TWISTED TENSOR PRODUCTS OF K^n WITH K^m

JACK ARCE, JORGE A. GUCCIONE, JUAN J. GUCCIONE, AND CHRISTIAN VALQUI

Abstract. We find three families of twisting maps of K^m with K^n. One of them is related to truncated quiver algebras, the second one consists of deformations of the first and the third one requires $m = n$ and yields algebras isomorphic to $M_n(K)$. Using these families and some exceptional cases we construct all twisting maps of K^3 with K^3.

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Introduction

Let A, C be unitary K-algebras, where K is a field. By definition, a twisted tensor product of A with C over K, is an algebra structure defined on $A\otimes_K C$, with unit $1\otimes 1$, such that the canonical maps $i_A\colon A\to A\otimes_K C$ and $i_C\colon C\to A\otimes_K C$ are algebra maps satisfying $a\otimes c=i_A(a)i_C(c)$. When K is a commutative ring this structure was introduced independently in [13] and [17], and it has been formerly studied by many people with different motivations (In addition to the previous references see also [1], [2], [4], [3], [5], [8], [14], [10], [18]). A number of examples of classical and recently defined constructions in ring theory fits into this construction. For instance, Ore extensions, skew group algebras, smash products, etcetera (for the definitions and properties of these structures we refer to [15] and [11]). On the other hand, it has been applied to braided geometry and it arises as a natural representative for the product of noncommutative spaces, this being based on the existing duality between the categories of algebraic affine spaces and commutative algebras, under which the cartesian product of spaces corresponds to the tensor product of algebras. And last, but not least, twisted tensor products arise as a tool for building algebras starting with simpler ones.

Given algebras A and C, a basic problem is to determine all the twisted tensor products of A with C. To our knowledge, the first paper in which this problem was attacked in a systematic way was [6], in which C. Cibils studied and completely solved the case $C:=K\times K$. In [9],the case $C:=K^n$ is analysed and some partial classification result were achieved.

In this paper we consider the case $A=K^m$ and $C=K^n$. It is well known that there is a canonical bijection between the twisted tensor products of A with C and the so called twisting maps $\chi\colon C\otimes_k A\to A\otimes_k C$. So each twisting map χ is associated with a twisted tensor product of A with C over K, which will be denoted by $A\otimes_\chi C$.

It is evident that each K-linear map $\chi\colon K^n\otimes K^m\to K^m\otimes K^n$ determines and is determined by unique scalars λ_{ij}^{kl}, such that

$$\chi(e_i\otimes f_j)=\sum_{k,l}\lambda_{ij}^{kl} f_k\otimes e_l \qquad \text{for all } e_i \text{ and } f_j.$$

Given such a map χ, for all $i,l\in\mathbb{N}_m^*$ and $j,k\in\mathbb{N}_n^*$, we let $A(i,l)\in M_n(K)$ and $B(j,k)\in M_m(K)$ denote the matrices defined by

$$A(i,l)_{kj}:=\lambda_{ij}^{kl}=:B(j,k)_{li}.$$

In Proposition 2.3 we show that χ is a twisting map if and only if these matrices satisfy certain (easily verifiable) conditions. This transforms the problem of finding all twisting maps into a problem of linear algebra. When one tries to find all twisting maps of K^3 with K^3 using this linear algebra approach, one encounters that nearly all cases of twisting maps have a very special form. We call these twisting maps standard and prove that the resulting twisted tensor product algebras are isomorphic to certain square zero radical truncated quiver algebras. Moreover, there arises a second type of twisting maps, which we call quasi-standard twisting maps, which yield algebras which corresponds to a formal deformation of the latter case, whenever the corresponding quiver has a triangle which is not a cycle. We also construct a third family of twisting maps when $n=m$, and we show that the resulting algebras are isomorphic to $M_n(K)$.

These three families cover nearly all twisting maps of K^3 with K^3. We find additionally some extensions of the algebras corresponding to the third family in the case K^2 with K^2, and one additional case.

The paper is organized as follows: in section 1 we make a quick review of twisting maps and the $n-1$-ary cross product of vectors. In section 2 we present the characterization in terms of matrices of the twisting maps of K^n with K^m and some basic results, specially on isomorphism of twisting maps and a basic representation on $M_n(K)$.

In section 3 we reprove the results of [6] in our language. In section 4 we prove some basic results on the idempotent matrices $A(i,l)$, and pay special attention to the case of rk $= 1$, where a family arises with algebras isomorphic to $M_n(K)$. In section 5 we define standard and quasi-standard twisting columns and twisting maps and prove several results about them. In section 6 we classify completely the case of reduced rank 1. In section 7 we explore the relation of standard twisting maps and quiver algebras, and also the case of quasi-standard twisting maps. In section 8 we use all results in order to classify the twisting maps in low dimensional cases, including all the twisting maps which are not quasi-standard in the case K^3 with K^3. In the appendix we list all standard and quasi-standard twisting maps of K^3 with K^3.

1 Preliminaries

Let K be a field. From now on we assume implicitly that all the maps whose domain and codomain are K-vector spaces are K-linear maps and that all the algebras are over K. Next we introduce some notations and make some comments.

- $K^\times \coloneqq K \setminus \{0\}$.
- For each natural number i, we set $\mathbb{N}_i^* \coloneqq \{1,\dots,i\}$.
- The tensor product over K is denoted by $\otimes$, without any subscript.
- Given a matrix X we let X^{T} denote the transpose matrix of X. Moreover, we denote with a juxtaposition the multiplication of two matrices and with a bullet the multiplication in K^n. So, $(a_1,\dots,a_n) \centerdot (b_1,\dots,b_n) = (a_1b_1,\dots,a_nb_n)$. Note that $\mathbf{a} = (a_1,\dots,a_n)$ is invertible respect to the multiplication map $\centerdot$ if and only if $\mu_n(\mathbf{a}) \coloneqq a_1\cdots a_n \neq 0$. In this case we let $\mathbf{a}^{\centerdot}$ denote the inverse $(a_1^{-1},\dots,a_n^{-1})$ of $\mathbf{a}$.
- We let $E^{ij} \in M_n(K)$ denote the matrix with 1 in the i,j-entry and 0 otherwise. So, $\{E^{ij} : 1 \le i,j \le n\}$ is the canonical basis of $M_n(K)$.
- For the sake of simplicity we write $\mathbb{1} = \mathbb{1}_n = \mathbb{1}_{K^n} \coloneqq (1,\dots,1)^T$.
- The symbol τ_{nm} denotes the flip $K^n \otimes K^m \longrightarrow K^n \otimes K^m$.

1.1 Twisting maps

Let A, C be unitary algebras. Let μ_A, η_A, μ_C and η_C be the multiplication and unit maps of A and C, respectively. A *twisted tensor product* of A with C is an algebra structure on the K-vector space $A \otimes C$, such that the canonical maps

$$i_A\colon A \longrightarrow A\otimes C \quad \text{and} \quad i_C\colon C \longrightarrow A \otimes C$$

are algebra homomorphisms and $\mu \circ (i_A \otimes i_C) = \mathrm{id}_{A\otimes C}$, where μ denotes the multiplication map of the twisted tensor product.

Assume we have a twisted tensor product of A with C. Then, the map

$$\chi\colon C\otimes A \longrightarrow A \otimes C,$$

defined by $\chi \coloneqq \mu \circ (i_C \otimes i_A)$, satisfies:

(1) $\chi \circ (\eta_C \otimes A) = A \otimes \eta_C$,
(2) $\chi \circ (C \otimes \eta_A) = \eta_A \otimes C$,
(3) $\chi \circ (\mu_C \otimes A) = (A \otimes \mu_C) \circ (\chi \otimes C) \circ (C \otimes \chi)$,
(4) $\chi \circ (C \otimes \mu_A) = (\mu_A \otimes C) \circ (A \otimes \chi) \circ (\chi \otimes A)$.

A map satisfying these conditions is called a *twisting map* of C with A. Conversely, if

$$\chi\colon C\otimes A\longrightarrow A\otimes C$$

is a twisting map, then $A\otimes C$ becomes a twisted tensor product via

$$\mu_\chi := (\mu_A\otimes\mu_C)\circ(A\otimes\chi\otimes C).$$

This algebra will be denoted by $A\otimes_\chi C$. Furthermore, these constructions are inverse one to each other.

Definition 1.1. Let $\chi\colon C\otimes A\longrightarrow A\otimes C$ and $\chi'\colon C'\otimes A'\longrightarrow A'\otimes C'$ be twisting maps. A *morphism* $F_{gh}\colon\chi\to\chi'$, from χ to χ', is a pair (g,h) of algebra maps $g\colon C\to C'$ and $h\colon A\to A'$ such that $\chi'\circ(g\otimes h)=(h\otimes g)\circ\chi$.

Remark 1.2. Let χ and χ' be as above. If $F_{gh}\colon\chi\to\chi'$ is a morphism of twisting maps, then the map $h\otimes g\colon A\otimes_\chi C\longrightarrow A'\otimes_{\chi'} C'$ is a morphism of algebras. Moreover this correspondence is functorial in an evident sense.

Remark 1.3. Let $h\colon A\to A'$ and $g\colon C\to C'$ be isomorphisms of algebras. If

$$\chi'\colon C'\otimes A'\longrightarrow A'\otimes C'$$

is a twisting map, then $\chi:=(h^{-1}\otimes g^{-1})\circ\chi'\circ(g\otimes h)$ is also. Moreover $F_{gh}\colon\chi\to\chi'$ is an isomorphism.

Proposition 1.4. *Let $\chi\colon(B\times C)\otimes A\longrightarrow A\otimes(B\times C)$ be a twisting map. Denote by ι_B, ι_C, p_B, p_C be the evident inclusions and projections. The map $\chi_B\colon B\otimes A\longrightarrow A\otimes B$, defined by*

$$\chi_B := (A\otimes p_B)\circ\chi\circ(\iota_B\otimes A),$$

is a twisting map if and only if $(A\otimes p_B)\circ\chi\circ(\iota_C\otimes A)=0$. Moreover in this case F_{p_B,id_A} is a morphism of twisting maps from χ to χ_B. We say that $p_B(\chi):=\chi_B$ is the twisting map of B with A induced by χ *and that χ is an* extension *of χ_B.*

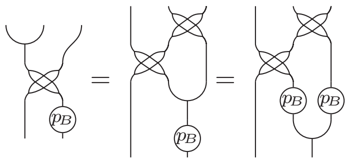
Proof. Since χ is a twisting map

$$\begin{aligned}\chi\big((1_B,0)\otimes a\big)&=\chi\big((1_B,1_C)\otimes a\big)-\chi\big((0,1_C)\otimes a\big)\\&=a\otimes(1_B,1_C)-\chi\big((0,1_C)\otimes a\big)\\&=a\otimes(1_B,1_C)+a\otimes(0,1_C)-\chi\big((0,1_C)\otimes a\big).\end{aligned}$$

Consequently, if χ_B is also a twisting map, then

$$a\otimes 1_B=\chi_B(1_B\otimes a)=a\otimes 1_B-(A\otimes p_B)\circ\chi\big((0,1_C)\otimes a\big),$$

or, equivalently, $(A\otimes p_B)\circ\chi\big((0,1n_C)\otimes a\big)=0$. Evaluating now the equalities



in $(0,1_C)\otimes(0,c)\otimes a$ for all $c\in C$ and $a\in A$, we conclude that $(A\otimes p_B)\circ\chi\circ(\iota_C\otimes A)=0$. We leave to the reader the task to check the other assertions. $\square$

1.2 Cross product

We recall that the *cross product* is the $(n-1)$-ary operation

$$(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}) \mapsto \mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1}$$

on K^n, determined by

$$(\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1})\mathbf{x}^{\mathrm{T}} = \det \begin{pmatrix} \mathbf{x} \\ \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_{n-1} \end{pmatrix}$$

for all $\mathbf{x} \in K^n$. From this definition it follows immediately that $\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1}$ is orthogonal to the subspace $\langle \mathbf{v}_1, \dots, \mathbf{v}_{n-1} \rangle$ generated by $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$, and that $\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1} = 0$ if $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$ are not linearly independent. It is well known (and very easy to check) that

$$\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1} = \det \begin{pmatrix} e_1 & \dots & e_n \\ v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{n-1,1} & \cdots & v_{n-1,n} \end{pmatrix},$$

where $\{e_1, \dots, e_n\}$ is the standard basis of K^n, $\mathbf{v}_i = (v_{i1}, \dots, v_{in})$ and the determinant is computed by the Laplace expansion along the first row. From this it follows immediately that if X is the matrix with rows $\mathbf{x}_1, \dots, \mathbf{x}_n$ and columns $\mathbf{y}_1, \dots, \mathbf{y}_n$, then

$$(\mathbf{y}_1^{\mathrm{T}} \times \cdots \times \widehat{\mathbf{y}_j^{\mathrm{T}}} \times \cdots \times \mathbf{y}_n^{\mathrm{T}}) \centerdot e_j = (\mathbf{x}_1 \times \cdots \times \widehat{\mathbf{x}_j} \times \cdots \times \mathbf{x}_n) \centerdot e_j \qquad \text{for all } j. \tag{1.1}$$

Proposition 1.5. *If $\mathbf{x} \in K^n$ is invertible, then*

$$\mathbf{x} \centerdot (\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1}) = \mu_n(\mathbf{x})\, (\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_1) \times \cdots \times (\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_{n-1}),$$

for all $\mathbf{v}_1, \dots, \mathbf{v}_{n-1} \in K^n$, where $\mu_n(\mathbf{x}) = x_1 \cdots x_n$, as in the introduction.

Proof. This assertion is an immediate consequence of the fact that

$$\begin{aligned} \mathbf{y}\big(\mathbf{x} \centerdot (\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1})\big)^{\mathrm{T}} &= (\mathbf{x} \centerdot \mathbf{y})(\mathbf{v}_1 \times \cdots \times \mathbf{v}_{n-1})^{\mathrm{T}} \\ &= \det\big((\mathbf{x} \centerdot \mathbf{y})^{\mathrm{T}} \quad \mathbf{v}_1^{\mathrm{T}} \quad \cdots \quad \mathbf{v}_{n-1}^{\mathrm{T}}\big) \\ &= \tau(\mathbf{x}) \det\big(\mathbf{y}^{\mathrm{T}} \quad (\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_1)^{\mathrm{T}} \quad \cdots \quad (\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_{n-1})^{\mathrm{T}}\big) \\ &= \tau(\mathbf{x})\, \mathbf{y}\big((\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_1) \times \cdots \times (\mathbf{x}^{\centerdot} \centerdot \mathbf{v}_{n-1})\big)^{\mathrm{T}} \end{aligned}$$

for all $\mathbf{y} \in K^n$. $\square$

2 Twisted tensor products of K^n with K^m

Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a map and let $\{e_1, \dots, e_m\}$ and $\{f_1, \dots, f_n\}$ be the canonical bases of K^m and K^n, respectively. There exist unique scalars λ_{ij}^{kl}, such that

$$\chi(e_i \otimes f_j) = \sum_{k,l} \lambda_{ij}^{kl} f_k \otimes e_l \qquad \text{for all } e_i \text{ and } f_j. \tag{2.2}$$

Given such a map χ, for all $i, l \in \mathbb{N}_m^*$ and $j, k \in \mathbb{N}_n^*$, we let $A(i,l) \in M_n(K)$ and $B(j,k) \in M_m(K)$ denote the matrices defined by

$$A(i,l)_{kj} \coloneqq \lambda_{ij}^{kl} \eqqcolon B(j,k)_{li}. \tag{2.3}$$

If necessary we will specify these map with a subscript, writing $A_\chi(i,l)$ and $B_\chi(j,k)$. Moreover, we let $\mathcal{A} = \mathcal{A}_\chi$ denote the family $(A(i,l))_{i,l\in\mathbb{N}_m^*}$ and $\mathcal{B} = \mathcal{B}_\chi$ denote the family $(B(j,k))_{j,k\in\mathbb{N}_n^*}$.

Notation 2.1. For each $i,l \in \mathbb{N}_m^*$ we set $J_i(l) \coloneqq \{j \in \mathbb{N}_n^* : A(i,l)_{jj} = 1\}$. If there is no danger of confusion (as is the case, for example, when we work with the matrices $A(1,l),\dots,A(m,l)$ of a fixed column of $\mathcal{A}$), we write J_i instead of $J_i(l)$. Similarly, for each $i,l \in \mathbb{N}_n^*$ we set $\tilde{J}_u(k) \coloneqq \{i \in \mathbb{N}_m^* : B_\chi(u,k)_{ii} = 1\}$, and we write $\tilde{J}_u$ instead of $\tilde{J}_u(k)$ whenever there is no danger of confusion.

Remark 2.2. Let $\tilde{\chi} \coloneqq \tau_{mn} \circ \chi \circ \tau_{nm}$ An immediate computation shows that

$$A_{\tilde{\chi}}(i,l)_{kj} = B_\chi(i,l)_{kj} \qquad \text{and} \qquad B_{\tilde{\chi}}(j,k)_{li} = A_\chi(j,k)_{li}$$

for each map $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$. Moreover $\tilde{\chi}$ is a twisting map if and only if χ is. In this case we say that χ and $\tilde{\chi}$ are *dual* of each other.

Proposition 2.3. *The map χ is a twisting map if and only if the following facts hold:*

1. $\delta_{ii'}A(i,l) = A(i,l)A(i',l)$ *for all i, i' and l,*
2. $\delta_{jj'}B(j,k) = B(j,k)B(j',k)$ *for all j, j' and k,*
3. $A(i,l)\mathbb{1} = \delta_{il}\mathbb{1}$ *for all i and l,*
4. $B(j,k)\mathbb{1} = \delta_{jk}\mathbb{1}$ *for all j and k.*

Proof. A direct computation shows that

$$\chi \circ (\mu_{K^m} \otimes K^n) = (K^n \otimes \mu_{K^m}) \circ (\chi \otimes K^m) \circ (K^m \otimes \chi)$$

if and only if

$$\delta_{ii'}\lambda_{ij}^{kl} = \sum_{u=1}^{n} \lambda_{iu}^{kl}\lambda_{i'j}^{ul} \quad \text{for all } i,i',j,k,l,$$

which is equivalent to condition (1), and that

$$\chi \circ (K^m \otimes \mu_{K^n}) = (\mu_{K^n} \otimes K^m) \circ (K^n \otimes \chi) \circ (\chi \otimes K^n)$$

if and only if

$$\delta_{jj'}\lambda_{ij}^{kl} = \sum_{u=1}^{m} \lambda_{ij}^{ku}\lambda_{uj'}^{kl} \quad \text{for all } i,j,j',k,l,$$

which is equivalent to condition (2). Finally it is easy to check that

$$\chi \circ (K^m \otimes \eta_{K^n}) = \eta_{K^n} \otimes K^m \qquad \text{and} \qquad \chi \circ (\eta_{K^m} \otimes K^n) = K^n \otimes \eta_{K^m}$$

if and only if conditions (3) and (4) are fulfilled. □

Remark 2.4. Statement (1) says that for each $l \in \mathbb{N}_m^*$, the matrices $A(1,l),\dots,A(m,l)$ are a family of orthogonal idempotents, and Statement (2) says that for each $k \in \mathbb{N}_n^*$, the matrices $B(1,k),\dots,B(n,k)$ are also a family of orthogonal idempotents. Statement (1) implies that Statement (3) holds if and only if $\mathbb{1}_{K^n}$ belongs to the image of $A(i,i)$ for all i. Similarly, if Statement (2) is fulfilled, then Statement (4) is true if and only if $\mathbb{1}_{K^m} \in \operatorname{Im} B(j,j)$ for all j.

Corollary 2.5. *The map χ is a twisting map if and only if the following conditions are fulfilled:*

1. $\delta_{ii'}A(i,l) = A(i,l)A(i',l)$ *for all i, i' and l,*
2. $A(i,l)\mathbb{1} = \delta_{il}\mathbb{1}$ *for all i and l,*
3. $\sum_{i=1}^{m} A(i,l) = \mathrm{id}$ *for all l,*
4. $\sum_{h=1}^{m} A(i,h)_{kj}A(h,l)_{kj'} = \delta_{jj'}A(i,l)_{kj}$ *for all i, j, j', k and l.*

Proof. Conditions (1) and (2) are conditions (1) and (3) of Proposition 2.3. Since by (2.3),

$$\sum_{i=1}^{m} A(l,i)_{kj} = \sum_{i=1}^{m} B(j,k)_{li},$$

condition (3) is equivalent to condition (4) of that proposition, and since, again by (2.3),

$$\sum_{u=1}^{m} A(u,l)_{kj} A(i,u)_{kj'} = \sum_{u=1}^{m} B(j,k)_{lu} B(j',k)_{ui},$$

condition (4) is equivalent to condition (2) of the same proposition. □

Remark 2.6. By Remark 2.2 and the fact that χ is a twisting map if and only if $\tilde{\chi}$ is, there is a similar corollary with the matrices $A(i,l)$ replaced by the matrices $B(j,k)$.

Remark 2.7. Corollary 2.5(4) says in particular that the vector $\big(A(i,1)_{kj},\dots,A(i,m)_{kj}\big)$ is orthogonal to the vector $\big(A(1,l)_{kj'},\dots,A(m,l)_{kj'}\big)$ for each i, j, j', k and l with $j \ne j'$.

Remark 2.8. Let $X_1,\dots,X_k \in M_n(K)$ such that $\sum_{j=1}^{k} X_j = \mathrm{id}_n$. A straightforward computation shows that if $\sum_{j=1}^{k} \mathrm{rk}(X_j) \le n$, then the X_i's are orthogonal idempotents, which means that $X_i X_j = \delta_{ij} X_i$ for all i and j.

Remark 2.9. Let $X_1,\dots,X_k \in M_n(K)$ be idempotent matrices such that $\sum_{i=1}^{k} X_i = \mathrm{id}_n$. Then the X_i's are orthogonal idempotents. In fact, since $\mathrm{rk}(X_i) = \mathrm{Tr}(X_i)$ and

$$\sum_i \mathrm{Tr}(X_i) = \mathrm{Tr}\left(\sum_i X_i\right) = \mathrm{Tr}(\mathrm{id}) = n,$$

this follows from the Remark 2.8.

Remark 2.10. Fix $k \in \mathbb{N}_n^*$ and assume that $\sum_j A(i,l)_{kj} = \delta_{il}$ for all i and l (which is Corollary 2.5 (2) for this k). If the equality in Corollary 2.5(4) holds for all i, l and $j = j'$, then it holds for all i, j, j' and l. In fact, the assumptions guarantee that $B(j,k)$ is idempotent for each $j \in \mathbb{N}_n^*$ and that $\sum_{j=1}^{n} B(j,k) = \mathrm{id}$. So, by Remark 2.9, the family of idempotent matrices $(B(j,k))_{j\in\mathbb{N}_n^*}$ is orthogonal, which is equivalent to Corollary 2.5(4) for this fixed k.

Definition 2.11. The matrices $\Gamma_\chi \in M_m(K)$, of $\mathcal{A}$-ranks, and $\tilde{\Gamma}_\chi \in M_n(K)$, of $\mathcal{B}$-ranks, are defined by

$$\Gamma_\chi \coloneqq \begin{pmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & \ddots & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{pmatrix} \quad\text{and}\quad \tilde{\Gamma}_\chi \coloneqq \begin{pmatrix} \tilde{\gamma}_{11} & \cdots & \tilde{\gamma}_{1n} \\ \vdots & \ddots & \vdots \\ \tilde{\gamma}_{n1} & \cdots & \tilde{\gamma}_{nn} \end{pmatrix},$$

where $\gamma_{il} \coloneqq \mathrm{rk}(A(i,l))$ and $\tilde{\gamma}_{jk} \coloneqq \mathrm{rk}(B(j,k))$.

Corollary 2.12. *If χ is a twisting map, then the rank matrices have the following properties:*

1. $\delta_{il} \le \gamma_{il} \le n$ *for all i and l.*
2. $\sum_{i=1}^{m} \gamma_{il} = n$ *for all l.*
3. $\delta_{jk} \le \tilde{\gamma}_{jk} \le m$ *for all j and k.*
4. $\sum_{j=1}^{n} \tilde{\gamma}_{jk} = m$ *for all k.*

Proof. Items (1) and (2) follow from Corollary 2.5 and (3) and (4) from the corresponding properties of the $B(j,k)$'s. □

2.1 Isomorphisms of twisting maps

Proposition 2.13. *Two twisting maps $\chi, \chi'\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ are isomorphic if and only if there exists $\sigma \in S_m$ and $\varsigma \in S_n$ such that*

$$A_{\chi'}(i,l)_{kj} = A_\chi(\sigma(i),\sigma(l))_{\varsigma(k)\varsigma(j)}$$

or, equivalently,

$$B_{\chi'}(j,k)_{li} = B_\chi(\varsigma(j),\varsigma(k))_{\sigma(l)\sigma(i)}.$$

Proof. By definition χ and χ' are isomorphic if and only if there are algebra automorphisms $g\colon K^m \to K^m$ and $h\colon K^n \to K^n$ such that $\chi' = (h^{-1}\otimes g^{-1})\circ\chi\circ(g\otimes h)$. Since the automorphisms of K^n and K^m are given by permutation of the entries, there exist $\varsigma \in S_n$ and $\sigma \in S_m$ such that $g(e_i) = e_{\sigma(i)}$ and $h(f_j) = f_{\varsigma(j)}$ for all $i \in \mathbb{N}_m^*$ and $j \in \mathbb{N}_n^*$, and so

$$\begin{aligned}\chi'(e_i \otimes f_j) &= (h^{-1}\otimes g^{-1})\chi(e_{\sigma(i)} \otimes f_{\varsigma(j)})\\ &= \sum_{k,l} \lambda_{\sigma(i)\varsigma(j)}^{\varsigma(k)\sigma(l)} (h^{-1}\otimes g^{-1})(f_{\varsigma(k)} \otimes e_{\sigma(l)})\\ &= \sum_{k,l} \lambda_{\sigma(i)\varsigma(j)}^{\varsigma(k)\sigma(l)} f_k \otimes e_l.\end{aligned}$$

Now the result follows immediately from (2.2) and (2.3). □

2.2 Representations in matrix algebras

In this subsection $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ denotes a twisting map and λ_{ij}^{kl}, $A(i,l)$ and $B(j,k)$ are as at the beginning of Section 2.

