Simultaneous state estimation and control for nonlinear systems subject to bounded disturbances ⋆

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Abstract

In this work, we address the output–feedback problem for nonlinear systems under bounded disturbances using a moving horizon approach. The controller is posed as an optimization-based problem that simultaneously estimates the state trajectory and computes the future of control inputs. It minimizes a criterion that involves finite forward and backward horizon with respect the unknown initial state, measurement noises and control input variables and it is maximized with respect the unknown disturbances. Under appropriate assumptions that encode stability and detectability, we show that the states of the closed-loop system remain bounded. A simulation example is included to show that the algorithm succeeds even in nonlinear problems.

Key words: Moving horizon estimation, Model predictive control, Robust stability, Nonlinear systems.

1 Introduction

One of the most popular control technique in both academia and industry is model predictive control (*MPC*) due its ability to explicitly accommodate

⋆ The material in this paper was not presented at any conference.

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hard state and input constraints (Bemporad & Morari 1999, Camacho & Alba 2004, Rawlings & Mayne 2009, Mayne 2014, among others). Thereon, much effort has been devoted to develop a stability theory for *MPC* (see i.e. Grüne & Pannek 2011, Rawlings & Mayne 2009, Mayne 2016). An overview of recent developments can be found in Mayne (2014). *MPC* involves the solution of an open–loop optimal control problem at each sampling time with the current state as initial condition. Each of these optimizations provides the sequences of future control actions and states. The first element of the control action sequence is applied to the system and, then the optimization problem is solved again at the next sampling time after updating the initial condition with the system state. *MPC* keeps constant the computational burden by optimizing the system behaviour within a finite length window. The system behaviour beyond the window is summarized in a term known as *cost–to–go*.

MPC is often formulated assuming that the system state can be measured. However, in many practical cases the only information available are noisy measurements of system output, so the use of independent algorithms for state estimation (including observers, filters and estimators) becomes necessary (see Rawlings & Bakshi 2006). Of all these methods, moving horizon estimation (*MHE*) is especially engaging for use with *MPC* because it can be formulated as a similar on–line optimization problem. Solving the *MHE* problem produces an estimated state that is compatible with a set of past measurements that recedes as current time advances (Schweppe 1973, Rao et al. 2001, 2003). This estimate is optimal in the sense that it maximizes a criterion that capture the likelihood of the measurements. Along the same time that relevant results on *MPC* were developed, research works on *MHE* begun. The works of Rao et al. (2001) and Rao et al. (2003) provide overviews of linear and nonlinear *MHE*. Recent results regarding *MHE* for nonlinear systems are given for robust stability and estimate convergence properties (Alessandri et al. 2005, 2008, 2012, Garcia-Tirado et al. 2016, Sánchez et al. 2017). In recent years several results have been obtained for different *MHE* formulations, advancing from idealistic assumptions, like observability and vanishing disturbances, to realistic situations like detectability and bounded disturbances (see Ji et al. 2015, Müller 2017, Deniz et al. 2019, Allan & Rawlings 2019).

When disturbances, model uncertainty and system constraints can be neglected, state and control sequences can be independently computed (see Duncan & Varaiya 1971, Bensoussan 2004, Åström 2012, Georgiou & Lindquist 2013). However, in practical applications, these conditions are very difficult to fulfil, i.e., process disturbances and measurement noise are usually present, as well as model uncertainty. In this context, it becomes necessary approaches that includes these information into the controller design.

State-feedback *MPC* is a mature field with results that considers model uncertainty, input disturbances, and noises (Magni et al. 2003, Bemporad et al.

2003, Raimondo et al. 2009, among others). However, these works did not consider robustness with respect to errors in state estimation. Fewer results are available for output-feedback MPC. An overview of nonlinear output-feedback MPC is given by Findeisen et al. (2003) and the references therein. Many of these approaches involve designing separate estimator and controller, using different estimation algorithm (Roset et al. 2006, Magni et al. 2009, Patwardhan et al. 2012, Zhang & Liu 2013, Ellis et al. 2017). Results on robust output-feedback MPC for constrained, linear, discrete-time systems with bounded disturbances and measurement noise can be found in Mayne et al. (2006, 2009) and Voelker et al. (2010, 2013). These approaches first solve the estimation problem and show convergence of the estimated state to a bounded set, and then take the uncertainty of the estimation into account when solving the *MPC* problem.

The approach of solving simultaneously *MHE*/*MPC* was originally introduced by Copp & Hespanha (2014) and later developed in several papers (Copp & Hespanha 2016*a*,*b*, 2017). In the first paper, Copp & Hespanha (2014) proposed an output feedback controller that combines state estimation and control into a single $min - max$ optimization problem that, under observability and controllability assumptions (Copp & Hespanha 2016*a*), guarantees the boundedness of state and tracking errors. Finally, in the last work reported by Copp & Hespanha (2017), the authors established the conditions for guaranteeing the boundedness of error for trajectory tracking problems. They also introduced a primal–dual interior point method that can be used to efficiently solve the $min - max$ optimization problem. The criterion used in these works involves finite forward and backward horizons that is minimized with respect to feedback control policies and maximized with respect to the unknown parameters in order to guaranty robustness in the worst-case scenario.

In the present work, we introduce an output–feedback controller for nonlinear systems subject to bounded disturbances using simultaneous *MHE*/*MPC* approach. The resulting optimization problem minimizes a criterion that involves finite forward and backward horizons with respect the unknown initial state, measurement noise and control input variables while it is maximized with respect the unknown disturbance variables. We show that the proposed controller results in closed–loop trajectories along which the state remains bounded. These results rely on three assumptions: The first assumption requires that the optimization criterion to include an adaptive arrival cost (Sánchez et al. 2017). This assumption allows to ensure the boundedness of the state estimate and to obtain a bound for the estimation error set if the parameters of the estimation problem are properly chosen (Deniz et al. 2019). The second assumption requires that the optimization criterion to include a terminal cost that is a control *ISS*–Lyapunov function with respect to the disturbance input. This type of assumption common in classical state–feedback robust *MPC*. The third assumption requires that the backward (estimation) and forward (control) horizons are sufficiently large so that enough information is obtained in order to find state estimates and control inputs compatible with dynamics, noises and constraints. This assumption is satisfied if the system is detectable, stabilizable and the parameters in the cost function (weights and horizons) are chosen appropriately.

The rest of the paper is organized as follows: Section 2 introduces the notation, definitions and properties that will be used through the paper. In Section 3 we formulate the estimation and control problem, and in Section 4 we analyze its closed-loop stability. Finally, we use this method to simulate a nonlinear example in Section 5 and discuss conclusions and future work in Section 6.

2 Preliminaries and setup

2.1 Notation

Let $\mathbb{Z}_{[a,b]}$ denotes the set of integers in the interval $[a,b] \subseteq \mathbb{R}$, and $\mathbb{Z}_{\geq a}$ denotes the set of integers greater or equal to a. Boldface symbols denote sequences of finite or infinite length, i.e., $\boldsymbol{w} := \{w_{k_1}, \ldots, w_{k_2}\}$ for some $k_1, k_2 \in \mathbb{Z}_{\geq 0}$ and $k_1 < k_2$, respectively. We denote $x_{j|k}$ the element at time j of the sequence $\boldsymbol{x}$ given at time $k \in \mathbb{Z}_{\geq 0}$ and $j \in [k_1, k_2]$ or $j \in [k_1, \infty)$ for finite or infinite sequences, respectively. By $|x|$ we denote the Euclidean norm of a vector $x \in \mathbb{R}^n$. Let $\|\boldsymbol{x}\| := \sup_{k \in \mathbb{Z}_{\geq 0}} |x_k|$ denote the supreme norm of the sequence $\boldsymbol{x}$ and $\|\boldsymbol{x}\|_{[a,b]} := \sup_{k \in \mathbb{Z}_{[a,b]}} |x_k|$. A function $\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{K}$ if γ is continuous, strictly increasing and $\gamma(0) = 0$. If γ is also unbounded, it is of class $\mathscr{K}_\infty$. A function $\zeta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{L}$ if $\zeta(k)$ is non increasing and $\lim_{k\to\infty} \zeta(k) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathscr{KL}$ if $\beta(\cdot, k)$ is of class $\mathscr{K}$ for each fixed $k \in \mathbb{Z}_{\geq 0}$, and $\beta(r, \cdot)$ of class $\mathscr{L}$ for each fixed $r \in \mathbb{R}_{\geq 0}$. Let us consider two sets A and B, the Minkowski addition is defined as $A \bigoplus B := \{a + b |\, a \in A, b \in B\}$. On the other hand, the Minkowski difference[1] is defined as $A \ominus B := \{d |\, d + b \in A\}$.

