Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER Linear Algebra and its Applications 393 (2004) 299–318
LINEAR ALGEBRA AND ITS APPLICATIONS
www.elsevier.com/locate/laa

# Generalized Schur complements and $P$-complementable operators

Pedro Massey *, Demetrio Stojanoff [1]

*Depto. de Matemática, FCE-UNLP, La Plata, Argentina and IAM-CONICET*

Received 3 April 2003; accepted 29 July 2003

Submitted by S. Fallat

**Abstract**

Let $A$ be a selfadjoint operator and $P$ be an orthogonal projection both operating on a Hilbert space $\mathscr{H}$. We say that $A$ is $P$-complementable if $A - \mu P \geqslant 0$ holds for some $\mu \in \mathbb{R}$. In this case we define $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}$. As a tool for computing $I_P(A)$ we introduce a natural generalization of the Schur complement or shorted operator of $A$ to $\mathscr{S} = R(P)$, denoted by $\Sigma(A, P)$. We give expressions and a characterization for $I_P(A)$ that generalize some known results for particular choices of $P$. We also study some aspects of the shorted operator $\Sigma(A, P)$ for $P$-complementable $A$, under the hypothesis of *compatibility* of the pair $(A, \mathscr{S})$. We give some applications in the finite dimensional context.


*AMS classification:* Primary 47A64; 15A60; 47B63

*Keywords:* Positive semidefinite operators; Shorted operator; Hadamard product; Completely positive maps

## 1. Introduction

Let $\mathscr{H}$ be a Hilbert space and $L(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$. Given a closed subspace $\mathscr{S} \subseteq \mathscr{H}$ and $P = P_{\mathscr{S}} \in L(\mathscr{H})$ the orthogonal

* Corresponding author.
*E-mail addresses:* massey@mate.unlp.edu.ar (P. Massey), demetrio@mate.unlp.edu.ar (D. Stojanoff).
[1] Partially supported CONICET (PIP 4463/96), Universidad de La Plata (UNLP 11 X350) and ANPCYT (PICT03-09521).

0024-3795/$ - see front matter 
doi:10.1016/j.laa.2003.07.010

projection onto $\mathscr{S}$, we study the following two problems: for any selfadjoint operator $A \in L(\mathscr{H})$,

1. determine whether there exists some $\mu \in \mathbb{R}$ such that
$$A - \mu P \geqslant 0; \tag{1}$$
2. in case Eq. (1) holds for some $\mu$, compute the optimum number
$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}. \tag{2}$$

The general solution of problem (1) is well known, see for example [14] or Proposition 3.3. Also, if $A \geqslant 0$, problem (2) has a known answer (see, for example, [7]). We state this case in the preliminary Section 2 (Corollary 2.2). Therefore our main interest is to study problem (2) in the non-positive case. It should be mentioned that the general case seems not to be easily reduced to the positive case (see Remark 5.1).

If condition (1) is satisfied by $A$, we shall say that $A$ is $P$-*complementable*, because in this case there exists the *shorted operator* (or Schur complement, see [1]) defined as follows:
$$\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, D(\mathscr{H}) \subseteq \mathscr{S}\}.$$
Using the identity $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$, in Section 3 we extend several known properties of shorted operators of positive operators to our case. On the other hand, in Section 5 we show that, if $A \ngeqslant 0$ but it is $P$-complementable, then
$$I_P(A) = \lambda_{\min}(\Sigma(A, P)),$$
where $\lambda_{\min}(C)$ denotes the minimum of the spectrum $\sigma(C)$ of $C \in L(\mathscr{H})$.

Although most applications of the problems mentioned above appear in matrix theory, i.e., when $\dim \mathscr{H} < \infty$, an additional hypothesis of the operator $A$ allows to extend all finite dimensional results to our setting. This hypothesis is the so called *compatibility* of the pair $(A, \mathscr{S})$. This notion, defined by Corach, Maestripieri and the second author in [4–6], is the following: The pair $(A, \mathscr{S})$ is compatible if there exists a $A$-selfadjoint projection onto $\mathscr{S}^\perp$, i.e., $Q \in L(\mathscr{H})$ such that $Q^2 = Q$, $AQ = Q^*A$ and $R(Q) = \mathscr{S}^\perp$.

There are several characterization of the compatibility of $(A, \mathscr{S})$ and general properties of such pairs in the case $A \geqslant 0$ (see, for example, [4]); some of them are stated in Section 4 of this paper, where we also extend these properties to the non-positive case. We say that compatibility is an additional condition because, if $(1 - P)A(1 - P) \geqslant 0$ and $(A, \mathscr{S})$ is compatible , then $A$ is $P$-complementable. The reverse implication is false in general, but it is true if $\dim \mathscr{S}^\perp < \infty$, in particular in the finite dimensional case.

Section 5 is devoted to the computation of the number $I_P(A)$ for $A$ selfadjoint, not necessarily positive. We first obtain the formula
$$I_P(A) = \inf\{\langle A\xi, \xi\rangle : \xi \in \mathscr{H}, \|P\xi\| = 1\}$$
so that, if $\mathscr{S}$ is the subspace generated by the unit vector $\xi \in \mathscr{H}$, then
$$I_P(A) = \inf\{\langle A\eta, \eta\rangle : \eta \in \mathscr{H}, \langle \eta, \xi\rangle = 1\}.$$

If $(A, \mathscr{S})$ is compatible , we show that the computation of $I_P(A)$ can be reduced to the case in which $\mathscr{S} \subseteq \overline{R(A)}$, by replacing $\mathscr{S}$ by $\overline{\mathscr{S} \cap R(A)}$. We state the results of the rest of this section in the following theorem:

**Theorem.** *Let* $A = A^* \in L(\mathscr{H})$, $A \ngeqslant 0$, *and* $P = P^* = P^2 \in L(\mathscr{H})$ *with* $R(P) = \mathscr{S}$. *Suppose that* $(1 - P)A(1 - P) \geqslant 0$ *and* $(A, \mathscr{S})$ *is compatible. Then*

1. $R(\Sigma(A, P)) = \mathscr{S} \cap R(A) \neq \{0\}$.
2. *If* $\mathscr{T} = \overline{\mathscr{S} \cap R(A)}$ *and* $Q = P_{\mathscr{T}}$, *then the pair* $(A, \mathscr{T})$ *is compatible*, $\Sigma(A, P) = \Sigma(A, Q)$ *and* $I_P(A) = I_Q(A)$.
3. *If* $R(A)$ *is closed*, *then* $\mathscr{T} = \mathscr{S} \cap R(A)$ *and*

$$I_P(A) = I_Q(A) = \lambda_{\min}\big((QA^\dagger Q)^\dagger\big), \tag{3}$$

*where* $C^\dagger$ *denotes the Moore–Penrose pseudoinverse of a closed range operator* $C$.

Formula (3) is the natural generalization of $I_P(A) = \|PA^\dagger P\|^{-1}$, which holds if $A$ is positive (semidefinite) with closed range (see Corollary 2.2). In Section 6 we study some applications of the mentioned results, particularly to problems posed by Fiedler–Markham [9] and Reams [15]. Given a completely positive map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$, we also compute the number

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu \cdot \mathrm{Id} \text{ is completely positive}\},$$

which can be considered as a notion of index for such maps.

## 2. Preliminary results

In this paper $\mathscr{H}$ denotes a Hilbert space, $L(\mathscr{H})$ is the algebra of all linear bounded operators on $\mathscr{H}$, $\mathrm{Gl}(\mathscr{H})$ is the group of invertible operators in $L(\mathscr{H})$ and $L(\mathscr{H})^+$ is the subset of $L(\mathscr{H})$ of all positive (semidefinite) operators. If $\dim \mathscr{H} = n < \infty$ we shall identify $\mathscr{H}$ with $\mathbb{C}^n$ and $L(\mathscr{H})$ with the space of $n \times n$ complex matrices $M_n(\mathbb{C})$. The elements of $\mathbb{C}^n$ are considered as column vectors. For simplicity we sometimes describe a column vector $\xi \in \mathbb{C}^n$ as $\xi = (\xi_1, \ldots, \xi_n)$.