Proposition 2.14. *For each $1 \le u \le m$ the formulas*

$$\rho_u(f_j \otimes 1) := E^{jj} \quad \textit{and} \quad \rho_u(1 \otimes e_i) := A(i,u)$$

define a representation $\rho_u\colon K^n \otimes_\chi K^m \longrightarrow M_n(K)$. Similarly, for each $1 \le v \le n$ the formulas

$$\tilde{\rho}_v(1 \otimes e_i) := E^{ii} \quad \textit{and} \quad \tilde{\rho}_v(f_j \otimes 1) := B(j,v)$$

define a representation $\tilde{\rho}_v\colon K^n \otimes_\chi K^m \longrightarrow M_m(K)$.

Proof. Clearly the restriction of ρ_u to $K^n \otimes K \cdot \mathbb{1}$ is a morphism of algebras. Moreover, by items (1) and (3) of Corollary 2.5, the restriction of ρ_u to $K \cdot \mathbb{1} \otimes K^m$ is also a morphism of algebras. So, in order to prove that ρ_u defines a representation, it only remains to check that

$$\rho_u\big((1 \otimes e_i)(f_j \otimes 1)\big) = \rho_u(1 \otimes e_i)\rho_u(f_j \otimes 1).$$

But this is true, since, on one hand,

$$(1 \otimes e_i)(f_j \otimes 1) = \sum_{k,l} \lambda_{ij}^{kl} f_k \otimes e_l,$$

and, on the other hand,

$$A(i,u)E^{jj} = \sum_{k,l} \lambda_{ij}^{kl} E^{kk} A(l,u),$$

because

$$\begin{aligned}\sum_{k,l} \lambda_{ij}^{kl} E^{kk} A(l,u) &= \sum_{k,l,s} \lambda_{ij}^{kl} A(l,u)_{ks} E^{ks} \\ &= \sum_{k,l,s} A(i,l)_{kj} A(l,u)_{ks} E^{ks} \\ &= \sum_{k} A(i,u)_{kj} E^{kj} \\ &= A(i,u) E^{jj},\end{aligned}$$

where the first and the last equality are straightforward, the second equality is true by (2.3) and the third one by Corollary 2.5(4). The proof for $\tilde{\rho}_v$ is similar. □

Remark 2.15. We can give a complete description of the image of ρ_u and $\tilde{\rho}_v$. For this, note that if $A(i,u)_{kj} \neq 0$ for some i, j and k, then $E^{kj} \in \operatorname{Im}(\rho_u)$. In fact,

$$E^{kj} A(i,u)_{kj} = E^{kk} A(i,u) E^{jj} = \rho_u((f_k \otimes 1)(1 \otimes e_i)(f_j \otimes 1)).$$

Hence

$$E^{kj} = \rho_u \left(\frac{(f_k \otimes 1)(1 \otimes e_i)(f_j \otimes 1)}{A(i,u)_{kj}} \right).$$

so, the image of ρ_u is the matrix incidence algebra of the preorder on $\{1, \dots, n\}$ given by $k \leq j$ if and only if $E^{kj} \in \operatorname{Im}(\rho_u)$. In particular, if for all k, j there exists a i with $A(i,u)_{kj} \neq 0$, then ρ_u is surjective. Similarly, the image of $\tilde{\rho}_v$ is the matrix incidence algebra of the preorder on $\{1, \dots, m\}$ given by $l \leq i$ if and only if $E^{li} \in \operatorname{Im}(\tilde{\rho}_u)$.

Remark 2.16. Set $x_{ji} \coloneqq f_j \otimes e_i$. A straightforward computation shows that in $K^n \otimes_\chi K^m$

$$x_{ki} x_{jl} = \lambda_{ij}^{kl} x_{kl} = A(i,l)_{kj} x_{kl} = B(j,k)_{li} x_{kl}.$$

We also can prove that all two sided ideals of the algebra $K^n \otimes_\chi K^m$ are generated by monomials. In fact, let I be an ideal and let $x = \sum_{r,s} \alpha_{rs} x_{rs}$. Then

$$(f_j \otimes 1) \left(\sum_{r,s} \alpha_{rs} x_{rs} \right) (1 \otimes e_i) = \sum_{r,s} \alpha_{rs} (f_j \otimes 1)(f_r \otimes 1)(1 \otimes e_s)(1 \otimes e_i) = \alpha_{ji} x_{ji},$$

and so, if $\alpha_{ji} \neq 0$ for some element $x \in I$, then $x_{ji} \in I$. This shows that the ideal I is linearly generated by a set of elements x_{ji}.

3 Twisting maps of K^m with K^2

The proofs given in this section could be lightly simplified using some of the results given in Section 4, but we prefer to use the least machinery possible in order to give a flavour of how our methods work, reproducing the beautiful result of Cibils in [6]. Therefore we restrict ourselves to use the results established in the previous sections and the following remark:

Remark 3.1. Let $A \in M_2(K)$ be such that $A^2 = A$, $A\mathbb{1} = \mathbb{1}$ and $\operatorname{rk}(A) = 1$. There exists $a \in K$ such that

$$A = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}.$$

The twisting maps of K^m with K^2 have been classified completely by Cibils, who shows that they correspond to colored quivers $Q_{f,\delta}$. The first step is to describe the quiver Q_f. We can obtain this quiver directly from our $\mathcal{A}$-rank matrix. Given an algebra map $f\colon C \to C$, where $C := K^m$, we let ${}^fC$ denote that C-bimodule structure on C given by $c \cdot c' \cdot c'' := f(c)c'c''$. Let $(e_i)_{i\in\mathbb{N}_m^*}$ be the canonical basis of C.

Consider a map

$$\chi\colon C \otimes \frac{K[X]}{\langle X(1-X)\rangle} \longrightarrow \frac{K[X]}{\langle X(1-X)\rangle} \otimes C.$$

In [6, Section 3] it was proved that χ is a twisting map if and only if there exists an algebra morphism $f\colon C \to C$ and an idempotent derivation $\delta\colon C \to {}^fC$, satisfying $f = f^2 + \delta f + f\delta$, such that

$$\chi(e_i \otimes X) = X \otimes f(e_i) + 1 \otimes \delta(e_i) = X \otimes (f+\delta)(e_i) + (1-X) \otimes \delta(e_i).$$

With our notations, we have

$$\chi(e_i \otimes f_1) = \sum_l \bigl(\lambda_{i1}^{1l} f_1 \otimes e_l + \lambda_{i1}^{2l} f_2 \otimes e_l\bigr) = \sum_l \bigl(A(i,l)_{11} f_1 \otimes e_l + A(i,l)_{21} f_2 \otimes e_l\bigr),$$

where f_1 is the class of X in $k[X]/\langle X(1-X)\rangle$ and f_2 is the class of $1-X$ in $k[X]/\langle X(1-X)\rangle$. Hence

$$f(e_i) = \sum_l \bigl(A(i,l)_{11} - A(i,l)_{21}\bigr)e_l \quad \text{and} \quad \delta(e_i) = \sum_l A(i,l)_{21} e_l. \tag{3.4}$$

The quiver Q_f in [6] is constructed in the following way. Since f is an algebra map, there exists a unique set map $\varphi\colon \mathbb{N}_m^* \to \mathbb{N}_m^*$, such that

$$f(e_l) = \sum_{\{i:\varphi(i)=l\}} e_i. \tag{3.5}$$

By definition, the quiver Q_f of f has set of vertices $\mathbb{N}_m^*$ and an arrow from i to $\varphi(i)$ for each $i \in \mathbb{N}_m^*$.

Proposition 3.2. *Let χ be a twisting map and let f be as above. The adjacency matrix of the quiver Q_f is $M(\chi) := (\Gamma_\chi - \mathrm{id})^T$, where Γ_χ is as in Definition 2.11.*

Proof. Let $l \in \mathbb{N}_m^*$. By Corollary 2.12 we know that $\mathrm{rk}(A(l,l)) = 2$ and $A(i,l) = 0$ for all $i \neq l$, or $\mathrm{rk}(A(l,l)) = 1$ and there exists a unique $i \neq l$ such that $\mathrm{rk}(A(i,l) = 1$ and $A(j,l) = 0$ for all $j \notin \{i,l\}$. Thus, if $\mathrm{rk}(A(l,l)) = 2$ then $A(l,l) = \mathrm{id}$, and so $A(l,l)_{11} - A(l,l)_{21} = 1$. On the other hand if $\mathrm{rk}(A(l,l)) = 1$, then by Proposition 2.3 and Remark 3.1 there exists $a_l \in K$ such that $A(l,l) = \left(\begin{smallmatrix} a_l & 1-a_l \\ a_l & 1-a_l \end{smallmatrix}\right)$, and hence $A(l,l)_{11} - A(l,l)_{21} = 0$. Moreover, since $A(i,l) + A(l,l) = \mathrm{id}$, we have $A(i,l) = \left(\begin{smallmatrix} 1-a_l & a_l-1 \\ -a_l & a_l \end{smallmatrix}\right)$, and so $A(i,l)_{11} - A(i,l)_{21} = 1$. Finally, if $\mathrm{rk}(A(j,l)) = 0$, then (of course) $A(j,l)_{11} - A(j,l)_{21} = 0$. Consequently, by the first equality in (3.4) and equality (3.5),

$$M(\chi)_{il} = \begin{cases} 1 & \text{if } \varphi(i) = l, \\ 0 & \text{otherwise,} \end{cases}$$

which finishes the proof. □

Corollary 3.3. *A vertex i of Q_f is a loop vertex if and only if* $\mathrm{rk}(A(i,i)) = 2$.

In the rest of this section, for each $i \in \mathbb{N}_m^*$ we let a_i denote $A(i,i)_{11}$. We want to determine the possible matrices $A(i,l)$ which can occur in a twisting map of K^m with K^2:

(1) If $\mathrm{rk}(A(l,l)) = 2$, then $A(l,l) = \mathrm{id}$ and $A(i,l) = 0$ for all $i \neq l$.

(2) If $\operatorname{rk}(A(l,l)) = 1$, then there exists $i \neq l$ such that
$$A(l,l) = \begin{pmatrix} a_l & 1-a_l \\ a_l & 1-a_l \end{pmatrix}, \quad A(i,l) = \begin{pmatrix} 1-a_l & a_l-1 \\ -a_l & a_l \end{pmatrix} \quad \text{and} \quad A(h,l) = 0 \text{ for } h \notin \{i,l\}.$$
Now we have several possibilities:
- If $\operatorname{rk}(A(i,i)) = 2$, then $A(l,i) = 0$, and so, by (2.3) and Proposition 2.3(2),
$$a_l - a_l^2 = B(1,1)_{ll} - (B(1,1)^2)_{ll} = 0, \tag{3.6}$$
which implies that $a_l \in \{0,1\}$.
- If $\operatorname{rk}(A(i,i)) = 1$, then we have $A(i,i) = \left(\begin{smallmatrix} a_i & 1-a_i \\ a_i & 1-a_i \end{smallmatrix}\right)$, and, again by (2.3) and Proposition 2.3(2),
$$(1-a_l)(1-a_i-a_l) = B(1,1)_{li} - (B(1,1)^2)_{li} = 0 \tag{3.7}$$
and
$$a_l(a_i+a_l-1) = B(2,2)_{li} - (B(2,2)^2)_{li} = 0. \tag{3.8}$$
Hence $a_i + a_l = 1$. If $A(l,i) \neq 0$, then we do not obtain additional conditions on a_l, while if $A(l,i) = 0$, then, by (3.6), we have $a_l \in \{0,1\}$, and so there are only two cases: $a_l = 0$ and $a_i = 1$ or $a_l = 1$ and $a_i = 0$.

Next we recall the definition of a coloration on Q_f in [6, Definition 3.12], but we take the opposite coloration.

Definition 3.4. A coloration of Q_f is an element $c = \sum_i c_i e_i \in C$ such that:

(1) For a connected component reduced to the round trip quiver with vertices i and j the coefficients c_i and c_j satisfy $c_i + c_j = 1$.
(2) For other connected components:
   (a) In case i is a non loop vertex $c_i \in \{0,1\}$.
   (b) For each arrow having no loop vertex target, one extremity value is 0 and the other is 1.
   (c) At a loop vertex i we have $c_i = 0$.

Given a twisting map $\chi\colon K^m \otimes K^2 \longrightarrow K^2 \otimes K^m$ consider the matrices $A(i,l) \coloneqq A_\chi(i,l)$. By Proposition 3.2 and the discussion above Definition 3.4, the element $c \coloneqq (c_1,\dots,c_m) \in C$ given by $c_l \coloneqq A(l,l)_{21}$ is a coloration. Conversely, given a coloration $c = (c_1,\dots,c_m) \in C$ on a one-valued quiver Q_f with set of vertices $\mathbb{N}_m^*$, we can construct matrices $A(i,l) \in M_2(K)$ in the following way: if l is a loop vertex, then $A(l,l) \coloneqq \mathrm{id}$ and $A(i,l) \coloneqq 0$ for $i \neq l$. Otherwise

- we set $A(l,l) \coloneqq \left(\begin{smallmatrix} a_l & 1-a_l \\ a_l & 1-a_l \end{smallmatrix}\right)$, where $a_l \coloneqq c_l$,
- for the target $t(l)$ of the arrow starting at l, we set $A(t(l),l) \coloneqq \left(\begin{smallmatrix} 1-a_l & a_l-1 \\ -a_l & a_l \end{smallmatrix}\right)$,
- for all $i \notin \{t(l),l\}$, we set $A(h,l) \coloneqq 0$.

In order to verify that these matrices define a twisting map, we must check the conditions of Proposition 2.3, where the matrices $B(j,k)$ are defined by (2.3). Conditions (1) and (3) are satisfied by construction. Condition (2) is equivalent to
$$\sum_i A(i,l)_{kj} = \delta_{jk} \qquad \text{for all } l,\ j \text{ and } k,$$
which holds, because
$$\sum_i A(i,l)_{kj} = \begin{cases} A(l,l)_{kj} = \delta_{jk}, & \text{if } \operatorname{rk}(A(l,l)) = 2 \\ A(l,l)_{kj} + A(t(l),l)_{kj} = \delta_{jk}, & \text{if } \operatorname{rk}(A(l,l)) = 1. \end{cases}$$

Finally we check condition (4), which is equivalent to

$$\delta_{jj'}A(i,l)_{kj} = \sum_u A(i,u)_{kj'}A(u,l)_{kj} \qquad \text{for all } i,\ j\ ,\ j',\ k \text{ and } l. \tag{3.9}$$

When $t(l) = l$, then $A(u,l) = \delta_{ul}\,\mathrm{id}$ for all u, which implies that equality (3.9) holds. Assume that $t(l) \neq l$. We consider three cases: $i = l$, $i = t(l)$ and $i \notin \{l, t(l)\}$. If $i = l$, then equality (3.9) reads

$$\delta_{jj'}A(l,l)_{kj} = A(l,l)_{kj'}A(l,l)_{kj} + A(l,t(l))_{kj'}A(t(l),l)_{kj} \qquad \text{for all } j,\ j' \text{ and } k;$$

if $i = t(l)$, then equality (3.9) reads

$$\delta_{jj'}A(t(l),l)_{kj} = A(t(l),l)_{kj'}A(l,l)_{kj} + A(t(l),t(l))_{kj'}A(t(l),l)_{kj} \qquad \text{for all } j,\ j' \text{ and } k;$$

and finally, if $i \notin \{l, t(l)\}$, then equality (3.9) reads

$$0 = A(i,t(l))_{kj'}A(t(l),l)_{kj} \qquad \text{for all } j,\ j' \text{ and } k.$$

All these equalities are easily verified using that, since $(c_1,\dots,c_m)$ is a coloration,

$$A(l,l) = \begin{pmatrix} a_l & 1-a_l \\ a_l & 1-a_l \end{pmatrix}, \qquad A(t(l),l) = \begin{pmatrix} 1-a_l & a_l-1 \\ -a_l & a_l \end{pmatrix},$$

and that

- if $t(t(l)) = t(l)$, then $A(t(l),t(l)) = \mathrm{id}$;
- if $t(t(l)) \neq t(l)$, then $A(t(l),t(l)) = \left(\begin{smallmatrix} 1-a_l & a_l \\ 1-a_l & a_l \end{smallmatrix}\right)$;
- if $t(t(l)) = l$, then $A(l,t(l)) = \left(\begin{smallmatrix} a_l & -a_l \\ a_l-1 & 1-a_l \end{smallmatrix}\right)$ and $A(u,t(l)) = 0$ for all $u \notin \{l, t(l)\}$;
- if $t(t(l)) \neq l$, then $a_l \in \{0,1\}$, $A(t(t(l)),t(l)) = \left(\begin{smallmatrix} a_l & -a_l \\ a_l-1 & 1-a_l \end{smallmatrix}\right)$ and $A(u,t(l)) = 0$ for all $u \notin \{t(l), t(t(l))\}$.

4 Miscellaneous results

Throughout this section $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ denotes a map and λ_{ij}^{kl}, $A(i,l)$ and $B(j,k)$ are as at the beginning of Section 2. We also assume that $A(i,l)$ and $B(j,k)$ are idempotent matrices for all $i, l \in \mathbb{N}_m^*$ and $j, k \in \mathbb{N}_n^*$. The following results are useful in our quest of classifying the twisted tensor products $K^n \otimes_\chi K^m$.

4.1 General properties

Remark 4.1. The rank matrices Γ_χ and $\tilde{\Gamma}_\chi$, introduced in Definition 2.11, have the same trace. In fact,

$$\mathrm{Tr}(\Gamma_\chi) = \sum_i \mathrm{rk}(A(i,i)) = \sum_{i,j} \lambda_{ij}^{ji} = \sum_j \mathrm{rk}(B(j,j)) = \mathrm{Tr}(\tilde{\Gamma}_\chi).$$

Remark 4.2. Since the matrices $A(i,l)$ are idempotent, we know that $\mathrm{rk}(A(i,l)) = \mathrm{Tr}(A(i,l))$. Consequently,

$$\mathrm{rk}(A(i,l)) = \sum_j A(i,l)_{jj} = \sum_j B(j,j)_{li}.$$

Similarly, $\mathrm{rk}(B(j,k)) = \sum_i A(i,i)_{kj}$.

4.2 Standard idempotent $0,1$-matrices

Definition 4.3. A $0,1$-matrix $A \in M_n(K)$ is called a *standard idempotent* $0,1$-*matrix* if there exist $r \in \mathbb{N}_n^*$ and a matrix $C \in M_{n-r\times r}(K)$ that has exactly one non-zero entry in each row, such that

$$A = \begin{pmatrix} \mathrm{id}_r & 0 \\ C & 0 \end{pmatrix}, \tag{4.10}$$

where id_r is the identity of $M_r(K)$.

Definition 4.4. Two matrices $A, A' \in M_n(K)$ are *equivalent via identical permutations in rows and columns* if there exists a permutation $\sigma \in S_n$ such that $A_{\sigma(k)\sigma(j)} = A'_{kj}$ for all k, j.

Remark 4.5. A matrix $A \in M_n(K)$ is equivalent via identical permutations in rows and columns to a standard idempotent $0,1$-matrix if and only if it is a $0,1$-matrix with exactly one nonzero entry in every row, that satisfies the following condition: for each j, if $A_{jj} = 0$, then $A_{kj} = 0$ for all k.

Notation 4.6. Let $A \in M_n(K)$ be a $0,1$-matrix such that $A\mathbb{1} = \mathbb{1}$. For each k such that $A_{kk} = 0$, we let $c_k = c_k(A)$ denote the unique index such that A_{kc_k} is non-zero.

Proposition 4.7. *Let $A \in M_n(K)$ be a $0,1$-matrix. If A is idempotent and $A\mathbb{1} = \mathbb{1}$, then A is equivalent via identical permutations in rows and columns to a standard idempotent $0,1$-matrix.*

Proof. Since $r \coloneqq \mathrm{rk}(A) = \mathrm{Tr}(A)$, we have r times the entry 1 and $n - r$ times the entry 0 on the diagonal of A. Applying an identical permutations in rows and columns we can assume that the 1's are in the first r entries. Since $A\mathbb{1} = \mathbb{1}$, each row of this matrix has only one 1, and the other entries are zero. Thus, the first r rows of A are as in (4.10). Now the fact that $\mathrm{rk}(A) = r$ implies that, again as in (4.10), the right lower block of A is the zero matrix and its left lower block is a matrix C that satisfies the required properties. □

Remark 4.8. By Proposition 4.7 we have $A_{c_kc_k} = 1$ for each k such that $A_{kk} = 0$.

Corollary 4.9. *Assume that χ is a twisting map. If $A(l,l)$ is a $0,1$-matrix, then $A(l,l)$ is equivalent via identical permutations in rows and columns to a standard idempotent $0,1$-matrix.*

Proposition 4.10. *Assume that χ is a twisting map and let $l \in \mathbb{N}_m^*$. If*

$$\mathrm{rk}(A(i,l))\,\mathrm{rk}(A(l,i)) = 0 \quad \textit{for all } i \neq l,$$

then $A(l,l)$ is a $0,1$-matrix.

Proof. By Corollary 2.5(4) and the fact that $A(i,l)A(l,i) = 0$ for all $i \neq l$,

$$A(l,l)_{kj} = \sum_{i=1}^{m} A(l,i)_{kj} A(i,l)_{kj} = A(l,l)_{kj}^2.$$

So, $A(l,l)_{kj} \in \{0,1\}$ for all k, j. □

Corollary 4.11. *If χ is a twisting map and Γ_χ is upper or lower triangular, then each of the matrices $A(l,l)$ is a $0,1$-matrix.*

Remark 4.12. Proposition 4.10 and Corollaries 4.9 and 4.11 are valid for the matrices $B(j,j)$ (in the second corollary we replace Γ_χ by $\tilde{\Gamma}_\chi$).

4.3 Rank 1 idempotent matrices

Remark 4.13. At the beginning of Section 3 we noted that if $A \in M_2(K)$ satisfies $A^2 = A$, $A\mathbb{1} = \mathbb{1}$ and $\mathrm{rk}(A) = 1$, then there exists $a \in K$ such that

$$A = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}.$$

More generally, if $A \in M_n(K)$ such that $A^2 = A$, $A\mathbb{1} = \mathbb{1}$ and $\mathrm{rk}(A) = 1$, then there exists $a_1, \dots, a_n \in K$ with $\sum a_j = 1$, such that

$$A = \begin{pmatrix} a_1 & \dots & a_n \\ \vdots & \ddots & \vdots \\ a_1 & \dots & a_n \end{pmatrix}.$$

Proposition 4.14. *If $\mathrm{rk}(A(i,i)) = 1$ for some $i \in \mathbb{N}_m^*$, then there exists $j \in \mathbb{N}_n^*$ such that $\tilde{\Gamma}_{jk} \neq 0$ for all k. Moreover, if such j is unique, then $A(i,i)_{st} = \delta_{tj}$ for all s,t. A similar statement holds for $B(j,j)$ and Γ.*

Proof. Since $\mathrm{Tr}(A(i,i)) = \mathrm{rk}(A(i,i)) = 1$, there exists j such that $A(i,i)_{jj} \neq 0$. By Remark 4.13

$$B(j,k)_{ii} = A(i,i)_{kj} = A(i,i)_{jj} \neq 0, \quad \text{for all } k.$$

This implies that $B(j,k) \neq 0$ for all k, and so $\tilde{\Gamma}_{jk} \neq 0$ for all k. If j is unique, then for each $l \neq j$ there exists k such that $\tilde{\Gamma}_{lk} = 0$, and so, again by Remark 4.13, we have

$$A(i,i)_{hl} = A(i,i)_{kl} = B(l,k)_{ii} = 0 \quad \text{for all } h.$$

The argument for $B(j,j)$ and Γ is the same. □

4.4 Columns of 1's in Γ_χ

Proposition 4.15. *Assume that χ is a twisting map and that $n = m$. If $\mathrm{Diag}(\Gamma_\chi) = (1,1,\dots,1)$, then $\Gamma_\chi = \tilde{\Gamma}_\chi$ is the matrix $\mathfrak{J}_n$ whose entries are all 1.*

Proof. By Remark 4.1 and Proposition 2.12(3), we know that $\mathrm{Diag}(\tilde{\Gamma}_\chi) = (1,\dots,1)$. In other words, $\mathrm{rk}(B(j,j)) = 1$ for all j. Assume by contradiction that $\Gamma_\chi \neq \mathfrak{J}_n$. Then by items (1) and (3) of Corollary 2.5 there exist i,l such that $A(i,l) = 0$. Hence, by Remark 4.13 the i-th column of $B(j,j)$ is zero for all j. But then $\mathrm{Diag}(A(i,i)) = (0,\dots,0)$, which, since $A(i,i)$ is idempotent, implies that $A(i,i) = 0$, a contradiction. For $\tilde{\Gamma}_\chi$ proceed in a similar way. □

Proposition 4.16. *Let $l \in \mathbb{N}_m^*$. Assume that χ is a twisting map, that $\Gamma_\chi = \tilde{\Gamma}_\chi$ is the matrix $\mathfrak{J}_n$ whose entries are all 1's, and that there exists k such that $A(l,l)_{kj} \neq 0$ for all j. Let $\mathbf{v}_i = (v_{i1}, \dots, v_{in}) \in K^n \setminus \{0\}$. If $\mathbf{v}_i^\perp \in \mathrm{Im}(A(i,l))$, then $v_{ik} \neq 0$ for all k.*

Proof. Since $\mathrm{rk}(A(i,l)) = 1$ there exists $\mathbf{w}_i = (w_{i1}, \dots, w_{in}) \in K^n$ such that $A(i,l) = \mathbf{v}_i^{\mathrm{T}} \mathbf{w}_i$. Assume by contradiction that there exists k such that $v_{ik} = 0$. Then $A(i,l)_{kj} = v_{ik} w_{ij} = 0$ for all j. By (2.3) this means that $B(j,k)_{li} = 0$ for all j, and so

$$\det \begin{pmatrix} B(1,k)_{l1} & \dots & B(1,k)_{li} & \dots & B(1,k)_{ln} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ B(n,k)_{l1} & \dots & B(n,k)_{li} & \dots & B(n,k)_{ln} \end{pmatrix} = 0. \tag{4.11}$$

On the other hand, By Remark 2.6 we know that $(B(1,k), \dots, B(n,k))$ is a complete family of orthogonal idempotent matrices of rank 1. But then, also $(B(1,k)^{\mathrm{T}}, \dots, B(n,k)^{\mathrm{T}})$ is. Since $B(j,k)_{ll} = A(l,l)_{kj} \neq 0$ implies that the vector $(B(j,k)_{l1}, \dots, B(j,k)_{ln})$ generates $\mathrm{Im}(B(j,k)^{\mathrm{T}})$, the determinant of (4.11) cannot be zero, a contradiction which concludes the proof. □

Theorem 4.17. *Let* $\mathbf{v}_1, \dots, \mathbf{v}_n$ *be* n *invertible elements of* K^n *with* $\mathbf{v}_1 = 1_{K^n}$*, such that*

$$\det\begin{pmatrix}\mathbf{v}_1^{\mathrm{T}} & \dots & \mathbf{v}_n^{\mathrm{T}}\end{pmatrix} = 1.$$

There exists a unique twisting map $\xi\colon K^n \otimes K^n \longrightarrow K^n \otimes K^n$ *with*

$$A_\xi(i,l) \coloneqq (-1)^{i-1}(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_i)^{\mathrm{T}}(\mathbf{v}_l \cdot (\mathbf{v}_1 \times \cdots \times \widehat{\mathbf{v}_i} \times \cdots \times \mathbf{v}_n)) \quad \text{for all } i,l,$$

where, as usual, $\widehat{\mathbf{v}_i}$ *means that the term* $\mathbf{v}_i$ *is omitted. Moreover, the twisted tensor product algebra* $K^n \otimes_\xi K^n$ *is isomorphic to* $M_n(K)$.

