

Uncertainty principle and geometry of the infinite Grassmann manifold

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August 3, 2018

Abstract

We study the pairs of projections

$$P_I f = \chi_I f, \quad Q_J f = (\chi_J \hat{f})^\sim, \quad f \in L^2(\mathbb{R}^n),$$

where $I, J \subset \mathbb{R}^n$ are sets of finite Lebesgue measure, χ_I, χ_J denote the corresponding characteristic functions and $\hat{\cdot}, \sim$ denote the Fourier-Plancherel transformation $L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R}^n)$ and its inverse. These pairs of projections have been widely studied by several authors in connection with the mathematical formulation of Heisenberg's uncertainty principle. Our study is done from a differential geometric point of view. We apply known results on the Finsler geometry of the Grassmann manifold $\mathcal{P}(\mathcal{H})$ of a Hilbert space \mathcal{H} to establish that there exists a unique minimal geodesic of $\mathcal{P}(\mathcal{H})$, which is a curve of the form

$$\delta(t) = e^{itX_{I,J}} P_I e^{-itX_{I,J}}$$

which joins P_I and Q_J and has length $\pi/2$. As a consequence we obtain that if H is the logarithm of the Fourier-Plancherel map, then

$$\|[H, P_I]\| \geq \pi/2.$$

The spectrum of $X_{I,J}$ is denumerable and symmetric with respect to the origin, it has a smallest positive eigenvalue $\gamma(X_{I,J})$ which satisfies

$$\cos(\gamma(X_{I,J})) = \|P_I Q_J\|.$$

2010 MSC: 58B20, 47B15, 42A38, 47A63.

Keywords: Projections, pairs of projections, Grassmann manifold, uncertainty principle.

1 Introduction

Consider the following example:

Example 1.1. Let $I, J \subset \mathbb{R}^n$ be Lebesgue-measurable sets of finite measure. Let P_I, Q_J be the projections in $L^2(\mathbb{R}^n, dx)$ given by

$$P_I f = \chi_I f \quad \text{and} \quad Q_J f = (\chi_J \hat{f})^\sim,$$

where χ_L denotes the characteristic function of the set L . Equivalently, denoting by $U_{\mathcal{F}}$ the Fourier transformation regarded as a unitary operator acting in $L^2(\mathbb{R}^n, dx)$ and by M_{φ} the multiplication by φ , then

$$P_I = M_{\chi_I} \quad \text{and} \quad Q_J = U_{\mathcal{F}}^* P_J U_{\mathcal{F}}.$$

The operator $P_I Q_J$ is Hilbert-Schmidt (see for instance [11], Lemma 2).

An intuitive formulation of Heisenberg's uncertainty principle says that a nonzero function and its Fourier transform cannot be (simultaneously) sharply localized (see [13], page 207). We give more precision to this statement below (see for instance [11], page 906).

According to Folland and Sitaram [13], the idea of using projections P_I and Q_J to obtain a form of the uncertainty principle is due to Fuchs [14], and it was developed later in a series of papers by Landau, Pollack and Slepian [20], [21], [25]. See the survey by Folland and Sitaram [13].

Donoho and Stark [11] proved that if $I, J \subset \mathbb{R}^n$ with finite Lebesgue measure and $f \in L^2(\mathbb{R}^n)$ with $\|f\|_2 = 1$ satisfy that

$$\int_{\mathbb{R}^n - I} |f(t)|^2 dt < \epsilon_I \quad \text{and} \quad \int_{\mathbb{R}^n - J} |\hat{f}(w)|^2 dw < \epsilon_J$$

then

$$|I||J| \geq (1 - (\epsilon_I + \epsilon_J))^2.$$

Donoho and Stark showed several applications of these ideas to signal processing (and the obstruction to the existence of an instantaneous frequency). Smith [26] generalized these results to a locally compact abelian group G where $I \subset G$ and $J \subset \hat{G}$, the dual group of G . The books by Havin and Jörnicke [17], Hogan and Lakey [18], and Gröchenig [15] among many others, contain further applications, generalizations and history of the different uncertainty principles.

By an elementary computation using Fubini's theorem, Donoho and Stark prove that

$$\|P_I Q_J\|_{HS} = \sqrt{|I||J|},$$

where $\|\cdot\|_{HS}$ is Hilbert-Schmidt norm. Next they prove that

$$\|P_I Q_J\| \geq 1 - \epsilon_I - \epsilon_J.$$

The fact that $\|P_I Q_J\| \leq \|P_I Q_J\|_{HS}$ is well known.

