ON A CLASS OF NON-HERMITIAN MATRICES WITH POSITIVE DEFINITE SCHUR COMPLEMENTS

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Abstract. Given Hermitian matrices $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$, and $\kappa > 0$, we characterize under which conditions there exists a matrix $K \in \mathbb{C}^{n\times m}$ with $\|K\| < \kappa$ such that the non-Hermitian block-matrix

$$\begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix}$$

has a positive (semi-)definite Schur complement with respect to its submatrix A. Additionally, we show that K can be chosen such that diagonalizability of the block-matrix is guaranteed and we compute its spectrum. Moreover, we show a connection to the recently developed frame theory for Krein spaces.

1. Introduction

Given a matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ assume it is partitioned as

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $C \in \mathbb{C}^{m\times n}$ and $D \in \mathbb{C}^{m\times m}$. If A is invertible, then the *Schur complement of A in S* is defined by

$$S_{/A} := D - CA^{-1}B.$$

This terminology is due to Haynsworth [12, 13], but the use of such a construction goes back to Sylvester [18] and Schur [17]. The Schur complement arises, for instance, in the following factorization of the block matrix S:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ CA^{-1} & I_m \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix}, \tag{1.1}$$

which is due to Aitken [1]; note that I_k denotes the identity matrix in $\mathbb{C}^{k\times k}$. It is a common argument in the proof of the *Schur determinant formula* [3]:

$$\det(S) = \det(A) \cdot \det(S_{/A}), \tag{1.2}$$

of the *Guttman rank additivity formula* [11], and of the *Haynsworth inertia additivity formula* [14].

The Schur complement has been generalized for example to non-invertible A. In this case, if $A^\dagger$ is the Moore-Penrose inverse of A, then the Schur complement $S_{/A}$ is defined by $S_{/A} = D - CA^\dagger B$. It is a key tool not only

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in matrix analysis but also in applied fields such as numerical analysis and statistics. For further details see [19].

If A is invertible and S is a Hermitian matrix, then $C = B^*$ and the Schur complement of A in S is $S_{/A} = D - B^*A^{-1}B$. Then (1.1) reads as

$$\begin{bmatrix} A & B \\ B^* & D \end{bmatrix} = \begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix}^* \begin{bmatrix} A & 0 \\ 0 & D - B^*A^{-1}B \end{bmatrix} \begin{bmatrix} I_n & A^{-1}B \\ 0 & I_m \end{bmatrix},$$

which implies the following well-known criteria: S is positive definite if and only if A and $S_{/A}$ are both positive definite. This equivalence is not true for positive semidefinite matrices, but Albert [2] showed that S is positive semidefinite if and only if A and $S_{/A}$ are both positive semidefinite and $R(B) \subseteq R(A)$, where $R(X)$ stands for the range of a matrix X.

In this paper, given $\kappa > 0$, a Hermitian matrix $A \in \mathbb{C}^{n\times n}$ with eigenvalues $\lambda_1 \geq \ldots \geq \lambda_k > 0 \geq \lambda_{k+1} \geq \ldots \geq \lambda_n$, and a Hermitian matrix $D \in \mathbb{C}^{m\times m}$ with eigenvalues $\mu_1 \leq \ldots \leq \mu_r \leq 0 < \mu_{r+1} \leq \ldots \leq \mu_m$ we investigate under which conditions there exists a matrix $K \in \mathbb{C}^{n\times m}$ with $\|K\| < \kappa$ such that

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} \tag{1.3}$$

has a positive (semi-)definite Schur complement $S_{/A}$ with respect to the submatrix A. Note that

$$S_{/A} = D + K^*(AA^\dagger A)K = D + K^*AK.$$

Interest in such non-Hermitian block-matrices arises, for instance, in the recently developed frame theory in Krein spaces, see [7, 9]. There, block-matrices as in (1.3) with a positive definite A, a Hermitian D and a positive definite $S_{/A}$ correspond to so-called J-frame operators, see Section 5.

In Theorem 3.3 below we show that this special *structured matrix completion problem* has a solution if and only if

$$r \leq k \quad \text{and} \quad \kappa^2\lambda_i + \mu_i > 0 \quad \text{for all } i = 1, \ldots, r - p,$$

where $p = \dim(\ker D)$; this condition may be slightly relaxed if only positive semidefinite $S_{/A}$ is required. We stress that S is not diagonalizable in general, not even if $S_{/A}$ is positive definite. Under the above conditions, we construct a particular matrix K, which depends on some parameters $\varepsilon_1, \ldots, \varepsilon_r$. In Theorems 4.2 and 4.4 we compute the eigenvalues of the corresponding block matrix S in terms of the eigenvalues of A and D and the parameters $\varepsilon_1, \ldots, \varepsilon_r$. A root locus analysis of the latter reveals that if each ε_i is small enough, then S is diagonalizable and has only real eigenvalues, although S is non-Hermitian.

2. Preliminaries

Given Hermitian matrices $A, B \in \mathbb{C}^{n\times n}$, various different relations between the eigenvalues of A, B and $A+B$ can be obtained, see e.g. [4, 15, 16]. The following result was first proved by Weyl, see e.g. [4].

Theorem 2.1. *Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian matrices. Then,*

$$\lambda_j^{\downarrow}(A+B) \le \lambda_i^{\downarrow}(A) + \lambda_{j-i+1}^{\downarrow}(B) \quad \text{for } i \le j;$$
$$\lambda_j^{\downarrow}(A+B) \ge \lambda_i^{\downarrow}(A) + \lambda_{j-i+n}^{\downarrow}(B) \quad \text{for } i \ge j;$$

where $\lambda_j^{\downarrow}(C)$ denotes the j-th eigenvalue of C (counted with multiplicities) if they are arranged in nonincreasing order.

For a rectangular matrix $A \in \mathbb{C}^{m\times n}$ with $\operatorname{rank}(A) = r$ denote by

$$\sigma_1(A) \ge \sigma_2(A) \ge \ldots \ge \sigma_r(A) > 0$$

the singular values of A. Recall that $\sigma_i(A) = \lambda_i^{\downarrow}(|A|)$ for $i = 1, \ldots, r$, where $|A| = (A^*A)^{1/2}$. In particular $\|A\| = \sigma_1(A)$ denotes the spectral norm of A.

Given $A, B \in \mathbb{C}^{m\times n}$, the following inequalities hold. If $i \in \{1, \ldots, \operatorname{rank}(A)\}$ and $j \in \{1, \ldots, \operatorname{rank}(B)\}$ are such that $i + j - 1 \le \operatorname{rank}(AB^*)$, then

$$\sigma_{i+j-1}(AB^*) \le \sigma_i(A)\sigma_j(B), \tag{2.1}$$

see e.g. [16, Theorem 3.3.16]. As a consequence of these inequalities we have the following well-known result; for completeness we include a short proof.