For every $C \in L(\mathscr{H})$ its range is denoted by $R(C)$, $\sigma(C)$ denotes the spectrum of $C$ and $\rho(C)$ the spectral radius of $C$. If $C^* = C$, we denote $\lambda_{\min}(C)$ the minimum of $\sigma(C)$. If $R(C)$ is closed, then $C^\dagger$ denotes the Moore–Penrose pseudoinverse of $C$. The orthogonal projection onto a closed subspace $\mathscr{S}$ is denoted by $P_{\mathscr{S}}$. We use the notations $\mathbb{Q} = \{Q \in L(\mathscr{H}) : Q^2 = Q\}$ for the set of idempotents and $\mathbb{P} = \{P \in \mathbb{Q} : P = P^*\}$ for the set of orthogonal projections. For every $P \in \mathbb{P}$, the decomposition $\mathscr{H} = R(1 - P) \oplus R(P)$ induces a $2 \times 2$ representation of $A \in L(\mathscr{H})$:

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix},$$

which we call the matrix representation induced by $P$.

Now we state the well known criterion due to Douglas [8] (see also [10]) about ranges and factorization of operators:

**Theorem 2.1.** *Let $A, B \in L(\mathcal{H})$. Then the following conditions are equivalent*:

1. $R(B) \subseteq R(A)$.
2. *There exists a positive number $\mu$ such that $BB^* \leqslant \mu AA^*$.*
3. *There exists $D \in L(\mathcal{H})$ such that $B = AD$.*

*Moreover, in this case there exists a unique solution $D$ of the equation $AX = B$ such that $R(D) \subseteq \overline{R(A)}$. The operator $D$ is called the reduced solution of the equation $AX = B$ and $\|D\|^2 = \min\{\mu : BB^* \leqslant \mu AA^*\}$. If $R(A)$ is closed, then $D = A^\dagger B$.*

**Corollary 2.2.** *Let $A, B \in L(\mathcal{H})^+$. Then there exists $\mu > 0$ such that $A - \mu B \geqslant 0$ if and only if $R(B^{1/2}) \subseteq R(A^{1/2})$. In this case, if $B \neq 0$ and $D$ is the reduced solution of the equation $A^{1/2}X = B^{1/2}$, we have*

$$\max\{\mu \geqslant 0 : A - \mu B \geqslant 0\} = \|D\|^{-2}.$$

*If $R(A)$ is closed, this number coincides with $\rho(A^\dagger B)^{-1} = \|B^{1/2}A^\dagger B^{1/2}\|^{-1}$.*

**Proof.** If $R(A)$ is closed, then $R(A) = R(A^{1/2})$ and $D = (A^{1/2})^\dagger B^{1/2}$. Hence

$$\|D\|^2 = \|D^*D\| = \|B^{1/2}A^\dagger B^{1/2}\| = \rho(B^{1/2}A^\dagger B^{1/2}) = \rho(A^\dagger B). \qquad \square$$

**Corollary 2.3.** *Let $A \in L(\mathcal{H})^+$ and $\xi \in \mathcal{H}$ with $\|\xi\| = 1$. Consider the rank one projection $P = \xi \otimes \xi = P_\xi$ onto the subspace generated by $\xi$. If*

$$I_\xi(A) = \max\{\mu \geqslant 0 : A - \mu P \geqslant 0\},$$

*then $I_\xi(A) \neq 0 \iff \xi \in R(A^{1/2})$. In this case, if $\eta \in \ker A^\perp$ satisfies $A^{1/2}\eta = \xi$, we get $I_\xi(A) = \|\eta\|^{-2}$. If $\xi \in R(A)$, then for every $\zeta \in \mathcal{H}$ such that $A\zeta = \xi$ it holds $I_\xi(A) = \langle A\zeta, \zeta\rangle^{-1}$. If $R(A)$ is closed, then $I_\xi(A) = \langle A^\dagger\xi, \xi\rangle^{-1}$.*

**Proof.** The first part follows from Corollary 2.2. Let $\eta \in \ker A^\perp$ such that $A^{1/2}\eta = \xi$. Then the reduced solution of the equation $A^{1/2}X = P$ is $\eta \otimes \xi$, the one rank operator defined by

$$\eta \otimes \xi(\gamma) = \langle \gamma, \xi\rangle\eta, \quad \gamma \in \mathcal{H}.$$

It is easy to see that $\|\eta \otimes \xi\| = \|\xi\|\,\|\eta\| = \|\eta\|$. If there exists $\zeta \in \mathcal{H}$ such that $A\zeta = \xi$, then $A^{1/2}\zeta = \eta$ and $\langle A\zeta, \zeta\rangle = \|\eta\|^2$. If $R(A)$ is closed, then $\eta = (A^{1/2})^\dagger\xi$, so that $\|\eta\|^2 = \langle \eta, \eta\rangle = \langle A^\dagger\xi, \xi\rangle$. $\square$

**2.4.** Suppose that $\dim \mathcal{H} = n < \infty$. We identify $L(\mathcal{H})$ with $M_n(\mathbb{C})$, the algebra of $n \times n$ complex matrices. Let $A \in L(\mathcal{H})^+$. In [16] the notion of minimal index for $A$ was defined as

$$
\begin{aligned}
I(A) &= \max\left\{\mu \geqslant 0 : A \circ B \geqslant \mu \cdot B \quad \forall\, B \in L(\mathscr{H})^+\right\} \\
&= \max\left\{\mu \geqslant 0 : \Phi_A - \mu \cdot \mathrm{Id} \geqslant 0 \quad \text{on } L(\mathscr{H})^+\right\} \\
&= \max\left\{\mu \geqslant 0 : A - \mu \cdot ee^t \geqslant 0\right\},
\end{aligned}
\tag{4}
$$

where $e = (1, \dots, 1)$, the symbol $\circ$ denotes the Hadamard product of matrices and $\Phi_A(C) = A \circ C$, $C \in L(\mathscr{H})$. The last equality follows from the fact that for $C \in L(\mathscr{H})$, $\Phi_C \geqslant 0 \Leftrightarrow C \geqslant 0$ (see [13]).

Note that, if $\xi = n^{-1/2}e$, then $\|\xi\| = 1$ and $I(A) = n^{-1} I_\xi(A)$. By Corollary 2.3, $I(A) > 0$ if and only if $e$ belongs to the range of $A$. In [7,16] it is shown that, in this case, for any vector $y$ such that $A(y) = e$,

$$
I(A) = \langle Ay, y\rangle^{-1} = \langle A^\dagger e, e\rangle^{-1} = \min\{\langle Az, z\rangle : \langle z, e\rangle = 1\}. \tag{5}
$$

Note that the first two equalities are particular cases of Corollary 2.3.

## 3. The shorted operator for selfadjoint operators

Let $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$. We first need a characterization of those pairs $(A, P)$ such that, for some $\mu \in \mathbb{R}$, it holds

$$
A - \mu P \geqslant 0. \tag{6}
$$

The solution of this problem is well known, see for example [14]. We shall give a brief survey of the characterization of pairs $(A, P)$ satisfying Eq. (6), for the sake of completeness.

Note that if $Px = 0$ then $\langle (A - \mu P)x, x\rangle = \langle Ax, x\rangle$. Thus, a necessary condition for $A$ and $P$ to satisfy condition (6) is that $(1 - P)A(1 - P) \geqslant 0$.

**Definition 3.1.** Let $A \in L(\mathscr{H})$ such that $A = A^*$ and let $P \in \mathbb{P}$. We shall say that $A$ is $P$-positive if $(1 - P)A(1 - P) \geqslant 0$.

**Remark 3.2.** Let $e = (1, \dots, 1) \in \mathbb{C}^n$ and let $P_e \in M_n(\mathbb{C})$ denote the orthogonal projection onto the subspace generated by $e$. A real symmetric matrix $A \in M_n(\mathbb{R})$ is called *almost positive* if $\langle A\xi, \xi\rangle \geqslant 0$ for all $\xi \in \mathbb{R}^n$ such that $\langle \xi, e\rangle = 0$. Therefore a real selfadjoint matrix $A \in M_n(\mathbb{C})$ is almost positive if and only if it is $P_e$-positive.

**Proposition 3.3.** *Let $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and $A \in L(\mathscr{H})$ be hermitian and $P$-positive. Let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ be the representation induced by $P$. Then the following conditions are equivalent*:

1. *There exists $\mu \in \mathbb{R}$ such that $A - \mu P \geqslant 0$.*
2. *The partial matrix $\begin{pmatrix} a & b \\ b^* & ? \end{pmatrix}$ admits a positive completion.*

3. *The set* $M(A, \mathcal{S}) = \{D \in L(\mathcal{H}) : D = D^*, D \leqslant A, R(D) \subseteq \mathcal{S}\}$ *is not empty.*
4. *There exists* $x \in L(\mathcal{S}, \mathcal{S}^\perp)$ *such that* $b = a^{1/2}x$.
5. $R(b) \subseteq R(a^{1/2})$.
   *and, if* $R(a)$ *is closed, also*
6. $\ker a = \ker A \cap \mathcal{S}^\perp$.