2.2 Problem statement

Consider a system described by a discrete-time nonlinear system

$$\begin{aligned} x_{k+1} &= f(x_k, u_k, w_k) \quad \forall\, k \in \mathbb{Z}_{\geq 0}, \\ y_k &= h(x_k) + v_k, \end{aligned} \tag{1}$$

[1] Also known as the Pontryagin difference.

in which $x \in \mathscr{X} \subset \mathbb{R}^{n_x}$ is the system state, $u \in \mathscr{U} \subset \mathbb{R}^{n_u}$ is the system's input and $w \in \mathscr{W} \subset \mathbb{R}^{n_w}$ is the unmeasured process disturbance posed as an additive input. The output of the system is $y \in \mathscr{Y} \subset \mathbb{R}^{n_y}$ and $v \in \mathscr{V} \subset \mathbb{R}^{n_v}$ is the measurement noise. The estimation and control problem attempts to find simultaneously the optimal past state trajectory which minimizes the process and measurements noises as well as to minimize the effects of uncertainties in the initial condition and computes the optimal sequence of control inputs that steer the actual state to the desired region. This results in an infinite-horizon optimization problem

$$
\begin{aligned}
\min_{\substack{\hat{x}_{0|k},\hat{w}_{j|k}\\ u_{j|k}}} \Psi_k^{\infty} := \sum_{j=0}^{j=k-1} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) + \sum_{j=k}^{\infty}\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) \\
\text{s.t.} \begin{cases} \hat{x}_{j+1|k} = f\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) + \hat{w}_{j|k},\ j \in \mathbb{Z}_{\geq 0} \\ y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \qquad j \in \mathbb{Z}_{[0,k-1]} \\ \hat{x}_{j|k} \in \mathscr{X},\ \hat{u}_{j|k} \in \mathscr{U}, \hat{w}_{j|k} \in \mathscr{W},\ \hat{v}_{j|k} \in \mathscr{V}. \end{cases}
\end{aligned} \tag{2}
$$

The functions $\ell_e(\cdot,\cdot), \ell_c(\cdot,\cdot)$ and $\ell_{w_c}(\cdot)$ are all assumed to take non-negative values. One can view ℓ_e and ℓ_{w_c} as measures of likelihood of the specific values of $\hat{w}_{j|k}$ and $\hat{v}_{j|k}$. Then the negative sign in front of ℓ_{w_c} penalizes the maximizer for using large values of $\hat{w}_{j|k}$ $j \geq k$. Problem (2) is valuable from a theoretical point of view since it guarantees the boundedness of cost function $\Psi_k^{\infty} \leq \gamma$ $\gamma > 0, \forall k \geq 0$ and therefore

$$
\sum_{j=0}^{j=k-1} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) + \sum_{j=k}^{\infty} \ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) \leq \gamma + \sum_{j=k}^{\infty} \ell_{w_c}\left(\hat{w}_{j|k}\right). \tag{3}
$$

If functions ℓ_e, ℓ_c and ℓ_{w_c} are defined using a norm–ℓ_p, problem (2) would guarantee that the state x_k and u_k are ℓ_p , provided that noises w_k and v_k are also ℓ_p. This would mean that the closed-loop system has a finite ℓ_p-induced gain.

The infinite-horizon problem (2) is intractable in practical situations, therefore, it is reformulated into a receding finite-horizon problem

$$\begin{aligned}\min_{\substack{\hat{x}_{k-N_e|k},\hat{\boldsymbol{w}}_k\\ \boldsymbol{u}_k}} \Psi_k^{N_e+N_c} := \Gamma_E\left(\hat{x}_{k-N_e|k}\right) + \sum_{j=k-N_e}^{j=k-1} \ell_e\left(\hat{w}_{j|k},\hat{v}_{j|k}\right) + \\ \sum_{j=k}^{k+N_c-1}\left(\ell_c\left(\hat{x}_{j|k},\hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) + \Gamma_C\left(\hat{x}_{k+N_c|k}\right)\\ \text{s.t.}\begin{cases}\hat{x}_{k-N_e|k} = \bar{x}_{k-N_e} + \hat{w}_{k-N_e-1|k}\\ \hat{x}_{j+1|k} = f\left(\hat{x}_{j|k},\hat{u}_{j|k}\right) + \hat{w}_{j|k},\ j\in\mathbb{Z}_{[k-N_e,k+N_c-1]}\\ y_j = h\left(\hat{x}_{j|k}\right) + \hat{v}_{j|k}, \qquad j\in\mathbb{Z}_{[k-N_e,k-1]}\\ \hat{x}_{j|k}\in\mathscr{X},\hat{u}_{j|k}\in\mathscr{U},\hat{w}_{j|k}\in\mathscr{W},\ \hat{v}_{j|k}\in\mathscr{V}.\end{cases}\end{aligned} \tag{4}$$

For computation tractability, the infinite summations of Ψ_k^∞ have been replaced by backward and forward windows of finite length, corresponding to the estimation Ψ_k^E and control Ψ_k^C part of criterion $\Psi_k^{N_e+N_c}$ respectively. Ψ_k^E includes N_e terms $\ell_e(\hat{w}_{j|k},\hat{v}_{j|k})$ backward in time from sample k and the extra term $\Gamma_E(\hat{x}_{k-N_e|k})$, known as *arrival-cost*, that summarizes information beyond the estimation window by penalizing the uncertainty in the initial state $\hat{x}_{k-N_e|k}$. On the other hand, Ψ_k^C includes N_c terms $\ell_c(\hat{x}_{j|k},\hat{u}_{j|k}) - \ell_{w_c}(\hat{w}_{j|k})$ forward in time from sample k and the extra term $\Gamma_C(\hat{x}_{k+N_c|k})$, known as *cost-to-go*, that summarizes the information beyond the control window by penalizing the error of the final state $\hat{x}_{k-N_c|k}$.

The objective of problem (4) is to compute the initial state $\hat{x}_{k-N_e|k}$ and disturbance $\hat{w}_{j|k} \quad j \leq k$ that provides an estimate $\hat{x}_{k|k}$ that allows to compute the control inputs $u_{j|k} \quad j \geq k$ that drive the system states to the desired region. Therefore, there is no point in penalizing the control cost ℓ_c along the estimation window Ψ_k^E. The variables $\hat{v}_{j|k}$ are not independent optimization variables as they are uniquely determined by the remaining optimization variables and the output equation

$$\hat{v}_{j|k} := y_j - h(\hat{x}_{j|k}) \qquad \forall j \in [k-N_e, k-N_e+1,\dots,k].$$

Since there is no measurement of future system output, $v_{j|k}$ will not be considered along the control window Ψ_k^C. However, the disturbance $w_{j|k}$ needs to be considered along both windows Ψ_k^E and Ψ_k^C because they affect all the state starting from $j = k - N_e - 1$.