Proposition 2.2. *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian with exactly k positive eigenvalues (counted with multiplicities) and let $K \in \mathbb{C}^{n\times m}$. Then,*

$$\lambda_j^{\downarrow}(K^*AK) \le \|K\|^2 \lambda_j^{\downarrow}(A) \quad \text{for } j = 1, \ldots, \min\{k, m, \operatorname{rank}(K^*AK)\}.$$

Proof. If $K = 0$, then the statement trivially holds, so assume that $K \ne 0$ and hence $\operatorname{rank}(K) \ge 1$. Then, for all $j = 1, \ldots, \min\{k, m, \operatorname{rank}(K^*AK)\}$

$$\lambda_j^{\downarrow}(K^*AK) \le \sigma_j(K^*AK) \le \sigma_j(K^*A)\sigma_1(K^*) \le \sigma_1(K^*)^2\sigma_j(A) = \|K\|^2\lambda_j^{\downarrow}(A),$$

because $\lambda_j^{\downarrow}(A)$ is positive for $j = 1, \ldots, k$. □

3. Positive (semi-)definiteness of the Schur complement

Throughout this work we consider non-Hermitian block matrices S as in (1.3), where $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ are Hermitian matrices and $K \in \mathbb{C}^{n\times m}$. In this section we characterize the existence of a matrix K such that S in (1.3) has a positive definite (positive semidefinite) Schur complement.

Assumption 3.1. Let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_k > 0 \ge \lambda_{k+1} \ge \ldots \ge \lambda_n$ denote the eigenvalues of A (counted with multiplicities) arranged in nonincreasing order. Further, let $\mu_1 \le \mu_2 \le \ldots \le \mu_r \le 0 < \mu_{r+1} \le \ldots \le \mu_m$ denote the eigenvalues of D (counted with multiplicities) arranged in nondecreasing order, and assume that $\dim(\ker D) = p$.

Lemma 3.2. *Let Assumption 3.1 hold. If $r > k$ then there is no $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive definite. Moreover, if $r - p > k$ then there is no $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive semidefinite.*

Proof. Assume that $r > k$. Given $K \in \mathbb{C}^{n\times m}$ let $\mathcal{S}_1 = \ker(K^*(A + |A|)K)$ and consider the subspace $\mathcal{S}_2$ of $\mathbb{C}^m$ spanned by all eigenvectors of D corresponding to non-positive eigenvalues. Observe that

$$\dim \mathcal{S}_1 = m - \operatorname{rank}(K^*(A + |A|)K) \geq m - \operatorname{rank}(A + |A|) = m - k.$$

By Assumption 3.1 we have that $\dim \mathcal{S}_2 = r$ and hence

$$\dim \mathcal{S}_1 + \dim \mathcal{S}_2 \geq (m - k) + r = m + (r - k) > m.$$

Thus, $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \{0\}$ and for any non-trivial vector $v \in \mathcal{S}_1 \cap \mathcal{S}_2$ we have

$$\langle (D + K^*AK)v, v\rangle = \langle Dv, v\rangle - \langle K^*|A|Kv, v\rangle \leq 0,$$

because $K^*AKv = -K^*|A|Kv$. Therefore, $D + K^*AK$ cannot be positive definite.

Moreover, assume that $r - p > k$ and consider the subspace $\mathcal{S}_3$ of $\mathbb{C}^m$ spanned by all eigenvectors of D corresponding to negative eigenvalues. Then, $\dim \mathcal{S}_3 = r - p$ and a similar argument shows that $D + K^*AK$ cannot be positive semidefinite. □

The next result characterizes under which conditions there exists a matrix $K \in \mathbb{C}^{n\times m}$ such that $D + K^*AK$ is positive (semi-)definite.

Theorem 3.3. *Let Assumption 3.1 hold. Given $\kappa > 0$, the following statements hold.*

(i) *There exists $K \in \mathbb{C}^{n\times m}$ with $\|K\| < \kappa$ such that $D + K^*AK$ is positive definite if and only if*

$$r \leq k \quad \text{and} \quad \kappa^2\lambda_i + \mu_i > 0 \quad \text{for all } i = 1, \dots, r - p. \tag{3.1}$$

(ii) *There exists $K \in \mathbb{C}^{n\times m}$ with $\|K\| \leq \kappa$ such that $D + K^*AK$ is positive semidefinite if and only if*

$$r - p \leq k \quad \text{and} \quad \kappa^2\lambda_i + \mu_i \geq 0 \quad \text{for all } i = 1, \dots, r - p. \tag{3.2}$$

Proof. We show (i). Assume that there exists a matrix $K \in \mathbb{C}^{n\times m}$ with $\|K\| < \kappa$ such that $D + K^*AK > 0$. By Lemma 3.2, it is necessary that $r \leq k$. On the other hand, by Theorem 2.1,

$$0 < \lambda_m^{\downarrow}(D + K^*AK) \leq \lambda_i^{\downarrow}(D) + \lambda_{m-i+1}^{\downarrow}(K^*AK),$$

for $i = 1, \dots, m$. In particular, for $i = m - r + p + 1, \dots, m$ we can combine the above inequalities with Proposition 2.2 and obtain

$$0 < \lambda_i^{\downarrow}(D) + \|K\|^2\lambda_{m-i+1}^{\downarrow}(A) < \mu_{m-i+1} + \kappa^2\lambda_{m-i+1}.$$

Equivalently, we have that $\mu_j + \kappa^2\lambda_j > 0$ for $j = 1, \dots, r - p$.

Conversely, assume that $r \leq k$ and $\kappa^2\lambda_i + \mu_i > 0$ for $i = 1, \ldots, r-p$. For each $i = 1, \ldots, r-p$ let $0 < \varepsilon_i < \kappa^2$ be such that $\varepsilon_i\lambda_i + \mu_i > 0$, and for $j = r-p+1, \ldots, r$ let $0 < \varepsilon_j < \kappa^2$ be arbitrary. Then, define $E \in \mathbb{C}^{n\times m}$ by

$$E = \begin{bmatrix} \mathrm{diag}\,(\sqrt{\varepsilon_1}, \ldots, \sqrt{\varepsilon_r}) & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix},$$

where $0_{p,q}$ is the null matrix in $\mathbb{C}^{p\times q}$. Further, let $U \in \mathbb{C}^{n\times n}$ and $V \in \mathbb{C}^{m\times m}$ be unitary matrices such that $A = UD_\lambda U^*$ and $D = VD_\mu V^*$, where

$$D_\lambda = \mathrm{diag}\,(\lambda_1, \ldots, \lambda_n) \quad \text{and} \quad D_\mu = \mathrm{diag}\,(\mu_1, \ldots, \mu_m).$$

Then, for

$$K := UEV^*, \tag{3.3}$$

it is straightforward to observe that $\|K\| < \kappa$ and

$$\begin{aligned} D + K^*AK &= V(D_\mu + E^*U^*AUE)V^* = V(D_\mu + E^*D_\lambda E)V^* \\ &= V \begin{bmatrix} \mathrm{diag}\,(\varepsilon_1\lambda_1 + \mu_1, \ldots, \varepsilon_r\lambda_r + \mu_r) & 0_{r,m-r} \\ 0_{m-r,r} & \mathrm{diag}\,(\mu_{r+1}, \ldots, \mu_m) \end{bmatrix} V^* \end{aligned}$$

is a positive definite matrix because ε_i was chosen in such a way that $\varepsilon_i\lambda_i + \mu_i > 0$ for $i = 1, \ldots, r-p$, and $\varepsilon_j\lambda_j + \mu_j = \varepsilon_j\lambda_j > 0$ for $j = r-p+1, \ldots, r$.