**Proof.** $1 \to 2$: Take $d = c - \mu P$. Then $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} = A - \mu P \geqslant 0$.

$2 \to 3$: Let $d \in L(\mathcal{S})$ such that $\begin{pmatrix} a & b \\ b^* & d \end{pmatrix} \geqslant 0$. Then $D = \begin{pmatrix} 0 & 0 \\ 0 & c-d \end{pmatrix} \in M(A, \mathcal{S})$.

$3 \to 1$: If $D \in M(A, \mathcal{S})$, take $\mu \in \mathbb{R}$ such that $-D \leqslant -\mu P$.

$4 \leftrightarrow 5$: It is a consequence of Douglas Theorem 2.1.

$2 \leftrightarrow 5$: It is well known (see [1] or [14]). For example, if $b = a^{1/2}x$ with $x \in L(\mathcal{S}, \mathcal{S}^\perp)$, then

$$\begin{pmatrix} a & b \\ b^* & x^*x \end{pmatrix} = \begin{pmatrix} a^{1/2} & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} a^{1/2} & x \\ 0 & 0 \end{pmatrix} \geqslant 0.$$

If $R(a)$ is closed, then $R(a^{1/2}) = R(a) = (\ker a)^\perp$. In this case

$$R(b) \subseteq R(a) \Leftrightarrow \ker a \subseteq \ker b^*$$
$$\Leftrightarrow (\forall\, \xi \in \mathcal{S}^\perp,\ a\xi = 0 \Rightarrow a\xi + b^*\xi = A\xi = 0),$$

i.e., condition 5 is equivalent to $\ker a \subseteq \ker A \cap \mathcal{S}^\perp$. Note that the reverse inclusion always holds. □

**Remark 3.4.** With the notations of Proposition 3.3, if $R(a)$ is not closed, then conditions 1–5 still imply, with the same proof, that $\ker a = \ker A \cap \mathcal{S}^\perp$.

**Definition 3.5.** Let $A \in L(\mathcal{H})$ be hermitian and $P \in \mathbb{P}$ such that $A$ is $P$-positive.

1. $A$ is called $P$*-complementable* if any of the conditions of Proposition 3.3 holds.
2. In this case we define: $I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}$.

If $A$ is $P$-complementable, the shorted operator can be defined for the pair $(A, P)$, and several results for shorted operators of positive operators (see [1]) remain true in this case. We show these properties in the rest of this section.

**Definition 3.6.** Let $P \in \mathbb{P}$ with $R(P) = \mathcal{S}$ and let $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix} \in L(\mathcal{H})$ be hermitian $P$-complementable. Let $d \in L(\mathcal{S}, \mathcal{S}^\perp)$ be the reduced solution of the equation $b = a^{1/2}x$. Then we define the Schur complement (or shorted operator) of $A$ with respect to $\mathcal{S}$ as

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - d^*d \end{pmatrix}.$$

**Proposition 3.7.** *Let $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and let A be $P$-complementable.*

1. *If $A \geqslant 0$, then $\Sigma(A, P)$ is the usual shorted operator for A and $\mathscr{S}$.*
2. *Let $\mu \in \mathbb{R}$. Then $\Sigma(A - \mu P, P) = \Sigma(A, P) - \mu P$.*
3. $\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, R(D) \subseteq \mathscr{S}\}$.
4. $\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathscr{S}\}$.
5. *Let $\xi \in \mathscr{S}$. Then*

$$\langle \Sigma(A, P)\xi, \xi\rangle = \inf\left\{\langle A(\xi + \eta), \xi + \eta\rangle,\ \eta \in \mathscr{S}^{\perp}\right\}.$$

6. *If $a = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ and $R(a)$ is closed, then*

$$\Sigma(A, P) = \begin{pmatrix} 0 & 0 \\ 0 & c - b^*a^{\dagger}b \end{pmatrix}.$$

*where $a^{\dagger}$ is the Moore–Penrose pseudoinverse of a in $L(\mathscr{S}^{\perp})$.*

**Proof**

1. It is shown in [1].
2. It is clear by definition.
3. If $A \geqslant 0$, then $\Sigma(A, P) = \max\{D \in L(\mathscr{H}) : D \geqslant 0,\ D \leqslant A$ and $R(D) \subseteq \mathscr{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2.
4. If $A \geqslant 0$, then $\Sigma(A, P) = \inf\{QAQ^* : Q = Q^2, R(Q) = \mathscr{S}\}$ (see [1]). The general case can be easily deduced from the positive case using item 2 and the fact that, if $Q \in \mathbb{Q}$ has $R(Q) = \mathscr{S}$, then $QPQ^* = P$.
5. The positive was shown in [1]. If $A \not\geqslant 0$, denote by $B = A - I_P(A)P \geqslant 0$. By item 2, $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$. Thus,

$$\begin{aligned}\langle \Sigma(B, P)\xi, \xi\rangle &= \inf\{\langle B(\xi + \eta), \xi + \eta\rangle, \eta \in \mathscr{S}^{\perp}\} \\ &= \inf\{\langle A(\xi + \eta), \xi + \eta\rangle, \eta \in \mathscr{S}^{\perp}\} - I_P(A)\|\xi\|^2.\end{aligned}$$

6. If $R(a)$ is closed, then $R(a^{1/2})$ is also closed, $(a^{1/2})^{\dagger} = (a^{\dagger})^{1/2}$ and $d = (a^{1/2})^{\dagger}b$ is the reduced solution of the equation $a^{1/2}x = b$. □

**Remark 3.8.** The following properties are easy consequences of Proposition 3.7 and the corresponding results for the positive case (see [1,11]):

1. Let $P, Q \in \mathbb{P}$ such that $P \leqslant Q$, let $A \in L(\mathscr{H})$ $P$-complementable and $B \in L(\mathscr{H})$ such that $A \leqslant B$. Then $B$ is $P$-complementable, $\Sigma(A, P) \leqslant \Sigma(B, P)$, $A$ is $Q$-complementable and $\Sigma(A, P) \leqslant \Sigma(A, Q)$.

2. Let $\{E_n\} \in L(H)$ be a monotone decreasing sequence of positive operators strongly convergent to 0 and let $A \in L(H)$ be $P$-complementable. Then $\Sigma(A + E_n, P)$ converges strongly to $\Sigma(A, P)$.
3. Let $A \in L(H)$ be an invertible $P$-complementable operator. Then $\|\Sigma(A + \epsilon, P) - \Sigma(A, P)\| \to 0$ as $\epsilon \to 0^+$.
4. Let $A \in L(H)$ be $P$-complementable. Then there exist unique operators $F$ and $G$ such that $A = F + G$ with $R(F) \subseteq \mathscr{S}$, $G \geqslant 0$ and $R(G^{1/2}) \cap \mathscr{S} = \{0\}$.
5. Let $A \in L(\mathscr{H})$ and $P \in \mathbb{P}$ such that $A$ is $P$-complementable. Let $f$ be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geqslant 0$. Then $\Sigma(f(A), P) \geqslant f(\Sigma(A, P))$.
6. Let $\{P_n\} \in \mathbb{P}$ be a decreasing sequence of projections such that $P_n \xrightarrow{\text{S.O.T}} P$ and let $A \in L(\mathscr{H})$ be $P$-complementable. Then $\{\Sigma(A, P_n)\}$ decreases to $\Sigma(A, P)$ (see [14] or [2]).

## 4. $A$-selfadjoint projections

Given $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ and $A \in L(\mathscr{H}) P$-positive, we shall consider a condition stronger than being $P$-complementable which is the existence of $A$-selfadjoint projections onto $\mathscr{S}^\perp$, i.e., $Q \in \mathbb{Q}$ such that $AQ = Q^*A$ and $R(Q) = \mathscr{S}^\perp$.

**Definition 4.1.** Let $A = A^* \in L(\mathscr{H})$ and $\mathscr{S} \subseteq \mathscr{H}$ a closed subspace. We denote by

$$\mathscr{P}(A, \mathscr{S}) = \left\{Q \in \mathbb{Q} : R(Q) = \mathscr{S}^\perp, AQ = Q^*A\right\}.$$

The pair $(A, \mathscr{S})$ is said to be *compatible* if $\mathscr{P}(A, \mathscr{S})$ is not empty.