Proof. We assert that the $A_\xi(i,j)$'s are idempotent matrices of rank 1 satisfying:

(1) $A_\xi(i,o)A_\xi(j,o) = \delta_{ij}A_\xi(i,o)$,
(2) $A_\xi(i,j)1_{K^n}^{\mathrm{T}} = \delta_{ij}1_{K^n}^{\mathrm{T}}$,
(3) $\sum_{i=1}^n A_\xi(i,o) = \mathrm{id}$.

In fact, since by Proposition 1.5

$$\mathbf{v}_l \cdot (\mathbf{v}_1 \times \cdots \times \widehat{\mathbf{v}_i} \times \cdots \times \mathbf{v}_n) = \tau(\mathbf{v}_l)\,(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_1) \times \cdots \times \widehat{(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_i)} \times \cdots \times (\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_n),$$

we have

$$\begin{aligned}
\big(\mathbf{v}_l \cdot (\mathbf{v}_1 \times \cdots \times \widehat{\mathbf{v}_i} \times \cdots \times \mathbf{v}_n)\big)(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_j)^{\mathrm{T}} &= \tau(\mathbf{v}_l)\det\begin{pmatrix}\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_j\\ \mathbf{v}_l^{\bullet} \cdot \mathbf{v}_1\\ \vdots\\ \mathbf{v}_l^{\bullet} \cdot \mathbf{v}_{i-1}\\ \mathbf{v}_l^{\bullet} \cdot \mathbf{v}_{i+1}\\ \vdots\\ \mathbf{v}_l^{\bullet} \cdot \mathbf{v}_n\end{pmatrix}\\
&= (-1)^{i-1}\tau(\mathbf{v}_l)\delta_{ij}\det\begin{pmatrix}\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_1\\ \vdots\\ \mathbf{v}_l^{\bullet} \cdot \mathbf{v}_n\end{pmatrix}\\
&= (-1)^{i-1}\delta_{ij}\det\begin{pmatrix}\mathbf{v}_1\\ \vdots\\ \mathbf{v}_n\end{pmatrix}\\
&= (-1)^{i-1}\delta_{ij}.
\end{aligned}$$

This implies that $A(i,l)$ is the idempotent with image $K(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_i)^{\mathrm{T}}$ and kernel $\langle(\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_j)^{\mathrm{T}} : j \neq i\rangle$, which implies items (1), (2) y (3) (for (2) use that $\mathbf{v}_l^{\bullet} \cdot \mathbf{v}_l = 1_{K^n}$).

Now we consider the vectors $\mathbf{w}_{\mathbf{u}}$ $(1 \le i \le n)$ determined by the equality

$$\begin{pmatrix}\mathbf{w}_1\\ \vdots\\ \mathbf{w}_n\end{pmatrix} \coloneqq \begin{pmatrix}\mathbf{v}_1^{\mathrm{T}} & \dots & \mathbf{v}_n^{\mathrm{T}}\end{pmatrix},$$

and we define the matrices

$$B_\xi(j,k) \coloneqq (-1)^{j-1}(\mathbf{w}_k^{\bullet} \cdot \mathbf{w}_j)^{\mathrm{T}}(\mathbf{w}_k \cdot (\mathbf{w}_1 \times \cdots \times \widehat{\mathbf{w}_j} \times \cdots \times \mathbf{w}_n))$$

One checks that $A_\xi(i,l)_{kj} = B_\xi(j,k)_{li}$. Moreover, arguing as above for the $A_\xi(i,j)$'s, it can be proven that

$$B_\xi(i,o)B_\xi(j,o) = \delta_{ij}B_\xi(i,o) \quad \text{for all } i,j,o.$$

From this it follows immediately that the matrices $A_\xi(i,l)$ satisfy condition (4) in Corollary 2.5, which finishes the proof of the existence of χ. The uniqueness is clear, so it remains to prove that $K^n \otimes_\xi K^n$ is isomorphic to $M_n(K)$. By Remark 2.15 for this it suffices to prove that for any l and all k, j there exists i such that $A(i,l)_{jk} \neq 0$, since then the representation ρ_l is a surjective morphism between two algebras of the same dimension, and hence is an isomorphism. So fix l, k, j. From $\sum_i A(i,l) = \mathrm{id}$ it follows that there exists i such that $A(i,l)_{kk} \neq 0$. But then

$$A(i,l)_{jk} = \frac{(v_i)_j}{(v_i)_k} A(i,l)_{kk} \neq 0,$$

as desired. □

Remark 4.18. The uniqueness part in Theorem 4.17 can be improved. If two twisting maps χ and $\check{\chi}$ with $\Gamma_\chi = \Gamma_{\check{\chi}} = \mathfrak{J}_n$ satisfy $A_\chi(i,l) = A_{\check{\chi}}(i,l)$ for a fixed l and all i, and all the entries of $A_\chi(i,l)$ are non null, then $\chi = \check{\chi}$. The proof is left to the reader (use (2.3), Proposition 4.15 and Remark 4.13).

5 Standard and quasi-standard columns

Definition 5.1. The *support* of a matrix $A \in M_n(K)$ is the set

$$\mathrm{Supp}(A) \coloneqq \{(i,j) \in \mathbb{N}_n^* \times \mathbb{N}_n^* : a_{ij} \neq 0\},$$

and the support of the k-th row of A is the set $\mathrm{Supp}(A_{k*}) \coloneqq \{j \in \mathbb{N} : a_{kj} \neq 0\}$.

Definition 5.2. A family $(A(i,l))_{i,l \in \mathbb{N}_m^*}$ of matrices $A(i,l) \in M_n(K)$, is called a *pre-twisting* of K^m with K^n if it satisfies conditions (1), (2) and (3) of Corollary 2.5.

Throughout this section $\mathcal{A} = (A(i,l))_{i,l \in \mathbb{N}_m^*}$ denotes a pre-twisting of K^m with K^n.

Definition 5.3. We say that the l_0-th column of $\mathcal{A}$ is a *standard column* if

(1) $A(l_0,l_0)$ is a $0,1$-matrix,
(2) $\mathrm{Supp}(A(i,l_0)) \subseteq \mathrm{Supp}(A(l_0,l_0)) \cup \mathrm{Supp}(\mathrm{id})$ for all i.

Remark 5.4. Assume that $(A(i,l_0))_{i \in \mathbb{N}_m^*}$ is a standard column of $\mathcal{A}$ and let $k \in \mathbb{N}_n^*$. The following facts hold:

(1) For each index i, we have $A(i,l_0)_{kk} \in \{0,1\}$.
(2) $A(i,l_0)_{kk} = 1$ for exactly one i. We let $i(k) = i(k,l_0)$ denote this index.
(3) If $i \neq i(k)$ and $i \neq l_0$, then $A(i,l_0)_{kj} = 0$ for all j.
(4) $A(i,l_0)_{kj} = -1$ if and only if $i = i(k) \neq l_0$ and $j = c_k(A(l_0,l_0))$. Moreover $A(i,l_0)_{kj'} = 0$ for all $j' \notin \{k, c_k(A(l_0,l_0))\}$.
(5) $A(i,l_0)_{kj} \in \{1,0,-1\}$ for all i, k, j, and $A(i,l_0)_{kj} = 1$ implies $i = l_0$ or $j = k$.

Remark 5.5. From Remark 5.4 it follows that each standard column $A(i,l_0)_{i \in \mathbb{N}_m^*}$ of a pre-twisting of K^m with K^n can be obtained in the following way:

(1) Take a matrix $A \in M_n(K)$, which is equivalent via identical permutations in rows and columns to a standard idempotent $0,1$-matrix, and set $A(l_0,l_0) \coloneqq A$.
(2) Set $J_{l_0} \coloneqq \{k \in \mathbb{N}_n^* : A(l_0,l_0)_{kk} = 1\}$.
(3) For all $i \in \mathbb{N}_m^* \setminus \{l_0\}$ choose $J_i \subseteq \mathbb{N}_n^* \setminus J_{l_0}$ such that

$$\bigcup_{i=1}^{m} J_i = \mathbb{N}_n^* \quad \text{and} \quad J_i \cap J_{i'} = \emptyset \quad \text{if } i \neq i'.$$

(4) For $i \neq l_0$ define $A(i, l_0) \in M_n(K)$ by

$$A(i,l_0)_{kj} := \begin{cases} 1 & \text{if } k \in J_i \text{ and } j = k, \\ -1 & \text{if } k \in J_i \text{ and } j = c_k, \\ 0 & \text{otherwise.} \end{cases}$$

Next we generalize the notation introduced in Remark 5.4(2).

Remark 5.6. Let $l_0 \in \mathbb{N}_m^*$ and $k \in \mathbb{N}_n^*$. If $A(i, l_0)_{kk} \in \{0, 1\}$ for all i, then there is a unique index i_0, which is denoted $i(k) = i(k, l_0) = i(k, l_0, \mathcal{A})$, such that $A(i_0, l_0)_{kk} = 1$. So, $A(i, l_0)_{kk} = \delta_{ii_0}$.

Definition 5.7. We say that a twisting map $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ is *standard* if the columns of $\mathcal{A}_\chi$ are standard columns. In this case we also say that the twisted tensor product $K^n \otimes_\chi K^m$ is *standard*.

Proposition 5.8. *A twisting map χ is a standard twisting map if and only if the map $\tilde{\chi}$, introduced in Remark 2.2, is.*

Proof. By Remark 2.2 we know that $\mathcal{A}_{\tilde{\chi}} = \mathcal{B}_\chi$. Thus, since $\tilde{\chi}$ is a twisting map, we only must check that the l_0-th column of $\mathcal{B}_\chi$ is a standard column for all $l_0 \in \mathbb{N}_n^*$. Item (1) of Definition 5.3 is an immediate consequence of Remark 5.4(1). For item (2) it suffices to consider the case $i \neq l_0$. By Remark 5.4(4), we know that $B_\chi(i, l_0)_{kj} \in \{1, 0, -1\}$ for all j, k and that $B_\chi(i, l_0)_{kj} \neq 1$ for $j \neq k$. Since $\sum_{j=1}^m B_\chi(i, l_0)_{kj} = 0$, this implies that if $B_\chi(i, l_0)_{kk} = 0$, then the k-th row vanishes. Else $B_\chi(i, l_0)_{kk} = 1$ and there exists exactly one index j' such that $B_\chi(i, l_0)_{kj'} = -1$. It remains to check that $j' = c_k(B_\chi(l_0, l_0))$. Using that $B_\chi(i, l_0)$ is idempotent, we obtain that

$$-1 = B_\chi(i,l_0)_{kj'} = \sum_{j=1}^m B_\chi(i,l_0)_{kj} B_\chi(i,l_0)_{jj'} = B_\chi(i,l_0)_{kj'} - B_\chi(i,l_0)_{j'j'} = -1 - B_\chi(i,l_0)_{j'j'}.$$

Set $i_0 := i\big(j', l_0, \mathcal{A}_{\tilde{\chi}}\big)$. Since $B_\chi(i_0, l_0)_{j'j'} = 1$ the above equality implies that $i \neq i_0$. Thus,

$$0 = \sum_{j=1}^m B_\chi(i,l_0)_{kj} B_\chi(i_0,l_0)_{jj'} = B_\chi(i_0,l_0)_{kj'} - B_\chi(i_0,l_0)_{j'j'} = B_\chi(i_0,l_0)_{kj'} - 1,$$

where the first equality holds because $B_\chi(i, l_0)B_\chi(i_0, l_0) = 0$. Therefore $B_\chi(i_0, l_0)_{kj'} = 1$, and so $i_0 = l_0$, because $j' \neq k$. Hence, $j' = c_k(B_\chi(l_0, l_0))$, as desired. $\square$

Remark 5.9. Let χ be a standard twisting map and let $i \neq l$ and $k \neq j$. Then $A_\chi(i, l)_{kj} = -1$ if and only if $B_\chi(k, k)_{li} = 1$ and $A_\chi(l, l)_{kj} = 1$. In fact, by Remark 5.4(4),

$$A_\chi(i,l)_{kj} = -1 \Rightarrow B_\chi(k,k)_{li} = A_\chi(i,l)_{kk} = 1.$$

Since, by Proposition 5.8 and Remark 2.2 we know that the map $\tilde{\chi}$ is a standard twisting map and $\mathcal{A}_{\tilde{\chi}} = (B_\chi(i, l))_{i,l \in \mathbb{N}_n^*}$, we also have $A_\chi(l, l)_{kj} = 1$. Conversely,

$$1 = B_\chi(k,k)_{li} = A_\chi(i,l)_{kk} \Rightarrow \exists! j \text{ such that } A_\chi(i,l)_{kj} = -1.$$

So $j = c_k(A_\chi(l, l))$.

Theorem 5.10. *Let $(A(i))_{i \in N_m^*}$ and $(B(k))_{k \in N_n^*}$ be two families of idempotent $0,1$-matrices $A(i) \in M_n(K)$ and $B(k) \in M_m(K)$, such that, for all i and k,*

(1) $A(i)\mathbb{1} = \mathbb{1}$ *and* $B(k)\mathbb{1} = \mathbb{1}$,

(2) $A(i)_{kk} = B(k)_{ii}$.

The family $\mathcal{A}_\chi = (A_\chi(i,l))_{i,l\in\mathbb{N}_m^*}$*, of matrices* $A_\chi(i,l) \in M_n(K)$ *defined by*

$$A_\chi(i,l)_{kj} := \begin{cases} A(l)_{kj} & \text{if } i = l,\\ B(k)_{li} & \text{if } k = j,\\ -1 & \text{if } i \ne l,\ k\ne j \text{ and } A(l)_{kj} = B(k)_{li} = 1,\\ \ 0 & \text{otherwise,}\end{cases}$$

gives the unique standard twisting map

$$\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m,$$

such that $A_\chi(i,i) = A(i)$ *and* $B_\chi(k,k) = B(k)$.

Proof. The uniqueness holds since the definition of $\mathcal{A}_\chi$ is forced. Set $B_\chi(j,k)_{li} := A_\chi(i,l)_{kj}$. Note that $B_\chi(k,k) = B(k)$. We must check that conditions (1)–(4) of Proposition 2.3 are fulfilled and that χ is standard. For condition (3) we must verify that

$$\delta_{il} = \sum_j A_\chi(i,l)_{kj} \qquad \text{for all } i,\ l \text{ and } k. \tag{5.12}$$

When $i = l$ this is true by assumption. When $i \ne l$ and $B(k)_{li} = 0$, we have $A_\chi(i,l)_{kj} = 0$ for all j, and thus equality (5.12) is true. Finally, when $i \ne l$ and $B(k)_{li} = 1$, we have $A_\chi(i,l)_{kk} = 1$, $A_\chi(i,l)_{kc_k} = -1$ (where $c_k = c_k(A(l))$) and $A_\chi(i,l)_{kj} = 0$ for $j \notin \{k, c_k\}$, and again equality (5.12) is true. The proof of condition (4) is similar. Since $B_\chi(j,k)_{li} = A_\chi(i,l)_{kj}$, conditions (3) and (4) say that $\sum_i A_\chi(i,l) = \mathrm{id}$ and $\sum_j B_\chi(j,k) = \mathrm{id}$ for all l and for all k. Hence, by Remark 2.8, in order to check condition (1) it suffices to prove that

$$\sum_i \mathrm{rk}\bigl(A_\chi(i,l)\bigr) \le n \qquad \text{for all } l. \tag{5.13}$$

Fix $l \in \mathbb{N}_m^*$. Since the $B(k)$'s are equivalent, via identical permutations in rows and columns, to a standard idempotent $0,1$-matrices, we know that for each k there exists a unique i such that $A_\chi(i,l)_{kk} = B(k)_{li} = 1$. Thus $\sum_i \#\{k : A_\chi(i,l)_{kk} = 1\} = n$. Consequently, to conclude that inequality (5.13) holds it is enough to show that

$$\mathrm{rk}\bigl(A_\chi(i,l)\bigr) \le \#\bigl\{k : A_\chi(i,l)_{kk} = 1\bigr\} \qquad \text{for all } i.$$

But, for $i = l$ we know that $\mathrm{rk}\bigl(A_\chi(l,l)\bigr) = \#\bigl\{k : A_\chi(l,l)_{kk} = 1\bigr\}$, because $A(l)$ is an idempotent $0,1$-matrix, while, for $i \ne l$, from the fact that

$$A_\chi(i,l)_{kk} \in \{0,1\} \quad \text{and} \quad A_\chi(i,l)_{kk} = 0 \text{ implies that } A_\chi(i,l)_{kj} = 0 \text{ for all } j,$$

it follows that $\#\bigl\{k : A_\chi(i,l)_{kk} = 1\bigr\}$ is the number of non zero rows of $A_\chi(i,l)$, which is greater than or equal to $\mathrm{rk}\bigl(A_\chi(i,l)\bigr)$. This concludes the proof of condition (1) of Proposition 2.3. The proof of condition (2) is similar. □

Notation 5.11. For all $l \in \mathbb{N}_m^*$, we set

$$F_0(\mathcal{A}, l) := \{k \in \mathbb{N}_n^* : A(i,l)_{kj} = \delta_{il}\delta_{kj}, \text{ for all } i \text{ and } j\}.$$

and for all $i, l \in \mathbb{N}_m^*$, we set $F(A(i,l)) := \{j \in \mathbb{N}_n^* : A(i,l)_{jj} = 1\}$.

Remark 5.12. The set $F(A(i,l))$ was introduced in Notation 2.1, where was denoted $J_i(l)$, but in some places we prefer to use the longer but more precise notation $F(A(i,l))$.

Definition 5.13. We will say that Corollary 2.5(4) is satisfied in the l_0-th column of $\mathcal{A}$ if

$$\sum_{h=1}^m A(i,h)_{kj} A(h,l_0)_{kj'} = \delta_{jj'} A(i,l_0)_{kj} \qquad \text{for all } i,\ j,\ j' \text{ and } k. \tag{5.14}$$

Proposition 5.14. *If the l_0-th column of $\mathcal{A}$ is a standard column, then Corollary 2.5(4) is satisfied in the l_0-th column of $\mathcal{A}$ if and only if $F(A(v,l_0)) \subseteq F_0(\mathcal{A},v)$ for all $v \in \mathbb{N}_m^*$.*

Proof. $\Rightarrow$) Let $v \in \mathbb{N}_m^*$ and $k \in \mathbb{N}_n^*$. If $k \in F(A(v,l_0))$, then $A(u,l_0)_{kk} = \delta_{uv}$ for all $u \in \mathbb{N}_m^*$ (see Remark 5.4). So, from (5.14) with $j = k$, we obtain that

$$A(i,v)_{kj} = \sum_{u=1}^{m} A(i,u)_{kj} A(u,l_0)_{kk} = \delta_{jk} A(i,l_0)_{kj} = \delta_{jk} A(i,l_0)_{kk} = \delta_{jk}\delta_{iv}$$

for all i, j, which says that $k \in F_0(\mathcal{A},v)$, as desired.

$\Leftarrow$) Fix $k \in \mathbb{N}_n^*$. If $i(k,l_0) = l_0$, then $k \in F(A(l_0,l_0)) \subseteq F_0(\mathcal{A},l_0)$, and so condition (5.14) holds if and only if

$$A(i,l_0)_{kj}\delta_{kj'} = \delta_{jj'} A(i,l_0)_{kj} \qquad \text{for all } i,\ j \text{ and } j'.$$

But this is true for $i \neq l_0$, since then $A(i,l_0)_{kj} = 0$, and also for $i = l_0$, since $A(l_0,l_0)_{kj} = \delta_{kj}$.

If $h_0 \coloneqq i(k,l_0) \neq l_0$, then equality (5.14) holds if and only if

$$A(i,h_0)_{kj} A(h_0,l_0)_{kj'} + A(i,l_0)_{kj} A(l_0,l_0)_{kj'} = \delta_{jj'} A(i,l_0)_{kj} \qquad \text{for all } i,\ j \text{ and } j', \tag{5.15}$$

since for $h \notin \{h_0, l_0\}$ we have $A(h,l_0)_{kj'} = 0$ for all j'. In order to prove that (5.15) is true, we consider the cases $j = k$, $j = c_k = c_k(A(l_0,l_0))$ and $j \notin \{k, c_k\}$. We will use that $A(i,h_0)_{kj} = \delta_{ih_0}\delta_{kj}$ for all i, j, which is true, because $k \in F(A(h_0,l_0)) \subseteq F_0(\mathcal{A},h_0)$.

- If $j = k$, then we must prove that

  $$A(i,h_0)_{kk} A(h_0,l_0)_{kj'} + A(i,l_0)_{kk} A(l_0,l_0)_{kj'} = \delta_{kj'} A(i,l_0)_{kk} \quad \text{for all } i \text{ and all } j'.$$

  But this is true, since by the above discussion, Remark 5.4 and Proposition 4.7,

  $$A(i,h_0)_{kk} = \delta_{ih_0}, \quad A(h_0,l_0)_{kj'} = \delta_{kj'} - \delta_{j'c_k}, \quad A(i,l_0)_{kk} = \delta_{ih_0} \quad \text{and} \quad A(l_0,l_0)_{kj'} = \delta_{j'c_k}.$$

- Since $A(i,h_0)_{kc_k} = 0$ for all i, when $j = c_k$ we are reduced to prove that

  $$A(i,l_0)_{kc_k} A(l_0,l_0)_{kj'} = \delta_{c_kj'} A(i,l_0)_{kc_k} \quad \text{for all } i \text{ and all } j'.$$

  But this is true, since $A(l_0,l_0)_{kj'} = \delta_{j'c_k}$.
- If $j \notin \{k, c_k\}$, then both sides of (5.15) vanish.

Thus, (5.14) holds in all the cases. □

Corollary 5.15. *Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a k-linear map such that $\mathcal{A}_\chi$ is a pre-twisting. If each column of $\mathcal{A}_\chi$ is standard, then χ is a twisting map if and only if $F(A(i,l)) \subseteq F_0(A,i)$ for all $i, l \in \mathbb{N}_m^*$.*

Given sets X, Y, in the sequel we let $M_{X,Y}(K)$ denote the set of functions from $X \times Y$ to K. We also denote by id_X the identity matrix in $M_X(K) \coloneqq M_{X,X}(K)$.

Proposition 5.16. *Let $l \in \mathbb{N}_k^*$ and let $A(1), \dots, A(k) \in M_n(K)$ be matrices such that $A(l)$ is an idempotent $0,1$-matrix with $A(l)\mathbb{1} = \mathbb{1}$. Set $J_l \coloneqq \{k : A(l)_{kk} = 1\}$ and $J_l^c \coloneqq \mathbb{N}_n^* \setminus J_l$. For each i set*

$$X_i \coloneqq A(i)|_{J_l \times J_l}, \quad Y_i \coloneqq A(i)|_{J_l \times J_l^c}, \quad U_i \coloneqq A(i)|_{J_l^c \times J_l} \quad \textit{and} \quad W_i \coloneqq A(i)|_{J_l^c \times J_l^c}.$$

The matrices $A(i)$'s are orthogonal idempotents satisfying $\sum_i A(i) = \mathrm{id}$ if and only if the following facts hold:

1. *$X_i = 0$ for all $i \neq l$,*
2. *$Y_i = 0$ for all i,*
3. *$W_i W_j = \delta_{ij} W_i$ for all i,*
4. *$U_i = -W_i U_l$ for all $i \neq l$,*

(5) $\sum_i W_i = \mathrm{id}_{J_l^c}$.