The proof of (ii) is analogous. If there is a matrix $K \in \mathbb{C}^{n\times m}$ with $\|K\| \leq \kappa$ such that $D + K^*AK$ is positive semidefinite, then $r - p \leq k$ (see Lemma 3.2) and following the same arguments as before it is easy to see that $\kappa^2\lambda_i + \mu_i \geq 0$ for $i = 1, \ldots, r-p$. The converse can also be proved in a similar way, but in this case ε_i may be equal to κ^2 for some $i = 1, \ldots, r-p$ (and ε_j can also be zero for $j = r-p+1, \ldots, r$). Therefore, $\|K\| \leq \kappa$ and $D + K^*AK$ is positive semidefinite. □

4. Spectrum of the block matrix

In the following, we consider the matrix K constructed in the proof of Theorem 3.3 and investigate the location of the eigenvalues of S in (1.3). The locations depend on the parameters $\varepsilon_1, \ldots, \varepsilon_r$ and hence their study resembles a root locus analysis. We start with a preliminary lemma.

Lemma 4.1. *Let Assumption 3.1 and* (3.2) *hold and set*

$$\alpha_i := \tfrac{(\lambda_i - \mu_i)^2}{4\lambda_i^2}, \quad i = 1, \ldots, r-p. \tag{4.1}$$

Then we have that

$$0 < \tfrac{-\mu_i}{\lambda_i} \leq \alpha_i \leq \left(\tfrac{\kappa^2+1}{2}\right)^2, \quad \textit{for all } i = 1, \ldots, r-p.$$

Proof. Given $i = 1, \ldots, r-p$ it is straightforward that $(\lambda_i - \mu_i)^2 \geq -4\mu_i\lambda_i$. If (3.2) holds, then $\lambda_i > 0$ for all $i = 1, \ldots, r-p$ and hence $\alpha_i \geq -\frac{\mu_i}{\lambda_i} > 0$. Furthermore,

$$\lambda_i - \mu_i = (\kappa^2 + 1)\lambda_i - (\kappa^2\lambda_i + \mu_i) \leq (\kappa^2 + 1)\lambda_i,$$

which implies that $\alpha_i \le \left(\frac{\kappa^2+1}{2}\right)^2$. □

In case that Assumption 3.1 and (3.2) hold, we describe the spectrum of the block matrix S given in (1.3) for the matrix K defined in (3.3).

Theorem 4.2. *Let Assumption 3.1 hold. Given $\kappa > 0$, assume that* (3.2) *also holds. For $i = 1, \dots, r-p$ choose $0 < \varepsilon_i \le \kappa^2$ such that $\varepsilon_i \lambda_i + \mu_i \ge 0$, and for $j = r-p+1, \dots, r$ set $\varepsilon_j = 0$.*

If $K \in \mathbb{C}^{n\times m}$ is as defined in (3.3)*, then $\|K\| \le \kappa$ and the spectrum of the block matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ given in* (1.3) *consists of the real numbers $\lambda_{r-p+1}, \dots, \lambda_n$, $\mu_{r-p+1}, \dots, \mu_m$, and*

$$\eta_i^{\pm} = \frac{\lambda_i+\mu_i}{2} \pm \lambda_i\sqrt{\alpha_i - \varepsilon_i}, \quad i = 1, \dots, r-p, \tag{4.2}$$

where α_i is given by (4.1)*. Moreover, for $i \in \{1, \dots, r-p\}$, we have*

a) *if $0 < \varepsilon_i < \frac{-\mu_i}{\lambda_i}$, then $\lambda_i > \eta_i^+ > 0 > \eta_i^- > \mu_i$;*
b) *if $\frac{-\mu_i}{\lambda_i} \le \varepsilon_i < \alpha_i$, then $\max\{\lambda_i+\mu_i, 0\} \ge \eta_i^+ > \eta_i^- \ge \min\{\lambda_i+\mu_i, 0\}$;*
c) *if $\alpha_i < \varepsilon_i \le \kappa^2$, then $\eta_i^+ = \overline{\eta_i^-} \in \mathbb{C}\setminus\mathbb{R}$;*
d) *if $\varepsilon_i = \alpha_i$, then $\eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i)$ and there exists a Jordan chain of length 2 corresponding to this eigenvalue.*

Additionally, if $\varepsilon_i \ne \alpha_i$ for all $i = 1, \dots, r-p$, then S is diagonalizable.

Proof. First note that by Lemma 4.1 the range for ε_i in case a) is non-empty independently of κ, but the same may not be true for cases b) and c). We will discuss this later in Remark 4.3.

Using the notation from the proof of Theorem 3.3 we obtain

$$S = \begin{bmatrix} A & -AK \\ K^*A & D \end{bmatrix} = \begin{bmatrix} UD_\lambda U^* & -UD_\lambda EV^* \\ VE^*D_\lambda U^* & VD_\mu V^* \end{bmatrix} =$$
$$= \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} D_\lambda & -B \\ B^* & D_\mu \end{bmatrix} \begin{bmatrix} U & 0 \\ 0 & V \end{bmatrix}^* = W \begin{bmatrix} D_\lambda & -B \\ B^* & D_\mu \end{bmatrix} W^*,$$

where $B \in \mathbb{C}^{n\times m}$ is given by

$$B := D_\lambda E = \begin{bmatrix} \operatorname{diag}(\lambda_1\sqrt{\varepsilon_1}, \dots, \lambda_{r-p}\sqrt{\varepsilon_{r-p}}) & 0_{r-p,m-r+p} \\ 0_{n-r+p,r-p} & 0_{n-r+p,m-r+p} \end{bmatrix},$$

and $W := \left[\begin{smallmatrix} U & 0 \\ 0 & V \end{smallmatrix}\right] \in \mathbb{C}^{(n+m)\times(n+m)}$ is unitary. Then, if $\{e_1, \dots, e_{n+m}\}$ denotes the standard basis of $\mathbb{C}^{n+m}$, it is easy to see that

$$\begin{aligned} SWe_i &= \lambda_i We_i && \text{for } i = r-p+1, \dots, n, \\ \text{and} \quad SWe_j &= \mu_{j-n} We_j && \text{for } j = n+r-p+1, \dots, n+m, \end{aligned} \tag{4.3}$$

which yields that $\lambda_{r-p+1}, \dots, \lambda_n$ and $\mu_{r-p+1}, \dots, \mu_m$ are eigenvalues of S.