The notion of a compatible pair was introduced in [4], where a characterization of compatible pairs $(A, \mathscr{S})$ in terms of the Schur complements $\Sigma(A, P)$ is given, in case that $A \geqslant 0$. The following two results are taken from [4]:

**Lemma 4.2.** *Let $A = A^* \in L(\mathscr{H})$ and $Q \in \mathbb{Q}$. Then the following conditions are equivalent*:

1. *$Q$ satisfies that $AQ = Q^*A$, i.e., $Q$ is $A$-selfadjoint.*
2. $\ker Q \subseteq A^{-1}(R(Q)^\perp)$.
   *and, if $A \geqslant 0$,*
3. $Q^*AQ \leqslant A$.

**Proposition 4.3.** *Given $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$, the following conditions are equivalent*:

1. *The pair $(A, \mathscr{S})$ is compatible (i.e., $\mathscr{P}(A, \mathscr{S})$ is not empty).*

2. *If* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *then* $R(b) \subseteq R(a)$.
3. $\mathscr{S}^\perp + A^{-1}(\mathscr{S}) = \mathscr{H}$.

*In this case, for every* $E \in \mathscr{P}(A, \mathscr{S})$, $\ker E \subseteq A^{-1}(\mathscr{S})$.

**Corollary 4.4.** *If* $(A, \mathscr{S})$ *is compatible and* $A$ *is* $P$*-positive, then* $A$ *is* $P$*-complementable.*

**Proof.** Just note that, if $a = (1 - P)A(1 - P) \geqslant 0$, then $R(a) \subseteq R(a^{1/2})$. □

**Remark 4.5.** Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that $A$ is $P$-positive and suppose that $R((1 - P)A(1 - P))$ is closed. Then $(A, \mathscr{S})$ is compatible if and only if $A$ is $P$-complementable. This last condition holds whenever $\dim \mathscr{S}^\perp < \infty$. Therefore if $\mathscr{H}$ is a finite dimensional space and $A$ is $P$-positive, the conditions $(A, \mathscr{S})$ is compatible and $A$ is $P$-complementable are equivalent.

**Proposition 4.6.** *Let* $A = A^* \in L(\mathscr{H})$ *such that* $A$ *is* $P$*-positive and the pair* $(A, \mathscr{S})$ *is compatible. Let* $E \in \mathscr{P}(A, \mathscr{S})$ *and* $Q = I - E$. *Then*

1. $\Sigma(A, P) = AQ = Q^*A = Q^*AQ$.
2. $\Sigma(A, P) = \min\{FAF^* : F \in \mathbb{Q}, R(F) = \mathscr{S}\}$.
3. $R(\Sigma(A, P)) \subseteq R(A) \cap S$.

**Proof.** The case $A \geqslant 0$ was shown in [4] (with equality in item 3). The general case follows from the fact that if $F \in \mathbb{Q}$ and $R(F) = \mathscr{S}$, then $FP = PF^* = FPF^* = P$. Recall that if $B = A - I_P(A)P$, then $\Sigma(A, P) = \Sigma(B, P) + I_P(A)P$; and $R((I - E)^*) = \ker(I - E)^\perp = \mathscr{S}$. Item 3 is clear because $R(AQ) \subseteq R(A)$. □

**Lemma 4.7.** *Let* $A = A^* \in L(\mathscr{H})$ *and* $P \in \mathbb{P}$ *with* $R(P) = \mathscr{S}$. *Suppose that* $A$ *is* $P$*-positive and* $(A, \mathscr{S})$ *is compatible. Let* $E \in \mathscr{P}(A, \mathscr{S})$ *and* $Q = 1 - E$. *Consider the operator* $T = (1 - P) + Q$. *Then*

1. $T \in \mathrm{Gl}(\mathscr{H})$ *with* $T^{-1} = E + P$.
2. *If* $A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ *in terms of* $P$, *then*

$$T^*AT = \begin{pmatrix} a & 0 \\ 0 & \Sigma(A, P) \end{pmatrix}. \tag{7}$$

3. *If* $A \in \mathrm{Gl}(\mathscr{H})$ *then* $a \in \mathrm{Gl}(\mathscr{S}^\perp)$ *and* $\Sigma(A, P) \in \mathrm{Gl}(\mathscr{S})$. *Moreover, if we view* $\Sigma(A, P) \in L(\mathscr{S})$, *then* $\Sigma(A, P)^{-1} = PA^{-1}P$ *or, in other words,*

$$\Sigma(A, P) = (PA^{-1}P)^\dagger. \tag{8}$$

**Proof**

1. Since $R(1-P) = R(E) = \ker P = \ker Q = \mathscr{S}^{\perp}$, then $(1-P)E = E$ and $QP = Q$. Thus $T(E+P) = E+Q = 1$. The other case is similar.
2. The fact that $R(Q) = \ker E \subseteq A^{-1}(\mathscr{S})$ implies that $Q^*A(1-P) = (1-P)AQ = 0$. By Proposition 4.6, $Q^*AQ = \Sigma(A, P)$.
3. Note that $(T^*AT)^{-1} = T^{-1}A^{-1}(T^*)^{-1} = (E+P)A^{-1}(E^*+P)$. But $PE = E^*P = 0$, so that $\Sigma(A, P)^{-1} = P(T^*AT)^{-1}P = PA^{-1}P$. □

**Proposition 4.8.** *Let $A = A^* \in L(\mathscr{H})$ and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$. Suppose that $A$ is $P$-positive and $(A, \mathscr{S})$ is compatible. Then $R(\Sigma(A, P)) = R(A) \cap \mathscr{S}$.*

**Proof.** We use the notations of Lemma 4.7. By formula (7), $R(T^*AT) \cap \mathscr{S} = R(\Sigma(A, P))$. On the other hand, if $\xi \in \mathscr{S}$, then $T^*\xi = Q^*\xi = \xi$, because $R(Q^*) = \ker Q^{\perp} = \mathscr{S}$ and $Q^* \in \mathbb{Q}$. Hence $R(A) \cap \mathscr{S} = R(AT) \cap \mathscr{S} \subseteq R(T^*AT) \cap \mathscr{S} = R(\Sigma(A, P))$. The reverse inclusion was shown in Proposition 4.6. □

## 5. Computation of $I_P(A)$

Let $P \in \mathbb{P}$ and $A = A^* \in L(\mathscr{H})$. Recall that, if $A$ is $P$-complementable, we have defined

$$I_P(A) = \max\{\mu \in \mathbb{R} : A - \mu P \geqslant 0\}.$$

**Remark 5.1.** If $A \geqslant 0$ then, by Corollary 2.2, $I_P(A) \neq 0$ if and only if $R(P) \subseteq R(A^{1/2})$ and, in this case, $I_P(A) = \|D\|^{-2}$, where $D$ is the reduced solution of the equation $A^{1/2}X = P$. Thus, if $R(A)$ is closed, then $I_P(A) = \rho(A^{\dagger}P)$.

Suppose now that $A \ngeqslant 0$. It is easy to see that if $B = A + \mu P$, then $I_P(B) = I_P(A) + \mu$. Therefore a way to compute $I_P(A)$ would be to find a lower bound $\mu \leqslant I_P(A)$ in order to compute firstly $I_P(B)$ for $B = A - \mu P \geqslant 0$, reducing the general case to the positive case. Nevertheless this way seems to be not applicable. For example, it is easy to get, for any $M > 0$, selfadjoint matrices $A \in M_2(\mathbb{C})$ with $\|A\| \leqslant 2$ such that $I_P(A) < -M$, where $P$ is a fixed projection of rank one. Indeed, take $P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $A = \begin{pmatrix} \varepsilon & 1 \\ 1 & 0 \end{pmatrix}$, for $\varepsilon < M^{-1}$.