Remark 1 *The sequence of process disturbances $\hat{w}_{j|k}$ is minimized when it is part of the estimator, i.e., $j \in [k-N_e-1, k-1]$, and it is maximized when the sequence is part of the controller, $j \in [k, k+N_c]$.*

2.3 *Relationship with* MPC *and* MHE

The criterion $\Psi_k^{N_e+N_c}$ can be rewritten as follows

$$\Psi_k^{N_e+N_c} := \theta\Psi_k^E + (1-\theta)\Psi_k^C \quad \theta \in [0,1], \tag{5}$$

with Ψ_k^E and Ψ_k^C are given by

$$\begin{aligned} \Psi_k^E &:= \Gamma_E\left(\hat{x}_{k-N_e|k}\right) + \sum_{j=k-N_e}^{j=k-1} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right), \\ \Psi_k^C &:= \Gamma_C\left(\hat{x}_{k+N_c|k}\right) + \sum_{j=k}^{k+N_c-1} \left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \ell_{w_c}\left(\hat{w}_{j|k}\right)\right). \end{aligned} \tag{6}$$

Ψ_k^E corresponds to the criterion implemented by *MHE* estimator while Ψ_k^C corresponds to the criterion implemented by a *min-max MPC* controller.

Equation (5) corresponds to a weighted sum multi-objective formulation of criterion (4), where θ controls the relative importance of Ψ_k^E within Ψ_k^N. When $\theta = 0$, $\Psi_k^{N_e+N_c} := \Psi_k^C$ and problem (4) becomes a *robust model predictive control* problem with terminal cost considered by Chen & Allgöwer (1998), given that x_k is measurable or it is provided by an estimator. On the other case, when $\theta = 1$, $\Psi_k^{N_e+N_c} := \Psi_k^E$ and problem (4) becomes a *moving horizon estimation* problem considered by Ji et al. (2016), Garcia-Tirado et al. (2016), Müller (2017), Deniz et al. (2019), given that the control input $u_{j|k}$ is computed by a controller. In these cases, the optimization problem problem (4) has only one objective and the separation principle needs to be applied since the estimator and controller a implemented independently.

When $0 < \theta < 1$, Ψ_k^E and Ψ_k^C are simultaneously considered by Ψ_k^N and the optimization problem (4) becomes multi-objective. The importance of Ψ_k^E, and therefore the one of Ψ_k^C, is defined by θ emphasizing or deemphasizing the influence of estimation problem on the solution. In the case of $\theta = 0.5$, Ψ_k^E and Ψ_k^C have similar influence on the solution of (4) and it becomes the problem proposed by Copp & Hespanha (2017).

Definition 1 *A point $z^o \in \mathscr{Z}$, is Pareto optimal iff there does not exist another point $z \in \mathscr{Z}$ such that $\Psi(z) \leq \Psi(z^o)$ and $\Psi^i(z) < \Psi^i(z^o)$ for at least one function (Miettinen 2012).*

According to this concept, problem (4) looks for solutions that neither Ψ_k^E nor Ψ_k^C can be improved without deteriorate of at least one of them. Any optimal solution of problem (4) with $0 < \theta < 1$ is Pareto optimal (Miettinen 2012), therefore it has an optimal trade-off between Ψ_k^E and Ψ_k^C. On the other cases, $\theta = 0$ or $\theta = 1$ the solutions of problem (4) are optimal in the sense of the selected objective (Ψ_k^C or Ψ_k^E, respectively). In these cases, the solutions obtained are not Pareto optimal and, therefore the overall system performance can be poorer than the one provided by solutions of multi-objective problem.

3 Robust stability of simultaneous state estimation and control under bounded disturbances

In this section, we introduce the results regarding feasibility and robust stability of the proposed algorithm. The properties of the estimator and controller parts are analyzed. Besides, feasibility conditions for the existence of a solution to (4), the minimum horizon lengths required to achieve the desired estimation and control performances, are analyzed.

3.1 Backward window

The simultaneous state estimation and control problem rely on a backward window of fixed length N_e to compute the optimal state estimate $\hat{x}_{k|k}$ that will be used by the controller in order to compute the optimal control inputs. The controller takes the estimate $\hat{x}_{k|k}$ as initial condition, which does not necessarily is equal to x_k. Previous results on robust output-feedback *MPC* with bounded disturbances firstly solve the estimation problem and show the convergence of estimated states to a bounded set, then take the uncertainty of estimation into account when solving the *MPC* problem (Mayne et al. 2006, 2009). The key idea in these works is to consider the estimation error as an additional, unknown but bounded uncertainty that must be accounted for guaranteeing stability and feasibility of the resulting closed–loop system. This idea is equivalent to assume that the set of all states belonging to the ball centred at $\hat{x}_{k|k} \in \mathscr{X}_{N_e}^E$ with radius $\mathscr{E}_{N_e}$ must be included within the robust controllable set ($\mathscr{X}_{N_c}^C$) for all time. The set $\mathscr{X}_{N_e}^E$ is the estimation set and $\mathscr{E}_{N_e}$ is the estimation error set is given by

$$\mathscr{E}_{N_e} := \left\{ x_k : |x_k - \hat{x}_{k|k}| \leq \varepsilon_e, \forall\, \hat{x}_{k|k} \right\}, \tag{7}$$

where the error bound ε_e

$$\varepsilon_e \leq \frac{|x_{k-N_e} - \bar{x}_{k-N_e}|^\zeta}{N_e^\eta} \Big(\mathbb{C}^\rho_{P^{-1}} \Big(c_\beta \; 18^p + c_1 \; 3^{\alpha_1} \; \lambda^{\alpha_1}_{\min} \left(P_0^{-1} \right) + c_2 \; 3^{\alpha_2} \; \lambda^{\alpha_1}_{\min} \left(P_0^{-1} \right) \Big) + c_\beta \; 2^p \Big) + \left(\frac{c_\beta \; 18^p \; \bar{\gamma}_w^{\frac{p}{a}} \left(\|\boldsymbol{w}\| \right)}{|P_0^{-1}|} + \gamma_1 \left(3 \left(\|\boldsymbol{w}\| + \underline{\gamma}_w^{-1} \left(3 \bar{\gamma}_w \left(\|\boldsymbol{w}\| \right) \right) \right) \right) + c_2 \; 3^{\alpha_2} \bar{\gamma}_w^{\alpha_2} \left(\|\boldsymbol{w}\| \right) \right) + \left(\frac{c_\beta \; 18^p \; \bar{\gamma}_v^{\frac{p}{a}} \left(\|\boldsymbol{v}\| \right)}{|P_0^{-1}|} + \gamma_2 \left(3 \left(\|\boldsymbol{v}\| + \underline{\gamma}_v^{-1} \left(3 \bar{\gamma}_v \left(\|\boldsymbol{v}\| \right) \right) \right) \right) + c_1 \; 3^{\alpha_1} \bar{\gamma}_v^{\alpha_1} \left(\|\boldsymbol{v}\| \right) \right). \tag{8}$$

whit the matrix P_k^{-1} related to the *arrival-cost* $\Gamma\left(\hat{x}_{k-N_e|k}\right)$

$$|P_0^{-1}||\hat{x}_{k-N_e|k} - \bar{x}_{k-N_e}|^a \leq \Gamma_E\left(\hat{x}_{k-N_e|k}\right) \leq |P_\infty^{-1}||\hat{x}_{k-N_e|k} - \bar{x}_{k-N_e}|^a, \tag{9}$$

and $\mathbb{C}_{P^{-1}} := \frac{\lambda_{\max}\left(P^{-1}\right)}{\lambda_{\min}\left(P^{-1}\right)}$, with $\lambda_{\min}\left(P^{-1}\right)$ and $\lambda_{\max}\left(P^{-1}\right)$ the minimum and maximal eigenvalues of matrix P_k^{-1} for $k \in \mathbb{Z}_{[0,\infty)}$, respectively. The constants ζ, ρ, η, p, α_1, α_2, a, c_β and the functions $\overline{\gamma}_w$, γ_1, γ_2 and $\overline{\gamma}_v$ are chosen properly to satisfy certain assumptions and inequalities. Moreover, if the length of the backwards horizon is chosen as

$$\mathscr{N} \geq \left(\delta^\zeta r_{\max}^{\zeta-1} \mathbb{C}^\rho_{P^{-1}} \left(c_\beta 18^p + \lambda^{\alpha_1}_{\min}\left(P_0^{-1}\right)\left(c_1 3^{\alpha_1} + c_2 3^{\alpha_2}\right) + c_\beta 2^p \right) \right)^{\frac{1}{\eta}} \tag{10}$$

the bound ε_e behaves contractively due to the effects of initial conditions vanishes. For a detailed explanation, the reader can visit Sánchez et al. (2017), Müller (2017) and Deniz et al. (2019).