They argue that any bound c such that

$$\|P_I Q_J\| \leq c < 1$$

is an expression of the uncertainty principle ([11], page 912).

Denote by $\mathcal{P}(\mathcal{H})$ the set of orthogonal projections of the Hilbert space \mathcal{H} , also called the Grassmann manifold of \mathcal{H} . It is indeed a differentiable manifold of $\mathcal{B}(\mathcal{H})$ (also in the infinite dimensional setting), with rich geometric structure (see for instance [24] or [7]). The pairs (P_I, Q_J) might be put in the broader context of the sets

$$\mathcal{C} = \{(P, Q) : P, Q \text{ are orthogonal projections and } PQ \text{ is compact}\}.$$

This set is a C^∞ -submanifold of $\mathcal{P}(\mathcal{H}) \times \mathcal{P}(\mathcal{H})$.

An application of these geometrical results facts is a form of the uncertainty principle (see Theorem 3.6 below).

Let us describe the content of the paper.

In Section 2 we recall the known facts on the geometry of $\mathcal{P}(\mathcal{H})$. In section 3 we apply known results [24], [7], [2] on the Finsler geometry of the Grassmann manifold of \mathcal{H} to the special case of pairs P_I, Q_J . We prove that there exists a unique minimal geodesic of the Grassmann manifold of length $\pi/2$ which joins P_I and Q_J . That is, there exists a unique selfadjoint operator $X_{I,J}$ of norm $\pi/2$, which is co-diagonal with respect both to P_I and Q_J , such that

$$e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J.$$

The spectrum of the operator $X_{I,J}$ is denumerable and symmetric with respect to the origin. The smallest positive eigenvalue $\gamma(X_{I,J})$ verifies

$$\cos(\gamma(X_{I,J})) = \|P_I Q_J\|.$$

As a consequence from the fact that the minimal geodesic has length $\pi/2$, we prove that H is the logarithm of the Fourier transform in $L^2(\mathbb{R}^n)$, and $I \subset \mathbb{R}^n$ is a set of finite Lebesgue measure, then

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

In Section 4 we show that for any pair of sets $I, J \subset \mathbb{R}^n$ of finite measure, one has

$$N(P_I) + N(Q_J) = L^2(\mathbb{R}^n),$$

where the sum is non-direct (the subspaces have infinite dimensional intersection).

2 Basic properties

2.1 Halmos decomposition

Let \mathcal{H} be a Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators in \mathcal{H} , $\mathcal{K}(\mathcal{H})$ the ideal of compact operators and $\mathcal{P}(\mathcal{H})$ the set of selfadjoint (orthogonal) projections, and $\mathcal{P}_\infty(\mathcal{H})$ the subset of projections whose nullspaces and ranges have infinite dimension.

A tool that will be useful in the study of the pairs P_I, Q_J is *Halmos decomposition* [16], which is the following orthogonal decomposition of \mathcal{H} : given a pair of projections P and Q , consider

$$\mathcal{H}_{11} = R(P) \cap R(Q), \quad \mathcal{H}_{00} = N(P) \cap N(Q), \quad \mathcal{H}_{10} = R(P) \cap N(Q), \quad \mathcal{H}_{01} = N(P) \cap R(Q)$$

and \mathcal{H}_0 the orthogonal complement of the sum of the above. This last subspace is usually called the *generic part* of the pair P, Q . Note also that

$$N(P - Q) = \mathcal{H}_{11} \oplus \mathcal{H}_{00}, \quad N(P - Q - 1) = \mathcal{H}_{10} \quad \text{and} \quad N(P - Q + 1) = \mathcal{H}_{01},$$

so that the generic part depends in fact of the difference $P - Q$.

Halmos proved that there is an isometric isomorphism between \mathcal{H}_0 and a product Hilbert space $\mathcal{L} \times \mathcal{L}$ such that in the above decomposition (putting $\mathcal{L} \times \mathcal{L}$ in place of \mathcal{H}_0), the projections are

$$P = 1 \oplus 0 \oplus 1 \oplus 0 \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$Q = 1 \oplus 0 \oplus 0 \oplus 1 \oplus \begin{pmatrix} C^2 & CS \\ CS & S^2 \end{pmatrix},$$

where $C = \cos(X)$ and $S = \sin(X)$ for some operator $0 < X \leq \pi/2$ in \mathcal{L} with trivial nullspace.