Moreover, if the $A(i)$'s satisfy the required conditions, then $A(i)\mathbb{1} = \delta_{il}\mathbb{1}$.

Proof. Without loss of generality we can assume that $J_l = \mathbb{N}_r^*$, where $r := \mathrm{rk}(A(l))$. Then $A(l) = \left(\begin{smallmatrix} \mathrm{id}_r & 0 \\ U_l & 0 \end{smallmatrix}\right)$ and $A(i) = \left(\begin{smallmatrix} X_i & Y_i \\ U_i & W_i \end{smallmatrix}\right)$. Let $i \neq l$. A direct computation shows that $A(l)A(i) = 0$ if and only if $X_i = 0$ and $Y_i = 0$. Under this condition, $A(i)A(l) = 0$ if and only if $U_i = -W_iU_l$. Assuming all the previous conditions for all $i \neq l$, we have $A(i)A(j) = \delta_{ij}A(i)$ if and only if $W_iW_j = \delta_{ij}W_i$, and, under the same conditions, $\sum_i A(i) = \mathrm{id}_n$ if and only if $\sum_i W_i = \mathrm{id}_{J_l^c}$. The last assertion follows from the fact that $U_i = -W_iU_l$ and $U_l\mathbb{1}_{J_l} = \mathbb{1}_{J_l^c}$. □

Definition 5.17. Let $l_0 \in \mathbb{N}_m^*$. For all $i, u, v \in \mathbb{N}_m^*$, set $D_{(i,l_0)}^{uv} = D_{(i)}^{uv} := A(i, l_0)|_{J_u \times J_v}$, where $J_i := J_i(l_0)$. We say that $(A(i, l_0))_{i \in \mathbb{N}_m^*}$ is a *quasi-standard column* of $\mathcal{A}$ if

(1) $A(l_0, l_0)$ is a $0,1$-matrix,

(2) $A(i, l_0)_{kk} \in \{0, 1\}$ for all i and k,

(3) $D_{(i)}^{uv} = 0$ if $u \neq i$ and $v \notin \{i, l_0\}$,

(4) For $u, i \in \mathbb{N}_m^*$, $v \in \mathbb{N}_m^* \setminus \{l_0\}$ and $k \in J_u$, we have $\#\operatorname{Supp}\big((D_{(i)}^{uv})_{k*}\big) \leq 1$. Moreover if $d \in \operatorname{Supp}\big((D_{(i)}^{uv})_{k*}\big)$, then $c_d = c_k$, where $c_d := c_d(A(l_0, l_0))$ and $c_k := c_k(A(l_0, l_0))$[1]. If necessary we will write $d^{(v)}$ or $d_k^{(v)}$ instead of d.

Remark 5.18. Let $k \in J_{l_0}$ and let $i \neq l_0$. By items (1) and (2) of Proposition 5.16 we know that $A(i, l_0)_{kj} = 0$ for all j. Consequently $D_{(i)}^{l_0 v} = 0$ for all $v \in \mathbb{N}_m^*$. Note that this implies that $F(A(l_0, l_0)) = F_0(\mathcal{A}, l_0)$.

Remark 5.19. Since $\sum_i A(i, l_0) = \mathrm{id}$, we have $\sum_i D_{(i)}^{uu} = \mathrm{id}$ for all $u \in \mathbb{N}_m^*$, which by condition (3) implies that $D_{(u)}^{uu} = \mathrm{id}$ for all $u \neq l_0$ (by Proposition 4.7, also $D_{(l_0)}^{l_0 l_0} = \mathrm{id}$).

Remark 5.20. Since $\sum_i A(i, l_0) = \mathrm{id}$, we have $\sum_i D_{(i)}^{uv} = 0$ for all $u \neq v$ in $\mathbb{N}_m^*$, which by condition (3) implies that $D_{(u)}^{uv} = -D_{(v)}^{uv}$ for all $u \in \mathbb{N}_m^*$ and $v \in \mathbb{N}_m^* \setminus \{u, l_0\}$.

Remark 5.18 is valid for pre-twistings that satisfy condition (1) of Definition 5.17, while Remarks 5.19 and 5.20 are true for pre-twistings that satisfy conditions (1) and (3) of the same definition.

Remark 5.21. From the fact that $A(l_0, l_0)$ is a $0,1$-matrix it follows immediately that $D_{(l_0)}^{uv} = 0$ for all $u \in \mathbb{N}_m^*$ and $v \in \mathbb{N}_m^* \setminus \{l_0\}$. Combining this with Remarks 5.18 and 5.20 we obtain that Conditions (3) and (4) in Definition 5.17 could be replaced by

(3') $D_{(i)}^{uv} = 0$ if $i \neq l_0$ and $u, v \notin \{i, l_0\}$,

(4') $\#\operatorname{Supp}\big((D_{(v)}^{uv})_{k*}\big) \leq 1$ for $u, v \in \mathbb{N}_m^* \setminus \{l_0\}$ and $k \in J_u$. Moreover if $d \in \operatorname{Supp}\big((D_{(v)}^{uv})_{k*}\big)$, then $c_d = c_k$, where $c_d := c_d(A(l_0, l_0))$ and $c_k := c_k(A(l_0, l_0))$,

respectively.

Remark 5.22. Each standard column of $\mathcal{A}$ is a quasi-standard column of $\mathcal{A}$.

Example 5.23. Assume for example that $n = 10$, $J_{l_0} = \{1, 2\}$ and $J_i = \{5, 6, 7\}$. If the l_0-th column of $\mathcal{A}$ is a quasi-standard column, then the only entries where the matrix $A(i, l_0)$ may have nonzero values are the entries indicated by stars. In this example and in Example 5.26 below, the elements of each family J_u are consecutive, but of course this need not be the case.

[1]Note that c_k exists since necessarily $u \neq l_0$ (see Remark 5.18 and the beginning of Remark 5.21)

$$
A(i,l_0) = \begin{array}{cc} & \begin{array}{cccccccccc} J_{l_0} & & & & J_i & & & & & \end{array} \\ \begin{array}{c} J_{l_0} \\ \\ \\ \\ J_i \\ \\ \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0\\ \star&\star&0&0&\star&\star&\star&0&0&0\\ \star&\star&0&0&\star&\star&\star&0&0&0\\ \star&\star&\star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&\star&\star&\star&\star&\star&\star&\star&\star\\ \star&\star&0&0&\star&\star&\star&0&0&0\\ \star&\star&0&0&\star&\star&\star&0&0&0\\ \star&\star&0&0&\star&\star&\star&0&0&0 \end{pmatrix} \end{array}
$$

Lemma 5.24. *Assume that the l_0-th column of $\mathcal{A}$ satisfies conditions (1)–(3) of Definition 5.17. Take $i,u \in \mathbb{N}_m^* \setminus \{l_0\}$ and $k \in J_u$. If $A(i,l_0)_{kc_k} \neq 0$, then there exist indices $v \in \mathbb{N}_m^* \setminus \{l_0\}$ and $j \in J_v$ such that $(D_{(i)}^{uv})_{kj} \neq 0$. Moreover, if $u \neq i$, then necessarily $v = i$.*

Proof. By Remark 5.18 we know that $A(i,l_0)_{jc_k} = 0$ for all $j \in J_{l_0}$. So

$$\sum_{v \in \mathbb{N}_m^* \setminus \{l_0\}} \sum_{j \in J_v} A(i,l_0)_{kj} A(i,l_0)_{jc_k} = \sum_{j \in \mathbb{N}_n^*} A(i,l_0)_{kj} A(i,l_0)_{jc_k} = A(i,l_0)_{kc_k} \neq 0.$$

Consequently, there exists $v \in \mathbb{N}_m^* \setminus \{l_0\}$ and $j \in J_v$ such that $(D_{(i)}^{uv})_{kj} = A(i,l_0)_{kj} \neq 0$. The last assertion is true by item (3) of Definition 5.17. □

For $u \in \mathbb{N}_m^* \setminus \{l_0\}$ and $k \in J_u = J_u(l_0)$, we set

$$\mathscr{X}_k \coloneqq \big\{v \in \mathbb{N}_m^* \setminus \{u,l_0\} : \operatorname{Supp}\big((D_{(u)}^{uv})_{k*}\big) \neq \emptyset\big\} \qquad \text{and} \qquad d^{(\mathscr{X}_k)} \coloneqq \{d^{(v)} : v \in \mathscr{X}_k\}.$$

Lemma 5.25. *Assume that the l_0-th column of $\mathcal{A}$ is quasi-standard. For each $k \in \mathbb{N}_n^* \setminus J_{l_0}$ and $v \in \mathbb{N}_m^*$, we have $\operatorname{Supp}\big(A(v,l_0)_{k*}\big) \subseteq \{k, c_k\} \cup d^{(\mathscr{X}_k)}$.*

Proof. When $v = l_0$ this is clear. So, we can assume that $v \neq l_0$. Let $u \coloneqq i(k,l_0)$. We consider two cases:

$u \neq v$) By the very definition of quasi-standard column and Remark 5.20

$$\operatorname{Supp}\big(A(v,l_0)_{k*}\big) \subseteq \operatorname{Supp}\big((D_{(v)}^{ul_0})_{k*}\big) \cup \operatorname{Supp}\big((D_{(v)}^{uv})_{k*}\big) \qquad \text{and} \qquad \operatorname{Supp}\big((D_{(v)}^{uv})_{k*}\big) \subseteq d^{(\mathscr{X}_k)}.$$

Hence it suffice to prove that $\operatorname{Supp}\big((D_{(v)}^{ul_0})_{k*}\big) \subseteq \{c_k\}$. Since $D_{(v)}^{ui} = 0$ for $i \notin \{v, l_0\}$,

$$D_{(v)}^{ul_0} D_{(l_0)}^{l_0l_0} + D_{(v)}^{uv} D_{(l_0)}^{vl_0} = A(v,l_0)A(l_0,l_0)|_{J_u \times J_{l_0}} = 0.$$

Since $D_{(l_0)}^{l_0l_0} = \mathrm{id}$, this yields

$$D_{(v)}^{ul_0} = -D_{(v)}^{uv} D_{(l_0)}^{vl_0}.$$

Thus, if $\operatorname{Supp}((D_{(v)}^{uv})_{k*}) = \emptyset$, then $\operatorname{Supp}((D_{(v)}^{ul_0})_{k*}) = \emptyset$. Else $\operatorname{Supp}((D_{(v)}^{uv})_{k*}) = \{d^{(v)}\}$ and so

$$\big(D_{(v)}^{ul_0}\big)_{k*} = -\big(D_{(v)}^{uv}\big)_{kd^{(v)}} \big(D_{(l_0)}^{vl_0}\big)_{d^{(v)}*}.$$

Combining this with the fact that

$$\operatorname{Supp}\big(\big(D_{(l_0)}^{vl_0}\big)_{d^{(v)}*}\big) = \operatorname{Supp}\big(A(l_0,l_0)_{d^{(v)}*}\big) = \{c_{d^{(v)}}\} = \{c_k\},$$

we obtain that $\operatorname{Supp}\big(\big(D_{(v)}^{ul_0}\big)_{k*}\big) = \{c_k\}$, as desired.

$u=v$) Using that $A(u,l_0)_{k*}=\delta_{k*}-\sum_{i\ne u}A(i,l_0)_{k*}$ we obtain that

$$\operatorname{Supp}\bigl(A(u,l_0)_{k*}\bigr)\subseteq\{k\}\cup\bigcup_{i\ne u}\operatorname{Supp}\bigl(A(i,l_0)_{k*}\bigr),$$

which finishes the proof. ☐

Example 5.26. The matrices

$$A(1,1):=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\end{pmatrix},\quad A(2,1):=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\-1-\lambda_1&0&1&0&0&\lambda_1&0&0\\-1-\lambda_2&0&0&1&0&\lambda_2&0&0\\0&-1&0&0&1&0&0&0\\0&0&0&0&0&0&0&0\\0&-\lambda_3&0&0&\lambda_3&0&0&0\\0&-\lambda_4&0&0&\lambda_4&0&0&0\end{pmatrix}$$

and $A(3,1):=\mathrm{id}-A(1,1)-A(2,1)$ form a quasi-standard column of each pre-twisting of K^3 with K^8 that include them (for instance we can take $A(1,2)=A(3,2)=A(1,3)=A(2,3)=0$ and $A(2,2)=A(3,3)=\mathrm{id}$). In this example $J_1=\{1,2\}$, $J_2=\{3,4,5\}$ and $J_3=\{6,7,8\}$.

Theorem 5.27. *Assume that the l_0-th column of $\mathcal{A}$ is quasi-standard. Then Corollary 2.5(4) is satisfied in the l_0-th column of $\mathcal{A}$ (that is, condition (5.14) is fulfilled) if and only if the following conditions hold:*

(1) *$J_i=F(A(i,l_0))\subseteq F_0(\mathcal{A},i)$ for all $i\in\mathbb{N}_m^*$.*
(2) *If $\bigl(D^{uv}_{(u)}\bigr)_{kd}\ne 0$ and $u\ne v\ne l_0$, then*
   (a) *$A(u,v)_{kj}=\delta_{kj}-\delta_{jd}$ for all j,*
   (b) *$A(v,v)_{kj}=\delta_{jd}$ for all j,*
   (c) *$A(i,v)_{kj}=0$ for $i\notin\{u,v\}$ and for all j.*

Proof. $\Rightarrow$) The arguments given in the proof of Proposition 5.14 show that condition (1) is fulfilled. So we only must prove condition (2). By Definition 5.17(3)

$$A(i,l_0)_{kd}=0\qquad\text{for } i\notin\{u,v\},\tag{5.16}$$

which, since $\sum_i A(i,l_0)=\mathrm{id}$ and $k\ne d$, implies that

$$A(v,l_0)_{kd}=-A(u,l_0)_{kd}=-\bigl(D^{uv}_{(u)}\bigr)_{kd}\ne 0.\tag{5.17}$$

Moreover, by condition (1) we know that $k\in J_u\subseteq F_0(\mathcal{A},u)$, and so

$$A(i,u)_{kj}=\delta_{iu}\delta_{kj}\qquad\text{for all } i \text{ and } j.\tag{5.18}$$

By (5.16) and (5.18), the equality (5.14) with $j'=d$ reads

$$\delta_{iu}\delta_{kj}A(u,l_0)_{kd}+A(i,v)_{kj}A(v,l_0)_{kd}=\delta_{jd}A(i,l_0)_{kd}\qquad\text{for all } i \text{ and } j.\tag{5.19}$$

When $i=u$, from (5.17) and (5.19), we obtain that

$$\delta_{kj}A(u,l_0)_{kd}-A(u,v)_{kj}A(u,l_0)_{kd}=\delta_{jd}A(u,l_0)_{kd}\qquad\text{for all } j,$$

which gives (a) since $A(u,l_0)_{kd}\ne 0$. On the other hand, when $i\ne u$, equality (5.19) reduces to

$$A(i,v)_{kj}A(v,l_0)_{kd}=\delta_{jd}A(i,l_0)_{kd}\qquad\text{for all } j,$$

which, combined with (5.16) and (5.17), gives items (b) and (c).

$\Leftarrow$) By Remark 2.10 it suffices to prove that

$$\sum_{h\in\mathbb{N}_m^*} A(i,h)_{kj}A(h,l_0)_{kj} = A(i,l_0)_{kj} \qquad \text{for all } i,\ k \text{ and } j. \tag{5.20}$$

Fix $k\in\mathbb{N}_n^*$ and set $u := i(k,l_0)$. If $k\in F(A(l_0,l_0)) = F_0(\mathcal{A},l_0)$, then $A(i,l_0)_{kj} = \delta_{il_0}\delta_{kj}$ for all i and j, and equality (5.20) is trivially true. Consequently we can assume that $u\neq l_0$. So, by Lemma 5.25,

$$\operatorname{Supp}\bigl(A(i,l_0)_{k*}\bigr)\subseteq\{k,c_k\}\cup d^{(\mathscr{X}_k)} \qquad \text{for all } i.$$

Thus, if $j\notin\{k,c_k\}\cup d^{(\mathscr{X}_k)}$ both sides of the equality (5.20) are zero.

Assume that $j=k$. By Remark 5.6 equality (5.20) reads

$$A(i,u)_{kk} = \delta_{iu} \qquad \text{for all } i.$$

But this is true since, by condition (1), we have $k\in J_u\subseteq F(\mathcal{A},u)$.

Suppose now that $j=c_k$. By Remark 5.20, Lemma 5.24 and conditions (1) and (2), equality (5.20) reduces to

$$A(i,l_0)_{kc_k}A(l_0,l_0)_{kc_k} = A(i,l_0)_{kc_k} \quad \text{for all } i,$$

which is true.

If $j=d^{(v)}$ for $v\notin\{u,l_0\}$, then $0\neq A(h,l_0)_{kd^{(v)}} = (D^{uv}_{(h)})_{kd^{(v)}}$ implies that $h\in\{u,v\}$, by item (3) of Definition 5.17. But by condition (1) we know that $A(i,u)_{kd^{(v)}} = \delta_{iu}\delta_{kd^{(v)}} = 0$. So equality (5.20) reduces to

$$A(i,v)_{kd^{(v)}}A(v,l_0)_{kd^{(v)}} = A(i,l_0)_{kd^{(v)}},$$

which can be verified easily using that

$$A(u,l_0)_{kd^{(v)}} = \bigl(D^{uv}_{(u)}\bigr)_{kd^{(v)}} = -\bigl(D^{uv}_{(v)}\bigr)_{kd^{(d)}} = -A(v,l_0)_{kd^{(v)}}\neq 0$$

and condition (2). □

Definition 5.28. We say that the l_0-th column of $\mathcal{A}$ has *reduced rank* r if there are exactly r indices $i\neq l_0$ such that $A(i,l_0)\neq 0$. In this case we write $\operatorname{rrank}_{\mathcal{A}}(l_0)=r$. If $\mathcal{A}$ is associated with a map χ as at the beginning of Section 2, then we use $\operatorname{rrank}_\chi(l_0)$ as a synonym of $\operatorname{rrank}_{\mathcal{A}}(l_0)$.

Remark 5.29. Let $l_0,u\in\mathbb{N}_m^*$ and let and $k\in J_u$. Assume that $\mathcal{A}$ is a family of matrices associated with a twisting map of K^m with K^n and that conditions (1) and (2) of Definition 5.17 are fulfilled for the l_0-th column of $\mathcal{A}$. By Remark 5.6 we have $A(v,l_0)_{kk}=\delta_{uv}$ for all v. Consequently, from Corollary 2.5(4) with $j'=k$, it follows that

$$A(i,u)_{kj} = \delta_{iu}\delta_{kj} \quad \text{for all } i \text{ and } j. \tag{5.21}$$

Proposition 5.30. *Let $l_0\in\mathbb{N}_m^*$. Assume that $\mathcal{A}$ is a family of matrices associated with a twisting map of K^m with K^n and that conditions (1) and (2) of Definition 5.17 are fulfilled for the l_0-th column of $\mathcal{A}$.*

- (a) *If condition (3) is also fulfilled, then the l_0-th column of $\mathcal{A}$ is quasi-standard.*
- (b) *If the reduced rank of the l_0-th column of $\mathcal{A}$ is lower than or equal to 2, then the l_0-th column of $\mathcal{A}$ is quasi-standard.*

Proof. As in Definition 5.17, for all $i,u,v\in\mathbb{N}_m^*$ we set $D^{uv}_{(i)} := A(i,l_0)|_{J_u\times J_v}$, where $J_i := J_i(l)$.

(a) We must prove that if condition (3) of Definition 5.17 is fulfilled, then condition (4') of Remark 5.21 is also. We begin by proving that

$$\#\operatorname{Supp}\bigl(\bigl(D^{uv}_{(v)}\bigr)_{k*}\bigr)\leq 1 \quad \text{for all } u,v\in\mathbb{N}_m^*\setminus\{l_0\} \text{ and } k\in J_u. \tag{5.22}$$

For $u=v=i$ it is true by Remark 5.19. So, we can assume that $u\neq v$. Assume on the contrary that there exist $d_1\neq d_2$ in $\operatorname{Supp}\bigl((D^{uv}_{(v)})_{k*}\bigr)$. Since $A(i,l_0)_{kd_1}=0$ for $i\notin\{u,v\}$ and $A(v,l_0)_{kd_1}=-A(u,l_0)_{kd_1}\neq 0$, Corollary 2.5(4) with $i=u$, $l=l_0$ and $j=j'=d_1$ gives

$$A(v,v)_{kd_1}-A(v,u)_{kd_1}=1.$$

A similar argument shows that Corollary 2.5(4) with $i=u$, $l=l_0$, $j=d_1$ and $j'=d_2$, gives

$$A(v,v)_{kd_1}-A(v,u)_{kd_1}=0,$$

a contradiction.

It remains to check that if $d\in\operatorname{Supp}\bigl((D^{uv}_{(v)})_{k*}\bigr)$, then $c_d=c_k$. When $v=u$ this follows again from Remark 5.19. Assume that $v\neq u$. We assert that

$$\operatorname{Supp}\bigl(A(v,l_0)_{k*}\bigr)=\{d,c_d\}. \tag{5.23}$$

Since $\operatorname{Supp}\bigl((D^{uv}_{(v)})_{k*}\bigr)=\{d\}$ and $D^{ui}_{(v)}=0$ for $i\notin\{v,l_0\}$, this is true if and only if

$$\operatorname{Supp}\bigl((D^{ul_0}_{(v)})_{k*}\bigr)=\{c_d\}.$$

In order to check this, note that $A(v,l_0)A(l_0,l_0)=0$ imply

$$D^{ul_0}_{(v)}D^{l_0l_0}_{(l_0)}+D^{uv}_{(v)}D^{vl_0}_{(l_0)}=\sum_i D^{ui}_{(v)}D^{il_0}_{(l_0)}=0.$$

Since $D^{l_0l_0}_{(l_0)}=\mathrm{id}$ and $\operatorname{Supp}\bigl((D^{uv}_{(v)})_{k*}\bigr)=\{d\}$, this yields

$$\bigl(D^{ul_0}_{(v)}\bigr)_{k*}=-\bigl(D^{uv}_{(v)}\bigr)_{kd}\bigl(D^{vl_0}_{(l_0)}\bigr)_{d*}.$$

Combining this with the fact that

$$\operatorname{Supp}\bigl((D^{vl_0}_{(l_0)})_{d*}\bigr)=\operatorname{Supp}\bigl(A(l_0,l_0)_{d*}\bigr)=\{c_d\},$$

we obtain that $\operatorname{Supp}\bigl((D^{ul_0}_{(v)})_{k*}\bigr)=\{c_d\}$, as we need.

By Lemma 5.24, if $A(h,l_0)_{kc_k}\neq 0$, then $h\in\{u,v,l_0\}$. So Corollary 2.5(4) with $j=d$, $j'=c_k$ and $i=v$ gives

$$A(v,l_0)_{kd}+A(v,v)_{kd}A(v,l_0)_{kc_k}+A(v,u)_kA(u,l_0)_{kc_k}=0,$$

where we use that $A(l_0,l_0)_{kc_k}=1$. But by (5.21) we have $A(v,u)_{kd}=\delta_{vu}\delta_{kd}=0$, and so, necessarily $A(v,l_0)_{kc_k}\neq 0$, which, by equality (5.23), implies $c_k=c_d$.