We first show the key relation between $I_P(A)$ and the shorted operator $\Sigma(A, P)$:

**Proposition 5.2.** *Let $A \in L(\mathscr{H})$ be hermitian, $A \ngeqslant 0$, and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that $A$ is $P$-complementable. Then*

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \min\{\langle \Sigma(A, P)\xi, \xi\rangle : \xi \in \mathscr{S}, \|\xi\| = 1\}. \tag{9}$$

**Proof.** Denote by $\mu = \lambda_{\min}(\Sigma(A, P))$. Since $A \not\geqslant 0$, it is easy to see that $\mu < 0$. In particular this shows the last equality in Eq. (9). Note that $\mu P \leqslant \Sigma(A, P)$, so that

$$A - \mu P \geqslant A - \Sigma(A, P) \geqslant 0 \quad \text{and} \quad \mu \leqslant I_P(A).$$

On the other hand, since $A - I_P(A)P \geqslant 0$, then $I_P(A)P \in M(A, \mathscr{S})$ and $I_P(A)P \leqslant \Sigma(A, P)$ (see Propositions 3.3 and 3.7), which implies that $I_P(A) \leqslant \mu$. $\square$

**Remark 5.3.** With the notations of Proposition 5.2, if $A \geqslant 0$, then the identity $I_P(A) = \min\{\langle \Sigma(A, P)\xi, \xi\rangle : \xi \in \mathscr{S}, \|\xi\| = 1\}$ remains true; and this number coincides with $\lambda_{\min}(\Sigma(A, P))$ if we consider the spectrum of $\Sigma(A, P)$ as an operator of $L(\mathscr{S})$ (in order to remove the number 0 if necessary).

The following properties of $I_P(A)$ follow immediately from Remark 3.8 and Proposition 5.2.

**Corollary 5.4.** *Let $A \in L(\mathscr{H})$ be hermitian and $P \in \mathbb{P}$ such that $A$ is $P$-complementable*:

1. *Let $Q \in \mathbb{P}$ such that $P \leqslant Q$ and suppose that $A \not\geqslant 0$. Then $I_P(A) \leqslant I_Q(A)$. If $A \geqslant 0$ this property may fail because of the fact observed in Remark* 5.3.
2. *Let $B \in L(\mathscr{H})$ such that $A \leqslant B$. Then $I_P(A) \leqslant I_P(B)$.*
3. *Let $\{E_n\} \in L(\mathscr{H})$ be a monotone* (*not necessary strictly*) *decreasing sequence of positive operators strongly convergent to* 0. *Then the sequence $\{I_P(A + E_n)\}$ decreases to $I_P(A)$.*
4. *Let $\{A_n\} \in L(\mathscr{H})$ be a sequence of $P$-complementable operators which is norm convergent to an invertible $P$-complementable operator $A$. Then $\{I_P(A_n)\}$ converges to $I_P(A)$.*
5. *Let $f$ be a operator monotone map defined on $\sigma(A) \cup \sigma(\Sigma(A, P))$ such that $f(0) \geqslant 0$. Then $I_P(f(A)) \geqslant f(I_P(A))$.*
6. *Let $\{P_n\}$ be a decreasing sequence of orthogonal projections such that $P_n \xrightarrow{\text{S.O.T}} P$. Then $\{I_{P_n}(A)\}$ decreases to $I_P(A)$.*

**Remark 5.5.** It was pointed out in [16] that the hypothesis in item 3 can not be relaxed, i.e the map $A \mapsto I_P(A)$ is not norm continuous in general, as we see in the following example.

**Example 5.6.** Let $a \neq 1$ and $\{b_n\} \subseteq \mathbb{R}_{>a}$ such that $\lim_{n\to\infty} b_n = a$. Then the sequence of positive matrices

$$A_n = \begin{pmatrix} a^2 + a^{-2} & ab_n + (ab_n)^{-1} \\ ab_n + (ab_n)^{-1} & b_n^2 + b_n^{-2} \end{pmatrix}$$

converges in norm to $A=(a^2+a^{-2})ee^t$, where $e=(1,1)$. Let $x_n=(a,b_n)$ and $y_n=(a^{-1},b_n^{-1})$. Note that $A_n=x_nx_n^*+y_ny_n^*$ and $e=\lambda_nx_n+\mu_ny_n$, with $\lambda_n=(a+b_n)^{-1}$ and $\mu_n=ab_n(a+b_n)^{-1}$. If a vector $z$ satisfies that $A_nz=e$, then

$$e=A_nz=(x_nx_n^*+y_ny_n^*)z=\langle z,x_n\rangle x_n+\langle z,y_n\rangle y_n$$

and $\langle z,e\rangle^{-1}=(\langle z,x_n\rangle^2+\langle z,y_n\rangle^2)^{-1}=\frac{(a+b_n)^2}{1+a^2b_n^2}$. Since $I_{P_e}(A_n)=2\langle A_n^{-1}e,e\rangle^{-1}=\frac{2(a+b_n)^2}{1+a^2b_n^2}$, we get

$$\lim_{n\to\infty}I_{P_e}(A_n)=\lim_{n\to\infty}\frac{2(a+b_n)^2}{a^2b_n^2+1}=\frac{8}{a^2+a^{-2}}\neq 2(a^2+a^{-2})=I_{P_e}(A).$$

The following results are the natural generalizations of formula (5) to our setting.

**Corollary 5.7.** *Let $A\in L(\mathscr{H})$ be hermitian and $P\in\mathbb{P}$ with $R(P)=\mathscr{S}$ such that $A$ is $P$-complementable. Then*

$$I_P(A)=\inf\{\langle A\xi,\xi\rangle:\xi\in\mathscr{H},\ \|P\xi\|=1\}. \tag{10}$$

**Proof.** It is a consequence of Eq. (9) in Proposition 5.2 (or Remark 5.3 in case that $A\geqslant 0$) and item 5 of Proposition 3.7. □

**Corollary 5.8.** *Let A and P be as above and suppose that $P=\xi\otimes\xi$ for some unit vector $\xi\in\mathscr{H}$. Then*

$$I_P(A)=\inf\{\langle A\eta,\eta\rangle:\eta\in\mathscr{H},\ \langle\eta,\xi\rangle=1\}. \tag{11}$$

**Proof.** Note that $P\eta=\langle\eta,\xi\rangle\xi$ and $\|P\eta\|=|\langle\eta,\xi\rangle|$. Also, if $\omega\in\mathbb{C}$ has $|\omega|=1$, then $\langle A\omega\eta,\omega\eta\rangle=\langle A\eta,\eta\rangle$. □

Throughout, we shall consider $P\in\mathbb{P}$ with $R(P)=\mathscr{S}$ and $A\in L(\mathscr{H})$ $P$-positive such that $(A,\mathscr{S})$ is compatible. In this case almost all results which can be shown for matrices can be extended to the infinite dimensional case.

**Remark 5.9.** Let $A\in L(\mathscr{H})$ be hermitian and $P\in\mathbb{P}$ with $R(P)=\mathscr{S}$ such that $(A,\mathscr{S})$ is compatible. Suppose that $I_P(A)\neq 0$. Then

$$R(A)\cap\mathscr{S}=R(\Sigma(A,P))\neq\{0\}.$$

Indeed, since $(A,\mathscr{S})$ is compatible, $R(\Sigma(A,P))=R(A)\cap\mathscr{S}$ by Proposition 4.8. On the other hand, $0\neq I_P(A)=\lambda_{\min}(\Sigma(A,P))$, by Proposition 5.2. Hence $\Sigma(A,P)\neq 0$.

**Theorem 5.10.** *Let $A\in L(\mathscr{H})$ be hermitian and $P\in\mathbb{P}$ with $R(P)=\mathscr{S}$, such that A is $P$-positive and $(A,\mathscr{S})$ is compatible with $I_P(A)\neq 0$. Denote by $\mathscr{T}=\overline{\mathscr{S}\cap R(A)}$ and $Q=P_{\mathscr{T}}$. Then*

1. *$A$ is $Q$-complementable. Moreover, the pair $(A, \mathscr{T})$ is compatible.*
2. $\Sigma(A, P) = \Sigma(A, Q)$.
3. $I_P(A) = I_Q(A)$.

**Proof.** If $A \geqslant 0$, by Remark 5.3, we know that $\Sigma(A, P)$ is invertible in $L(\mathscr{S})$. On the other hand, since $(A, \mathscr{S})$ is compatible, $\mathscr{S} = R(\Sigma(A, P)) = R(A) \cap \mathscr{S} \subseteq R(A)$.