Aparently, the pair (P, Q) belongs to \mathcal{C} if and only if \mathcal{H}_{11} is finite dimensional and $C = \cos(X)$ is compact.

Remark 2.1. If $(P, Q) \in \mathcal{C}$, then the spectral resolution of X can be easily described. Since $0 < \cos(X)$ is compact, it follows that

$$X = \sum_n \gamma_n P_n + \frac{\pi}{2} E,$$

where $0 < \gamma_n < \pi/2$ is an increasing (finite or infinite) sequence. For all n , $\dim R(P_n) < \infty$, and

$$R(E) \oplus (\oplus_{n \geq 1} R(P_n)) = \mathcal{L}.$$

2.2 Finsler geometry of the Grassmann manifold of \mathcal{H}

Let us recall some basic facts on the differential geometry of the set $\mathcal{P}(\mathcal{H})$ (see for instance [7], [24], [2]).

1. The space $\mathcal{P}(\mathcal{H})$ is a homogeneous space under the action of the unitary group $\mathcal{U}(\mathcal{H})$ by inner conjugation: if $U \in \mathcal{U}(\mathcal{H})$ and $P \in \mathcal{P}(\mathcal{H})$, the action is given by

$$U \cdot P = UPU^*.$$

This action is locally transitive: it is well known that two projections P_1, P_2 such that $\|P_1 - P_2\| < 1$, are conjugate. Therefore, since the unitary group $\mathcal{U}(\mathcal{H})$ is connected, the orbits of the action coincide with the connected components of $\mathcal{P}(\mathcal{H})$, which are: for $n \in \mathbb{N}$, $\mathcal{P}_{n,\infty}(\mathcal{H})$ (projections of nullity n), $\mathcal{P}_{\infty,n}(\mathcal{H})$ (projections of rank n) and $\mathcal{P}_{\infty}(\mathcal{H})$ (projections of infinite rank and nullity). These components are C^∞ -submanifolds of $\mathcal{B}(\mathcal{H})$.

2. There is a natural linear connection in $\mathcal{P}(\mathcal{H})$. If $\dim \mathcal{H} < \infty$, it is the Levi-Civita connection of the Riemannian metric which consists of considering the Frobenius inner product at every tangent space. It is based on the diagonal / co-diagonal decomposition of $\mathcal{B}(\mathcal{H})$. To be more specific, given $P_0 \in \mathcal{P}(\mathcal{H})$, the tangent space of $\mathcal{P}(\mathcal{H})$ at P_0 consists of all self-adjoint co-diagonal matrices (in terms of P_0). The linear connection in $\mathcal{P}(\mathcal{H})$ is induced by a reductive structure, where the horizontal elements at P_0 (in the Lie algebra of $\mathcal{U}(\mathcal{H})$: the space of antihermitian elements of $\mathcal{B}(\mathcal{H})$) are the co-diagonal antihermitian operators. The geodesics of \mathcal{P} which start at P_0 are curves of the form

$$\delta(t) = e^{itX} P_0 e^{-itX}, \tag{1}$$

with $X^* = X$ co-diagonal with respect to P_0 . Observe that X is co-diagonal with respect to every $P_t = \delta(t)$. It was proved in [24] that if $P_0, P_1 \in \mathcal{P}(\mathcal{H})$ satisfy $\|P_0 - P_1\| < 1$, then there exists a unique geodesic (up to reparametrization) joining P_0 and P_1 . This condition is not necessary for the existence of a unique geodesic.

3. There exists a unique geodesic joining two projections P and Q if and only if

$$R(P) \cap N(Q) = N(P) \cap R(Q) = \{0\},$$

(see [2]).