From the controller point of view, let us define the robust controllable set in one step via the two-steps recursion (see Kerrigan & Maciejowski (2000) and the references therein) as:

$$\begin{aligned} X_k^* &= \{x | \, \forall \, w \in \mathscr{W} : x + w \in X_k\} \\ X_{k-1} &= \{x | \, \exists \, u : (x, u) \in \mathscr{X} \times \mathscr{U}, f(x, u) \in X_k^*\}. \end{aligned} \tag{11}$$

Note that computing (11) recursively, one can calculate the robust controllable set in N_c steps, i.e., $\mathscr{X}^C_{N_c}$.

Assumption 1 *In order to guaranty the feasibility of the simultaneous state estimation and control algorithm, we assume that* $\mathscr{E}_{N_e} \bigoplus \mathscr{X}^E_{N_e} \subseteq \mathscr{X}^C_{N_c} \quad \forall k \geq 0.$

This assumption guarantees the feasibility of the optimization problem (4) and it fulfillment depends on the parameters of Ψ^C and Ψ^E (see i.e. Kerrigan

& Maciejowski 2000, Müller 2017, Deniz et al. 2019, among others), therefore it can be verified a prior by design. The key idea is that every vector belonging to $\mathscr{X}^E_{N_e} \bigoplus \mathscr{E}_{N_e}$ can be steered in N_c steps to the set X_f of the final constraints control problem. Note that if $\mathscr{X}^E_{N_e} \bigoplus \mathscr{E}_{N_e} \nsubseteq \mathscr{X}^C_{N_c}$, one can try to enlarge either N_e or N_c. When N_e is enlarged, the volume of the set $\mathscr{E}_{N_e}$ is reduced due to the fastest vanishing behaviour of the error due to uncertainties in the initial condition, i.e., ε_e decreases. On the other hand, when N_c is enlarged, the volume of $\mathscr{X}^C_{N_c}$ can be expanded. However, it will depend on the dynamic of the system and the constraints.

3.2 Forward window

The forward window corresponds to the model predictive control problem, which from an estimate $\hat{x}_{k|k}$ computes the future optimal control inputs. The length of the control window is selected to guarantee the stability of the system. A common approach is to select a *large enough* window length. However, in order to keep computational burden low, we are interested in computing the minimum horizon length which guarantees stability. The work of Tuna et al. (2006) goes a step forward in this direction and develop a procedure which allows computing the required value of N_c for the nominal case. Here we extend some of these ideas for the case where process disturbance are acting as unmeasurable input of bounded amplitude on the system. Before to compute the minimum forward window length that guarantee stability for the system, let us state the following assumptions

Assumption 2 *There exist a constant $\gamma \in \mathbb{R}_{\geq 0}$ such that the terminal cost and the stage cost satisfy the following relation:*

$$\Gamma_C\left(f\left(x,u,w\right)\right) + \ell\left(x,u\right) \leq \Gamma_C\left(x\right)\left(1+\gamma\right) + \ell_w\left(w\right). \tag{12}$$

Remark 2 *A similar assumption was already used in Tuna et al. (2006), where the constant γ is introduced in order to relax ($\gamma > 0$) the requirement on the function $\Gamma_C\left(\cdot\right)$ to be a control Lyapunov function for the nominal case. Here we extend it to the more general case where noise and disturbances are affecting the system.*

Assumption 3 *The stage cost $\ell_c\left(x,u\right)$ is lower bounded by a function $\sigma\left(x\right) \in \mathscr{K}_\infty$, such that $\sigma\left(x\right) \leq \ell_c\left(x,u\right) - \ell_{w_c}\left(w\right)$, $\forall\, x \in \mathscr{X}$, $\forall\, u \in \mathscr{U}$ and $\forall\, w \in \mathscr{W}$. Moreover, there exists functions $\underline{\gamma}_x\left(x\right)$, $\underline{\gamma}_u\left(u\right)$, $\overline{\gamma}_x\left(x\right)$ and $\overline{\gamma}_u\left(u\right) \in \mathscr{K}_\infty$ such that $\underline{\gamma}_x\left(x\right) + \underline{\gamma}_u\left(u\right) \leq \ell_c\left(x,u\right) \leq \overline{\gamma}_x\left(x\right) + \overline{\gamma}_u\left(u\right)$, and $\underline{\gamma}_w\left(w\right)$, $\overline{\gamma}_w\left(w\right) \in \mathscr{K}_\infty$ such that $\underline{\gamma}_w\left(w\right) \leq \ell_w\left(w\right) \leq \overline{\gamma}_w\left(w\right)$.*

Assumption 4 *The cost to go $\Gamma_C\left(x\right)$ is lower and upper bounded: $\alpha_\Gamma\left(x\right) \leq \Gamma_C\left(x\right) \leq \beta_\Gamma\left(x\right)$, with $\alpha_\Gamma\left(\cdot\right) \in \mathscr{K}_\infty$, $\beta_\Gamma\left(\cdot\right) \in \mathscr{K}_\infty$.*

Assumption 5 *There exists a sequence* $\{L_i\}$, $i \in \mathbb{Z}_{\geq 0}$, *such that* $L_i \in \mathbb{R}$ *and* $1 \leq L_i \leq L$, $L_0 = 1$ *with* $L \in \mathbb{R}$ *that verifies*

$$\Psi_k^C\left(\hat{x}, \hat{u}, \hat{w}, i\right) \leq L_i \, \sigma\left(\hat{x}\right),$$

where $\Psi_k^C\left(\hat{x}, \hat{u}, \hat{w}, i\right)$ *is the value function with decision variables* $\hat{x}$, $\hat{u}$, $\hat{w}$ *and horizon length* i.

With all the elements stated former, we claim that under the fulfilment of Assumptions 1 and 2, the cost function $\Psi_k^{N_e+N_c}$ of the optimization problem (4) is a regional ISS-Lyapunov function for the closed loop system (1).

Proof. Let us assume that Assumptions 1 and 2 are fulfilled. Recalling Bellman's principle of optimality (Bellman & Kalaba 1965), $\Psi_k^{N_e+N_c}$ and $\Psi_{k+1}^{N_e+N_c}$ are compared using the same sequences of states, control actions and disturbances, since the optimal sequences at time k are feasible at time $k+1$

$$\begin{aligned}\Psi_{k+1}^{N_e+N_c} - \Psi_k^{N_e+N_c} = {} & \Gamma_E\left(\hat{x}_{k-N_e+1|k+1}\right) + \sum_{j=k-N_e+1}^{j=k} \ell_e\left(\hat{w}_{j|k+1}, \hat{v}_{j|k+1}\right) + \\ & \sum_{j=k+1}^{k+N_c+1}\left(\ell_c\left(\hat{x}_{j|k+1}, \hat{u}_{j|k+1}\right) - \ell_{w_c}\left(\hat{w}_{j|k+1}\right)\right) + \\ & \Gamma_C\left(\hat{x}_{k+N_c+1|k+1}\right) - \Gamma_E\left(\hat{x}_{k-N_e|k}\right) - \\ & \sum_{j=k-N_e}^{j=k-1} \ell_e\left(\hat{w}_{j|k}, \hat{v}_{j|k}\right) - \sum_{j=k}^{k+N_c}\left(\ell_c\left(\hat{x}_{j|k}, \hat{u}_{j|k}\right) - \right. \\ & \left. \ell_{w_c}\left(\hat{w}_{j|k}\right)\right) - \Gamma_C\left(\hat{x}_{k+N_c|k}\right)\end{aligned} \tag{13}$$