Suppose now that $A \not\geqslant 0$. By Remark 5.9, $R(\Sigma(A, P)) = R(A) \cap \mathscr{S} \subseteq \mathscr{T}$. Hence

$$\Sigma(A, P) \in M(A, \mathscr{T}) = \{D \in L(\mathscr{H}) : D = D^*, D \leqslant A, R(D) \subseteq \mathscr{T}\} \neq \emptyset.$$

Therefore, by Proposition 3.3, $A$ is $Q$-complementable and, by Proposition 3.7, $\Sigma(A, P) \leqslant \Sigma(A, Q)$. The inequality $\Sigma(A, Q) \leqslant \Sigma(A, P)$ follows by Remark 3.8. Then,

$$I_P(A) = \lambda_{\min}(\Sigma(A, P)) = \lambda_{\min}(\Sigma(A, Q)) = I_Q(A).$$

Using Proposition 4.3 item 3, in order to show that the pair $(A, \mathscr{T})$ is compatible, it suffices to verify that $\mathscr{T}^\perp + A^{-1}(\mathscr{T}) = \mathscr{H}$, which follows from the following facts: $\mathscr{S}^\perp + A^{-1}(\mathscr{S}) = \mathscr{H}$ (since $(A, \mathscr{S})$ is compatible), $\mathscr{S}^\perp \subseteq \mathscr{T}^\perp$ and $A^{-1}(\mathscr{S}) = A^{-1}(\mathscr{S} \cap R(A)) \subseteq A^{-1}(\mathscr{T})$. □

**Remark 5.11.** When $\dim \mathscr{S} = 1$, if $A$ is $P$-positive and $P$-compatible we can deduce that $\mathscr{S} \subseteq R(A)$. More generally, if $\dim \mathscr{S} < \infty$, $A$ is injective and $(A, \mathscr{S})$ is compatible , then $\mathscr{S} \subseteq R(A)$. Indeed, note that $\dim A^{-1}(\mathscr{S}) = \dim \mathscr{S} \cap R(A)$, and $A^{-1}(\mathscr{S})$ must be a supplement of $\mathscr{S}^\perp$. Nevertheless, if we remove the condition $(A, \mathscr{S})$ is compatible , this is not true, even if $\dim \mathscr{S} = 1$ and $A$ is injective and $P$-complementable, as the following example shows.

**Example 5.12.** Let $A \in L(\mathscr{H})^+$ be injective non-invertible. Let $\xi \in \mathscr{H} \setminus R(A)$ be a unit vector. Denote by $\mathscr{S}$ the subspace generated by $\xi$, $P = P_{\mathscr{S}}$. If

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$$

in terms of $P$ and $A\xi = \lambda\xi + \eta$ with $\eta \in \mathscr{S}^\perp$, then $\lambda = \langle A\xi, \xi\rangle \neq 0$ and $\eta \neq 0$ (otherwise $\xi \in R(A)$). Therefore $c = \lambda P$ and $b(\mu\xi) = \mu\eta$, $\mu \in \mathbb{C}$.

Suppose that $\eta \in R(a)$, i.e., there exists $\nu \in \mathscr{S}^\perp$ which verifies $a\nu = b\xi$. Then $(1 - P)A(\nu - \xi) = a\nu - b\xi = 0$, so $A(\nu - \xi)$ is a multiple of $\xi$, which must be 0 ($\xi \notin R(A)$). So $\nu = \xi$, a contradiction. Therefore $R(b) \not\subseteq R(a)$ and the pair $(A, \mathscr{S})$ is incompatible.

Now consider $B = A + \mu P$, for any $\mu \in \mathbb{R}$. It is clear that $B$ must be $P$-complementable ($B - \mu P = A \geqslant 0$). But the facts that $A$ is injective and $\xi \notin R(A)$, clearly imply that $B$ is injective and $\xi \notin R(B)$.

**5.13.** Fix $E \in \mathbb{P}$ with range $\mathscr{M}$. Denote by $L(\mathscr{H})_{\mathscr{M}} = \{C \in L(\mathscr{H}) : ECE = C\}$. For $C \in L(\mathscr{H})_{\mathscr{M}}$, denote by $C_0 \in L(\mathscr{M})$ the compression of $C$ to $\mathscr{M}$. With respect to the matrix representation induced by $E$

$$C = \begin{pmatrix} 0 & 0 \\ 0 & C_0 \end{pmatrix} \begin{matrix} \mathscr{M}^{\perp} \\ \mathscr{M} \end{matrix}.$$

The following properties of this compression are easy to see:

1. The map $L(\mathscr{H})_{\mathscr{M}} \ni C \mapsto C_0 \in L(\mathscr{M})$ is a $*$-isomorphism of $C^*$-algebras, i.e., it is isometric and compatible with sums, products and adjoints.
2. If $C = C^* \in L(\mathscr{H})_{\mathscr{M}}$ and $R(C) = \mathscr{M}$, then $C_0 \in \mathrm{Gl}(\mathscr{M})$ and $(C_0)^{-1} = (C^{\dagger})_0$. If $R(C)$ is closed, then $(C_0)^{\dagger} = (C^{\dagger})_0$.

**Theorem 5.14.** *Let $A \in L(\mathscr{H})$ be hermitian, $A \not\geqslant 0$, and $P \in \mathbb{P}$ with $R(P) = \mathscr{S}$ such that $A$ is $P$-positive and $(A, \mathscr{S})$ is compatible. Suppose that $R(A)$ is closed. Denote by $\mathscr{T} = \mathscr{S} \cap R(A)$ and $Q = P_{\mathscr{T}}$. Then*

$$I_P(A) = I_Q(A) = \lambda_{\min}(QA^{\dagger}Q)^{\dagger}. \tag{12}$$

**Proof.** Since we only need to prove the equality $I_Q(A) = \lambda_{\min}(QA^{\dagger}Q)^{\dagger}$, we shall directly suppose that $R(P) \subseteq R(A)$. Denote $\mathscr{M} = R(A)$ and $E = P_{\mathscr{M}}$. Using the notations of 5.13, we have that $A$, $P$ and $\Sigma(A, P) \in L(\mathscr{H})_{\mathscr{M}}$. It is clear that $\Sigma(A, P)_0 = \Sigma(A_0, P_0)$, $I_P(A) = I_{P_0}(A_0)$ and $A_0$ is invertible. Therefore, by Lemma 4.7,

$$\Sigma(A_0, P_0) = (P_0(A_0)^{-1}P_0)^{\dagger} = (PA^{\dagger}P)_0{}^{\dagger} = ((PA^{\dagger}P)^{\dagger})_0$$

and

$$\begin{aligned} I_P(A) = I_{P_0}(A_0) &= \lambda_{\min}\Sigma(A_0, P_0) = \lambda_{\min}((PA^{\dagger}P)^{\dagger})_0 \\ &= \lambda_{\min}(PA^{\dagger}P)^{\dagger}. \qquad \square \end{aligned}$$

## 6. Some applications

The problem of calculating $I_P(A)$ of a $P$-complementable operator $A$ with respect to a projection $P$ has already been considered for certain projections $P$, mainly in the finite dimensional case. Reams [15] showed that if $A \in M_n(\mathbb{R})$ is invertible and almost positive (see Remark 3.2), then $A$ is $P_e$-complementable and $I_{P_e}(A) = n \cdot \langle A^{-1}e, e\rangle^{-1}$, where $e = (1, \ldots, 1) \in \mathbb{C}^n$ and $P_e$ denotes the orthogonal projection onto the subspace generated by $e$. We obtain a generalization of this result in the non-positive case. The general positive case was already considered in [7] and Corollary 2.3 (for every unit vector $\xi \in \mathbb{C}^n$).