(b) If the reduced rank of the l_0-th column of $\mathcal{A}$ is lower than 2, then that column is standard and the result is trivial (see Remark 5.22). So we can assume that its reduced rank is 2. By item (a) and Remark 5.21 it suffices to prove that $D^{uv}_{(i)}=0$, if $i\neq l_0$ and $u,v\notin\{i,l_0\}$. Since the reduced rank of the l_0-th column is 2, there exist two indices $i_0,i_1\neq l_0$ such that $A(i,l_0)\neq 0$ if and only if $i\in\{l_0,i_0,i_1\}$. So we must prove that $D^{i_bi_b}_{(i_a)}=0$ for $a\in\{0,1\}$ and $b:=1-a$. Take $k\in J_{i_b}$. We first prove that either

$$\operatorname{Supp}(A(i_b,l_0)_{k*})\subseteq\{k,c_k\}\quad\text{or}\quad \exists!\ d\text{ such that } d\in\operatorname{Supp}(A(i_b,l_0)_{k*})\setminus\{k,c_k\}. \tag{5.24}$$

Assume by contradiction that there exist $d\neq e\in\operatorname{Supp}(A(i_b,l_0)_{k*})\setminus\{k,c_k\}$. First note that since $A(l_0,l_0)+A(i_a,l_0)+A(i_b,l_0)=\mathrm{id}$ and $\operatorname{Supp}(A(l_0,l_0)_{k*})=\{c_k\}$, if $f\notin\{k,c_k\}$, then

$$A(i_a,l_0)_{kf}=-A(i_b,l_0)_{kf}. \tag{5.25}$$

By equation (5.21) we know that $A(i_b,i_b)_{kd}=0$. Moreover, since $\operatorname{Supp}(A(l_0,l_0)_{k*})=\{c_k\}$, we have

$$A(l_0,l_0)_{kd}=A(l_0,l_0)_{ke}=0.$$

Consequently from Corollary 2.5(4) with $j = j' = d$ and $i = i_b$, we obtain that

$$A(i_b, i_a)_{kd} A(i_a, l_0)_{kd} = \sum_u A(i_b, u)_{kd} A(u, l_0)_{kd} = A(i_b, l_0)_{kd} \neq 0,$$

which implies $A(i_b, i_a)_{kd} \neq 0$. On the other hand from Corollary 2.5(4) with $j = d$, $j' = e$ and $i = i_b$, we obtain that

$$A(i_b, i_a)_{kd} A(i_a, l_0)_{ke} = \sum_u A(i_b, u)_{kd} A(u, l_0)_{ke} = 0,$$

and so, necessarily $A(i_a, l_0)_{ke} = 0$. But this is impossible since $A(i_a, l_0)_{ke} = -A(i_b, l_0)_{ke} \neq 0$. Hence condition (5.24) is satisfied. We claim that if it exists, then $d \in J_{i_a}$. In fact, since $k \in J_{i_b}$ we have $A(i_a, l_0)_{kk} = 0$, and thus, by equality (5.25), if $d \in \operatorname{Supp}(A(i_b, l_0)_{k*}) \setminus \{k, c_k\}$, then $\operatorname{Supp}(A(i_a, l_0)_{k*}) = \{c_k, d\}$. Using now that $A(i_a, l_0)$ is idempotent, we obtain that

$$\begin{aligned} A(i_a, l_0)_{kd} &= A(i_a, l_0)_{k*} A(i_a, l_0)_{*d} \\ &= A(i_a, l_0)_{kc_k} A(i_a, l_0)_{c_k d} + A(i_a, l_0)_{kd} A(i_a, l_0)_{dd} \\ &= A(i_a, l_0)_{kd} A(i_a, l_0)_{dd}, \end{aligned}$$

since $A(i_a, l_0)_{c_k d} = 0$ by Remark 5.18. But then $A(i_a, l_0)_{dd} = 1$, which means that $d \in J_{i_a}$, as we claim. Thus

$$\operatorname{Supp}(A(i_a, l_0)_{k*}) \subseteq J_{l_0} \cup J_{i_a},$$

which implies that $\operatorname{Supp}((D^{i_b i_b}_{(i_a)})_{k*}) = \operatorname{Supp}(A(i_a, l_0)_{k*}) \cap J_{i_b} = \emptyset$ for all $k \in J_{i_b}$, as desired. □

Definition 5.31. We say that a twisting map $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ is *quasi-standard* if the columns of $\mathcal{A}_\chi$ are quasi-standard.

Proposition 5.32. *A twisting map $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ is quasi-standard if and only if the map $\tilde{\chi}$, introduced in Remark 2.2, is a quasi-standard twisting map.*

Proof. By Proposition 5.30, Remark 2.2 and the fact that χ is a twisting map if and only if $\widetilde{\chi}$ is, in order to prove the proposition it suffices to check that if every column of $\mathcal{A}_\chi$ is quasi-standard, then each column of $\mathcal{A}_{\widetilde{\chi}} = \mathcal{B}_\chi$ satisfies items (1), (2) and (3) of Definition 5.17. Assume that each column of $\mathcal{A}_\chi$ is quasi-standard. Using equality (2.3) it is easy to check that items (1) and (2) are satisfied by the columns of $\mathcal{B}_\chi$. Consequently, by Remark 5.21 we only must prove that $\widetilde{D}^{uv}_{(j)} := B_\chi(j,k)|_{\tilde{J}_u \times \tilde{J}_v}$ (where $\tilde{J}_u := \tilde{J}_u(k)$) are null matrices for $j \neq k$ and $u, v \notin \{j, k\}$. So we are reduced to prove that

$$B_\chi(j,k)_{ls} = A_\chi(s,l)_{kj} = 0 \quad \text{for all } k \notin \{j,u,v\},\ j \notin \{u,v\},\ l \in \tilde{J}_u \text{ and } s \in \tilde{J}_v.$$

But, since

$$l \in \tilde{J}_u \text{ and } s \in \tilde{J}_v \quad \text{ if and only if } \quad A_\chi(l,l)_{ku} = 1 \text{ and } A_\chi(s,s)_{kv} = 1,$$

and, in that case,

$$j \notin \{u,v\} \quad \text{ if and only if } \quad A_\chi(l,l)_{kj} = 0 \text{ and } A_\chi(s,s)_{kj} = 0,$$

for this it suffices to check that if $k \notin \{j,u,v\}$ and the l-th column of $\mathcal{A}_\chi$ is quasi-standard, then

$$\left.\begin{aligned} A_\chi(l,l)_{ku} &= 1 \\ A_\chi(s,s)_{kv} &= 1 \\ A_\chi(l,l)_{kj} &= 0 \\ A_\chi(s,l)_{kj} &\neq 0 \end{aligned}\right\} \Rightarrow A_\chi(s,s)_{kj} \neq 0.$$

Clearly $s \neq l$. Moreover $k \in J_w(l)$ with $w \neq s$ since, otherwise $A_\chi(s,l)_{kj} = 0$ by Theorem 5.27(1). Suppose that $k \notin J_l(l)$ and $j \notin \{k, c_k(A_\chi(l,l))\}$. Then, by Lemma 5.25 we have $j \notin J_l(l) \cup J_w(l)$. Consequently, by Definition 5.17(3) and Remark 5.20,

$$j \in \operatorname{Supp}((D^{ws}_{(s)})_{k*}) = \operatorname{Supp}((D^{ws}_{(w)})_{k*}) \qquad \text{and} \qquad w \neq s \neq l.$$

Thus, from Theorem 5.27(2b) we obtain that $A_\chi(s,s)_{kj} = \delta_{jj} \neq 0$, as desired. So, in order to finish the proof we must check that $k \notin J_l(l)$ and $j \notin \{k, c_k(A_\chi(l,l))\}$. But $k \notin J_l$, because $A_\chi(l,l)_{ku} = 1$ implies that $A_\chi(l,l)_{kk} = 0$; $j \neq k$, because, by Theorem 5.27(1), if $j = k$, then $A_\chi(s,s)_{kv} = \delta_{ss}\delta_{kv} = 0$; and $j \neq c_k\big(A_\chi(l,l)\big)$, since $A_\chi(l,l)_{kj} = 0$. □

Proposition 5.33. *Each quasi-standard column $A(i,l_0)_{i\in\mathbb{N}^*_m}$ of a pre-twisting of K^m with K^n can be obtained in the following way:*

1. *Take a matrix $A \in M_n(K)$, which is equivalent via identical permutations in rows and columns to a standard idempotent $0,1$-matrix, and set $A(l_0,l_0) \coloneqq A$.*
2. *Set $J_{l_0} \coloneqq \{k \in \mathbb{N}^*_n : A(l_0,l_0)_{kk} = 1\}$ and $J^c_{l_0} \coloneqq \mathbb{N}^*_n \setminus J_{l_0}$.*
3. *For all $i \in \mathbb{N}^*_m \setminus \{l_0\}$ choose $J_i \subseteq \mathbb{N}^*_n \setminus J_{l_0}$ such that*
$$\bigcup_{i=1}^m J_i = \mathbb{N}^*_n \quad \text{and} \quad J_i \cap J_{i'} = \emptyset \quad \text{if } i \neq i'.$$
4. *Set $F \coloneqq \{i \in \mathbb{N}^*_m : J_i \neq \emptyset\}$ and choose $D^{ij}_{(i)} \in M_{J_i\times J_j}(K)$ for $i \neq j$ in $F \setminus \{l_0\}$, such that*
   (a) *$D^{ri}_{(r)} D^{ij}_{(i)} = 0$ for all $r \neq i \neq j$,*
   (b) *$\#\operatorname{Supp}\big((D^{ij}_{(i)})_{k*}\big) \leq 1$ for all $i \neq j$ and $k \in J_i$,*
   (c) *If $d \in \operatorname{Supp}\big((D^{ij}_{(i)})_{k*}\big)$, then $c_d = c_k$, where $c_d \coloneqq c_d(A(l_0,l_0))$ and $c_k \coloneqq c_k(A(l_0,l_0))$.*
5. *Set*
   (a) *$D^{ij}_{(j)} \coloneqq -D^{ij}_{(i)}$ for all $i \neq j$ in $F \setminus \{l_0\}$,*
   (b) *$D^{ii}_{(i)} \coloneqq \mathrm{id}_{J_i}$ for all $i \in F \setminus \{l_0\}$,*
   (c) *$D^{rj}_{(i)} \coloneqq 0$ for all $i,j,r \in F \setminus \{l_0\}$ such that $i \notin \{j,r\}$,*
6. *For each $i \in F \setminus \{l_0\}$ define $W^{(i)} \in M_{J^c_{l_0}\times J^c_{l_0}}(K)$ by*
$$W^{(i)}_{kj} \coloneqq \big(D^{uv}_{(i)}\big)_{kj} \qquad \text{for } k \in J_u \text{ and } j \in J_v \text{ (Note that } u,v \neq l_0).$$
7. *Set $C \coloneqq A(l_0,l_0)|_{J^c_{l_0}\times J_{l_0}}$. For each $i \in F \setminus \{l_0\}$, define $A(i,l_0)$ to be the unique matrix satisfying*
$$A(i,l_0)|_{J^c_{l_0}\times J^c_{l_0}} = W^{(i)}, \quad A(i,l_0)|_{J^c_{l_0}\times J_{l_0}} = -W^{(i)}C \quad \text{and} \quad A(i,l_0)|_{J_{l_0}\times \mathbb{N}^*_n} = 0.$$
8. *For $i \notin F$ set $A(i,l_0) \coloneqq 0$.*

Proof. We first prove that the construction yields a quasi-standard column of a pre-twisting. We begin by checking that conditions (1), (2) and (3) of Corollary 2.5 are satisfied in the l_0-th column. By Remark 2.9 and Proposition 5.16 for this it suffices to prove that $\sum_{i\in F\setminus\{l_0\}} W^{(i)} = \mathrm{id}_{J^c_{l_0}}$ and ${W^{(i)}}^2 = W^{(i)}$ for all $i \in F \setminus \{l_0\}$. But the first quality follows from item (5), while the second one, from items (4)(a) and (5). It remains to check that conditions (1) and (4) of Definition 5.17 and conditions (3') and (4') of Remark 5.21 are satisfied. Condition (1) is clear; condition (2) follows from (5)(b), (5)(c) and (7); condition (3'), from (5)(c); and condition (4'), from (4)(b), (4)(c) and (5)(b).

Now we are going to check that any quasi-standard column of idempotent matrices can be constructed as above. For this note that applying an identical permutations in rows and columns we can assume that $A(l_0, l_0)$ is a standard idempotent $0,1$-matrix. Using Proposition 5.16, Definition 5.17 and Remarks 5.18, 5.19 and 5.20, a straightforward verification shows that the $A(i, l_0)$'s can be constructed following the given receipt. $\square$

Remark 5.34. Suppose we have performed the steps indicated in items (1)–(3) of Proposition 5.33. An algorithm for the construction of matrices $D_{(i)}^{ij}$ satisfying item (4) of the previous proposition is the following:

- Set $\overline{F} := F \setminus \{l_0\}$ and fix a total order in $\overline{\Delta}_{\overline{F}} := (\overline{F} \times \overline{F}) \setminus \{(x, x) : x \in \overline{F}\}$.
- For increasing $(i, j) \in \overline{\Delta}_{\overline{F}}$ perform the following construction for all $k \in J_i$, which produce the matrix $D_{(i)}^{ij}$:
  (a) If $k \in \operatorname{Supp}\bigl(\bigl(D_{(r)}^{ri}\bigr)_{t*}\bigr)$ for some $t \in J_r$ and $(r, i) < (i, j)$, then set $\bigl(D_{(i)}^{ij}\bigr)_{k*} := 0$.
  (b) Let $\mathscr{D}_i^j := \bigl\{d \in J_j : c_d = c_k \text{ and } \bigl(D_{(j)}^{jr}\bigr)_{d*} = 0 \text{ for all } (j, r) < (i, j)\bigr\}$. If $\mathscr{D}_i^j = \emptyset$, then set $\bigl(D_{(i)}^{ij}\bigr)_{k*} := 0$. Else choose $d \in \mathscr{D}_i^j$ and $\lambda \in K$ and set $\bigl(D_{(i)}^{ij}\bigr)_{kv} := \lambda\delta_{vd}$ for all $v \in J_j$.

It is clear that the above construction guarantees that for a given (i, j) we have $D_{(r)}^{ri} D_{(i)}^{ij} = 0$ for all $(r, i) < (i, j)$ and $D_{(i)}^{ij} D_{(v)}^{jr} = 0$ for $(j, r) < (i, j)$. Also it is clear that this construction, performed with d and λ arbitraries, gives all the possible families $\bigl(D_{(i)}^{ij}\bigr)_{(i,j)\in\overline{\Delta}_{\overline{F}}}$ that satisfy item (4) of Proposition 5.33.

Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a twisting map and let $r < m$. By Proposition 1.4 we know that there exists a twisting map

$$\check{\chi}\colon K^r \otimes K^n \longrightarrow K^n \otimes K^r$$

such that $\mathcal{A}_{\check{\chi}} = \bigl(A_\chi(i, l)\bigr)_{1\le i,l\le r}$ if and only if $A_\chi(i, l) = 0$ for all $i > r$ and $l \le r$. Now suppose that we have a twisting map

$$\check{\chi}\colon K^r \otimes K^n \longrightarrow K^n \otimes K^r.$$

Let $\mathcal{A} = (A(i, l))_{1\le i,l\le m}$ be a pre-twisting which is a extension of the family $\mathcal{A}_{\check{\chi}} = (A_{\check{\chi}}(i, l))_{1\le i,l\le r}$ such that

- $A(i, l) = 0$ if $i > r$ and $l \le r$,
- for $l > r$, the l-th column of $\mathcal{A}$ is a quasi-standard column.

In the following theorem we give necessary and sufficient conditions in order that $\mathcal{A}$ defines a twisting map.

Theorem 5.35. *Let $\mathcal{A}$ be as above. For all $u, v, l \in \mathbb{N}_m^*$ with $l > r$, set $D_{(i,l)}^{uv} := A(i, l)|_{J_u \times J_v}$. The family $\mathcal{A}$ defines a twisting map if and only if*

(1) *for all $i \in \mathbb{N}_m^*$,*

$$\bigcup_{l>r} F(A(i, l)) \subseteq F_0(\mathcal{A}, i).$$

(2) *If $\bigl(D_{(u,l)}^{uv}\bigr)_{kd} \ne 0$, with $u \ne v \ne l$, then*
  (*a*) *$A(u, v)_{kj} = \delta_{kj} - \delta_{jd}$ for all j,*
  (*b*) *$A(v, v)_{kj} = \delta_{jd}$ for all j,*
  (*c*) *$A(i, v)_{kj} = 0$ for $i \notin \{u, v\}$ and for all j.*

Moreover there exist $u \neq v \neq l$ such that $D^{uv}_{(u,l)} \neq 0$ if and only if the l-th column is not a standard column.

Proof. The last assertion follows immediately from the definition of standard column. Next we prove the main part of the theorem.

$\Leftarrow$) We only must show that condition (4) of Corollary 2.5 is satisfied. For $l \leq r$ this is true since

$$\sum_{h=1}^{m} A(i,h)_{kj} A(h,l)_{kj'} = \sum_{h=1}^{r} A(i,h)_{kj} A(h,l)_{kj'} = \delta_{jj'} A(i,l)_{kj},$$

because $A(h,l) = 0$ if $h > r$ and $\check{\chi}$ is a twisting map; while for $l > r$ this follows from Theorem 5.27

$\Rightarrow$) This follows immediately from Theorem 5.27. □

Proposition 5.36. *Let $\mathcal{A}$ be a pre-twisting of K^m with K^n. Assume that $A(i,i) = \mathrm{id}$ for all $i \in \mathbb{N}^*_{m-1}$. Then $\mathcal{A}$ is the family $\mathcal{A}_\chi$ of matrices associated with a twisting map χ if and only if $\big(A(l,m)\big)_{l\in\mathbb{N}^*_m}$ is a standard column.*

Proof. $\Rightarrow$) By the assumptions it is clear that the rank matrix Γ_χ introduced in Definition 2.11 has the form

$$\Gamma_\chi = \begin{pmatrix} n\,\mathrm{id}_{m-1} & * \\ 0 & * \end{pmatrix}, \tag{5.26}$$

Consequently, Γ_χ satisfies the hypothesis of Proposition 4.10 for $l = m$, and so $A(m,m)$ is a $0,1$-matrix. It remain to check that item (2) of Definition 5.3 is fulfilled for $l_0 = m$, i.e., that

$$A(k,m)_{ij} \neq 0 \Rightarrow A(m,m)_{ij} \neq 0 \qquad \text{for } k < m \text{ and } i \neq j;$$

but this follows immediately from the fact that

$$A(m,m)_{ij} = \sum_t A(t,t)_{ij} = \sum_t B_\chi(j,i)_{tt} = \mathrm{rk}((B_\chi(j,i)) \qquad \text{for all } i \neq j,$$

and $B_\chi(j,i)_{mk} = A(k,m)_{ij}$.

$\Leftarrow$) This follows from Theorem 5.35, since by Remark 5.18 we know that $F(A(m,m)) = F_0(\mathcal{A},m)$ and $A(i,i) = \mathrm{id}_n$ implies that $F_0(\mathcal{A},i) = \mathbb{N}^*_n$ for all $i < m$. □

6 Reduced rank 1

In [9] the case of twisting maps χ in which all the columns of $\mathcal{A}_\chi$ have reduced rank less than or equal to 1 (see Definition 5.28) is analysed. In this section we use our tools, that are completely different to the ones used in [9], in order to describe these twisting maps.

Proposition 6.1. *Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a twisting map. Assume that* $\mathrm{rrank}_\chi(l) = 1$ *and $A_\chi(i,l) \neq 0$ where $i \neq l$. The following facts hold:*

1. *If $A_\chi(l,i) = 0$, then the l-th column of $\mathcal{A}_\chi$ is standard. Moreover, if $A_\chi(l,l)_{kk} = 0$, then $A_\chi(i,i)_{kj} = \delta_{kj}$ for all j.*
2. *If $A_\chi(l,i) \neq 0$ and* $\mathrm{rrank}_\chi(i) = 1$, *then there is a twisting map $\psi\colon K^2 \otimes K^n \longrightarrow K^n \otimes K^2$ with $A_\psi(a,b) := A_\chi(f(a),f(b))$, where $a,b \in \{1,2\}$, $f(1) := i$ and $f(2) := l$.*

Proof. (1) By Proposition 4.10 we know that $A_\chi(l,l)$ is a $0,1$-matrix, and clearly

$$A_\chi(l,l) + A_\chi(i,l) = \mathrm{id} \Rightarrow \mathrm{Supp}\big(A_\chi(i,l)\big) \subseteq \mathrm{Supp}\big(A_\chi(l,l)\big) \cup \mathrm{Supp}(\mathrm{id}).$$

So the l-th column of $\mathcal{A}_\chi$ is standard. The last assertion follows from Proposition 5.14, since $A_\chi(l,l)_{kk} = 0$ implies that $k \in F(A_\chi(i,l))$.

(2) The family of matrices $(A_\psi(a,b))_{1\le a,b\le 2}$ satisfies the conditions of Corollary 2.5. In fact, this is clear for the three first conditions, whereas the last one follows easily from the fact that

$$\sum_{h=1}^{m} A_\chi(u,h)_{kj}A_\chi(h,v)_{kj'} = A_\chi(u,i)_{kj}A_\chi(i,v)_{kj'} + A_\chi(u,l)_{kq}A_\chi(l,v)_{kj'},$$

if $v\in\{i,l\}$. □

Proposition 6.2. *Let $\mathcal{A}=(A(i,l))_{1\le i,l\le m}$ be a pre-twisting of K^m with K^n. For each l whose reduced rank is* 1*, let $i(l)$ denote the unique $i(l)\ne l$ such that $A(i(l),l)\ne 0$. If $\operatorname{rrank}_{\mathcal{A}}(l)\le 1$ for all l, the there exists a twisting map $\chi\colon K^m\otimes K^n\longrightarrow K^n\otimes K^m$ with $\mathcal{A}_\chi=\mathcal{A}$ if and only if for each $l\in\mathbb{N}^*_m$ such that $\operatorname{rrank}_{\mathcal{A}}(l)=1$ the following facts hold:*

(1) *If $A(l,i(l))=0$, then:*
   (*a*) *$A(l,l)$ is equivalent to a standard idempotent $0,1$-matrix via identical permutations in rows and columns,*
   (*b*) *$A(i(l),i(l))_{kj}=\delta_{kj}$ for all j, whenever $A(j,j)_{kk}=0$.*

(2) *If $A(l,i(l))\ne 0$, then there is a twisting map $\psi\colon K^2\otimes K^n\longrightarrow K^n\otimes K^2$ with $A_\psi(a,b):= A_\chi(f(a),f(b))$, where $a,b\in\{1,2\}$, $f(1):=i$ and $f(2):=j$.*

Proof. The conditions are necessary by Proposition 6.1 and Corollary 4.9. On the other hand, it is straightforward to check that if $\mathcal{A}$ satisfies items (1) and (2), then it also fulfills condition (4) of Corollary 2.5. □

We associate a quiver Q_χ with a twisting map $\chi\colon K^m\otimes K^n\longrightarrow K^n\otimes K^m$ in the following way. The vertices are $1,\dots,m$ and the adjacency matrix of Q_χ is the $0,1$-matrix with 1 in the entry (i,l) if and only if $i\ne l$ and $A_\chi(i,l)\ne 0$.

Remark 6.3. Proposition 6.2 allows to construct all the twisting maps $\chi\colon K^m\otimes K^n\longrightarrow K^n\otimes K^m$ of reduced rank 1 (this means that each column of $\mathcal{A}_\chi$ has reduced rank lesser than or equal to 1, and at least one of its columns has reduced rank 1). For this it suffices to consider twisting maps with connected quivers, since every twisting map is the direct sum of the twisting maps restricted to the connected components. Each connected component of the quiver Q_χ has at most one proper oriented cycle. This follows from the fact that each vertex of the quiver is the head of at most one arrow from another vertex, since the reduced rank of χ is 1. So, in order to construct such a twisting map χ take a quiver Q fulfilling this condition and fix a connected component. There are three possible cases: the connected component is a 2-cycle, the connected component contains no 2-cycle or the connected component contains properly a 2-cycle. The two first cases were treated in [9], and in our setting are very easy to describe: In the first one χ is obtained from a twisting map $\psi\colon K^2\otimes K^n\longrightarrow K^n\otimes K^2$, as in Proposition 6.2(2). In the second one by Proposition 6.1 all columns are standard, so it suffices to consider standard twisting maps compatible with the chosen quiver in the sense that $A_\chi(i,l)\ne 0$ if and only if the adjacency matrix of Q has 1 in the entry (i,l).

In the third case assume that the 2-cycle is at the vertices i,j. Suppose that there is a reduced rank 1 twisting map χ such that $Q_\chi=Q$. By Proposition 6.1 we know that the l-th columns of $\mathcal{A}_\chi$ is standard for all $l\notin\{i,j\}$. This implies that if χ has an arrow from i to l, then $F_0(\mathcal{A}_\chi,i)\ne\emptyset$ (and similarly for j). In fact, we have

$$\emptyset\ne F(A(i,l))\subseteq F_0(\mathcal{A}_\chi,i),$$

where inequality holds since $A(i,l)=\mathrm{id}-A(l,l)$ and $A(l,l)$ is an idempotent $(0,1)$-matrix, while the inclusion is true by Proposition 5.14. Thus, in order to obtain such a twisting map χ we first construct a twisting map

$$\psi\colon K^2 \otimes K^n \longrightarrow K^n \otimes K^2,$$

such that

- $A_\psi(1,2) \neq 0 \neq A_\psi(2,1)$,
- $F_0(\mathcal{A}_\chi, i) \neq \emptyset$ if Q has an arrow that starts at i and does not end at j,
- $F_0(\mathcal{A}_\chi, j) \neq \emptyset$ if Q has an arrow that starts at j and does not end at i.

Then we set $A_\chi(h,i) := 0$ and $A_\chi(h,j) := 0$ for $h \notin \{i,j\}$, and $A_\chi(f(a), f(b)) := A_\psi(a,b)$, where $f(1) := i$ and $f(2) := j$. After that, for each vertex $l \in Q_0 \setminus \{i,j\}$, we take a standard column $(A_\chi(u,l))_{u\in Q_0}$ such that

- $A_\chi(u,l) \neq 0$ if and only Q has an arrow from u to l,
- $F(A(v,l)) \subseteq F_0(\mathcal{A}, v)$ for all $v \in Q_0$ and $l \in Q_0 \setminus \{i,j\}$.

By Proposition 5.14, Corollary (4) is satisfied for all $l \notin \{i,j\}$. Since an straightforward computation shows that it is satisfied for also for i and j, this method produces all the twisting maps of reduced rank 1 with quiver Q.

7 Quiver associated with standard and quasi-standard twisting maps

In this section we will construct quivers that characterize completely the standard twisting maps. Moreover, the quiver indicates how one could possibly generate quasi-standard twisting maps out of a standard one.