Using Inequality (12) from Assumption 2, Equation (13) can be rewritten as follows

$$\begin{aligned}\Psi_{k+1}^N - \Psi_k^N \leq {} & -\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right) + \pi_E\left(\hat{w}, \hat{v}\right) + \ell_{w_c}\left(\hat{w}_{k|k}\right) + \\ & \Gamma_C\left(\hat{x}_{k+N_c|k}\right)\gamma,\end{aligned} \tag{14}$$

or

$$\begin{aligned}\Psi_{k+1}^N - \Psi_k^N \leq {} & -\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)\left(1 - \gamma\left(\frac{\Gamma_C\left(\hat{x}_{k+N_c|k}\right)}{\ell\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \right.\right. \\ & \left.\left.\frac{1}{\gamma}\frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}\right)\right) + \pi_E\left(\hat{w}, \hat{v}\right),\end{aligned} \tag{15}$$

where $\pi_E\left(\hat{w}, \hat{v}\right)$ is given by

$$\begin{aligned}\pi_E\left(\hat{w}, \hat{v}\right) \coloneqq {} & \Gamma_E\left(\hat{x}_{k-N_e+1|k}\right) - \Gamma_E\left(\hat{x}_{k+N_e|k}\right) + \ell_e\left(\hat{w}_{k|k}, \hat{v}_{k|k}\right) - \\ & \ell_e\left(\hat{w}_{k-N_e|k}, \hat{v}_{k-N_e|k}\right).\end{aligned} \tag{16}$$

Note that when $\gamma = 0$, the function $\Gamma_C(\cdot)$ become a ISS-Lyapunov function in the sense defined in Sontag & Wang (1997) (see also Sontag (2008)) $\square$.

From the first term in the right hand side of (15), one can see that if

$$\gamma \left(\frac{\Gamma_C\left(\hat{x}_{k+N_c|k}\right)}{\ell\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \frac{1}{\gamma} \frac{\ell_{w_c}\left(\hat{w}_{k|k}\right)}{\ell\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} \right) < 1, \tag{17}$$

then, there exists a invariant space which its volume depends on both estimation and control terms and parameters, and the objective function is effectively a regional ISS-Lyapunov function. Assuming that the left hand side of (17) belong to the interval $[0, 1)$, then, the first term of the right hand side of (15) will be always negative. When this term become dominating, the sequence of cost will present a contractive behaviour until it reaches the value of $\pi_E(\hat{w}, \hat{v})$.

We are concerned with analysing inequality (17) as a function of N_c. If such relation exists, we would like to compute the minimum value of N_c that satisfies Inequality (17). As the term $\pi_E(w, v)$ in Inequality (15) regard with the estimation error, it does not affect Inequality (17). Before to continue, let us define a_i, b_i and $c_i \quad \forall j \in \mathbb{Z}_{[0,N_c]}$ as follows

$$\begin{aligned}
a_i &:= \frac{\ell_c\left(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}, \quad a_0 = 1, \quad a_{N_c} = \frac{\Gamma_C\left(\hat{x}_{k+Nc|k}\right)}{\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}, \\
b_i &:= \frac{\ell_{w_c}\left(\hat{w}_{k+i|k}\right)}{\gamma\, \ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)}, \qquad b_{N_c} = 0, \\
c_i &:= a_i - \gamma\, b_i.
\end{aligned} \tag{18}$$

Then, the control cost at time k with a horizon length equal to N_c, i.e., $\Psi_k^C(\hat{x}, \hat{u}, \hat{w}, N_c)$ can be written as follows

$$\Psi_k^C(\hat{x}, \hat{u}, \hat{w}, N_c) = \ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right) \sum_{j=0}^{N_c} c_j \tag{19}$$

Note that

$$\begin{aligned}
\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right) \sum_{j=i}^{N_c} c_j &= \Psi_k^C(\hat{x}, \hat{u}, \hat{w}, N_c) - \ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right) \sum_{j=0}^{i-1} c_j, \\
&=: \Psi_k^C\left(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}, \hat{w}_{k+i|k}, N_c - i\right).
\end{aligned} \tag{20}$$

By mean of Assumption 3 and 5, the cost function $\Psi_k^C(\hat{x}, \hat{u}, \hat{w}, N_c - i)$ is bounded by

$$
\begin{aligned}
\Psi_k^C\left(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}, \hat{w}_{k+i|k}, N_c - i\right) &\leq L_{N_c-i}\,\sigma\left(\hat{x}_{k+i|k}\right)\\
&\leq L_{N_c-i}\left(\ell_c\left(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}\right) - \ell_w\left(\hat{w}_{k+i|k}\right)\right)\\
&\leq L_{N_c-i}\left(\ell_c\left(\hat{x}_{k+i|k}, \hat{u}_{k+i|k}\right) + \frac{1}{\gamma}\ell_w\left(\hat{w}_{k+i|k}\right)\right),\\
&\leq L_{N_c-i}\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)(a_i + b_i)\\
\sum_{j=i}^{N_c} c_j &\leq L_{N_c-i}\,(a_i + b_i)
\end{aligned}
\tag{21}
$$

Where the third Inequality in (21) holds since $\frac{1}{\gamma}\ell_w\left(\hat{w}_{k+N_c-i|k}\right) > -\ell_w\left(\hat{w}_{k+N_c-i|k}\right)$ and the last follows from (20). Defining $\bar{d}_j$ as follows

$$
\bar{d}_j \coloneqq a_j + \max\left\{b_j\right\} = a_j + \frac{\max\left\{\ell_w\left(\hat{w}_{[k|k,\,k+N_c|k]}\right)\right\}}{\gamma\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} = a_j + \frac{\ell_w\left(\|\hat{\boldsymbol{w}}\|_{[k|k,\,k+N_c|k]}\right)}{\gamma\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)},
\tag{22}
$$

therefore, $a_j + b_j \leq \bar{d}_j$. Replacing $a_i + b_i$ with $\bar{d}_i$ in Inequality (21)

$$
\sum_{j=i}^{N_c} c_j \leq \sum_{j=i}^{N_c} \bar{d}_j = (N_c - i + 1)\,\frac{\ell_w\left(\|\hat{\boldsymbol{w}}\|_{[k|k,\,k+N_c|k]}\right)}{\gamma\ell_c\left(\hat{x}_{k|k}, \hat{u}_{k|k}\right)} + \sum_{j=i}^{N_c} a_j \leq L_{N_c-i}\,\bar{d}_i,
\tag{23}
$$

or

$$
\sum_{j=i}^{N_c} \bar{d}_j \leq L_{N_c-i}\,\bar{d}_i,
\tag{24}
$$

with a suitable value of L_{N_c-i}. We assume a function $\Omega\left(L_i, i\right)$ such that $\bar{d}_{N_c} \leq \Omega\left(L_{N_c}, N_c\right)$. Moreover, assuming the existence of such function and given the sequence L_i, we are interested in the values of $i \geq N_c$ such that $\gamma\Omega\left(L_{N_c}, N_c\right) \leq 1$, since it will guarantee that Inequality (17) holds due to the fact that $\gamma\bar{d}_{N_c}$ is greater than the value of the left hand side of Inequality (17). Defining $\Omega\left(L_i, i\right)$ with the same behaviour as in Tuna et al. (2006), i.e.,

$$
\Omega\left(L_i, N_c\right) \coloneqq \bar{d}_0 \prod_{i=1}^{N_c} \frac{L_i - 1}{L_{i-1}}.
\tag{25}
$$

and taking $N_c = 1$ with $i = 0$, Inequality (24) is reduced to $\bar{d}_0 + \bar{d}_1 \leq L_1\bar{d}_0$, i.e., $\bar{d}_1 \leq \bar{d}_0\left(L_1 - 1\right)$, and Equation (25) holds. For values of $N_c > 1$, Equation (25) is verified too, a proof of this property can be found in Tuna et al. (2006). Then, $\bar{d}_{N_c}$ is upper bounded by

$$
\bar{d}_{N_c} \leq \bar{d}_0 \prod_{i=1}^{N_c} \frac{L_i - 1}{L_{i-1}} \leq \bar{d}_0 \prod_{i=1}^{N_c} \frac{L - 1}{L} \leq \bar{d}_0\,(L - 1)\left(\frac{L-1}{L}\right)^{N_c - 1},
\tag{26}
$$

where the last inequality holds since the sequence $(L_i - 1)/L_i$ is strictly increasing.