**Corollary 6.1.** *Let $\xi \in \mathbb{C}^n$ be a unit vector. Let $A \in M_n(\mathbb{C})$ be non-positive but $P_\xi$-positive. Then $A$ is $P_\xi$-complementable if and only if*

$$\forall \eta \in \mathbb{C}^n,\ \langle \eta, \xi \rangle = 0 \quad \text{and} \quad \langle A\eta, \eta \rangle = 0 \Rightarrow A\eta = 0. \tag{13}$$

*In this case $\xi \in R(A)$ and*

$$I_{P_\xi}(A) = \langle A^\dagger \xi, \xi \rangle^{-1} = \min\{\langle Az, z \rangle : \langle z, \xi \rangle = 1\}. \tag{14}$$

**Proof.** Condition (13) is equivalent to $\operatorname{Ker}((1 - P_\xi)A(1 - P_\xi)) \cap \{\xi\}^\perp = \operatorname{Ker}(A) \cap \{\xi\}^\perp$. By Proposition 3.3, this is equivalent to the fact that $A$ is $P_\xi$-complementable, since $R(A)$ is closed. Note that $I_{P_\xi}(A) < 0$, since $A \not\geqslant 0$. By Remarks 4.5 and 5.9 we get $R(A) \cap R(P_\xi) \neq \{0\}$. Therefore $\xi \in R(A)$ and $\langle A^\dagger \xi, \xi \rangle \neq 0$. By Eq. (12) in Theorem 5.14,

$$I_{P_\xi}(A) = \lambda_{\min}(P_\xi A^\dagger P_\xi)^\dagger = \lambda_{\min}(\langle A^\dagger \xi, \xi \rangle P_\xi)^\dagger = \langle A^\dagger \xi, \xi \rangle^{-1}.$$

In order to prove Eq. (14), it only remains to show that the infimum in Eq. (11) is actually a minimun. Let $\zeta = A^\dagger \xi$ and $\eta = \langle A^\dagger \xi, \xi \rangle^{-1} \zeta$. Then

$$\langle A\eta, \eta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle A\zeta, \zeta \rangle = \langle A^\dagger \xi, \xi \rangle^{-2} \langle \xi, A^\dagger \xi \rangle = \langle A^\dagger \xi, \xi \rangle^{-1},$$

and the minimum is attained at $\eta$. □

It was also noted in [15] that the problem of calculating $I_P(A)$ with respect to $P = P_e$ is equivalent to a problem posed by Fiedler and Markham in [9], that is to calculate

$$\max\{\lambda_{\min}((A \circ C)C^{-1}), C > 0\}$$

for a positive matrix $A \in M_n(\mathbb{C})$, where $A \circ B$ denotes the Hadamard product of $A$ and $B$. The corollary above complements the results obtained in [9] in the non-positive, non-invertible case.

Recall that given a positive matrix $A \in M_n(\mathbb{C})$, the minimal index was introduced in [16] as

$$I_A = \max\{\mu \geqslant 0 : A \circ B \geqslant \mu B, B \geqslant 0\}.$$

Given $P \in M_n(\mathbb{C})$ an orthogonal projection and a $P$-complementable matrix $A$, there is a relation between $I_P(A)$ and the Schur multiplier induced by $A$.

**Corollary 6.2.** *Let $M = \{x_1, \ldots, x_k\} \subseteq \mathbb{C}^n$ be an orthonormal set and let $P$ be the orthogonal projection onto the subspace spanned by $M$. Suppose that $A \in M_n(\mathbb{C})$ is $P$-complementable. Then*

$$I_P(A) = \max\left\{\mu \in \mathbb{R} : A \circ B \geqslant \mu \sum_{i=1}^{k} D_{x_i} B D_{x_i}^*, B \geqslant 0\right\}, \tag{15}$$

*where $D_x$ denotes the diagonal matrix with main diagonal $x \in \mathbb{C}^n$.*

**Proof.** First note that $P = \sum_{i=1}^k x_i x_i^*$. Thus $A - \mu P \geqslant 0$ if and only if every $B \geqslant 0$ satisfies $(A - \mu \sum_{i=1}^k x_i x_i^*) \circ B \geqslant 0$, which is equivalent to $A \circ B \geqslant \mu \sum_{i=1}^k D_{x_i}$

$BD_{x_i}^*$, since a simple calculation shows that $C \circ xx^* = D_x C D_x^*$ for every $C \in M_n(\mathbb{C})$ and $x \in \mathbb{C}^n$. This shows formula (15). □

### *6.1. Completely positive maps on $M_n(\mathbb{C})$*

**Definition 6.3.** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. $\Phi$ is positive if $\Phi(A) \geqslant 0$ whenever $A \geqslant 0$. $\Phi$ is selfadjoint if $\Phi(A^*) = \Phi(A)^*$ or equivalently if $\Phi(A)$ is selfadjoint whenever $A$ is selfadjoint.

Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. If $m \in N$, we denote $\Phi^{(m)}$: $M_m(M_n(\mathbb{C})) \to M_m(M_n(\mathbb{C}))$ the map given by

$$\Phi^{(m)}((a_{ij})_{ij}) = (\Phi(a_{ij}))_{ij}, \quad (a_{ij})_{ij} \in M_m(M_n(\mathbb{C})),$$

and call it the *inflation* of order $m$ of $\Phi$.

**Definition 6.4.** The linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called *completely positive* if $\Phi^{(m)}$ is positive for every $m \in N$.

In the following, $\{e_{ij}\} \subseteq M_n(\mathbb{C})$ denotes the canonical basis for $M_n(\mathbb{C})$. Now we state a result due to Choi [3].

**Theorem 6.5.** *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map. Then $\Phi$ is completely positive if and only if $\Phi^{(n)}((e_{ij})_{ij}) = (\Phi(e_{ij}))_{ij} \in M_n(M_n(\mathbb{C}))$ is positive.*

**Remark 6.6.** Note that the matrix $E = ((e_{ij})_{ij}) \in M_n(M_n(\mathbb{C})) \simeq M_{n^2}(\mathbb{C})$ is a scalar multiple of a rank one projection. Indeed, if $\{e_i\}$ denotes the canonical basis of $\mathbb{C}^n$ and $v \in \mathbb{C}^{n^2}$ is the vector $v = (e_1, \dots, e_n)$, then $(e_{ij})_{ij} = vv^*$. Thus $E = \frac{1}{n} P_v$, where $P_v$ is the projection onto the subspace generated by $v$.

**Remark 6.7.** Let $A \in M_n(A)$. Then the linear map $\Phi_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by $\Phi_A(B) = A \circ B$ is selfadjoint (resp. positive) if and only if $A$ is selfadjoint (resp. positive). Moreover, if $A \geqslant 0$, then $\Phi_A$ is completely positive, since the inflated matrix $A^{(n)} \geqslant 0$ and $\Phi_A^{(n)} = \Phi_{A^{(n)}}$ (see [13]). Therefore $\Phi_A - \mu\mathrm{Id}$ is completely positive if and only if $A - \mu ee^* \geqslant 0$, where $e \in \mathbb{C}^n$ is given by $e = (1, \dots, 1)$. Note that $ee^* = nP_e$, since $\|e\| = n^{1/2}$. Therefore we conclude that for every $P_e$-complementable matrix $A$,

$$I(A) = \max\{\mu \in \mathbb{R} : \Phi_A - \mu\mathrm{Id} \text{ is completely positive}\} = \frac{1}{n} I_{P_e}(A),$$

where $I(A)$ is the minimal index of $A$ defined in 2.4 (in fact, its natural generalization for $A$ not necesarily positive, but $P_e$-complementable).

**Definition 6.8.** Let $\Phi : M_n(C) \to M_n(C)$ be a selfadjoint map. We say that $\Phi$ is complementable if there exists $\mu \in \mathbb{R}$ such that $\Phi - \mu \mathrm{Id}$ is completely positive. In this case we define:

$$I(\Phi) = \max\{\mu \in \mathbb{R} : \Phi - \mu\mathrm{Id} \text{ is completely positive}\}.$$

Note that all completely positive maps $\Phi$ are complementable and $I(\Phi) \geqslant 0$. But in general not all selfadjoint maps are complementable. For example, if $A \in M_n(\mathbb{C})$ is selfadjoint, then $\Phi_A$ is complementable if and only if $A$ is $P_e$-complementable.

**Theorem 6.9.** *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. Then, with the notations of Remark* 6.6,

1. *Suppose that $\Phi$ is not completely positive. In this case $\Phi$ is complementable if and only if for all $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$*

$$\sum_{i=1}^{n} (\eta_i)_i = 0 \Rightarrow \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i \rangle \geqslant 0 \tag{16}$$

*and*

$$\sum_{i=1}^{n} (\eta_i)_i = 0 \quad and \quad \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i \rangle = 0 \Rightarrow \sum_{j=1}^{n} \Phi(e_{ij})\eta_j = 0, \quad i = 1, \ldots, n \tag{17}$$

*or, equivalently, if $A_\Phi = \Phi^{(n)} E = (\Phi(e_{ij}))_{ij} \in M_{n^2}(\mathbb{C})$ is $P_v$-complementable.*

2. *In this case $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and we have*

$$I(\Phi) = \min \left\{ \sum_{i,j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i \rangle : \eta_1, \ldots, \eta_n \in \mathbb{C}^n \ \ and \ \sum_{i=1}^{n} (\eta_i)_i = 1 \right\}. \tag{18}$$

3. *If conditions* (16) *and* (17) *hold, there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that*

$$\sum_{j=1}^{n} \Phi(e_{ij})\eta_j = e_i, \quad i = 1, \ldots, n \tag{19}$$

*where $\{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{C}^n$. For any such vectors,*

$$I(\Phi)^{-1} = \sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i \rangle. \tag{20}$$