7.1 Characterization of standard twisted tensor products

The aim of this section is to completely characterize the standard twisted tensor products of K^n with K^m. In particular we will prove that they are algebras with square zero Jacobson radical. Our main result generalizes [6, Theorem 4.2]. Let

$$\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$$

be a standard twisting map. As in Remark 2.16, for each $j \in \mathbb{N}_n^*$ and $i \in \mathbb{N}_m^*$ we let x_{ji} denote $f_j \otimes e_i$. In that remark we saw that

$$x_{ki}x_{jl} = A_\chi(i,l)_{kj}x_{kl}.$$

Remark 7.1. By Remark 5.4 we know that

$$A_\chi(i,l)_{kj} = \begin{cases} 1 & \text{if } k \in J_l(l),\ i = l \text{ and } j = k, \\ 1 & \text{if } k \notin J_l(l),\ i = l \text{ and } j = c_k(A_\chi(l,l)), \\ 1 & \text{if } k \notin J_l(l),\ i = i(k,l,\mathcal{A}_\chi) \text{ (which means that } k \in F(A_\chi(i,l))) \text{ and } j = k, \\ -1 & \text{if } k \notin J_l(l),\ i = i(k,l,\mathcal{A}_\chi) \text{ and } j = c_k(A_\chi(l,l)), \\ 0 & \text{otherwise,} \end{cases}$$

which implies that

$$x_{ki}x_{jl} = \begin{cases} x_{kl} & \text{if } k \in J_l(l),\ i = l \text{ and } j = k, \\ x_{kl} & \text{if } k \notin J_l(l),\ i = l \text{ and } j = c_k(A(l,l)), \\ x_{kl} & \text{if } k \notin J_l(l),\ i = i(k,l,\mathcal{A}) \text{ and } j = k, \\ -x_{kl} & \text{if } k \notin J_l(l),\ i = i(k,l,\mathcal{A}) \text{ and } j = c_k(A(l,l)), \\ 0 & \text{otherwise.} \end{cases}$$

Remark 7.2. If $j \notin J_l(l)$, then

$$x_{ki}x_{jl} = A_\chi(i,l)_{kj}x_{kl} = 0 \quad \text{for all } k \neq j \text{ and all } i$$

and

$$x_{jl}x_{ki} = B_\chi(k,j)_{il}x_{ji} = 0 \quad \text{for all } i \neq l \text{ and all } k,$$

since $A_\chi(i,l)_{kj} \neq 0$ for some i if and only if $j = c_k\big(A_\chi(l,l)\big)$, $B(k,j)_{il} \neq 0$ for some k if and only if $l = c_i\big(B_\chi(j,j)\big)$, $c_k\big(A_\chi(l,l)\big)$ belongs to $J_l(l)$ and $c_i\big(B_\chi(j,j)\big)$ belongs to $\tilde{J}_j(j)$. From these facts it follows that $I := \bigoplus_{j\notin J_l(l)} Kx_{jl}$ is a square zero two-sided ideal of $K^n \otimes_\chi K^m$. Furthermore, each two-sided ideal including properly I has an idempotent element x_{jl}, and therefore it is not a nilpotent ideal. So, I is the Jacobson ideal $\mathrm{J}(K^n \otimes_\chi K^m)$ of $K^n \otimes_\chi K^m$.

Let ${}^\chi Q$ be the quiver with set of vertices ${}^\chi Q_0 := \{(j,i) \in \mathbb{N}_n^* \times \mathbb{N}_m^* : j \in J_i(i)\}$, set of arrows ${}^\chi Q_1 := \{\alpha_{jl} : l \in \mathbb{N}_m^* \text{ and } j \in \mathbb{N}_n^* \setminus J_l(l)\}$, and source and target maps $s,t\colon {}^\chi Q_1 \longrightarrow {}^\chi Q_0$ given by

$$s(\alpha_{jl}) := (j, i(j,l,\mathcal{A}_\chi)) \quad \text{and} \quad t(\alpha_{jl}) := (i(l,j,\mathcal{B}_\chi),l) = (c_j(A_\chi(l,l)),l).$$

Note that s and t are well defined by Proposition 5.14 and Remark 4.8, respectively.

Remark 7.3. By Remarks 4.8 and 5.4(4), Proposition 5.14 and the definition of ${}^\chi Q$, in ${}^\chi Q$ there is an arrow from (k,i) to (j,l) if and only if $A_\chi(i,l)_{kj} = -1$.

Now we compute the products $x_{ki}x_{jl}$ in terms of the maps s and t. By the computations made in Remark 7.1, the following facts hold:

(a) If $k \in J_i(i)$ and $j \in J_l(l)$, then $x_{ki}x_{jl} = \begin{cases} x_{jl} & \text{if } (k,i) = (j,l), \\ -x_{kl} & \text{if } (k,i) = s(\alpha_{kl}) \text{ and } (j,l) = t(\alpha_{kl}), \\ 0 & \text{otherwise.} \end{cases}$

(b) If $k \notin J_i(i)$ and $j \in J_l(l)$, then $x_{ki}x_{jl} = \begin{cases} x_{ki} & \text{if } (j,l) = t(\alpha_{ki}), \\ 0 & \text{otherwise.} \end{cases}$

(c) If $k \in J_i(i)$ and $j \notin J_l(l)$, then $x_{ki}x_{jl} = \begin{cases} x_{jl} & \text{if } (k,i) = s(\alpha_{jl}), \\ 0 & \text{otherwise.} \end{cases}$

(d) If $k \notin J_i(i)$ and $j \notin J_l(l)$, then $x_{ki}x_{jl} = 0$.

Theorem 7.4. *The twisted tensor product $K^n \otimes_\chi K^m$ is isomorphic to the radical square zero algebra $K\,{}^\chi Q/\langle {}^\chi Q_1^2\rangle$.*

Proof. An algebra morphism from $K\,{}^\chi Q$ to $K^n \otimes_\chi K^m$ is determined by a coherent choice of images of the vertices and the arrows of ${}^\chi Q$, since $K\,{}^\chi Q$ is a tensor algebra on the vertices set algebra of the arrows bimodule. For each $(j,l) \in {}^\chi Q_0$ set $\mathrm{In}(j,l) := \{\alpha_{ki} \in {}^\chi Q_1 : (j,l) = t(\alpha_{ki})\}$. A straightforward computation using (a)–(d) shows that

$$\phi((j,l)) := x_{jl} + \sum_{\mathrm{In}(j,l)} x_{ki} \quad \text{if } j \in J_l(l) \qquad \text{and} \qquad \phi(\alpha_{jl}) := x_{jl} \quad \text{if } j \notin J_l(l)$$

is a coherent choice and hence defines an algebra morphism $\phi\colon K\,{}^\chi Q \longrightarrow K^n \otimes_\chi K^m$. Since the elements x_{jl}'s generate linearly $K^n \otimes_\chi K^m$, the morphism ϕ is surjective. Clearly a path of length two of ${}^\chi Q$ has zero image. Since both algebras $K\,{}^\chi Q/\langle {}^\chi Q_1^2\rangle$ and $K^n \otimes_\chi K^m$ have the same dimension, the induced map

$$\overline{\phi}\colon K\,{}^\chi Q/\langle {}^\chi Q_1^2\rangle \longrightarrow K^n \otimes_\chi K^m$$

is an algebra isomorphism, as desired. $\square$

The following remark generalizes the correct version of [6, Theorem 4.6].

Remark 7.5. The quiver ${}^\chi Q = ({}^\chi Q_0, {}^\chi Q_1)$ associated with a standard twisting map χ of K^m with K^n fulfill the following properties:

(1) ${}^\chi Q_0 \subseteq \mathbb{N}_n^* \times \mathbb{N}_m^*$ and for all $l \in \mathbb{N}_m^*$ there exists $j \in \mathbb{N}_n^*$ such that $(j,l) \in {}^\chi Q_0$,
(2) ${}^\chi Q^1 = \{\alpha_{jl} : (j,l) \in (\mathbb{N}_n^* \times \mathbb{N}_m^*) \setminus {}^\chi Q_0\}$,
(3) for all $(j,l) \in (\mathbb{N}_n^* \times \mathbb{N}_m^*) \setminus {}^\chi Q_0$ there exist $i \in \mathbb{N}_m^*$ and $k \in \mathbb{N}_n^*$ such that $s(j,l) = (j,i)$ and $t(j,l) = (k,l)$.

Conversely if $Q = (Q_0, Q_1)$ is a quiver that satisfies conditions (1), (2) and (3), then there exists a unique standard twisting map χ of K^m with K^n, such that $Q = {}^\chi Q$. Indeed, by Theorem 5.10 in order to construct χ out of Q it suffices to determine families $(A(l))_{l\in N_m^*}$ and $(B(j))_{j\in N_n^*}$ of idempotent $0,1$-matrices $A(l) \in M_n(K)$ and $B(j) \in M_m(K)$ satisfying conditions (1) and (2) of that theorem. For this we define the jth row of $A(l)$ and the lth row of $B(j)$ as follows:

(1) If $(j,l) \in Q_0$, then we set $A(l)_{jh} := \delta_{jh}$,
(2) if $(j,l) \notin Q_0$, then we set $A(l)_{jh} := \delta_{kh}$, where k is defined by $t(j,l) = (k,l)$,
(3) if $(j,l) \in Q_0$, then we set $B(j)_{lh} := \delta_{lh}$,
(4) if $(j,l) \notin Q_0$, then we set $B(j)_{lh} := \delta_{ih}$, where i is defined by $s(j,l) = (j,i)$.

7.2 Iterative construction of quasi-standard twisted tensor products

The aim of this subsection is to give a method to construct the quasi-standard twisting tensor products of K^m with K^n. Through it we use the notations of the previous sections, specially those introduced in the fifth one. By Theorem 5.10 we can associate in an evident way a standard twisting map $\hat\chi$ to each quasi-standard twisting map χ. This allow us to associate a quiver ${}^\chi Q := {}^{\hat\chi} Q$ with each quasi-standard twisting tensor product $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ (Actually it is clear that the definition of ${}^\chi Q$ introduced below Remark 7.2 has perfect sense for a quasi-standard twisting map χ and that ${}^\chi Q = {}^{\hat\chi} Q$).

Proposition 7.6. *Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a quasi-standard twisting map and let $k,d \in \mathbb{N}_n^*$. Assume that $\lambda := A_\chi(u,l)_{kd} \neq 0$, or, which is equivalent, that there exist $w,v \in \mathbb{N}_m^*$ such that $d \in \operatorname{Supp}\big((D^{wv}_{(u,l)})_{k*}\big)$. If $A_\chi(u,l)_{kk} = 1$ and $A_{\hat\chi}(u,l)_{kd} = 0$, then*

$$A_\chi(u,l)_{kd} = -A_\chi(v,l)_{kd} = A_\chi(v,l)_{kc_k}, \tag{7.27}$$

where $c_k = c_k(A_\chi(l,l))$. Moreover, there are the following arrows in the quiver of $\hat\chi$:

- *α_{kv}, from (k,u) to (d,v),*
- *α_{kl}, from (k,u) to (c_k,l),*
- *α_{dl}, from (d,v) to (c_d,l), (where $c_d = c_d(A_\chi(l,l))$).*

Proof. In order to prove this result it suffices to verify that $u \neq l$, $w = u$, $d \notin \{k, c_k\}$ and $v \neq u$. In fact, by Lemma 5.25, from the fact that $d \notin \{k,c_k\}$ it follows that $v \neq l$, and hence equalities (7.27) hold by Remark 5.20 and Definition 5.17(3). Moreover, α_{kv}, α_{kl} and α_{dl} are arrows of ${}^\chi Q$ since $k \notin J_v(v)$ by Theorem 5.27, $k \notin J_l(l)$ by Theorem 5.27 and the fact that $A_\chi(u,l)_{kd}$, and $d \notin J_l(l)$ by Lemma 5.25, and the starting and target vertices of these arrow are those ones given in the statement because:

- $i(k,v,\mathcal{A}_{\hat\chi}) = u$, since $A_{\hat\chi}(u,v)_{kk} = B_{\hat\chi}(k,k)_{vu} = A_\chi(u,v)_{kk} = 1$,
- $c_k\big(A_{\hat\chi}(v,v)\big) = d$, since $A_{\hat\chi}(v,v)_{kd} = A_\chi(v,v)_{kd} = 1$,
- $i(k,l,\mathcal{A}_{\hat\chi}) = u$, since $A_{\hat\chi}(u,l)_{kk} = B_\chi(k,k)_{lu} = A_\chi(u,l)_{kk} = 1$,

- $c_k\big(A_{\hat{\chi}}(l,l)\big) = c_k$, since $A_{\hat{\chi}}(l,l) = A_\chi(l,l)$,
- $i(d,l,\mathcal{A}_{\hat{\chi}}) = v$, since $A_{\hat{\chi}}(v,l)_{dd} = B_\chi(d,d)_{lv} = A_\chi(v,l)_{dd} = 1$,
- $c_d\big(A_{\hat{\chi}}(l,l)\big) = c_d$, since $A_{\hat{\chi}}(l,l) = A_\chi(l,l)$,

where in the first and second item the last equality hold by Theorem 5.27.

So, we are reduced to prove that the facts pointed out at the beginning of the proof are true. But $u \neq l$, because otherwise

$$A_\chi(l,l)_{kd} = \delta_{kd} \Rightarrow d = k \Rightarrow A_{\hat{\chi}}(l,l)_{kd} = A_\chi(l,l)_{kk} = 1,$$

which contradicts $A_{\hat{\chi}}(u,l)_{kd} = 0$; the equality $A_\chi(u,l)_{kk} = 1$ say that $w = u$; the equalities

$$A_{\hat{\chi}}(u,l)_{kk} = 1 \quad \text{and} \quad A_{\hat{\chi}}(u,l)_{kc_k} = -1$$

imply that $d \notin \{k, c_k\}$ because $A_{\hat{\chi}}(u,l)_{kd} = 0$; and by Remark 5.19 we have $u \neq v$. □

Definition 7.7. Let $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ be a quasi-standard twisting map, $u, v, l \in \mathbb{N}_m^*$, $k \in J_u(u)$ and $d \in J_v(v)$. Assume that there are the following arrows in the quiver of $\hat{\chi}$:

- α_{kv}, from (k,u) to (d,v),
- α_{kl}, from (k,u) to (c_k,l) (where $c_k = c_k(A_\chi(l,l))$),
- α_{dl}, from (d,v) to (c_d,l) (where $c_d = c_d(A_\chi(l,l))$).

If $\operatorname{Supp}\big(D^{uv}_{(u,l)}\big) = \emptyset$, then for each $\lambda \in K$ we define the map $\chi_1\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$, by

$$\begin{aligned}
A_{\chi_1}(u,l)_{kd} &:= \lambda, \\
A_{\chi_1}(v,l)_{kd} &:= -\lambda, \\
A_{\chi_1}(v,l)_{kc_k} &:= \lambda, \\
A_{\chi_1}(u,l)_{kc_k} &:= A_\chi(u,l)_{kc_k} - \lambda, \\
A_{\chi_1}(i,t)_{js} &:= A_\chi(i,t)_{js} \qquad \text{if } (i,t,j,s) \notin \{(u,l,k,d),(v,l,k,d),(v,l,k,c_k),(u,l,k,c_k)\}.
\end{aligned}$$

If necessary to be more precise the map χ_1 will be denoted by $\Lambda^\lambda_{(k,u),(d,v),(c_k,l)}(\chi)$.

Remark 7.8. Note that if $\lambda = 0$, then $\chi_1 = \chi$.

Remark 7.9. By Remark 7.3, if the twisting map χ satisfies the assumptions made in Definition 7.7, then $A_{\hat{\chi}}(u,l)_{kc_k} = -1$, which by Remark 5.9 implies that $A_{\chi_1}(u,l)_{kk} = A_{\hat{\chi}}(u,l)_{kk} = 1$. Moreover, by the very definition of ${}^\chi Q$, the existence of the arrows α_{kv}, α_{kl} and α_{dl} implies that u, v and l are three different elements of $\mathbb{N}_m^*$. Since $k \in J_u(u) \setminus (J_v(v) \cup J_l(l))$, $c_k \in J_l(l)$ and $d \in J_v(v) \setminus J_l(l)$, from this fact it follows that k, c_k and d are three different elements of $\mathbb{N}_n^*$, which implies that $A_{\hat{\chi}}(u,l)_{kd} = 0$, because $\operatorname{Supp}(A_{\hat{\chi}}(u,l)_{k*}) = \{k, c_k\}$. So, the hypothesis of Proposition 7.6 are satisfied by χ_1, provided that it is a quasi-standard twisting map.

Remark 7.10. Assume that the twisting map χ satisfies the assumptions made in Definition 7.7. If χ_1 is a (quasi-standard) twisting map, then $\Gamma_{\chi_1} = \Gamma_\chi$ and $\tilde{\Gamma}_{\chi_1} = \tilde{\Gamma}_\chi$.

Proposition 7.11. *Let χ, χ_1, u, v, l, k, d, α_{kv}, α_{kl}, α_{kd} and λ be as in Definition 7.7. Assume that $\lambda \neq 0$. If χ_1 is a quasi-standard twisting map, then*

(1) *$A_{\chi_1}(i,u)_{kj} = A_{\hat{\chi}}(i,u)_{kj}$ and $A_{\chi_1}(i,v)_{kj} = A_{\hat{\chi}}(i,v)_{kj}$ for all i, j,*
(2) *$A_{\chi_1}(u,l)$, $A_{\chi_1}(v,l)$, $B_{\chi_1}(d,k)$ and $B_{\chi_1}(c_k,k)$ are idempotent matrices.*

Moreover, condition (2) implies that χ_1 is a (quasi-standard) twisting map.

Proof. By Remark 7.9 we know that $A_{\chi_1}(u,l)_{kk} = A_\chi(u,l)_{kk} = 1$, and so, by Theorem 5.27(1),

$$A_{\chi_1}(i,u)_{kj} = \delta_{iu}\delta_{kj} = A_{\hat{\chi}}(i,u)_{kj} \quad \text{for all } i,j.$$

On the other hand, since $A_{\chi_1}(u,l)_{kd} \neq 0$, by Theorem 5.27(2) we have

$$A_{\chi_1}(i,v)_{kj} = \begin{cases} \delta_{kj} - \delta_{jd} & \text{if } i = u,\\ \delta_{jd} & \text{if i=v},\\ 0 & \text{otherwise,} \end{cases}$$

and a direct computation using Theorem 5.10 shows that $A_{\hat{\chi}}(i,v)_{kj}$ is given by the same formula (for this computation is can be useful to see the proof of Proposition 7.6). This finishes the proof of condition (1). By items (1) and (2) of Proposition 2.3, condition (2) is also satisfied. Finally, by Remark 2.9 condition (2) is sufficient for χ_1 to be a twisting map, since $\sum_i A_{\chi_1}(i,l) = \mathrm{id}_{K^n}$ and $\sum_j B_{\chi_1}(j,k) = \mathrm{id}_{K^m}$. □

Corollary 7.12. *Under the assumptions made in Definition 7.7, if χ is standard, then χ_1 is a quasi-standard twisting map.*

Proof. When $\lambda = 0$ this is evident, whereas when it is different from 0 a straightforward computation shows that $A_{\chi_1}(u,l)$, $A_{\chi_1}(v,l)$, $B_{\chi_1}(d,k)$ and $B_{\chi_1}(c_k,k)$ are idempotent matrices. □

Remark 7.13. A straightforward computation shows that if χ_1 of Definition 7.7 is a twisting map, then the construction χ_1 out of χ corresponds to a formal deformation in the sense of Gerstenhaber. To be more precise, the multiplication map μ_{χ_1} of χ_1 is given by

$$\mu_{\chi_1}(a\otimes b) = \mu_0(a\otimes b) + \lambda\mu_1(a\otimes b),$$

where μ_0 is the multiplication in $D := K^n \otimes_\chi K^m$ and $\mu_1\colon D\otimes D \longrightarrow D$ is the map defined by

$$\begin{aligned} \mu_1(x_{ku}\otimes x_{dl}) &= 1\\ \mu_1(x_{kv}\otimes x_{c_kl}) &= 1\\ \mu_1(x_{kv}\otimes x_{dl}) &= -1\\ \mu_1(x_{ku}\otimes x_{c_kl}) &= -1 \end{aligned}$$

and

$$\mu_1(x_{pq}\otimes x_{rs}) = 0 \qquad \text{if } (x_{pq},x_{rs}) \notin \{(x_{ku},x_{dl}),(x_{kv},x_{dl}),(x_{ku},x_{c_kl}),(x_{kv},x_{c_kl})\}.$$

Remark 7.14. Each quasi-standard twisting map can be obtained from a standard twisting map by applying repeatedly the construction of Definition 7.7, thus adding parameters $\lambda_1,\lambda_2,\lambda_3,\dots$, obtaining quasi-standard twisting maps $\chi_1,\chi_2,\chi_3,\dots$.

Remark 7.15. Let $\chi\colon K^m\otimes K^n \longrightarrow K^n\otimes K^m$ be a quasi-standard twisting map. For each $u,l,v\in \mathbb{N}_m^*$ and $k,d\in\mathbb{N}_n^*$ such that u, l and v are three different elements of $\mathbb{N}_m^*$, $k\in J_u(u)$, $d\in J_v(v)$ and $A_\chi(u,l)_{kd}\neq 0$, the quiver of $\hat{\chi}$ has a triangle with vertices (k,u), (d,v) and (c_k,l), and arrows α_{kv}, α_{kl} and α_{dl}, from (k,u) to (d,v), (k,u) to (c_k,l), and (d,v) to (c_k,l), respectively. In fact, this follows from the previous results and the fact that $c_k = c_d$ by Definition 5.17(4).

7.3 Jacobson radical of quasi-standard twisted tensor products

Let $\chi\colon K^m\otimes K^n \longrightarrow K^n\otimes K^m$ be a quasi-standard twisting map. For each $j\in\mathbb{N}_n^*$ and $l\in\mathbb{N}_m^*$, let x_{jl} be as in Remark 2.16. In this subsection we prove that, as in the case when χ is standard, the Jacobson ideal $\mathrm{J}(C)$ of $C := K^n\otimes_\chi K^m$ is the ideal $I := \bigoplus_{j\notin J_l(l)} Kx_{jl}$ of C (however unlike what happens in the standard case, when χ is not standard I can be not a square zero ideal). As a consequence there exists a subalgebra $A \simeq \frac{C}{\mathrm{J}(C)}$ of C such that $C = A\bigoplus \mathrm{J}(C)$.

Theorem 7.16. *Let χ, C and I be as above. Then I is the Jacobson ideal of C.*

Proof. For each $j,k \in \mathbb{N}_n^*$ and $i,l \in \mathbb{N}_m^*$. If $i \neq l$ or $k \neq j$, then

$$x_{ki}x_{jl} = A_\chi(i,l)_{kj}x_{kl} \in I.$$

In fact, if $k \notin J_l(l)$, then this is true by the very definition of I, and if $k \in J_l(l)$, then it is true because $A_\chi(i,l)_{kj} = 0$ by Theorem 5.27(1). So, I is a two-sided ideal of C. To finish the proof it suffices to show that

$$x_{j_1l_1}\cdots x_{j_{n+1}l_{n+1}} = 0 \qquad \text{for each } x_{j_1l_1},\dots,x_{j_{n+1}l_{n+1}} \in I.$$

By the above argument this is true if there exist $s < t$ such that $j_s \in J_{l_t}(l_t)$. So, assume that this is not the case. Then, since $j_1,\dots,j_{n+1} \in \mathbb{N}_n^*$ there exist $u < v$ such that $j_u = j_v$, and so

$$x_{j_ul_u}\cdots x_{j_vl_v} = \prod_{h=0}^{v-u-1} A_\chi(l_{u+h}, l_{u+h+1})_{j_uj_{u+h+1}}x_{j_ul_v} = 0,$$

because $A_\chi(l_{v-1}, l_v)_{j_uj_v} = 0$ by the fact that $j_u \notin J_{l_{v-1}}(l_{v-1})$ and Theorem 5.27(1). □

Corollary 7.17. *Under the hypothesis of Theorem7.16, the quotient algebra $\frac{C}{\mathrm{J}(C)}$ is isomorphic to the direct product $\prod_{j\in J_l(l)} Kx_{jl}$ of fields, and there exists a subalgebra $A \simeq \frac{C}{\mathrm{J}(C)}$ of C such that $C = A \bigoplus \mathrm{J}(C)$.*

Proof. The first assertion follows from the fact that for each $i,l \in \mathbb{N}_m^*$, $k \in J_i(i)$ and $j \in J_l(l)$, with $j \neq k$,

$$\begin{aligned} x_{jl}x_{jl} &= A(l,l)_{jj}x_{jl} = x_{jl},\\ x_{ki}x_{jl} &= A(i,l)_{kj}x_{kl} \in I \text{ if } k \notin J_l(l) \end{aligned}$$

and

$$x_{ki}x_{jl} = A(i,l)_{kj}x_{kl} = 0 \text{ if } k \in J_l(l).$$

The second assertion follows now by a direct application of the Principal Theorem of Wedderburn-Malcev ([16, Chapter 11]). □

Remark 7.18. It is easy to check that if $\chi\colon K^m \otimes K^n \longrightarrow K^n \otimes K^m$ is a quasi-standard twisting map that it is not standard, then $\mathrm{J}(C)^2 \neq 0$.

8 Low dimensional cases

In this section we determine the twisting maps of K^3 with K^3. To achieve this it is convenient first to describe in detail the twisting maps of K^2 with K^2 and the twisting map of K^2 with K^3.

8.1 Twisting maps of K^2 with K^2

We first use our results to obtain a classification of all twisting maps $\chi\colon K^2 \otimes K^2 \longrightarrow K^2 \otimes K^2$ in a direct way. This classification was already obtained by [12]. By Corollary 2.12 and Proposition 2.13 we can assume that the $\mathcal{A}_\chi$-rank matrix is one of the following:

$$\Gamma_1 = \begin{pmatrix} 2 & 0\\ 0 & 2\end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 2 & 1\\ 0 & 1\end{pmatrix} \quad \text{or} \quad \Gamma_3 = \begin{pmatrix} 1 & 1\\ 1 & 1\end{pmatrix}.$$

First case If the $\mathcal{A}_\chi$-rank matrix is Γ_1, then $A(1,1) = A(2,2) = \mathrm{id}$. Consequently χ is the flip and $K^2 \otimes_\chi K^2 \cong K^4$.