We have now all the necessary elements to state the following Theorem.

Theorem 1 *Suppose Assumptions 2-5 hold, then, choosing the control horizon length N_c as follows*

$$N_c \geq \left\lceil 1 + L \ln\left(\gamma\, \bar{d}_0\, (L-1)\right)\right\rceil \tag{27}$$

the value function of the controller in (15) *is a regional ISS-Lyapunov function such that*

$$\Gamma_C\left(f\left(x,u,w\right)\right) - \Gamma_C\left(x,u\right) \leq -\ell_c\left(x,u\right)\left(1 - \gamma\,\Omega\left(L,N_c\right)\right) + \ell_{w_c}\left(w\right), \tag{28}$$

Proof. Defining the control horizon N_c^* as follows

$$N_c^* := \lceil 1 + L \ln\left(\gamma \bar{d}_0\, (L-1)\right)\rceil, \tag{29}$$

then, from the definition (29)

$$\begin{aligned} N_c^* &\geq 1 + L\ln\left(\gamma\bar{d}_0\,(L-1)\right) \\ \left(N_c^* - 1\right)\ln\left(1 + (L-1)^{-1}\right) &> \ln\left(\gamma\bar{d}_0\,(L-1)\right) \\ 0 &> \ln\left(\gamma\bar{d}_0\,(L-1)\right) + \left(N_c^* - 1\right)\ln\left(\tfrac{L-1}{L}\right) \end{aligned} \tag{30}$$

where the second Inequality in (30) holds since

$$\frac{1}{L} < \ln\left(1 + (L-1)^{-1}\right), \tag{31}$$

Taking anti-log function on both side of last inequality of (30)

$$\gamma\bar{d}_0\,(L-1)\left(\frac{L-1}{L}\right)^{N_c^*-1} < 1 \tag{32}$$

or

$$\gamma\,\Omega\left(L,N_c\right) < \gamma\bar{d}_0\,(L-1)\left(\frac{L-1}{L}\right)^{N_c^*-1} < 1. \tag{33}$$

Hence, from Inequality (32) it follows that $0 < 1 - \gamma\,\Omega\left(L,N_c\right)$, and $\gamma\,\Omega\left(L,N_c\right) < 1$. Therefore, $\Omega\left(L,N_c\right) \in (0,1)$ and equation (28) is a regional ISS-Lyapunov function for the controller $\forall N_c \geq N_c^*$. □

3.2.1 *Computing the value of L*

In order to compute the value of L, essential to calculate the length of the horizon from Equation (29), let us note that

$$\begin{aligned}\Psi_k^C\left(\hat{x},\hat{u},\hat{w},N_c+1\right) &= \sum_{j=k}^{k+N_c}\left\{\ell_c\left(\hat{x}_{j|k},\hat{u}_{j|k}\right)-\ell_{w_c}\left(\hat{w}_{j|k}\right)\right\}+\Gamma_C\left(\hat{x}_{k+N_c+1|k}\right),\\ &\overset{(12)}{\leq} \Psi_k^C\left((\hat{x},\hat{u},\hat{w}),N_c\right)+\gamma\Gamma_C\left(\hat{x}_{k+N_c|k}\right),\end{aligned}$$

when $\gamma\to 0$ (see (Magni et al. 2006)), $\Psi_k^{N_c}\left(\hat{x},\hat{u},\hat{w},N_c+1\right)\leq\Psi_k^{N_c}\left(\hat{x},\hat{u},\hat{w},,N_c\right)$. Moreover

$$\Psi_k^C\left(\hat{x},\hat{u},\hat{w},N_c+1\right)\leq\Psi_k^C\left(\hat{x},\hat{u},\hat{w},N_c\right)\leq\ldots\leq\Psi_k^C\left(\hat{x},\hat{u},\hat{w},1\right)$$

since $\Psi_k^C\left(\hat{x},\hat{u},\hat{w},1\right)=\Gamma_C\left(x\right)\leq\beta_\Gamma\left(x\right)$ from Assumption 4.

Recalling that $\sigma\left(x\right)\leq\Psi_k^C\left(x,u,w,N_c\right)\leq L\sigma\left(x\right)$, one can approximate L as the quotient between the upper and lower bound functions as follows

$$L\geq\left\lceil\frac{\beta_\Gamma\left(x\right)}{\sigma\left(x\right)}\right\rceil, \tag{34}$$

or by mean of Assumption 3

$$L\geq\left\lceil\frac{\beta_\Gamma\left(x\right)}{\underline{\gamma}_x\left(x\right)+\underline{\gamma}_u\left(u\right)-\underline{\gamma}_w\left(w\right)}\right\rceil. \tag{35}$$

4 Examples

In this section, we demonstrate how the proposed framework improve the overall performance of a nonlinear system via simulations. The nonlinear model was originally used by Rao et al. (2003), whose dynamic behavior is given by

$$\begin{aligned}x_{k+1}^{(1)}&=0.99x_k^{(1)}+0.2x_k^{(2)}+u_k\\ x_{k+1}^{(2)}&=-0.1x_k^{(1)}+\frac{0.5x^{(2)}}{1+\left(x^{(2)}\right)^2}+w_k\\ y_k&=x_k^{(1)}-3x_k^{(2)}+v_k.\end{aligned}$$

The control inputs u_k is constrained to the set $|u_k|\leq 0.25\quad\forall k\in\mathbb{Z}_{\geq 0}$ and the additive process and measurement noises w and v are drawn from normal distributions with zero mean and covariances $S_w^2=1$ and $S_v^2=0.5$, respectively.

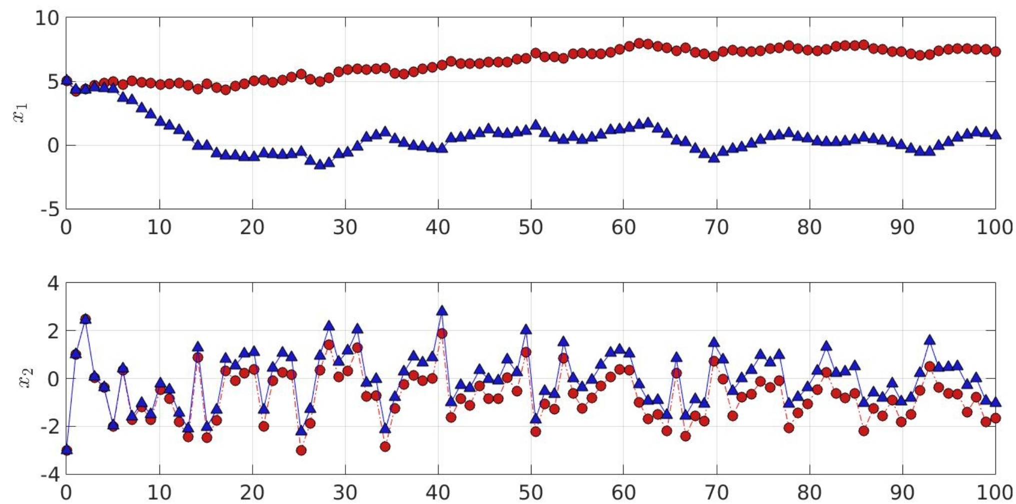

Fig. 1. Comparison of system state x_k between simultaneous (blue triangles) and separate (red circles) *MHE-MPC* algorithms.