Now, define the following $(r-p)\times(r-p)$ diagonal matrices:

$$F_\lambda := \operatorname{diag}(\lambda_1, \dots, \lambda_{r-p}), \qquad F_\mu := \operatorname{diag}(\mu_1, \dots, \mu_{r-p}),$$
$$G := \operatorname{diag}(\lambda_1\sqrt{\varepsilon_1}, \dots, \lambda_{r-p}\sqrt{\varepsilon_{r-p}}),$$

and observe that the remaining $2(r-p)$ eigenvalues of S coincide with the spectrum of the submatrix $\tilde{S}$ of W^*SW given by

$$\tilde{S} := \begin{bmatrix} F_\lambda & -G \\ G & F_\mu \end{bmatrix}.$$

In order to calculate the eigenvalues of $\tilde{S}$, consider the matrix $P_\sigma \in \mathbb{C}^{2(r-p)\times 2(r-p)}$ associated to the following permutation of the integers $\{1, 2, \ldots, 2(r-p)\}$:

$$\sigma(j) = \begin{cases} 2j-1, & \text{for } j = 1, \ldots, r-p, \\ 2(j-r+p), & \text{for } j = r-p+1, \ldots, 2(r-p). \end{cases}$$

Then, we have that $P_\sigma^2 = I_{2(r-p)}$ and $P_\sigma \tilde{S} P_\sigma$ is a block diagonal matrix, with $r-p$ blocks of size 2×2 along the main diagonal:

$$\begin{bmatrix} \lambda_j & -\lambda_j\sqrt{\varepsilon_j} \\ \lambda_j\sqrt{\varepsilon_j} & \mu_j \end{bmatrix}, \quad j = 1, \ldots, r-p.$$

Thus, the characteristic polynomial of $\tilde{S}$ is given by

$$q(\eta) = \prod_{i=1}^{r-p} \left((\mu_i - \eta)(\lambda_i - \eta) + \varepsilon_i \lambda_i^2\right),$$

and $\eta \in \mathbb{C}$ is a root of $q(\eta)$ if and only if

$$\eta^2 - (\lambda_i + \mu_i)\eta + \lambda_i(\mu_i + \varepsilon_i\lambda_i) = 0$$

for some $i \in \{1, \ldots, r-p\}$. This leads to the following eigenvalues of $\tilde{S}$:

$$\eta_i^{\pm} = \tfrac{\lambda_i+\mu_i}{2} \pm \tfrac{1}{2}\sqrt{(\lambda_i-\mu_i)^2 - 4\varepsilon_i\lambda_i^2} = \tfrac{\lambda_i+\mu_i}{2} \pm \lambda_i\sqrt{\alpha_i - \varepsilon_i} \tag{4.4}$$

for $i = 1, \ldots, r-p$. Hence, (4.2) follows and statement c) holds.

For statement a) we observe that if $0 < \varepsilon_i < \frac{-\mu_i}{\lambda_i}$, then $\sqrt{\alpha_i - \varepsilon_i} > \frac{|\lambda_i+\mu_i|}{2\lambda_i}$. Therefore,

$$\eta_i^+ > \tfrac{\lambda_i+\mu_i}{2} + \lambda_i\tfrac{|\lambda_i+\mu_i|}{2\lambda_i} \geq 0 \quad \text{and} \quad \eta_i^- < \tfrac{\lambda_i+\mu_i}{2} - \lambda_i\tfrac{|\lambda_i+\mu_i|}{2\lambda_i} \leq 0.$$

Furthermore,

$$\begin{aligned} \eta_i^+ &< \tfrac{\lambda_i+\mu_i}{2} + \lambda_i\sqrt{\alpha_i} = \tfrac{\lambda_i+\mu_i}{2} + \lambda_i\tfrac{\lambda_i-\mu_i}{2\lambda_i} = \lambda_i, \\ \eta_i^- &> \tfrac{\lambda_i+\mu_i}{2} - \lambda_i\sqrt{\alpha_i} = \tfrac{\lambda_i+\mu_i}{2} - \lambda_i\tfrac{\lambda_i-\mu_i}{2\lambda_i} = \mu_i. \end{aligned}$$

On the other hand, if $\frac{-\mu_i}{\lambda_i} \leq \varepsilon_i < \alpha_i$, then $\sqrt{\alpha_i - \varepsilon_i} \leq \frac{|\lambda_i+\mu_i|}{2\lambda_i}$ and

$$\begin{aligned} \eta_i^- &\geq \tfrac{\lambda_i+\mu_i}{2} - \lambda_i\tfrac{|\lambda_i+\mu_i|}{2\lambda_i} = \min\{\lambda_i+\mu_i, 0\}, \\ \eta_i^+ &\leq \tfrac{\lambda_i+\mu_i}{2} + \lambda_i\tfrac{|\lambda_i+\mu_i|}{2\lambda_i} = \max\{\lambda_i+\mu_i, 0\}, \end{aligned}$$

and, clearly, $\eta_i^+ > \frac{\lambda_i+\mu_i}{2} > \eta_i^-$, which proves b).

To show d), assume that $\varepsilon_i = \alpha_i$ for some $i \in \{1, \dots, r-p\}$. Since $\eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i)$ and $\sqrt{\varepsilon_i} = \frac{\lambda_i - \mu_i}{2\lambda_i}$, it is straightforward to compute

$$\left(\tilde{S} - \tfrac{1}{2}(\lambda_i + \mu_i) I_{2(r-p)}\right) \begin{pmatrix} \left(1 + \frac{2}{\lambda_i - \mu_i}\right) f_i \\ f_i \end{pmatrix} = \begin{pmatrix} f_i \\ f_i \end{pmatrix},$$

$$\left(\tilde{S} - \tfrac{1}{2}(\lambda_i + \mu_i) I_{2r}\right) \begin{pmatrix} f_i \\ f_i \end{pmatrix} = 0,$$

using the standard basis $\{f_1, \dots, f_{r-p}\}$ of $\mathbb{C}^{r-p}$. The vectors above form a Jordan chain of length 2 of $\tilde{S}$ corresponding to the eigenvalue $\frac{1}{2}(\lambda_i + \mu_i)$. Hence, a Jordan chain of S corresponding to the eigenvalue $\frac{1}{2}(\lambda_i + \mu_i)$ can also be constructed.

