 0 ($\xi \notin R(A)$). So $\nu = \xi$, a contradiction. Therefore $R(b) \not\subseteq R(a)$ and the pair $(A, \mathscr{S})$ is incompatible, while $\dim \mathscr{S}^\perp = 1 < \infty$.

Let $B = A_{\mathscr{S}}$. Since $PA = PB$, then the pair $(B, \mathscr{S})$ is also incompatible. By Proposition 2.4 and the fact that $\xi \notin R(A)$, $\ker B = A^{-1}(\mathscr{S}^\perp) = \{0\}$ and also B is injective. Recall that $B_{/\mathscr{S}} = 0$ so that $R(B_{/\mathscr{S}}) \subseteq R(B)$. This example shows that the condition $\ker B_{/\mathscr{S}} = \mathscr{S} + \ker B$ (which does not hold in this example, because $\ker B_{/\mathscr{S}} = \mathscr{H}$ and $\ker B = \{0\}$) is indeed necessary in Theorem 3.8.

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