For the simultaneous *MHE* and *MPC* algorithm (*MHE*/*MPC*) (4) the estimation $\ell_e(\cdot,\cdot)$, control $\ell_c(\cdot,\cdot)$ and disturbance $\ell_{w_c}(\cdot)$ stage costs are chosen as follows

$$\begin{aligned} \ell_e\left(\hat{w},\hat{v}\right) &= w_{j|k}^T Q_w w_{j|k} + R_v v_{j|k}^2 \qquad j\in[k-N_e,k-1], \\ \ell_c\left(\hat{x},\hat{u}\right) &= x_{j|k}^T Q_x x_{j|k} + R_u u_{j|k}^2 \qquad j\in[k,k+N_c], \\ \ell_{w_c}\left(\hat{w}\right) &= w_{j|k}^T Q_{w_c} w_{j|k}, \end{aligned} \tag{36}$$

with $Q_w = 50I_2, R_v = 50, Q_x = 50I_2, R_u = 1$ and $Q_{w_c} = (1/N_c)I_2$, respectively. The arrival-cost $\Gamma_E(\hat{x}_{k-N_e|k})$ is computed using the adaptive algorithm proposed by Sánchez et al. (2017) with $\sigma = 0.1$ and $c = 1e6$. The cost-to-go $\Gamma_C(\hat{x}_{k-N_c|k})$ is chosen as $\Gamma_C(\hat{x}_{k-N_c|k}) = \hat{x}_{k-N_c|k}^T P_c \hat{x}_{k-N_c|k}$ with $P_c = 100I_2$. The remaining parameters of the cost function Ψ_k^N are chosen as $N_e = 5$ and $N_c = 5$. The independent *MHE* and *MPC* algorithms (*MHE*+*MPC*) use the same parameters than the simultaneous algorithm. The parameters were chosen in this way in order to evaluate the effect of simultaneously solve, or not, the estimation and control problems. For both simulations we use a multiple shooting strategy with sampling time of $\Delta = 1$ and we add the restriction $|\hat{x}_{k-N_c|k}| \leq \delta$, where δ is equal to the double of the sum of process and measurement noises bounds.

Figure 1 shows the time evolution of the system states x_k controlled by the simultaneous and independent *MHE/MPC* algorithms. Both states of the system controlled by the simultaneous algorithm (blue triangles) converge to zero and then regulate the effect of process disturbance. On the other hand, $x_k^{(2)}$ of the system controlled by the independent *MHE* and *MPC* algorithms (red circles) converges to zero and then regulate the effect of process disturbance. The other state $x_k^{(1)}$ slowly drifts away until the system lost controllability due to the saturation of the manipulated variable u_k (see Figure 2). These phenomenon is due to problems in the estimator, who only aims to minimizes its cost function Ψ_k^E without regarding the effects on the control problem. This approach leads to the saturation of the control input and the lose of system controllability.

Figure 2 shows the time evolution of the manipulated variable u_k for both systems: the simultaneous and independent *MHE/MPC* algorithm. The inputs computed by the independent *MHE/MPC* algorithms hit the input constraint almost permanently throughout the simulation, forcing the lost of system controllability. On the other hand, the inputs computed by the simultaneous *MHE/MPC* algorithm hit the input constraint only several times throughout the simulation.

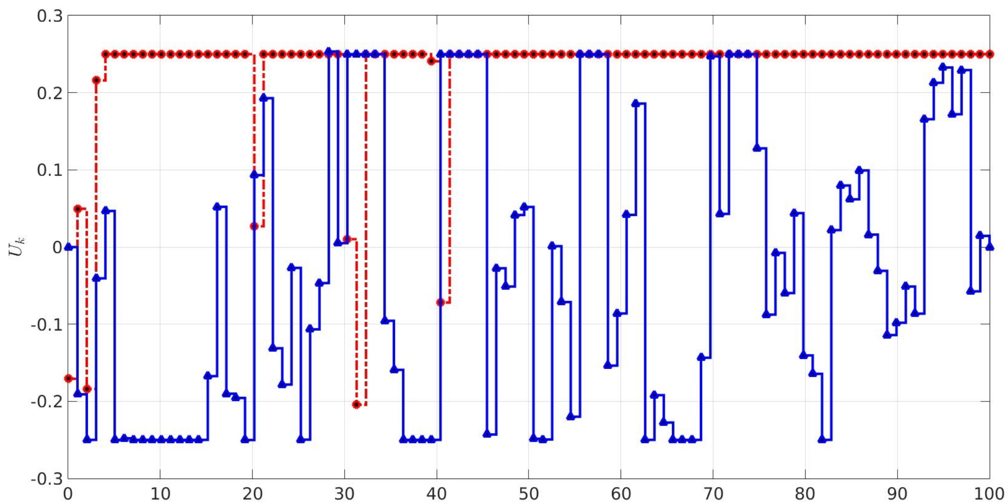

Fig. 2. Comparison of control inputs u_k between simultaneous (blue triangles) and separate (red circles) *MHE-MPC* algorithms.

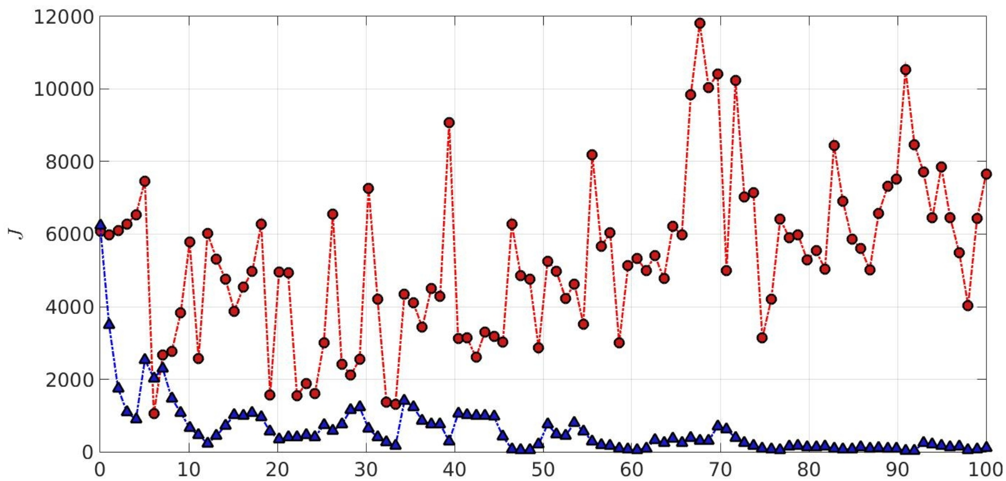

Fig. 3. Comparison of optimization costs between simultaneous Ψ_k^N (blue triangles) and the sum of separate (red circles) *MHE-MPC* $\Psi_k^E + \Psi_k^C$ algorithms.

Figure 3 shows the time evolution of the optimization costs Ψ_k^N computed by the simultaneous and independent *MHE/MPC* algorithms. The optimization costs of the simultaneous algorithm Ψ_k^N show a decreasing behaviour through the samples with occasional jumps throughout the simulation introduced by large values of the process noise w_k. On the other hand, the sum optimization costs $\Psi_k^E + \Psi_k^C$ of the independent algorithms show a continuous drift in its mean value with superimpose jumps introduced by the process noise

5 Conclusions

In this work, we address the challenge to solve simultaneously the problem of estimation and control for nonlinear systems subject to bounded disturbances. We have investigated the necessary conditions to guaranty the feasibility of the problem. Moreover, the minimum horizon length required for the estimator in order to neglect the effects of uncertainty in the initial conditions is given. The effects of the length of the control horizon are analyzed as well, and an expression for the minimum length of the control horizon required to guaranty stability is given.

Acknowledgements

The authors wish to thank the Consejo Nacional de Investigaciones Cientificas y Tecnicas (CONICET) from Argentina, for their support.

References

Alessandri, A., Baglietto, M. & Battistelli, G. (2005), 'Robust receding-horizon state estimation for uncertain discrete-time linear systems', *Systems & Control Letters* **54**(7), 627–643.

Alessandri, A., Baglietto, M. & Battistelli, G. (2008), 'Moving-horizon state estimation for nonlinear discrete-time systems: New stability results and approximation schemes', *Automatica* **44**(7), 1753–1765.

Alessandri, A., Baglietto, M. & Battistelli, G. (2012), 'Min-max moving-horizon estimation for uncertain discrete-time linear systems', *SIAM Journal on Control and Optimization* **50**(3), 1439–1465.

Allan, D. A. & Rawlings, J. B. (2019), A lyapunov-like function for full information estimation, *in* '2019 American Control Conference (ACC)', IEEE, pp. 4497–4502.

Åström, K. J. (2012), *Introduction to stochastic control theory*, Courier Corporation.