4. If \mathcal{H} is infinite dimensional, the Frobenius metric is not available. However, if one endows each tangent space of $\mathcal{P}(\mathcal{H})$ with the usual norm of $\mathcal{B}(\mathcal{H})$, one obtains a continuous (non regular) Finsler metric,

$$d(P_0, P_1) = \inf\{\ell(\gamma) : \gamma \text{ a continuous piecewise smooth curve in } \mathcal{P}(\mathcal{H}) \text{ joining } P_0 \text{ and } P_1\}$$

where $\ell(\gamma)$ denotes the length of γ (parametized in the interval I):

$$\ell(\gamma) = \int_I \|\dot{\gamma}(t)\| dt.$$

In [24] it was shown that the geodesics (1) remain minimal among their endpoints for all t such that

$$|t| \leq \frac{\pi}{2\|X\|}.$$

It can be shown that $d(P_0, P_1) < \pi/2$ if and only if $\|P_0 - P_1\| < 1$. In other words, $\|P_0 - P_1\| = 1$ if and only if $d(P_0, P_1) = \pi/2$.

3 Geometry of the pairs P_I, Q_J

Lenard proved in [22] that the projections $P_I, Q_J \in \mathcal{P}(L^2(\mathbb{R}^n, dx))$ defined in Example (1.1), satisfy

$$R(P_I) \cap N(Q_J) = R(Q_J) \cap N(P_I) = \{0\}. \quad (2)$$

Moreover, $\|P_I - Q_J\| = 1$.

Therefore one obtains the following:

Theorem 3.1. *Let I, J be measurable subsets of \mathbb{R}^n of finite measure, and P_I, Q_J the above projections. Then there exists a unique selfadjoint operator $X_{I,J}$ satisfying:*

1. $\|X_{I,J}\| = \pi/2$.
2. $X_{I,J}$ is P_I and Q_J co-diagonal. In other words, $X_{I,J}$ maps functions in $L^2(\mathbb{R}^n, dx)$ with support in I to functions with support in $\mathbb{R}^n - I$, and functions such that \hat{f} has support in J to functions such that the Fourier transform has support in $\mathbb{R}^n - J$.
3. $e^{iX_{I,J}} P_I e^{-iX_{I,J}} = Q_J$.
4. If $P(t), t \in [0, 1]$ is a smooth curve in $\mathcal{P}(\mathcal{H})$ with $P(0) = P_I$ and $P(1) = Q_J$, then

$$\ell(P) = \int_0^1 \|\dot{P}(t)\| dt \geq \pi/2.$$

Proof. By the condition (2) above ([22]), it follows from [2] that there exists a unique minimal geodesic of $\mathcal{P}(\mathcal{H})$, of the form

$$\delta_{I,J}(t) = e^{itX_{I,J}} P_I e^{itX_{I,J}}$$

with $X_{I,J}^* = X_{I,J}$ co-diagonal with respect to P_I (and Q_J) such that

$$\delta_{I,J}(1) = Q_J.$$

Condition 4. above is the minimality property of $\delta_{I,J}$. Finally, the fact that $\|P_I - Q_J\| = 1$ means that $\|X_{I,J}\| = \pi/2$. \square

Remark 3.2. It is known [13] that $\lambda_1 = \|P_I Q_J P_I\| = \|P_I Q_J\|^2 < 1$, and moreover $\sqrt{\lambda_1}$ equals the cosine of the angle between the subspaces $R(P_I)$ and $R(Q_J)$.

One can also relate this number λ_1 with the operator $X_{I,J}$. Using Halmos decomposition (recall that it consists only of \mathcal{H}_{00} and the generic part \mathcal{H}_0 in this case),

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C^2 & 0 \\ 0 & 0 \end{pmatrix}$$

and thus $\lambda_1 = \|\cos(X)\|^2$. We shall see below that the spectrum of X is a strictly increasing sequence of positive eigenvalues $\gamma_n \rightarrow \pi/2$, with finite multiplicity. Moreover, since $P_I Q_J P_I$ belongs to $\mathcal{B}_1(\mathcal{H})$, it follows that $C \in \mathcal{B}_2(\mathcal{L})$. Thus

$$\{\cos(\gamma_n)\} \in \ell^2.$$

For a given $P \in \mathcal{P}(\mathcal{H})$, let \mathcal{A}_P be

$$\mathcal{A}_P = \{X \in \mathcal{B}(\mathcal{H}) : [X, P] \text{ is compact}\}.$$

Apparently \mathcal{A}_P is a C^* -algebra.