Second case If the $\mathcal{A}_\chi$-rank matrix is Γ_2, then χ is a standard twisting map (use Proposition 4.10), and one verifies readily that χ is equivalent via identical permutations in rows and columns to the twisting map χ' with quiver

$$
{}^{\chi'}Q = \begin{tikzcd} \bullet(1,1) & \bullet(1,2) \\ \bullet(2,1) \arrow[ur, "\alpha_{22}"'] & \circ(2,2) \end{tikzcd}.
$$

Here the bullets represent the vertices of ${}^{\chi'}Q$, and the white circle in the coordinate $(2,2)$, indicates that the arrow α_{22} starts at the 2-th row and ends at the 2-th column. It is easy to recover the matrices of $\mathcal{A}_{\chi'}$ from ${}^{\chi'}Q$. We have:

$$
A(1,1) = \mathrm{id}, \quad A(2,1) = 0, \quad A(2,2) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad A(1,2) = \begin{pmatrix} 0 & 0 \\ -1 & 1 \end{pmatrix}.
$$

In the sequel we will simply represent the quivers of this twisting map and of its equivalent twisting maps as

$$
\begin{tikzcd} \bullet & \bullet \\ \bullet \arrow[ur] & \circ \end{tikzcd} \qquad
\begin{tikzcd} \bullet \arrow[dr] & \circ \\ \bullet & \bullet \end{tikzcd} \qquad
\begin{tikzcd} \circ & \bullet \arrow[dl] \\ \bullet & \bullet \end{tikzcd} \qquad
\begin{tikzcd} \bullet & \bullet \\ \circ & \bullet \arrow[ul] \end{tikzcd}
$$

where there is a bullet in the position (j,i) if (j,i) is a vertex (thus $j \in J_i(i)$); and there is a white circle in the position (j,l) if the quiver has an arrow α_{jl} that starts at the j-th row and ends at the l-th column (it is unique). The quivers associated with standard twisting maps of K^3 with K^2 and of K^3 with K^3 will be represented by diagrams constructed following the same instructions.

Third case If the $\mathcal{A}_\chi$-rank matrix is Γ_3, then by Remark 3.1, there exist $a, a' \in K$, such that

$$
A_\chi(1,1) = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \qquad A_\chi(2,1) = \begin{pmatrix} 1-a & a-1 \\ -a & a \end{pmatrix},
$$

$$
A_\chi(1,2) = \begin{pmatrix} 1-a' & a'-1 \\ -a' & a' \end{pmatrix}, \qquad A_\chi(2,2) = \begin{pmatrix} a' & 1-a' \\ a' & 1-a' \end{pmatrix}.
$$

Thus, by (2.3) we have $B_\chi(1,1) = \left(\begin{smallmatrix} a & 1-a \\ 1-a' & a' \end{smallmatrix}\right)$. Therefore $a' = 1-a$ by Proposition 4.15 and Remark 3.1, and so

$$
A_\chi(1,1) = \begin{pmatrix} a & 1-a \\ a & 1-a \end{pmatrix}, \qquad A_\chi(2,1) = \begin{pmatrix} 1-a & a-1 \\ -a & a \end{pmatrix},
$$

$$
A_\chi(1,2) = \begin{pmatrix} a & -a \\ a-1 & 1-a \end{pmatrix}, \qquad A_\chi(2,2) = \begin{pmatrix} 1-a & a \\ 1-a & a \end{pmatrix}.
$$

Now a direct computation using (2.3) shows that $B_\chi(i,j) = A_\chi(i,j)$ for $i,j = 1,2$, which enables one to check easily that the conditions of Proposition 2.3 are satisfied. Hence we have a family of twisting maps parameterized by $a \in K$. Applying Proposition 2.13 we see that the twisting maps corresponding to a and $1-a$ are isomorphic. Moreover, using again the same proposition, we check that these are the only isomorphisms between these twisting maps. If $a \in \{0,1\}$, then the twisting map is standard and the quiver is one of

$$
\begin{tikzcd} \circ & \bullet \arrow[dl, bend right] \\ \bullet \arrow[ur, bend right] & \circ \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} \bullet \arrow[dr, bend right] & \circ \\ \circ & \bullet \arrow[ul, bend right] \end{tikzcd}.
$$

On the other hand, by Proposition 2.14 and Remark 2.15 we know that for $a \notin \{0,1\}$, the map $\rho_1\colon K^2 \otimes_\chi K^2 \longrightarrow M_2(K)$, given by

$$\rho_1(f_j \otimes 1) := E^{jj} \quad \text{and} \quad \rho_1(1 \otimes e_i) := A(i,1),$$

is an algebra isomorphism. So we obtain in this case, modulo isomorphism, four different algebras.

8.2 Twisting maps of K^3 with K^2

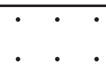
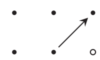
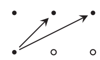
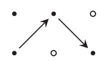
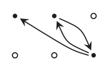
Now we use our results to classify all the twisting maps χ of K^3 with K^2 (By Remark 2.2, Proposition 5.8 and Proposition 5.32, this immediately gives a similar classification for the twisting maps of K^2 with K^3). By Corollary 2.12 and Proposition 2.13 we can assume that the $\mathcal{A}_\chi$-rank matrix is one of the following:

$$\Gamma_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Gamma_4 = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Gamma_5 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad \Gamma_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \Gamma_7 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

By Proposition 4.10, except perhaps in the cases Γ_5 and Γ_6, the matrices $A_\chi(l,l)$ are $0,1$-matrices, which (since the reduced rank of χ is less than or equal to 1) implies that χ is a standard twisting map. So we list all the possible standard twisting maps (for this we use the method given in Remark 7.5):

TABLE 1. Standard twisting maps of K^3 with K^2

#	$\sum \mathrm{Tr}$	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.
1.	6		$\begin{pmatrix} 2&0&0\\0&2&0\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0\\0&3 \end{pmatrix}$	1
2.	5		$\begin{pmatrix} 2&0&0\\0&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1\\0&2 \end{pmatrix}$	12
3.	4		$\begin{pmatrix} 2&1&1\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&2\\0&1 \end{pmatrix}$	6
4.	4		$\begin{pmatrix} 2&1&0\\0&1&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1\\1&2 \end{pmatrix}$	12
5.	4		$\begin{pmatrix} 2&0&0\\0&1&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1\\1&2 \end{pmatrix}$	6
6.	3		$\begin{pmatrix} 1&0&0\\0&1&1\\1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&2\\1&1 \end{pmatrix}$	12

Here $\sum \mathrm{Tr} := \sum_i \mathrm{Tr}(A(i,i)) = \sum_j \mathrm{Tr}(B(j,j))$ and # equiv. indicates how many equivalent standard twisting maps there are (Here and in the sequel we say that two standard twisting maps K^m with K^n are equivalent if they are isomorphic).

If $\Gamma_\chi = \Gamma_5$, then χ is a direct sum of two twisting maps, and the twisted tensor product algebra is isomorphic to $K^2 \oplus A$, where A is a twisted tensor product $K^2 \otimes_{\chi'} K^2$ with $\Gamma_{\chi'} = \left(\begin{smallmatrix} 1&1\\1&1 \end{smallmatrix}\right)$, so either it is standard (recovering the case #5 in the list), or it corresponds to a value of $a \notin \{0,1\}$ in the third case of Subsection 8.1, and we obtain an algebra isomorphic to $K^2 \oplus M_2(K)$.

If $\Gamma_\chi = \Gamma_6$, then by Proposition 4.10 the first column of $\mathcal{A}_\chi$ is a standard column, so that either $A_\chi(1,1) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$ and $A_\chi(3,1) = \left(\begin{smallmatrix} 0 & 0 \\ -1 & 1 \end{smallmatrix}\right)$, or $A_\chi(1,1) = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A_\chi(3,1) = \left(\begin{smallmatrix} 1 & -1 \\ 0 & 0 \end{smallmatrix}\right)$ (by Proposition 2.13 we can assume, and we do it, that $A_\chi(1,1) = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)$). Moreover, by Proposition 1.4 the matrices $A_\chi(i,j)$ for $i,j \in \{2,3\}$ define a 2 times 2 twisting map χ' with $\Gamma_{\chi'} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$, which is either standard, or has $1-a, a \notin \{0,1\}$ on the diagonal of $A_{\chi'}(3,3) = A_\chi(3,3)$. But Theorem 5.35 shows that

$$\{2\} = F(A(3,1)) \subseteq F_0(\mathcal{A},3).$$

So $A(3,3)_{22} = 1$ and the twisting map is standard, corresponding to the sixth case on the list.

If $\Gamma_\chi = \Gamma_7$, then the twisting map should be standard, but no standard twisting map χ yields $\Gamma_\chi = \Gamma_7$, so there is no twisting map in this case.

8.3 Twisting maps of K^3 with K^3

We next aim is to construct (up to isomorphisms) all the twisting maps of K^3 with K^3. Since in the appendix we list all standard and quasi-standard twisting maps of K^3 with K^3, for this purpose in this section we only need construct the twisting maps that are not quasi-standard. In order to carry out this task in addition to the previous results, we will use the following ones:

Remark 8.1. Let $\mathcal{A} = (A(i,l))_{i,l \in \mathbb{N}_m^*}$ be a pre-twisting of K^m with K^n. If the l-th column of $\mathcal{A}$ has reduced rank 1 and $A(l,l)$ is a $0,1$-matrix, then the l-th column of $\mathcal{A}$ is standard.

Proposition 8.2. *Let $\chi\colon K^m \otimes K^3 \longrightarrow K^3 \otimes K^m$ be a twisting map and let i_1, i_2 and i_3 be three different elements of $\mathbb{N}_m^*$ such that $A_\chi(i_2,i_1) \neq 0 \neq A_\chi(i_3,i_1)$ and $A_\chi(i_1,i_1)$ is equivalent to the matrix*

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

via identical permutations in rows and columns. If the i_1-th column of $\mathcal{A}_\chi$ is not quasi-standard, then the following facts hold:

1. *$A_\chi(i_2,i_3) \neq 0 \neq A_\chi(i_3,i_2)$ and neither the i_2-th nor the i_3-th column of $\mathcal{A}_\chi$ are quasi-standard columns.*
2. *If $A_\chi(i_1,i_1) = \left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right)$, then*

   $$A_\chi(i_2,i_1)_{22} = A_\chi(i_2,i_2)_{22} = A_\chi(i_2,i_3)_{22}, \qquad A_\chi(i_3,i_1)_{22} = A_\chi(i_3,i_2)_{22} = A_\chi(i_3,i_3)_{22},$$

   and there exist $z \in K^\times$ and $\alpha \in K^\times \setminus \{1\}$ such that

   $$A_\chi(i_2,i_1) = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha - z & \alpha & z \\ \alpha - 1 - \frac{\alpha(1-\alpha)}{z} & \frac{\alpha(1-\alpha)}{z} & 1-\alpha \end{pmatrix}$$

   and

   $$A_\chi(i_3,i_1) = \begin{pmatrix} 0 & 0 & 0 \\ \alpha + z - 1 & 1-\alpha & -z \\ \frac{\alpha(1-\alpha)}{z} - \alpha & -\frac{\alpha(1-\alpha)}{z} & \alpha \end{pmatrix}.$$

Proof. Without loss of generality we can assume that $A_\chi(i,1)=0$ for $i>3$ and that $i_1=1$, $i_2=2$ and $i_3=3$. By items (1) and (3) of Corollary 2.5 we know that $A_\chi(1,1)A_\chi(i,1)=0$ for all $i>1$ and that $A_\chi(1,1)+A_\chi(2,1)+A_\chi(3,1)=\mathrm{id}_3$. Hence there exists $\alpha\in K$ such that

$$A_\chi(2,1)=\begin{pmatrix}0&0&0\\ *&\alpha&*\\ *&*&1-\alpha\end{pmatrix}\quad\text{and}\quad A_\chi(3,1)=\begin{pmatrix}0&0&0\\ *&1-\alpha&*\\ *&*&\alpha\end{pmatrix},$$

where the $*$'s denote arbitrary elements of K. Moreover, by Proposition 5.30(2) we know that $\alpha\notin\{0,1\}$. Let $z\coloneqq A_\chi(3,1)_{23}$. Since the sum of the entries of each row of $A_\chi(2,1)$ and $A_\chi(3,1)$ is zero,

$$A_\chi(2,1)=\begin{pmatrix}0&0&0\\ -\alpha-z&\alpha&z\\ *&*&1-\alpha\end{pmatrix}\quad\text{and}\quad A_\chi(3,1)=\begin{pmatrix}0&0&0\\ \alpha+z-1&1-\alpha&-z\\ *&*&\alpha\end{pmatrix}.$$

Furthermore, since lower triangular idempotent matrices have 0 or 1 in each diagonal entry, necessarily $z\neq 0$. Now it is clear that, since $\mathrm{rk}\big(A_\chi(2,1)\big)=\mathrm{rk}\big(A_\chi(3,1)\big)=1$, both matrices have the desired form. But then the first row of $B_\chi(2,2)$ is $(0,\alpha,1-\alpha,0,\dots,0)$, the first row of $B_\chi(1,2)$ is $(1,-(\alpha+z),(\alpha+z)-1,0,\dots,0)$ and the first row of $B_\chi(3,2)$ is $(0,z,-z,0,\dots,0)$. An easy computation using these facts, that by Remark 2.6 we have $B(1,2)+B(2,2)+B(3,2)=\mathrm{id}_3$, and that Proposition 2.3(2) the columns of $B(2,2)$ are orthogonal to the first rows of $B(1,2)$ and $B(3,2)$, shows that

$$B(2,2)=\begin{pmatrix}0&\alpha&1-\alpha&0&\dots&0\\ 0&\alpha&1-\alpha&0&\dots&0\\ 0&\alpha&1-\alpha&0&\dots&0\\ *&*&*&*&&*\\ \vdots&&&\vdots&&\vdots\\ *&*&*&*&&*\end{pmatrix},$$

which finishes the proof of item (2) via (2.3).

Item (1) follows from the fact that $A(3,2)_{22}=1-\alpha\notin\{0,1\}$ and $A(2,3)_{22}=\alpha\notin\{0,1\}$. □

Our next aim is to determine up to isomorphisms all twisting maps $\chi\colon K^3\otimes K^3\longrightarrow K^3\otimes K^3$ which are not quasi-standard. For this, we can and we will assume that the values of the diagonal of Γ_χ are non increasing. So in the rest of this subsection χ denotes an arbitrary twisting map satisfying this restriction and we look for conditions in order that χ be not quasi-standard. We organize our search according to the values of $\sum\mathrm{Tr}\coloneqq\sum_i\mathrm{Tr}(A_\chi(i,i))=\sum_j\mathrm{Tr}(B_\chi(j,j))$.

8.3.1 $\sum\mathrm{Tr}=9,8$ or 7

Here the values of the diagonal of Γ_χ may be $(3,3,3)$, $(3,3,2)$, $(3,3,1)$ or $(3,2,2)$. By Proposition 5.36, in the first three cases necessarily χ is a standard twisting map. In the last case Γ_χ is equivalent via identical permutations in rows and columns to one of the following matrices:

$$\begin{pmatrix}3&1&1\\0&2&0\\0&0&2\end{pmatrix},\quad\begin{pmatrix}3&1&0\\0&2&1\\0&0&2\end{pmatrix}\quad\text{or}\quad\begin{pmatrix}3&0&0\\0&2&1\\0&1&2\end{pmatrix}.$$

By Proposition 4.10 in the two first cases the diagonal matrices are $0,1$-matrices, and so by Remark 8.1 the obtained twisting maps are standard. In the last one χ is a direct sum of the flip of K with K^3 and a twisting map $\chi'\colon K^2\otimes K^3\longrightarrow K^3\otimes K^2$. Moreover, the analysis made out in Subsection 8.2 shows that if χ' is not quasi-standard, then $K^3\otimes_{\chi'}K^2$ is isomorphic to $K^2\times M_2(K)$. Thus, in this case $K^3\otimes_\chi K^3\simeq K^5\times M_2(K)$.

8.3.2 $\sum \mathrm{Tr} = 6$

The diagonal of Γ_χ is either $(2,2,2)$ or $(3,2,1)$. We treat each case separately:

$\mathbf{Diag}(\boldsymbol{\Gamma_\chi}) = \mathbf{(2,2,2)}$ By Proposition 2.13 we can assume that the first column is $(2,1,0)^\perp$, or, in other words, that Γ_χ it is one of the following matrices:

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

Moreover, by Proposition 4.10 and Remark 8.1, each twisting map whose rank matrix is the last one is standard, and, again by Proposition 2.13, each twisting map whose rank matrix is the first or the second one is isomorphic to one twisting map whose rank matrix is the third one. So we only must consider the case

$$\Gamma_\chi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Since, by Proposition 4.10 and Remark 8.1, the first column is standard, the hypothesis of Theorem 5.35 are satisfied. By this theorem and Proposition 1.4, we know that χ is twisting map if and only if the first column of $\mathcal{A}_\chi$ is standard and the matrices $A_\chi(2,2)$, $A_\chi(3,2)$, $A_\chi(3,3)$ and $A_\chi(2,3)$ define a twisting map of K^2 with K^3 such that $F(A_\chi(2,1)) \subseteq F_0(\mathcal{A}_\chi, 2)$ (In fact, we also need that $F(A(i,1)) \subseteq F_0(\mathcal{A}, i)$ for $i \in \{1,3\}$, but for $i = 3$ this is trivial and for $i = 1$ its follows from Remark 5.18). Since we are looking for non quasi-standard twisting maps, by the discussion in Subsection 8.2 we may assume that

$$A_\chi(2,2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1-a \\ 0 & a & 1-a \end{pmatrix}, \qquad A_\chi(3,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1-a & a-1 \\ 0 & -a & a \end{pmatrix},$$

$$A_\chi(3,3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-a & a \\ 0 & 1-a & a \end{pmatrix}, \qquad A_\chi(2,3) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -a \\ 0 & a-1 & 1-a \end{pmatrix}.$$

But then $F_0(\mathcal{A}_\chi, 2) = \{1\}$, so necessarily

$$A_\chi(2,1) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad A_\chi(2,1) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In both cases setting $A_\chi(1,1) \coloneqq \mathrm{id}_3 - A_\chi(2,1)$, $A_\chi(3,1) = 0$, $A_\chi(1,2) = 0$ and $A_\chi(1,3) = 0$ (which is forced), we obtain a twisting map which is not quasi-standard. In the first one

$$\tilde{\Gamma}_\chi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix},$$

whereas in the second one

$$\tilde{\Gamma}_\chi = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

Taking into account Proposition 5.32, and applying the same arguments to $\tilde{\chi}$, we conclude that χ is a non quasi-standard twisting map with $\mathrm{Diag}(\Gamma_\chi) = (2,2,2)$ if and only if $\tilde{\chi}$ is.

$\mathrm{Diag}(\Gamma_\chi) = (3,2,1)$ Assume that χ is a not quasi-standard twisting map. Then, by the last assertion we know that $\mathrm{Diag}(\tilde{\Gamma}_\chi) = (3,2,1)$. The rank matrix Γ_χ is one of the following matrices:

$$\begin{pmatrix}3&0&0\\0&2&2\\0&1&1\end{pmatrix},\quad \begin{pmatrix}3&0&2\\0&2&0\\0&1&1\end{pmatrix},\quad \begin{pmatrix}3&0&1\\0&2&1\\0&1&1\end{pmatrix},\quad \begin{pmatrix}3&1&0\\0&2&2\\0&0&1\end{pmatrix},\quad \begin{pmatrix}3&1&1\\0&2&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}3&1&2\\0&2&0\\0&0&1\end{pmatrix}.$$

By Proposition 4.14, both Γ_χ and $\tilde{\Gamma}_\chi = \Gamma_{\tilde{\chi}}$ must be one of the last two matrices. But by Corollary 4.11, Proposition 5.8 and Remark 8.1, if Γ_χ or $\tilde{\Gamma}_\chi$ is the last matrix, then χ is a standard twisting map. So the only chance of being not standard for the twisting map χ is that both Γ_χ and $\tilde{\Gamma}_\chi$ be the second last matrix. In that case by Propositions 4.10 and 5.30(2) we recover the family of quasi-standard twisting maps listed in number 20 in the appendix.

8.3.3 $\sum \mathbf{Tr} = 5$

The diagonal of Γ is either $(2,2,1)$ or $(3,1,1)$. We treat each case separately:

$\mathrm{Diag}(\Gamma_\chi) = (2,2,1)$ By Proposition 2.13 we can assume that the rank matrix Γ_χ is one of the following matrices:

$$\begin{gathered}\begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix},\quad \begin{pmatrix}2&1&1\\0&2&1\\1&0&1\end{pmatrix},\quad \begin{pmatrix}2&0&1\\0&2&1\\1&1&1\end{pmatrix},\quad \begin{pmatrix}2&1&0\\1&2&2\\0&0&1\end{pmatrix}\\ \begin{pmatrix}2&1&0\\0&2&2\\1&0&1\end{pmatrix},\quad \begin{pmatrix}2&0&0\\1&2&2\\0&1&1\end{pmatrix},\quad \begin{pmatrix}2&0&0\\0&2&2\\1&1&1\end{pmatrix}.\end{gathered} \tag{8.28}$$

Since $\tilde{\Gamma}$ has at least one 1 in the diagonal, By Proposition 4.14 the rank matrix Γ_χ can not be the first of the second row. Assume first that

$$\Gamma_\chi = \begin{pmatrix}2&1&1\\1&2&1\\0&0&1\end{pmatrix} \tag{8.29}$$

and that the twisting map χ is not standard. So, χ is an extension of a twisting map χ' of K^2 with K^3. Clearly, if the third column of $\mathcal{A}_\chi$ is quasi-standard, then χ' must be a non quasi-standard twisting map. But by Proposition 8.2(1) this is also the case if the third column of $\mathcal{A}_\chi$ is quasi-standard. Thus, by the analysis made out in subsection 8.2, we can assume that there exists $a \in K \setminus \{0,1\}$, such that

$$A_{\chi'}(1,1) = \begin{pmatrix}1&0&0\\0&a&1-a\\0&a&1-a\end{pmatrix},\qquad A_{\chi'}(1,2) = \begin{pmatrix}0&0&0\\0&a&-a\\0&a-1&1-a\end{pmatrix}$$

$$A_{\chi'}(2,1) = \begin{pmatrix}0&0&0\\0&1-a&a-1\\0&-a&a\end{pmatrix},\qquad A_{\chi'}(2,2) = \begin{pmatrix}1&0&0\\0&1-a&a\\0&1-a&a\end{pmatrix}.$$

If the third column of $\mathcal{A}_\chi$ is quasi-standard, then by Theorem 5.35, we have

$$\{1\} = F_0(\mathcal{A}_\chi, 1) \supseteq F(A_\chi(1,3)) \neq F(A_\chi(2,3)) \subseteq F_0(\mathcal{A}_\chi, 2) = \{1\},$$

a contradiction. Hence it is not quasi-standard. Moreover, by Proposition 4.10 necessarily $A_\chi(3,3)$ is one of the following matrices:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}. \tag{8.30}$$

In the two last cases a straightforward computation using Propositions 2.13 and 8.2(2) leads to the contradiction $A(1,2)_{11} \neq 0$. Hence $A_\chi(3,3)$ is the first matrix. Now applying Proposition 8.2, we obtain a family of not quasi-standard twisting maps parameterized by $\alpha \in K \setminus \{0,1\}$ and $z \in K^\times$. Moreover, we have

$$\tilde{\Gamma}_\chi = \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}. \tag{8.31}$$

If Γ_χ is not the matrix at the right side of equality (8.29) , then $\tilde{\Gamma}_\chi$ can not be that matrix because Γ_χ would be the matrix at the right side of equality (8.31). But all the other possible matrices for $\tilde{\Gamma}_\chi$ (including those with diagonal $(3,1,1)$) have exactly one row without zeroes, and so, by Propositions 4.14 and 2.13 we can assume that $A_\chi(3,3)$ is the first matrix in (8.30). By Proposition 8.2, if

$$\Gamma_\chi = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \Gamma_\chi = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

then χ is a quasi-standard twisting map. If

$$\Gamma_\chi = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

then χ must be quasi-standard. In fact, otherwise it is an extension of a not quasi-standard twisting map of K^3 with K^2, and we know that in this case $\# F_0(\mathcal{A}_\chi, 2) = 1$, which contradict the fact that $\# F(A_\chi(2,3)) = 2$ and $F(A_\chi(2,3)) \subseteq F_0(\mathcal{A}_\chi, 2)$ by Theorem 5.35. Finally, if

$$\Gamma_\chi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{or} \quad \Gamma_\chi = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 2 \\ 0 & 1 & 1 \end{pmatrix},$$

then χ must be a standard twisting map, since the first column is standard and it is the extension of a standard twisting map of K^3 with K^2 (which follows from Proposition 5.8 and the analysis made out in Subsection 8.2).