Bellman, R. & Kalaba, R. E. (1965), *Dynamic programming and modern control theory*, Vol. 81, Citeseer.

Bemporad, A., Borrelli, F. & Morari, M. (2003), 'Min-max control of constrained uncertain discrete-time linear systems', *IEEE Transactions on automatic control* **48**(9), 1600–1606.

Bemporad, A. & Morari, M. (1999), Robust model predictive control: A survey, *in* 'Robustness in identification and control', Springer, pp. 207–226.

Bensoussan, A. (2004), *Stochastic control of partially observable systems*, Cambridge University Press.

Camacho, E. & Alba, B. (2004), 'Model predictive control'.

Chen, H. & Allgöwer, F. (1998), 'A quasi-infinite horizon nonlinear model predictive control scheme with guaranteed stability', *Automatica* **34**(10), 1205–1217.

Copp, D. A. & Hespanha, J. P. (2014), Nonlinear output-feedback model predictive control with moving horizon estimation, *in* '53rd IEEE conference on decision and control', IEEE, pp. 3511–3517.

Copp, D. A. & Hespanha, J. P. (2016*a*), Conditions for saddle-point equilibria in output-feedback mpc with mhe, *in* '2016 American Control Conference (ACC)', IEEE, pp. 13–19.

Copp, D. A. & Hespanha, J. P. (2017), 'Simultaneous nonlinear model predictive control and state estimation', *Automatica* **77**, 143–154.

Copp, D. & Hespanha, J. (2016*b*), Addressing adaptation and learning in the context of model predictive control with moving-horizon estimation, *in* 'Control of Complex Systems', Elsevier, pp. 187–209.

Deniz, N. N., Murillo, M. H., Sanchez, G., Genzelis, L. M. & Giovanini, L. (2019), 'Robust stability of moving horizon estimation for nonlinear systems with bounded disturbances using adaptive arrival cost', *arXiv preprint arXiv:1906.01060* .

Duncan, T. & Varaiya, P. (1971), 'On the solutions of a stochastic control system', *SIAM Journal on Control* **9**(3), 354–371.

Ellis, M., Liu, J. & Christofides, P. D. (2017), State estimation and empc, *in* 'Economic Model Predictive Control', Springer, pp. 135–170.

Findeisen, R., Imsland, L., Allgower, F. & Foss, B. A. (2003), 'State and output feedback nonlinear model predictive control: An overview', *European journal of control* **9**(2-3), 190–206.

Garcia-Tirado, J., Botero, H. & Angulo, F. (2016), 'A new approach to state estimation for uncertain linear systems in a moving horizon estimation setting', *International Journal of Automation and Computing* **13**(6), 653–664.

Georgiou, T. T. & Lindquist, A. (2013), 'The separation principle in stochastic control, redux', *IEEE Transactions on Automatic Control* **58**(10), 2481–2494.

Grüne, L. & Pannek, J. (2011), 'Nonlinear model predictive control. communications and control engineering', *Springer. doi* **10**, 978–0.

Ji, L., Rawlings, J. B., Hu, W., Wynn, A. & Diehl, M. (2015), 'Robust stability of moving horizon estimation under bounded disturbances', *IEEE Transactions on Automatic Control* **61**(11), 3509–3514.

Ji, L., Rawlings, J. B., Hu, W., Wynn, A. & Diehl, M. (2016), 'Robust stability of moving horizon estimation under bounded disturbances', *IEEE Transactions on Automatic Control* **61**(11), 3509–3514.

Kerrigan, E. C. & Maciejowski, J. M. (2000), Invariant sets for constrained nonlinear discrete-time systems with application to feasibility in model predictive control, *in* 'Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)', Vol. 5, IEEE, pp. 4951–4956.

Magni, L., De Nicolao, G., Scattolini, R. & Allgöwer, F. (2003), 'Robust model predictive control for nonlinear discrete-time systems', *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal* **13**(3-4), 229–246.

Magni, L., Raimondo, D. M. & Allgöwer, F. (2009), 'Nonlinear model predictive control', *Lecture Notes in Control and Information Sciences* (384).

Magni, L., Raimondo, D. M. & Scattolini, R. (2006), 'Regional input-to-state stability for nonlinear model predictive control', *IEEE Transactions on automatic control* **51**(9), 1548–1553.

Mayne, D. (2016), 'Robust and stochastic model predictive control: Are we going in the right direction?', *Annual Reviews in Control* **41**, 184–192.

Mayne, D. Q. (2014), 'Model predictive control: Recent developments and future promise', *Automatica* **50**(12), 2967–2986.

Mayne, D. Q., Raković, S., Findeisen, R. & Allgöwer, F. (2006), 'Robust output feedback model predictive control of constrained linear systems', *Automatica* **42**(7), 1217–1222.

Mayne, D. Q., Raković, S., Findeisen, R. & Allgöwer, F. (2009), 'Robust output feedback model predictive control of constrained linear systems: Time varying case', *Automatica* **45**(9), 2082–2087.

Miettinen, K. (2012), *Nonlinear multiobjective optimization*, Vol. 12, Springer Science & Business Media.

Müller, M. A. (2017), 'Nonlinear moving horizon estimation in the presence of bounded disturbances', *Automatica* **79**, 306–314.

Patwardhan, S. C., Narasimhan, S., Jagadeesan, P., Gopaluni, B. & Shah, S. L. (2012), 'Nonlinear bayesian state estimation: A review of recent developments', *Control Engineering Practice* **20**(10), 933–953.

Raimondo, D. M., Limon, D., Lazar, M., Magni, L. & ndez Camacho, E. F. (2009), 'Min-max model predictive control of nonlinear systems: A unifying overview on stability', *European Journal of Control* **15**(1), 5–21.

Rao, C. V., Rawlings, J. B. & Lee, J. H. (2001), 'Constrained linear state estimationa moving horizon approach', *Automatica* **37**(10), 1619–1628.

Rao, C. V., Rawlings, J. B. & Mayne, D. Q. (2003), 'Constrained state estimation for nonlinear discrete-time systems: Stability and moving horizon approximations', *IEEE transactions on automatic control* **48**(2), 246–258.

Rawlings, J. B. & Bakshi, B. R. (2006), 'Particle filtering and moving horizon estimation', *Computers & chemical engineering* **30**(10-12), 1529–1541.

Rawlings, J. B. & Mayne, D. Q. (2009), *Model predictive control: Theory and design*, Nob Hill Pub. Madison, Wisconsin.

Roset, B., Lazar, M., Nijmeijer, H. & Heemels, W. (2006), Stabilizing output feedback nonlinear model predictive control: An extended observer approach, *in* '17th Symposium on Mathematical Theory for Networks and Systems. Kyoto, Japan', Citeseer.

Sánchez, G., Murillo, M. & Giovanini, L. (2017), 'Adaptive arrival cost update for improving moving horizon estimation performance', *ISA transactions* **68**, 54–62.

Schweppe, F. C. (1973), *Uncertain dynamic systems*, Prentice Hall.

Sontag, E. D. (2008), Input to state stability: Basic concepts and results, *in* 'Nonlinear and optimal control theory', Springer, pp. 163–220.

Sontag, E. D. & Wang, Y. (1997), 'Output-to-state stability and detectability of nonlinear systems', *Systems & Control Letters* **29**(5), 279–290.

Tuna, S. E., Messina, M. J. & Teel, A. R. (2006), Shorter horizons for model predictive control, *in* '2006 American Control Conference', IEEE, pp. 6–pp.

Voelker, A., Kouramas, K. & Pistikopoulos, E. N. (2010), 'Unconstrained moving horizon estimation and simultaneous model predictive control by multi-parametric programming'.

Voelker, A., Kouramas, K. & Pistikopoulos, E. N. (2013), 'Moving horizon estimation: Error dynamics and bounding error sets for robust control', *Automatica* **49**(4), 943–948.

Zhang, J. & Liu, J. (2013), 'Lyapunov-based mpc with robust moving horizon estimation and its triggered implementation', *AIChE Journal* **59**(11), 4273–4286.