Theorem 3.3. *Let I, J be measurable subsets of \mathbb{R}^n of finite Lebesgue measure.*

1. *The selfadjoint operator $X_{I,J}$ has closed infinite dimensional range, in particular it is not compact.*
2. *Let I_0 be another measurable set with finite measure such that $|I \cap I_0| = 0$, and let $P_0 = P_{I_0}$. Then, the commutant $[X_{I,J}, P_0]$ is compact.*

Proof. Easy matrix computations ([2]) show that, in the decomposition $\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$, $X_{I,J}$ is of the form

$$X_{I,J} = 0 \oplus \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Note that the spectrum of this operator is symmetric with respect to the origin. Indeed, if V equals the symmetry

$$V = 1 \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then apparently $V X_{I,J} V = -X_{I,J}$. Also note that

$$X_{I,J}^2 = 0 \oplus \begin{pmatrix} X^2 & 0 \\ 0 & X^2 \end{pmatrix}.$$

Therefore the spectrum of $X_{I,J}$ is

$$\sigma(X_{I,J}) = \{0\} \cup \{\gamma_n : n \geq 1\} \cup \{-\gamma_n : n \geq 1\},$$

with 0 of infinite multiplicity, and the multiplicity of γ_n equal to the multiplicity of $-\gamma_n$, and finite. What matters here, is that the set $\{\gamma_n : n \geq 1\}$ is infinite, and is therefore an increasing sequence converging to $\pi/2$. This holds because otherwise, the operator C would have finite rank, and therefore $P_I Q_J P_I$ would be of finite rank, which is not the case (see [22]). Thus $X_{I,J}$ has closed range. of infinite dimension.

Note that P_I and Q_J satisfy that $P_I P_0 = 0$ and $Q_J P_0 = Q_J P_{I_0}$ is compact, and therefore $P_I, Q_J \in \mathcal{A}_{P_0}$. Thus the symmetries S_{P_I}, S_{Q_J} belong to \mathcal{A}_{P_0} . Since $S_{Q_J} = e^{i2X_{I,J}} S_{P_I}$, this implies that

$$e^{i2X_{I,J}} \in \mathcal{A}_{P_0}.$$

By the spectral picture of $X_{I,J}$ it is clear that $X_{I,J}$ can be obtained as an holomorphic function of $e^{i2X_{I,J}}$. Since \mathcal{A}_{P_0} is a C^* -algebra, this implies that $X_{I,J} \in \mathcal{A}_{P_0}$. \square

Let us relate the operator $X_{I,J}$ with the mathematical version of the uncertainty principle, according to [11] and [13].

Let $A \in \mathcal{B}(\mathcal{H})$ be an operator with closed range, the *reduced minimum modulus* γ_A of A is the positive number

$$\gamma_A = \min\{\|A\xi\| : \xi \in N(A)^\perp, \|\xi\| = 1\} = \min\{|\lambda| : \lambda \in \sigma(A), \lambda \neq 0\}.$$

Donoho and Stark [11] underline the role of the number $\|Q_J P_I\|$ and consider any constant c such that $\|Q_J P_I\| \leq c$ a manifestation of the (mathematical) uncertainty principle. By the above Remark, we have:

Corollary 3.4. *With the current notations,*

$$\|Q_J P_I\| = \cos(\gamma_{X_{I,J}}).$$

Proof. Indeed, in the above description of the spectrum of $X_{I,J}$, the reduced minimum modulus $\gamma_{X_{I,J}}$ of $X_{I,J}$ coincides with γ_1 . \square

Let $X_{I,J}^0$ be the restriction of $X_{I,J}$ to the generic part of P_I and Q_J , i.e., its restriction to $N(X_{I,J})^\perp$. In Halmos decomposition

$$X_{I,J}^0 = \begin{pmatrix} 0 & -iX \\ iX & 0 \end{pmatrix}.$$

Recall the formula by Donoho and Stark [11]

$$\|P_I Q_J\|_{HS} = |I|^{1/2} |J|^{1/2}.$$

From the preceding facts, it also follows:

Corollary 3.5. *With the current notations*

$$|I|^{1/2} |J|^{1/2} = \|\cos(X)\|_{HS} = \frac{1}{\sqrt{2}} \|\cos(X_{I,J}^0)\|_{HS} = \left\{ \sum_{n=1}^{\infty} \frac{1}{2} \cos(\gamma_n)^2 \right\}^{1/2}.$$

Proof.

$$|I||J| = \|P_I Q_J\|_{HS}^2 = \text{Tr}(P_I Q_J P_I) = \text{Tr}(C^2) = \frac{1}{2} \text{Tr} \begin{pmatrix} C^2 & 0 \\ 0 & C^2 \end{pmatrix} = \frac{1}{2} \text{Tr}(\cos(X_{I,J}^0)^2).$$