4. *If $\Phi$ is completely positive then it is complementable and $I(\Phi) \geqslant 0$. Moreover, $I(\Phi) > 0$ if and only if there exist $\eta_1, \ldots, \eta_n \in \mathbb{C}^n$ such that Eq.* (19) *holds. For any such vectors, Eq.* (20) *holds. Also Eq.* (18) *is true in this case.*

**Proof.** From Theorem 6.5 we conclude that the map $\Phi$ is complementable if and only if the matrix $A_\Phi = (\Phi(e_{ij}))_{ij}$ is $P_v$ complementable. It is easy to see that in fact $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$. Thus we can apply Corollary 6.1 to the matrix $A_\Phi \in M_{n^2}(\mathbb{C})$ and the projection $P_v$. Note that Eq. (16) holds if and only if $A_\Phi$ is $P_v$-positive and condition (13) is equivalent to condition (17). Indeed, if $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ with $\eta_i \in \mathbb{C}^n$ $(i = 1, \dots, n)$, then $\langle \eta, v\rangle = \sum_{i=1}^n (\eta_i)_i$ and $\langle A_\Phi \eta, \eta\rangle = \sum_{i,\, j=1}^n \langle \Phi(e_{ij})\eta_j, \eta_i\rangle$.

Note that condition (19) is equivalent to the fact that $v \in \mathrm{R}(A_\Phi)$, so this condition and Eq. (18) follow from Eq. (14). Similarly, $I(\Phi) = n \cdot I_{P_v}(A_\Phi) = \langle v, A_\Phi^\dagger v\rangle$.

Let $\zeta = (\zeta_1, \dots, \zeta_n) = A_\Phi^\dagger v$. If $\eta = (\eta_1, \dots, \eta_n) \in \mathbb{C}^{n^2}$ satisfy condition (19) (i.e., $A_\Phi \eta = v$), then $P_{R(A_\Phi)}\eta = \zeta$. Therefore

$$\sum_{i,\, j=1}^{n} \langle \Phi(e_{ij})\eta_j, \eta_i\rangle = \langle A_\Phi \eta, \eta\rangle = \langle A_\Phi \zeta, \zeta\rangle = \langle v, A_\Phi^\dagger v\rangle = I(\Phi)^{-1}.$$

Suppose now that $\Phi$ is completely positive. It is clear that $\Phi$ is complementable. By Corollary 2.3, it follows that $I(\Phi) = n \cdot I_{P_v}(A_\Phi) > 0$ if and only if $v \in R(A_\Phi)$, since $R(A_\Phi)$ is closed. This is equivalent to condition (19), and using Corollary 2.3, we can also deduce Eqs. (20) and (18) in this case. □

**Example 6.10.** Consider the map $T : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ given by

$$T(A) = \frac{1}{n} Tr(A) I_n = \frac{1}{n} \sum_{i,j=i}^{n} e_{ij}^* A e_{ij},$$

where $Tr(A) = \sum A_{ii}$ is the usual trace. Then $T$ is completely positive; morever it is a conditional expectation. Note that the matrix

$$A_T = (T(e_{ij}))_{ij} = \frac{1}{n} I_{n^2}.$$

Then $I(T) > 0$, since $A_T(n\, e_i) = e_i$, $1 \leqslant i \leqslant n$, and $T$ satisfies condition (19). Therefore, since $T(e_{ij}) = 0$ if $i \neq j$ and $T(e_{ii}) = \frac{1}{n} I_n$, using Eq. (20),

$$I(T)^{-1} = \sum_{i=1}^{n} \langle T(e_{ii}) n e_i, n e_i\rangle = \sum_{i=1}^{n} n = n^2.$$

This result is actually known in index theory of condidional expectations (using that $T^{(n)}(P_v) = n^{-1} A_T = n^{-2} I_{n^2}$, see [12]). Note that the number

$$J(T) = \max\{\lambda \in \mathbb{R} : T - \lambda \text{ Id is positive (not completely)}\}$$
$$= n^{-1} \neq n^{-2} = I(T).$$

Indeed, it is easy to see that $A \geqslant 0$ implies that $Tr(A) \geqslant \rho(A) = \|A\|$, so that

$$T(A) = \frac{1}{n} Tr(A) I_n \geqslant \frac{1}{n} A.$$

Taking $A = e_{11}$ we get $T(A) \ngeqslant \lambda A$ if $\lambda > \frac{1}{n}$; so that $J(T) = n^{-1}$.

**Remark 6.11.** Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a selfadjoint map. The formulation of Theorem 6.9 intends to characterize complementability and to compute $I(\Phi)$ in terms of $\Phi$ itself instead of doing it in terms of the "inflated" matrix $A_\Phi$. Another way would be to recall the identity $I(\Phi) = n \cdot I_{P_v}(A_\Phi)$ and use all the previous results of the paper. For example, let $U_1, \ldots, U_m \in M_n(\mathbb{C})$, and suppose that $\Phi$ is given by

$$\Phi(A) = \sum_{k=1}^{m} U_k^* A U_k, \quad A \in M_n(\mathbb{C}),$$

a prototypical completely positive map (see [3]). Denote by $V_k \in M_{n^2}(\mathbb{C})$ the block diagonal matrix with copies of $U_k$ in its diagonal. Denote by $v = (e_1, \ldots, e_n) \in \mathbb{C}^{n^2}$ and $E = (e_{ij})_{ij} = vv^*$. Note that $\|V_k v\| = \|U_k\|_2$ and $V_k^* E V_k = (V_k v)(V_k v)^*$. Therefore

$$A_\Phi = (\Phi(e_{ij}))_{ij} = \sum_{k=1}^{m} V_k^* E V_k = \sum_{k=1}^{m} \|U_k\|_2^2 P_{V_k v}.$$

Thus $I(\Phi)$ can be computed using this expression and Corollaries 6.1 and 2.3.

## References


[1] W.N. Anderson, G.E. Trapp, Shorted operators II, SIAM J. Appl. Math. 28 (1975) 60–71.

[2] J.A. Antezana, Shorted de Operadores, grade thesis UNLP, 2002.

[3] M.D. Choi, Completely positive maps on complex matrices, Linear Algebra Appl. 10 (1975) 285–292.

[4] G. Corach, A. Maestripieri, D. Stojanoff, Schur complements and oblique projections, Acta Sci. Math. (Szeged) 67 (2001) 439–459.

[5] G. Corach, A. Maestripieri, D. Stojanoff, Generalized Schur complements and oblique projections, Linear Algebra Appl. 341 (2002) 259–272.

[6] G. Corach, A. Maestripieri, D. Stojanoff , Oblique projections and abstract splines, J. Approx. Theory 117 (2002) 189–206.

[7] G. Corach, D. Stojanoff, Index of Hadamard multiplication by positive matrix II, Linear Algebra Appl. 332–334 (2001) 504–517.

[8] R.G. Douglas, On majorization, factorization and range inclusion of operators in Hilbert space, Proc. Amer. Math. Soc. 17 (1966) 413–416.

[9] M. Fiedler, T.L. Markham, An observation on the Hadamard product of Hermitian matrices, Linear Algebra Appl. 215 (1995) 179–182.

[10] P.A. Fillmore, J.P. Williams, On operator ranges, Adv. Math. 7 (1971) 254–281.

[11] C.K. Li, R. Mathias, Extrernal characterizations of the Schur complement and resulting inequalities, SIAM Rev. 42 (2) (2000) 233–246.

[12] R. Longo, Index of subfactors and statistics of quantum fields, I, Comm. Math. Phys. 126 (1989) 217–247.

[13] V.I. Paulsen, S.C. Power, R.R. Smith, Schur products and matrix completions, J. Funct. Anal. 85 (1989) 151–178.

[14] E.L. Pekarev, Shorts of operators and some extremal problems, Acta Sci. Math. (Szeged) 56 (1992) 147–163.

[15] R. Reams, Hadamard inverses, squares roots and products of almost semidefinite matrices, Linear Algebra Appl. 288 (1999) 35–43.

[16] D. Stojanoff, Index of Hadamard multiplication by positive matrices, Linear Algebra Appl. 290 (1999) 95–108.

[17] I. Schur, Bemerkungen zur Theorie der beschränkten Bilinearformen mit unendlich vielen Veränderlichen, J. Reine Angew. Math. 140 (1911) 1–28.