Finally, assume that $\varepsilon_i \neq \alpha_i$ for all $i = 1, \dots, r-p$. In this case, the space $\mathbb{C}^{n+m}$ has a basis consisting of eigenvectors of S. Indeed, this follows from (4.3) together with

$$\left(\tilde{S} - \eta_i^+ I_{2(r-p)}\right) \begin{pmatrix} f_i \\ -\frac{\lambda_i \sqrt{\varepsilon_i}}{\mu_i - \eta_i^+} f_i \end{pmatrix} = 0, \quad \left(\tilde{S} - \eta_i^- I_{2(r-p)}\right) \begin{pmatrix} f_i \\ -\frac{\lambda_i \sqrt{\varepsilon_i}}{\mu_i - \eta_i^-} f_i \end{pmatrix} = 0$$

for $i = 1, \dots, r-p$. □

We emphasize that if for all $i = 1, \dots, r-p$ the parameter ε_i in Theorem 4.2 is chosen such that a) or b) holds, then the block matrix S in (1.3) is diagonalizable and has only real eigenvalues, cf. Lemma 4.1.

Remark 4.3. Given $\kappa > 0$, note that $\left(\frac{\kappa^2+1}{2}\right)^2 \geq \kappa^2$ and equality holds if and only if $\kappa = 1$. Hence, if $\kappa \neq 1$ and $\kappa^2 < \alpha_i \leq \left(\frac{\kappa^2+1}{2}\right)^2$ for some $i \in \{1, \dots, r-p\}$, then the corresponding eigenvalues η_i^+ and η_i^- are real, because the range of values for ε_i in case c) is empty.

For $\kappa = 1$, if there exists $i \in \{1, \dots, r-p\}$ such that $\lambda_i + \mu_i > 0$, then

$$\lambda_i - \mu_i = -(\lambda_i + \mu_i) + 2\lambda_i < 2\lambda_i,$$

hence $\alpha_i < 1$ and we can choose the corresponding parameter ε_i such that S has non-real eigenvalues.

Furthermore, if A is positive semidefinite, $\kappa \leq 1$ and $\varepsilon_i \geq \frac{-\mu_i}{\lambda_i}$ for each $i = 1, \dots, r-p$, then $\lambda_i + \mu_i \geq 0$ and hence the eigenvalues of S are contained in the (closed) complex right half-plane.

In the remainder of this section, we calculate the eigenvalues of the block matrix S under the assumption that its Schur complement is positive definite. Note that if Assumption 3.1 and (3.1) hold we may define α_i as in (4.1) for all $i = 1, \dots, r$. In this case, $0 < \frac{-\mu_i}{\lambda_i} \leq \alpha_i < \left(\frac{\kappa^2+1}{2}\right)^2$ for $i = 1, \dots, r-p$, and $\alpha_j = \frac{1}{4}$ for $j = r-p+1, \dots, r$.

Theorem 4.4. *Let Assumption 3.1 hold. Given $\kappa > 0$, assume that (3.1) also holds. For $i = 1, \dots, r-p$ choose $0 < \varepsilon_i < \kappa^2$ such that $\varepsilon_i\lambda_i + \mu_i > 0$, and for $j = r-p+1, \dots, r$ let $0 \le \varepsilon_j < \kappa^2$ be arbitrary.*

If $K \in \mathbb{C}^{n\times m}$ is as defined in (3.3), then $\|K\| < \kappa$ and the spectrum of the block matrix $S \in \mathbb{C}^{(n+m)\times(n+m)}$ given in (1.3) consists of the real numbers $\lambda_{r+1}, \dots, \lambda_n$, $\mu_{r+1}, \dots, \mu_m$, and

$$\eta_i^{\pm} = \tfrac{\lambda_i+\mu_i}{2} \pm \lambda_i\sqrt{\alpha_i - \varepsilon_i}, \quad i = 1, \dots, r, \tag{4.5}$$

where α_i is given by (4.1). Moreover, for $i = 1, \dots, r$, we have

a) *if $0 < \varepsilon_i < \frac{-\mu_i}{\lambda_i}$, then $\lambda_i > \eta_i^+ > 0 > \eta_i^- > \mu_i$;*
b) *if $\frac{-\mu_i}{\lambda_i} \le \varepsilon_i < \alpha_i$, then $\max\{\lambda_i+\mu_i, 0\} \ge \eta_i^+ > \eta_i^- \ge \min\{\lambda_i+\mu_i, 0\}$;*
c) *if $\alpha_i < \varepsilon_i < \kappa^2$, then $\eta_i^+ = \overline{\eta_i^-} \in \mathbb{C}\setminus\mathbb{R}$;*
d) *if $\varepsilon_i = \alpha_i$, then $\eta_i^+ = \eta_i^- = \frac{1}{2}(\lambda_i + \mu_i)$ and there exists a Jordan chain of length 2 corresponding to this eigenvalue;*
e) *if $i > r-p$ and $\varepsilon_i = 0$, then $\eta_i^+ = \lambda_i > 0$ and $\eta_i^- = \mu_i = 0$.*

Additionally, if $\varepsilon_i \ne \alpha_i$ for all $i = 1, \dots, r$, then S is diagonalizable.

Proof. The proof is analogous to the proof of Theorem 4.2, the main difference is that in this case $S = W \begin{bmatrix} D_\lambda & -B \\ B^* & D_\mu \end{bmatrix} W^*$, where

$$B := D_\lambda E = \begin{bmatrix} \operatorname{diag}(\lambda_1\sqrt{\varepsilon_1}, \dots, \lambda_r\sqrt{\varepsilon_r}) & 0_{r,m-r} \\ 0_{n-r,r} & 0_{n-r,m-r} \end{bmatrix} \in \mathbb{C}^{n\times m},$$

which yields that $\lambda_{r+1}, \dots, \lambda_n$ and $\mu_{r+1}, \dots, \mu_m$ are eigenvalues of S. The remaining $2r$ eigenvalues of S can be calculated in the same way as before. Also, the only difference in the characterization of the eigenvalues $\eta_i^{\pm}$ appears in the case in which $i = r-p+1, \dots, r$ and $\varepsilon_i = 0$. But the proof of this last case is straightforward. □

Example 4.5. We illustrate Theorem 4.4 with a simple example. Let $n = m = 1$, $D = [0]$ and $A = [a]$ with $a > 0$. Then $r = 1$ and choosing K as in (3.3) with $0 < \varepsilon < 1 = \kappa^2$ gives $K = [\sqrt{\varepsilon}]$. In this case $\alpha = \frac{1}{4}$.

By Theorem 4.4, for $\varepsilon = \frac{1}{4}$ there is a Jordan chain of length 2 corresponding to the only eigenvalue $\frac{a}{2}$, and indeed we find that $\begin{pmatrix} \frac{1}{a} \\ \frac{-1}{a} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ form a Jordan chain of S, hence S is not diagonalizable.