□

This co-diagonal exponent $X_{I,J}$ (with respect both to P_I and Q_J) has interesting features when $I = J$ and $|I| < \infty$. In this case denote by $X_I = X_{I,I}$; then, we have two unitary operators intertwining P_I and Q_I . Namely, the Fourier transform $U_{\mathcal{F}}$ and the exponential e^{iX_I} ,

$$U_{\mathcal{F}}^* P_I U_{\mathcal{F}} = Q_I = e^{iX_I} P_I e^{-iX_I}.$$

Let $H = H^*$ be the natural logarithm of the Fourier transform, $e^{iH} = U_{\mathcal{F}}$. Namely, writing E_1, E_{-1}, E_i and E_{-i} the eigenprojections of $U_{\mathcal{F}}$,

$$H = -\pi E_{-1} + \frac{\pi}{2} E_i - \frac{\pi}{2} E_{-i}.$$

Note that $\|H\| = \pi$. Thus, one obtains a smooth path joining P_I and Q_I :

$$\varphi(t) = e^{-itH} P_I e^{itH}.$$

and, apparently, $\varphi(1) = Q_I$.

Since the Fourier transform intertwines P_I and Q_J , the norm of its commutant with either of these projections can be regarded as a measure of non commutativity between P_I and Q_J :

Theorem 3.6. *For any Lebesgue measurable set $I \subset \mathbb{R}^n$ with $|I| < \infty$, one has*

$$\|[H, P_I]\| = \|[H, Q_I]\| \geq \pi/2.$$

Proof. The geodesic δ_I with exponent X_I is the shortest curve in $\mathcal{P}(\mathcal{H})$ joining P_I and Q_I . Its length is $\pi/2$. Then

$$\pi/2 \leq \ell(\varphi) = \int_0^1 \|\dot{\varphi}(t)\| dt = \int_0^1 \|e^{itH} [H, P_I] e^{-itH}\| dt = \|[H, P_I]\|.$$

Note that

$$U_{\mathcal{F}}^* [H, P_I] U_{\mathcal{F}} = [H, U_{\mathcal{F}}^* P_I U_{\mathcal{F}}] = [H, Q_I]$$

because $U_{\mathcal{F}}$ and H commute. □

Remark 3.7.

1. We may write H in terms of $U_{\mathcal{F}}$ using the well known formulas

$$E_{-1} = \frac{1}{4}(1 - U_{\mathcal{F}} + U_{\mathcal{F}}^2 - U_{\mathcal{F}}^3), \quad E_i = \frac{1}{4}(1 - iU_{\mathcal{F}} - U_{\mathcal{F}}^2 + iU_{\mathcal{F}}^3), \quad E_{-i} = \frac{1}{4}(1 + iU_{\mathcal{F}} - U_{\mathcal{F}}^2 - iU_{\mathcal{F}}^3),$$

and thus

$$H = \frac{\pi}{4} \{-1 + (1+i)U_{\mathcal{F}} - U_{\mathcal{F}}^2 + (1+i)U_{\mathcal{F}}^3\}.$$

Then

$$[H, P_I] = \frac{\pi}{4} \{(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\}.$$

The inequality in Corollary 3.6 can be written

$$\|(1+i)[U_{\mathcal{F}}, P_I] - [U_{\mathcal{F}}^2, P_I] + (1+i)[U_{\mathcal{F}}^3, P_I]\| \geq 2.$$

2. In the special case when the set I is (essentially) symmetric with respect to the origin, P_I commutes with $U_{\mathcal{F}}^2$, so that

$$[U_{\mathcal{F}}^2, P_I] = 0 \quad \text{and} \quad [U_{\mathcal{F}}^3, P_I] = [U_{\mathcal{F}}, P_I]U_{\mathcal{F}}^2 = U_{\mathcal{F}}^2[U_{\mathcal{F}}, P_I]$$

one has

$$[H, P_I] = \frac{(1+i)\pi}{4}[U_{\mathcal{F}}, P_I](1 + U_{\mathcal{F}}^2).$$

The operator $U_{\mathcal{F}}^2 f(x) = f(-x)$ is a symmetry, then $\frac{1}{2}(1 + U_{\mathcal{F}}^2)$ is the orthogonal projection E_e onto the the subspace of essentially even functions ($f(x) = f(-x)$ a.e.). Then one can write

$$[H, P_I] = \frac{(1+i)\pi}{2}[U_{\mathcal{F}}, P_I]E_e = \frac{(1+i)\pi}{2}E_e[U_{\mathcal{F}}, P_I].$$