$\mathbf{Diag(\Gamma_\chi) = (3,1,1)}$ By Proposition 2.13 we can assume that the rank matrix Γ_χ is one of the following matrices:

$$\begin{pmatrix} 3 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 3 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

By Proposition 4.10, if Γ_χ is the first matrix, then χ is a standard twisting map. Assume that Γ_χ is not the first matrix, which by Propositions 4.14 implies that $\tilde{\Gamma}_\chi$ has one row without zeroes. By the arguments given above, if Γ_χ is not the last matrix, then $\tilde{\Gamma}_\chi$ can not be equivalent via identical permutations in rows and columns to the first matrix in the second row of (8.28). Hence, $\tilde{\Gamma}_\chi$ has exactly one row without zeroes, and by Propositions 4.14 and 8.2, if χ is not quasi-standard, then $A_\chi(2,1) \neq 0 \neq A_\chi(1,2)$. So, necessarily χ is quasi-standard, and in fact there is quasi-standard twisting maps with Γ_χ the fifth matrix (see the appendix). But if Γ_χ

is the second, third or fourth matrix, then condition (1) in Theorem 5.35 is not fulfilled, and thus there is not twisting maps in these cases. Finally, there is a family of not quasi-standard twisting maps χ with Γ_χ the last matrix, dual to the family found above, when analyzing the case $\mathrm{Diag}\big(\Gamma_\chi\big) = (2,2,1)$.

8.3.4 $\sum \mathrm{Tr} = 4$

We claim that in this case all twisting maps are quasi-standard. By Proposition 2.13 in order to prove this it suffices to check that χ is quasi-standard if its rank matrix Γ_χ is one of the following matrices:

$$\begin{pmatrix} 2 & 2 & 2\\ 1 & 1 & 0\\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 2\\ 1 & 1 & 0\\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 0\\ 1 & 1 & 2\\ 0 & 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 0 & 0\\ 1 & 1 & 2\\ 0 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 2 & 1\\ 1 & 1 & 1\\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 & 1\\ 1 & 1 & 1\\ 0 & 2 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 1 & 0\\ 1 & 1 & 2\\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 2\\ 1 & 1 & 0\\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 1 & 1\\ 1 & 1 & 1\\ 0 & 1 & 1 \end{pmatrix}.$$

If Γ_χ is the first or the third matrix of the first row, then χ is a extension of a standard twisting map χ' of K^2 with K^3, whose added column is standard, and so it is standard. If Γ_χ is the second matrix of the second row, then χ is an extension of a standard twisting map of K^2 with K^3. Moreover, by Proposition 4.10 we know that $A_\chi(3,3)$ is equivalent via identical permutations in rows and columns to a standard idempotent $(0,1)$-matrix, and hence, by proposition 8.2, the twisting map χ is quasi-standard. By Proposition 4.14, the rank matrix Γ_χ can not be the second matrix in the first row. Also Γ_χ can not be the first matrix in the second row, because otherwise it would be the extension of a twisting map χ' of K^2 with K^3 with $\Gamma_{\chi'} = \left(\begin{smallmatrix} 1 & 2\\ 2 & 1 \end{smallmatrix}\right)$, but $\sum \mathrm{Tr} = 2$ is impossible. So, we are left with the last four matrices. Assume first that Γ_χ is the last one. By Proposition 2.13 we also can assume that $\mathrm{Diag}(\Gamma_{\tilde{\chi}}) = (2,1,1)$. In this case $B_\chi(2,1) = 0$ or $B_\chi(3,1) = 0$, both cases being equivalent via Proposition 2.4 with $\sigma = \mathrm{id}$ and $\tau = (2,3)$. So, assume that $B_\chi(2,1) = 0$, which by Remark 4.13 implies that

$$A_\chi(2,2) = \begin{pmatrix} * & 0 & *\\ * & 0 & *\\ * & 0 & * \end{pmatrix}.$$

Moreover, again by Remark 4.13, $A_\chi(3,1) = 0$ implies that

$$B_\chi(3,3) = \begin{pmatrix} * & * & 0\\ * & * & 0\\ * & * & 0 \end{pmatrix} \quad \text{and} \quad B_\chi(2,2) = \begin{pmatrix} * & * & 0\\ * & * & 0\\ * & * & 0 \end{pmatrix}.$$

Hence $\mathrm{Diag}(A_\chi(3,2)) = (*,0,0)$ and

$$A_\chi(3,3) = \begin{pmatrix} 1 & 0 & 0\\ 1 & 0 & 0\\ 1 & 0 & 0 \end{pmatrix},$$

where for the last equality we use once more Remark 4.13. Since $\mathrm{rk}(A_\chi(3,2)) = 1$ it follows that $\mathrm{Diag}(A_\chi(3,2)) = (1,0,0)$, and therefore

$$A_\chi(3,2) = \begin{pmatrix} 1 & 0 & -1 \\ * & 0 & * \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$A_\chi(1,2) = \mathrm{id} - A_\chi(2,2) - A_\chi(3,2) = \begin{pmatrix} * & 0 & * \\ * & 1 & * \\ * & 0 & * \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ * & 1 & * \\ 0 & 0 & 0 \end{pmatrix},$$

where for the last equality we use that $\mathrm{rk}\big(A_\chi(1,2)\big) = 1$, and so

$$A_\chi(2,2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now, from Proposition 8.2(1) it follows that χ is quasi-standard. For the remaining three cases, the only way that χ can be not quasi-standard is that $\tilde{\Gamma}_\chi = \Gamma_{\tilde{\chi}}$ has exactly one row without zeroes. But then Propositions 4.14 and 8.2 shows that the twisting map χ is quasi-standard.

8.3.5 $\sum \mathrm{Tr} = 3$

By Proposition 4.15 we know that $\Gamma_\chi = \tilde{\Gamma}_\chi = \mathfrak{J}_3$. We have the following possibilities for each matrix $A_\chi(i,i)$ and each $B_\chi(j,j)$. It is equivalent to a standard idempotent $0,1$-matrix via identical permutations in rows and columns, it has all entries non-zero, or it has two non-zero columns and one zero column. If two of $A_\chi(1,1)$, $A_\chi(2,2)$ and $A_\chi(3,3)$ are $0,1$-matrices, then all the matrices $B_\chi(j,k)$ have zeroes and ones in its diagonal entries, and, moreover, by Remark 4.13 each $B_\chi(j,j)$ is a $(0,1)$-matrix. Therefore the hypothesis of Proposition 5.30 are fulfilled, and we have a quasi-standard twisting map. On the other hand, if one of the $A_\chi(i,i)$ (say $A_\chi(1,1)$) has all its entries non-zero, then χ is a non quasi-standard twisting map which yields a tensor product algebra isomorphic to $M_3(K)$. In fact, the existence follows from Proposition 4.16 and Theorem 4.17 (with $\mathbf{v}_2$ and $\mathbf{v}_3$ vectors that generate the images of $A_\chi(2,1)$ and $A_\chi(3,1)$, respectively), and the uniqueness follows from Remark 4.18). So there are two cases left:

- All three matrices $A_\chi(1,1)$, $A_\chi(2,2)$ and $A_\chi(3,3)$ have exactly one zero column.
- One of them (for example $A_\chi(1,1)$) is a $0,1$-matrix, the other two have exactly one zero column.

In the first case a straightforward computation shows that the resulting twisting map (up to an isomorphism) is given by

$$A(1,1) = \begin{pmatrix} a & b & 0 \\ a & b & 0 \\ a & b & 0 \end{pmatrix}, \quad A(1,2) = \begin{pmatrix} a & -a & 0 \\ -b & b & 0 \\ -b & b & 0 \end{pmatrix}, \quad A(1,3) = \begin{pmatrix} a & -a & 0 \\ -b & b & 0 \\ a & -a & 0 \end{pmatrix}$$

$$A(2,1) = \begin{pmatrix} b & 0 & -b \\ -a & 0 & a \\ -a & 0 & a \end{pmatrix}, \quad A(2,2) = \begin{pmatrix} b & 0 & a \\ b & 0 & a \\ b & 0 & a \end{pmatrix}, \quad A(2,3) = \begin{pmatrix} b & 0 & -b \\ b & 0 & -b \\ -a & 0 & a \end{pmatrix}$$

$$A(3,1) = \begin{pmatrix} 0 & -b & b \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}, \quad A(3,2) = \begin{pmatrix} 0 & a & -a \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}, \quad A(3,3) = \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & a & b \end{pmatrix},$$

for some $a \notin \{0,1\}$ and $b \coloneqq 1-a$, which gives a family of twisting maps parameterised by $a \in K$. In the second case we can assume that

$$A(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

and that the first column is not quasi-standard. A straightforward computation along the lines of the proof of Proposition 8.2 show that there exists $a \notin \{0,1\}$, $b \coloneqq 1-a$ and $x, y \in k^\times$ such that

$$A(1,1) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad A(1,2) = \begin{pmatrix} 1 & p & q \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad A(1,3) = \begin{pmatrix} 1 & t & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A(2,1) = \begin{pmatrix} 0 & 0 & 0 \\ r & a & y \\ u & \frac{ab}{y} & b \end{pmatrix}, \qquad A(2,2) = \begin{pmatrix} 0 & a & b \\ 0 & a & b \\ 0 & a & b \end{pmatrix}, \qquad A(2,3) = \begin{pmatrix} 0 & \frac{ab}{x} & -\frac{ab}{x} \\ 0 & a & -a \\ 0 & -b & b \end{pmatrix}$$

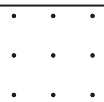
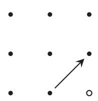
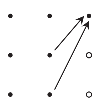
$$A(3,1) = \begin{pmatrix} 0 & 0 & 0 \\ s & b & -y \\ w & -\frac{ab}{y} & a \end{pmatrix}, \quad A(3,2) = \begin{pmatrix} 0 & x & -x \\ 0 & b & -b \\ 0 & -a & a \end{pmatrix}, \quad A(3,3) = \begin{pmatrix} 0 & b & a \\ 0 & b & a \\ 0 & b & a \end{pmatrix},$$

where $p \coloneqq -a-x$, $q \coloneqq x-b$, $r \coloneqq -a-y$, $s \coloneqq y-b$, $t \coloneqq -\frac{b(a+x)}{x}$, $u \coloneqq -\frac{b(a+y)}{y}$, $v \coloneqq \frac{ab}{x} - a$ and $w \coloneqq \frac{ab}{y} - a$.

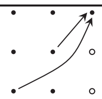
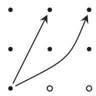
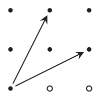
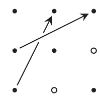
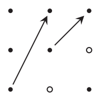
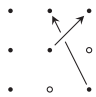
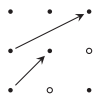
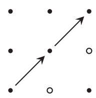
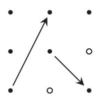
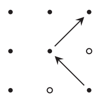
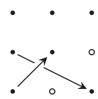
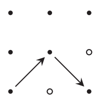
Appendix: Quasi-standard twisting maps of K^3 with K^3

Next we list the quasi-standard twisting maps of K^3 with K^3. For this, first we construct the standard ones using the method given in Remark 7.5, and then we construct the remaining quasi-standard twisting maps using the recursive method developed in Subsection 7.2. Is not possible iterate arbitrarily the steps in this construction because the conditions in item (2) of Proposition 7.11 would not be satisfied (see for instance the last item in following list).

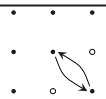
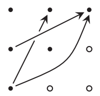
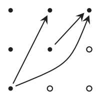
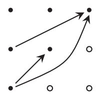
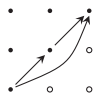
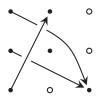
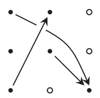
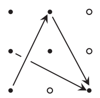
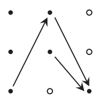
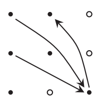
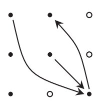
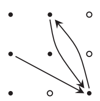
Table 2. Quasi-standard twisting maps of K^3 with K^3

#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
1.	9		$\begin{pmatrix} 3&0&0\\0&3&0\\0&0&3 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\0&3&0\\0&0&3 \end{pmatrix}$	1	–
2.	8		$\begin{pmatrix} 3&0&0\\0&3&1\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\0&3&1\\0&0&2 \end{pmatrix}$	36	–
3.	7		$\begin{pmatrix} 3&0&0\\0&3&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\0&2&0\\0&0&2 \end{pmatrix}$	18	–

continued on next page ...

#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
4.	7		$\begin{pmatrix} 3&0&1\\ 0&3&1\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	18	–
5.	7		$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&2\\ 0&3&0\\ 0&0&1 \end{pmatrix}$	18	–
6.	7		$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&1\\ 0&3&1\\ 0&0&1 \end{pmatrix}$	18	–
7.	7		$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	18	–
8.	7		$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	36	–
9.	7		$\begin{pmatrix} 3&0&0\\ 0&2&1\\ 0&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	18	–
10.	7		$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	36	–
11.	7		$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	36	–
12.	7		$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&1\\ 0&2&0\\ 0&1&2 \end{pmatrix}$	36	–
13.	7		$\begin{pmatrix} 3&0&0\\ 0&2&1\\ 0&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	36	–
14.	7		$\begin{pmatrix} 3&1&1\\ 0&2&0\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\ 0&2&1\\ 0&1&2 \end{pmatrix}$	18	–
15.	7		$\begin{pmatrix} 3&1&0\\ 0&2&1\\ 0&0&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\ 0&2&1\\ 0&1&2 \end{pmatrix}$	36	–

continued on next page ...

#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
16.	7		$\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\0&2&1\\0&1&2 \end{pmatrix}$	18	–
17.	6		$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	36	–
18.	6		$\begin{pmatrix} 3&1&0\\0&2&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	36	–
19.	6		$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	36	–
20.	6		$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,1),(2,2),(1,3)}(\chi)$
21.	6		$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
22.	6		$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
23.	6		$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
24.	6		$\begin{pmatrix} 3&1&0\\0&2&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
25.	6		$\begin{pmatrix} 3&0&2\\0&2&0\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
26.	6		$\begin{pmatrix} 3&0&1\\0&2&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
27.	6		$\begin{pmatrix} 3&0&1\\0&2&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–

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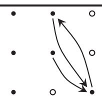
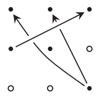
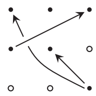
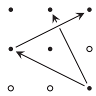
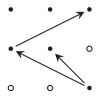
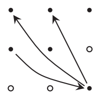
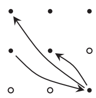
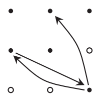
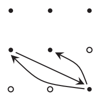
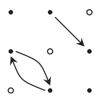
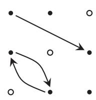
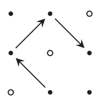
#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
28.	6		$\begin{pmatrix} 3&0&0\\0&2&2\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	36	–
29.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&2\\0&2&0\\0&0&1 \end{pmatrix}$	36	–
30.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	36	–
31.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\0&2&1\\0&0&1 \end{pmatrix}$	36	–
32.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&1&0\\0&2&2\\0&0&1 \end{pmatrix}$	36	–
33.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&2\\0&2&0\\0&1&1 \end{pmatrix}$	36	–
34.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&1\\0&2&1\\0&1&1 \end{pmatrix}$	36	–
35.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&1\\0&2&1\\0&1&1 \end{pmatrix}$	36	–
36.	6		$\begin{pmatrix} 2&0&1\\0&2&0\\1&1&2 \end{pmatrix}$	$\begin{pmatrix} 3&0&0\\0&2&2\\0&1&1 \end{pmatrix}$	36	–
37.	6		$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&1\\0&1&2 \end{pmatrix}$	36	–
38.	6		$\begin{pmatrix} 2&1&1\\1&2&0\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&1\\0&1&2 \end{pmatrix}$	36	–
39.	6		$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	36	–

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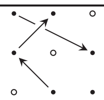
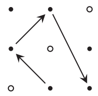
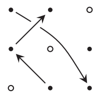
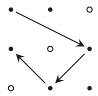
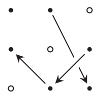
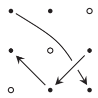
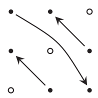
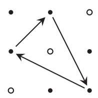
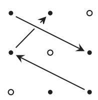
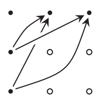
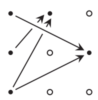
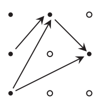
#	$\sum \mathrm{Tr}$	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
40.	6		$\begin{pmatrix} 2&1&1\\1&2&0\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	36	–
41.	6		$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix}$	36	–
42.	6		$\begin{pmatrix} 2&1&1\\1&2&0\\0&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix}$	36	–
43.	6		$\begin{pmatrix} 2&0&1\\1&2&0\\0&1&2 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&1\\0&1&2 \end{pmatrix}$	36	–
44.	6		$\begin{pmatrix} 2&0&0\\1&2&1\\0&1&2 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\0&2&1\\1&1&2 \end{pmatrix}$	36	–
45.	6		$\begin{pmatrix} 2&0&1\\1&2&0\\0&1&2 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\0&2&1\\1&1&2 \end{pmatrix}$	36	–
46.	6		$\begin{pmatrix} 2&0&1\\1&2&0\\0&1&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix}$	12	–
47.	6		$\begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\0&2&1\\1&0&2 \end{pmatrix}$	12	–
48.	6		$\begin{pmatrix} 2&1&1\\0&2&0\\1&0&2 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\1&2&1\\0&0&2 \end{pmatrix}$	36	–
49.	5		$\begin{pmatrix} 3&2&2\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&2&2\\0&1&0\\0&0&1 \end{pmatrix}$	9	–
50.	5		$\begin{pmatrix} 3&2&2\\0&1&0\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	18	–
51.	5		$\begin{pmatrix} 3&2&1\\0&1&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,1)(1,2)(2,3)}(\chi)$

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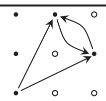
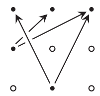
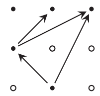
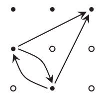
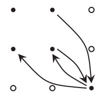
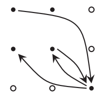
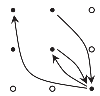
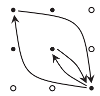
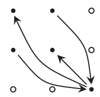
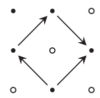
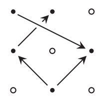
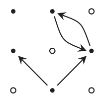
#	$\sum \mathrm{Tr}$	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
52.	5		$\begin{pmatrix} 3&1&1\\ 0&1&1\\ 0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\ 1&2&1\\ 0&0&1 \end{pmatrix}$	18	$\chi_1 = \Lambda^{\lambda_1}_{(3,1)(1,2)(2,3)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(3,1)(2,3)(1,2)}(\chi)$
53.	5		$\begin{pmatrix} 2&1&1\\ 1&2&1\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&2&2\\ 0&1&0\\ 0&0&1 \end{pmatrix}$	18	–
54.	5		$\begin{pmatrix} 2&1&1\\ 1&2&1\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&2&1\\ 0&1&1\\ 0&0&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,2)(2,1)(1,3)}(\chi)$
55.	5		$\begin{pmatrix} 2&1&1\\ 1&2&1\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 3&1&1\\ 0&1&1\\ 0&1&1 \end{pmatrix}$	18	$\chi_1 = \Lambda^{\lambda_1}_{(2,1)(3,2)(1,3)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(3,2)(2,1)(1,3)}(\chi)$
56.	5		$\begin{pmatrix} 2&0&0\\ 0&2&2\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\ 0&2&2\\ 1&1&1 \end{pmatrix}$	36	–
57.	5		$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\ 0&2&2\\ 1&1&1 \end{pmatrix}$	36	–
58.	5		$\begin{pmatrix} 2&0&0\\ 0&2&2\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	36	–
59.	5		$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	18	–
60.	5		$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\ 0&2&1\\ 1&1&1 \end{pmatrix}$	18	–
61.	5		$\begin{pmatrix} 2&1&0\\ 1&2&2\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\ 1&2&2\\ 0&0&1 \end{pmatrix}$	36	–
62.	5		$\begin{pmatrix} 2&1&1\\ 1&2&1\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\ 1&2&2\\ 0&0&1 \end{pmatrix}$	36	–
63.	5		$\begin{pmatrix} 2&0&0\\ 1&2&2\\ 0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\ 1&2&2\\ 0&0&1 \end{pmatrix}$	36	–

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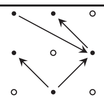
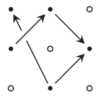
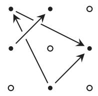
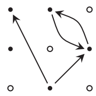
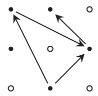
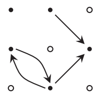
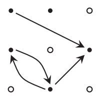
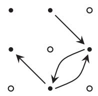
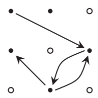
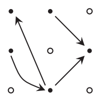
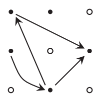
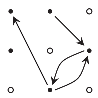
#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
64.	5		$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&0\\1&2&2\\0&0&1 \end{pmatrix}$	36	–
65.	5		$\begin{pmatrix} 2&1&0\\1&2&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	36	–
66.	5		$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,2)(1,1)(2,3)}(\chi)$
67.	5		$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	36	–
68.	5		$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,2)(1,1)(2,3)}(\chi)$
69.	5		$\begin{pmatrix} 2&1&0\\1&2&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	36	–
70.	5		$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	36	–
71.	5		$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	36	–
72.	5		$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	36	–
73.	5		$\begin{pmatrix} 2&1&0\\1&2&2\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	36	–
74.	5		$\begin{pmatrix} 2&1&1\\1&2&1\\0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,2)(1,1)(2,3)}(\chi)$
75.	5		$\begin{pmatrix} 2&0&0\\1&2&2\\0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\1&2&1\\0&1&1 \end{pmatrix}$	36	–

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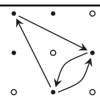
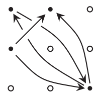
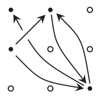
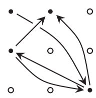
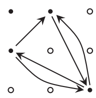
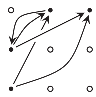
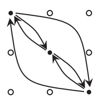
#	$\sum$ Tr	quiver	Γ_χ	$\tilde{\Gamma}_\chi$	# equiv.	quasi-st.
76.	5		$\begin{pmatrix} 2&0&1\\ 1&2&1\\ 0&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&0&1\\ 1&2&1\\ 0&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(3,2)(1,1)(2,3)}(\chi)$
77.	4		$\begin{pmatrix} 2&1&2\\ 0&1&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&2\\ 0&1&0\\ 1&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(2,1)(3,3)(1,2)}(\chi)$
78.	4		$\begin{pmatrix} 2&1&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&2\\ 0&1&0\\ 1&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(2,1)(3,3)(1,2)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(2,1)(1,2)(3,3)}(\chi)$
79.	4		$\begin{pmatrix} 2&1&2\\ 0&1&0\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(2,1)(3,3)(1,2)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(3,3)(2,1)(1,2)}(\chi)$
80.	4		$\begin{pmatrix} 2&1&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 2&1&1\\ 0&1&1\\ 1&1&1 \end{pmatrix}$	36	$\chi_1 = \Lambda^{\lambda_1}_{(2,1)(3,3)(1,2)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(3,3)(2,1)(1,2)}(\chi)$ $\chi_3 = \Lambda^{\lambda_3}_{(2,1)(1,2)(3,3)}(\chi_2)$
81.	4		$\begin{pmatrix} 2&2&2\\ 1&1&0\\ 0&0&1 \end{pmatrix}$	$\begin{pmatrix} 2&2&2\\ 1&1&0\\ 0&0&1 \end{pmatrix}$	36	–
82.	3		$\begin{pmatrix} 1&1&1\\ 1&1&1\\ 1&1&1 \end{pmatrix}$	$\begin{pmatrix} 1&1&1\\ 1&1&1\\ 1&1&1 \end{pmatrix}$	6	$\chi_1 = \Lambda^{\lambda_1}_{(1,1)(2,2)(3,3)}(\chi)$ $\chi_2 = \Lambda^{\lambda_2}_{(1,1)(3,3)(2,2)}(\chi)$ $\chi_3 = \Lambda^{\lambda_3}_{(3,3)(1,1)(2,2)}(\chi)$ $\chi_4 = \Lambda^{\lambda_4}_{(3,3)(1,1)(2,2)}(\chi_1)$ $\chi_5 = \Lambda^{\lambda_5}_{(3,3)(2,2)(1,1)}(\chi_1)$ $\chi_6 = \Lambda^{\lambda_6}_{(3,3)(2,2)(1,1)}(\chi_2)$ $\chi_7 = \Lambda^{\lambda_7}_{(2,2)(3,3)(1,1)}(\chi_2)$ $\chi_8 = \Lambda^{\lambda_8}_{(2,2)(1,1)(3,3)}(\chi_3)$ $\chi_9 = \Lambda^{\lambda_9}_{(2,2)(3,3)(1,1)}(\chi_4)$ $\chi_{10} = \Lambda^{\lambda_{10}}_{(2,2)(1,1)(3,3)}(\chi_6)$

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Pontificia Universidad Católica del Perú, Sección Matemáticas, PUCP, Av. Universitaria 1801, San Miguel, Lima 32, Perú.

E-mail address: jarcef@pucp.edu.pe

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales-UBA, Pabellón 1-Ciudad Universitaria, Intendente Guiraldes 2160 (C1428EGA) Buenos Aires, Argentina.

Instituto de Investigaciones Matemáticas "Luis A. Santaló", Facultad de Ciencias Exactas y Naturales-UBA, Pabellón 1-Ciudad Universitaria, Intendente Guiraldes 2160 (C1428EGA) Buenos Aires, Argentina.

E-mail address: vander@dm.uba.ar

Departamento de Matemática, Facultad de Ciencias Exactas y Naturales-UBA, Pabellón 1-Ciudad Universitaria, Intendente Guiraldes 2160 (C1428EGA) Buenos Aires, Argentina.

Instituto Argentino de Matemática-CONICET, Savedra 15 3er piso, (C1083ACA) Buenos Aires, Argentina.

E-mail address: jjgucci@dm.uba.ar

Pontificia Universidad Católica del Perú - Instituto de Matemática y Ciencias Afines, Sección Matemáticas, PUCP, Av. Universitaria 1801, San Miguel, Lima 32, Perú.

E-mail address: cvalqui@pucp.edu.pe