On the other hand, for $\varepsilon \ne \frac{1}{4}$ the block matrix S has eigenvalues $\eta^+ = \frac{a}{2} + a\sqrt{\frac{1}{4} - \varepsilon}$ and $\eta^- = \frac{a}{2} - a\sqrt{\frac{1}{4} - \varepsilon}$. They are positive if $\varepsilon < \frac{1}{4}$, and they are non-real if $\frac{1}{4} < \varepsilon < 1$. In these last two cases S is diagonalizable.

5. Application to J-frame operators

In this section, we exploit Theorems 3.3 and 4.4 to investigate whether a block matrix S as in (1.3) represents a so-called J-frame operator and when it is similar to a Hermitian matrix. In the following we briefly recall the concept of J-frame operators, which arose in [7, 9] in the context of frame theory in Krein spaces.

In a finite-dimensional setting, every indefinite inner product space is a (finite-dimensional) Krein space, see [10]. A map $[\cdot,\cdot] : \mathbb{C}^k \times \mathbb{C}^k \to \mathbb{C}$ is called an indefinite inner product in $\mathbb{C}^k$, if it is a non-degenerate Hermitian sesquilinear form. The indefinite inner product allows a classification of vectors: $x \in \mathbb{C}^k$ is called positive if $[x,x] > 0$, negative if $[x,x] < 0$ and neutral if $[x,x] = 0$. Also, a subspace $\mathcal{L}$ of $\mathbb{C}^k$ is positive if every $x \in \mathcal{L} \setminus \{0\}$ is a positive vector. Negative and neutral subspaces are defined analogously. A positive (negative) subspace of maximal dimension will be called maximal positive (maximal negative, respectively).

It is well-known that there exists a Gramian (or Gram matrix) $G \in \mathbb{C}^{k\times k}$, which is Hermitian, invertible and represents $[\cdot,\cdot]$ in terms of the usual inner product in $\mathbb{C}^k$, i.e., $[x,y] = \langle Gx, y\rangle$ for all $x, y \in \mathbb{C}^k$. The positive (resp. negative) index of inertia of $[\cdot,\cdot]$ is the number of positive (resp. negative) eigenvalues of the Gramian G, and it equals the dimension of any maximal positive (resp. negative) subspace of $\mathbb{C}^k$. It is clear that the sum of the inertia indices equals the dimension of the space.

A finite family of vectors $\mathcal{F} = \{f_i\}_{i=1}^q$ in $\mathbb{C}^k$ is a *frame for* $\mathbb{C}^k$, if

$$\operatorname{span}(\{f_i\}_{i=1}^q) = \mathbb{C}^k$$

(see, e.g., [5]) or, equivalently, if there exist $0 < \alpha \le \beta$ such that

$$\alpha\|f\|^2 \le \sum_{i=1}^q \left| \langle f, f_i\rangle \right|^2 \le \beta\|f\|^2 \qquad \text{for every } f \in \mathbb{C}^k.$$

The optimal set of constants $0 < \alpha \le \beta$ (the biggest α and the smallest β) are called the frame bounds of $\mathcal{F}$. If

$$F : \mathbb{C}^k \to \mathbb{C}^k,\ f \mapsto \sum_{i=1}^q \langle f, f_i\rangle f_i \tag{5.1}$$

is the associated *frame operator*, then the frame bounds of $\mathcal{F}$ are

$$\alpha = \|F^{-1}\|^{-1} = \lambda_k^{\downarrow}(F) \quad \text{and} \quad \beta = \|F\| = \lambda_1^{\downarrow}(F),$$

see e.g. [5] and the references therein.

Roughly speaking, a J-frame is a frame which is compatible with the indefinite inner product $[\cdot,\cdot]$.

Definition 5.1. Let $(\mathbb{C}^k, [\cdot,\cdot])$ be an indefinite inner product space. Then, a frame $\mathcal{F} = \{f_i\}_{i=1}^q$ in $\mathbb{C}^k$ is called a *J-frame for* $\mathbb{C}^k$, if

$$\mathcal{M}_+ := \operatorname{span}\{f \in \mathcal{F} \mid [f,f] \ge 0\} \quad \text{and} \quad \mathcal{M}_- := \operatorname{span}\{f \in \mathcal{F} \mid [f,f] < 0\}$$

are a maximal positive and a maximal negative subspace of $\mathbb{C}^k$, respectively.

If $[\cdot,\cdot]$ is an indefinite inner product with positive and negative index of inertia n and m, respectively, then the maximality of $\mathcal{M}_+$ and $\mathcal{M}_-$ is equivalent to $\dim \mathcal{M}_+ = n$ and $\dim \mathcal{M}_- = m$. Note that if $\mathcal{F}$ is a J-frame for $\mathbb{C}^k$, then there are no (non-trivial) $f \in \mathcal{F}$ with $[f, f] = 0$.

Given a J-frame $\mathcal{F} = \{f_i\}_{i=1}^q$ for $\mathbb{C}^k$, its associated J-frame operator $S : \mathbb{C}^k \to \mathbb{C}^k$ is defined by

$$Sf = \sum_{i=1}^{q} \sigma_i \, [f, f_i] \, f_i, \quad f \in \mathbb{C}^k,$$

where $\sigma_i = \operatorname{sgn}[f_i, f_i]$ is the signature of the vector f_i. S is an invertible symmetric operator with respect to $[\cdot, \cdot]$, i.e.,

$$[Sf, g] = [f, Sg] \quad \text{for all} \quad f, g \in \mathbb{C}^k.$$

Its relevance follows from the indefinite sampling-reconstruction formula: Given an arbitrary $f \in \mathbb{C}^k$,

$$f = \sum_{i=1}^{q} \sigma_i \left[f, S^{-1} f_i\right] f_i = \sum_{i=1}^{q} \sigma_i \, [f, f_i] \, S^{-1} f_i,$$

i.e., it plays a role analogous to the fame operator F in equation (5.1).

In the following, we aim to apply the results from Sections 3 and 4, hence we restrict ourselves to the following inner product on $\mathbb{C}^k = \mathbb{C}^{n+m}$,

$$[(x_1, \ldots, x_{n+m}), (y_1, \ldots, y_{n+m})] = \sum_{i=1}^{n} x_i \overline{y_i} - \sum_{j=1}^{m} x_{n+j} \overline{y_{n+j}}. \tag{5.2}$$

In [7, Theorem 3.1] a criterion was provided to determine if an (invertible) symmetric operator is a J-frame operator. In our setting it says that an invertible operator S in $(\mathbb{C}^k, [\cdot, \cdot])$, which is symmetric with respect to $[\cdot, \cdot]$, is a J-frame operator if and only if there exists a basis of $\mathbb{C}^k$ such that S can be represented as a block-matrix

$$S = \begin{bmatrix} A & -AK \\ K^* A & D \end{bmatrix}, \tag{5.3}$$

where $A \in \mathbb{C}^{n \times n}$ is positive definite, $K \in \mathbb{C}^{n \times m}$ is strictly contractive, and $D \in \mathbb{C}^{m \times m}$ is a Hermitian matrix such that $D + K^* A K$ is also positive definite. Any block-matrix $S \in \mathbb{C}^{(n+m) \times (n+m)}$ of the form (5.3), which satisfies these conditions will be called *J-frame matrix*.

Throughout this section we consider the following hypothesis.

Assumption 5.2. Assume that $A \in \mathbb{C}^{n \times n}$ is positive definite and $D \in \mathbb{C}^{m \times m}$ is a Hermitian matrix. Let $\mu_1 \le \mu_2 \le \ldots \le \mu_r \le 0 < \mu_{r+1} \le \ldots \le \mu_m$ denote the eigenvalues of D (counted with multiplicities) arranged in nondecreasing order, and let $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n > 0$ denote the eigenvalues of A (counted with multiplicities) arranged in nonincreasing order.

Theorem 3.3 (for $\kappa = 1$) provides a criterion to determine whether there exists a strictly contractive matrix $K \in \mathbb{C}^{n \times m}$ (i.e., $\|K\| < 1$) such that S as in (5.3) is a J-frame matrix.

Theorem 5.3. *Let Assumption 5.2 hold. Then there exists $K \in \mathbb{C}^{n\times m}$ with $\|K\| < 1$ such that S as in* (5.3) *is a J-frame matrix if and only if*

$$r \le n \quad \text{and} \quad \lambda_i + \mu_i > 0 \quad \text{for } i = 1, \dots, r. \tag{5.4}$$

We mention that the study of the spectral properties of a J-frame operator is quite recent, see [7, 8]. In the case of J-frame matrices, for given A and D, we always find conditions such that a strictly contractive K exists which turns S into a matrix similar to a Hermitian one. The following result is a direct consequence of Theorem 4.4 and Lemma 4.1.

Theorem 5.4. *Let Assumption 5.2 and* (5.4) *hold. Then, there exists a strictly contractive matrix K such that the matrix S given in* (5.3) *is a J-frame matrix which is similar to a Hermitian matrix. In this case, all eigenvalues of S are positive and there exists a basis of $\mathbb{C}^{n+m}$ consisting of eigenvectors of S.*

In the next paragraphs we recall how to construct J-frames for $\mathbb{C}^{n+m}$ with a prescribed J-frame matrix S. For $K \in \mathbb{C}^{n\times m}$ with $\|K\| < 1$ define

$$\mathcal{M}_- := \{0\} \times \mathbb{C}^m, \quad \mathcal{M}_+ := \left\{ \begin{pmatrix} f \\ K^* f \end{pmatrix} \;\middle|\; f \in \mathbb{C}^n \right\}. \tag{5.5}$$

If $\mathbb{C}^{n+m} = \mathbb{C}^n \times \mathbb{C}^m$ is endowed with the indefinite inner product given in (5.2), then it is immediate that $\mathcal{M}_-$ is a maximal negative subspace in $\mathbb{C}^{n+m}$ and $\mathcal{M}_+$ is maximal positive in $\mathbb{C}^{n+m}$. The contraction $K \in \mathbb{C}^{n\times m}$ represents the angle between the two subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$.

Moreover, if K with $\|K\| < 1$ is such that the block matrix S given in (5.3) is a J-frame matrix, consider $S = S_+ + S_-$ with

$$S_+ = \begin{bmatrix} A & -AK \\ K^*A & -K^*AK \end{bmatrix} \quad \text{and} \quad S_- = \begin{bmatrix} 0 & 0 \\ 0 & D + K^*AK \end{bmatrix}. \tag{5.6}$$

Then, the restriction of S_+ to $(\mathcal{M}_+, [\cdot,\cdot])$ is a positive definite matrix. Indeed, if $f \in \mathbb{C}^n \setminus \{0\}$, then

$$\begin{aligned} \left[S_+ \begin{pmatrix} f \\ K^* f \end{pmatrix}, \begin{pmatrix} f \\ K^* f \end{pmatrix} \right] &= \left[\begin{pmatrix} A(I - KK^*)f \\ K^*A(I - KK^*)f \end{pmatrix}, \begin{pmatrix} f \\ K^* f \end{pmatrix} \right] \\ &= \langle A(I - KK^*)f, f\rangle - \langle KK^*A(I - KK^*)f, f\rangle \\ &= \langle (I - KK^*)A(I - KK^*)f, f\rangle > 0. \end{aligned} \tag{5.7}$$

On the other hand, it is evident that the restriction of S_- to $(\mathcal{M}_-, -[\cdot,\cdot])$ is just $D + K^*AK$, which is also a positive definite matrix.

Therefore, it is possible to construct frames $\mathcal{F}_\pm$ for the (finite-dimensional) Hilbert spaces $(\mathcal{M}_\pm, \pm[\cdot,\cdot])$ with these matrices as frame operators, see [6]. Moreover, the family $\mathcal{F}_+ \cup \mathcal{F}_-$ is a J-frame for $\mathbb{C}^{n+m}$ with S as its J-frame operator, see [9, Theorem 5.6].

Proposition 5.5. *Let Assumption 5.2 hold and let $K \in \mathbb{C}^{n\times m}$ with $\|K\| < 1$ be such that S as in (5.3) is a J-frame matrix. Further, let $\mathcal{M}_\pm$ be as in (5.5) and let $\mathcal{F}_\pm$ be frames for $(\mathcal{M}_\pm, \pm[\cdot,\cdot])$ with frame operator $S_\pm$ given in (5.6). Then, the frame bounds of $\mathcal{F}_-$ are*

$$\alpha_- = \lambda_m^\downarrow(D + K^*AK) \qquad \text{and} \qquad \beta_- = \lambda_1^\downarrow(D + K^*AK), \tag{5.8}$$

and the frame bounds of $\mathcal{F}_+$ are the boundary values of the numerical range of the positive definite matrix $C := (I - KK^*)^{1/2}A(I - KK^*)^{1/2} \in \mathbb{C}^{n\times n}$,

$$\alpha_+ = \lambda_n^\downarrow(C) \qquad \text{and} \qquad \beta_+ = \lambda_1^\downarrow(C). \tag{5.9}$$

Proof. Recall $g \in \mathcal{M}_+$ if and only if $g = \begin{pmatrix} f \\ K^*f \end{pmatrix}$ for some $f \in \mathbb{C}^n$. Then,

$$\|g\|^2 = \left[\begin{pmatrix} f \\ K^*f \end{pmatrix}, \begin{pmatrix} f \\ K^*f \end{pmatrix}\right] = \langle (I - KK^*)f, f\rangle = \|(I - KK^*)^{1/2}f\|^2.$$

On the other hand, if $h = (I - KK^*)^{1/2}f \in \mathbb{C}^n$, by (5.7) we have that

$$\begin{aligned}[S_+g, g] &= \langle (I - KK^*)A(I - KK^*)f, f\rangle \\ &= \left\langle C(I - KK^*)^{1/2}f, (I - KK^*)^{1/2}f\right\rangle = \langle Ch, h\rangle .\end{aligned}$$

Since $\|h\| = \|g\|$, it is immediate that the numerical ranges of S_+ and C coincide. Therefore, (5.9) holds.

On the other hand, the desired characterization of the frame bounds α_- and β_- of $\mathcal{F}_-$ has been obtained in [7, Proposition 4.1]. □

Using Weyl's inequalities and the inequalities for the singular values of a product of matrices presented in (2.1) we can obtain the following a priori estimates for the frame bounds of $\mathcal{F}_+$ and $\mathcal{F}_-$.

Proposition 5.6. *Let Assumption 5.2 and (5.4) hold and let $K \in \mathbb{C}^{n\times m}$ with $\|K\| < 1$ be such that S as in (5.3) is a J-frame matrix. Further, let $\mathcal{M}_\pm$ be as in (5.5) and let $\mathcal{F}_\pm$ be frames for $(\mathcal{M}_\pm, \pm[\cdot,\cdot])$ with frame operator $S_\pm$ given in (5.6). If $\sigma_1 \geq \ldots \geq \sigma_l > 0$ are the singular values of K, then the frame bounds of $\mathcal{F}_-$ satisfy*

$$0 < \alpha_- \leq \beta_- \leq \sigma_1^2\lambda_1 + \mu_m.$$

Furthermore, the frame bounds of $\mathcal{F}_+$ satisfy

$$(1 - \sigma_1^2)\lambda_n \leq \alpha_+ \leq \beta_+ \leq (1 - \sigma_l^2)\lambda_1.$$

Proof. By Proposition 5.5 we have $\alpha_- = \lambda_m^\downarrow(D + K^*AK) > 0$. Furthermore, by Theorem 2.1 and Proposition 2.2 we have

$$\begin{aligned}\beta_- &= \lambda_1^\downarrow(D + K^*AK) \leq \lambda_1^\downarrow(D) + \lambda_1^\downarrow(K^*AK) \\ &\leq \lambda_1^\downarrow(D) + \|K\|^2\lambda_1^\downarrow(A) = \mu_m + \|K\|^2\lambda_1 = \sigma_1^2\lambda_1 + \mu_m.\end{aligned}$$

On the other hand, if $C = (I - KK^*)^{1/2}A(I - KK^*)^{1/2}$, then $\alpha_+ = \lambda_n^{\downarrow}(C)$ and $\beta_+ = \lambda_1^{\downarrow}(C)$. Hence, using (2.1) we obtain

$$\begin{aligned}\alpha_+ &= \lambda_n^{\downarrow}(C) = \sigma_n(A^{1/2}(I - KK^*)^{1/2})^2 = \tfrac{\sigma_1(A^{-1/2})^2}{\sigma_1(A^{-1/2})^2}\sigma_n(A^{1/2}(I - KK^*)^{1/2})^2 \\ &\geq \tfrac{\sigma_n((I-KK^*)^{1/2})^2}{\sigma_1(A^{-1/2})^2} = \lambda_n^{\downarrow}(I - KK^*)\lambda_n^{\downarrow}(A) = (1 - \sigma_1^2)\lambda_n,\end{aligned}$$

and further

$$\begin{aligned}\beta_+ &= \lambda_1^{\downarrow}(C) = \sigma_1(A^{1/2}(I - KK^*)^{1/2})^2 \leq \sigma_1(A^{1/2})^2\sigma_1((I - KK^*)^{1/2})^2 \\ &= \lambda_1^{\downarrow}(A)\lambda_1^{\downarrow}(I - KK^*) = \lambda_1(1 - \sigma_l^2),\end{aligned}$$

which completes the proof. □

Finally, let $A \in \mathbb{C}^{n\times n}$ and $D \in \mathbb{C}^{m\times m}$ satisfy Assumption 5.2 and assume that (5.4) holds. For $i = 1, \dots, r$ choose $0 < \varepsilon_i < 1$ such that $\varepsilon_i\lambda_i + \mu_i > 0$. If $A = UD_\lambda U^*$, $D = VD_\mu V^*$ and $K \in \mathbb{C}^{n\times m}$ is given by (3.3) then $\|K\| < 1$,

$$\begin{aligned}C &= (I - KK^*)^{1/2}A(I - KK^*)^{1/2} = \\ &= U \begin{bmatrix} \operatorname{diag}((1-\varepsilon_1)\lambda_1, \dots, (1-\varepsilon_r)\lambda_r) & 0_{r,m-r} \\ 0_{n-r,r} & \operatorname{diag}(\lambda_{r+1}, \dots, \lambda_n) \end{bmatrix} U^*,\end{aligned}$$

and

$$D + K^*AK = V \begin{bmatrix} \operatorname{diag}(\varepsilon_1\lambda_1 + \mu_1, \dots, \varepsilon_r\lambda_r + \mu_r) & 0_{r,m-r} \\ 0_{m-r,r} & \operatorname{diag}(\mu_{r+1}, \dots, \mu_m) \end{bmatrix} V^*.$$

Then, we can explicitly compute the frame bounds for $\mathcal{F}_+$ and $\mathcal{F}_-$:

- $\alpha_+ = \min\{(1-\varepsilon_1)\lambda_1, \dots, (1-\varepsilon_r)\lambda_r, \lambda_n\}$;
- $\beta_+ = \max\{(1-\varepsilon_1)\lambda_1, \dots, (1-\varepsilon_r)\lambda_r, \lambda_{r+1}\}$;
- $\alpha_- = \min\{\varepsilon_1\lambda_1 + \mu_1, \dots, \varepsilon_r\lambda_r + \mu_r, \mu_{r+1}\}$;
- $\beta_- = \max\{\varepsilon_1\lambda_1 + \mu_1, \dots, \varepsilon_r\lambda_r + \mu_r, \mu_m\}$.

Observe that, since $(1 - \varepsilon_i)\lambda_i < \lambda_i + \mu_i$ and $\varepsilon_i\lambda_i + \mu_i < \lambda_i + \mu_i$ for each $i = 1, \dots, r$, we can obtain the following a priori estimates for the lower frame bounds of $\mathcal{F}_+$ and $\mathcal{F}_-$:

$$\alpha_+ \leq \min\{\lambda_1 + \mu_1, \dots, \lambda_r + \mu_r, \lambda_n\},$$

and

$$\alpha_- \leq \min\{\lambda_1 + \mu_1, \dots, \lambda_r + \mu_r, \mu_{r+1}\},$$

which are independent of the strictly contractive matrix K given in (3.3), i.e. independent of the angle between the subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$.

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