Corollary 3.8. *Suppose that I is essentially symmetric, with finite measure.*

1.

$$\|E_e[U_{\mathcal{F}}, P_I]\| = \|E_e[U_{\mathcal{F}}, P_I]E_e\| \geq \frac{1}{\sqrt{2}}.$$

2.

$$\|E_e P_I - E_e Q_I\| \geq \frac{1}{\sqrt{2}},$$

where $E_e P_I = P_I E_e$ and $E_e Q_I = Q_I E_e$ are orthogonal projections.

Proof. Recall that E_e and $U_{\mathcal{F}}$ commute. Then

$$\begin{aligned} E_e[U_{\mathcal{F}}, P_I]E_e &= E_e(U_{\mathcal{F}}P_I - P_I U_{\mathcal{F}})E_e = U_{\mathcal{F}}E_e(P_I - U_{\mathcal{F}}^*P_I U_{\mathcal{F}})E_e \\ &= U_{\mathcal{F}}E_e(P_I - Q_I)E_e. \end{aligned}$$

where E_e , as well as $U_{\mathcal{F}}$, and thus also $Q_I = U_{\mathcal{F}}^*P_I U_{\mathcal{F}}$ commute with E_e . \square

The ranges of these two orthogonal projections $E_e P_I$ and $E_e Q_I$ consist of the elements of L^2 which are essentially even and vanish (essentially) outside I , and the analogous subspace for the Fourier transform.

4 Spatial properties of P_I and Q_J

Let us return to the general setting (I not necessarily equal to J). The ranges and nullspaces of P_I and Q_J have several interesting properties. First we need the following lemma:

Lemma 4.1. *Let P, Q be orthogonal projections such that $\|P - Q\| = 1$. Then one and only one of the following conditions hold:*

1. $N(P) + R(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $R(P) + N(Q)$ being a direct sum and a closed proper subspace of \mathcal{H}).
2. $R(P) + N(Q) = \mathcal{H}$, with non direct sum (and this is equivalent to $N(P) + R(Q)$ being a direct sum and a closed proper subspace of \mathcal{H}).

3. $R(P) + N(Q)$ is non closed (and this is equivalent to $N(P) + R(Q)$ being non closed).

Proof. By the Krein-Krasnoselskii-Milman formula (see for instance [19])

$$\|P - Q\| = \max\{\|P(1 - Q)\|, \|Q(1 - P)\|\},$$

we have that one and only one of the following hold:

1. $\|P(1 - Q)\| < 1$ and $\|Q(1 - P)\| = 1$,
2. $\|P(1 - Q)\| = 1$ and $\|Q(1 - P)\| < 1$, or
3. $\|P(1 - Q)\| = 1$ and $\|Q(1 - P)\| = 1$.

This alternative corresponds precisely with the three conditions in the Lemma. It is known [9] that for two orthogonal projections E and F , $\|EF\| < 1$ holds if and only if $R(E) \cap R(F) = \{0\}$ and $R(E) + R(F)$ closed. The sum $\mathcal{M} + \mathcal{N}$ of two subspaces is closed if and only if the sum $\mathcal{M}^\perp + \mathcal{N}^\perp$ is closed (see [9]). Therefore, $\|EF\| < 1$ is also equivalent to $N(E) + N(F) = \mathcal{H}$.

If we apply these facts to $E = P$ and $F = 1 - Q$, we obtain that the first alternative is equivalent to $R(P) \cap N(Q) = \{0\}$ and $R(P) + N(Q)$ closed, or to $N(P) + R(Q) = \mathcal{H}$.

Analogously, the second alternative is equivalent to $R(Q) \cap N(P) = \{0\}$ and $R(Q) + N(P)$ closed, or to $N(Q) + R(P) = \mathcal{H}$.

Note that in the first case, $R(P) + N(Q)$ is proper, otherwise its orthogonal complement would be $N(P) \cap R(Q) = \{0\}$, which together with the fact that $N(P) + R(Q) = \mathcal{H}$ (closed!), would lead us to the second alternative.

Analogously in the second alternative, $N(P) + R(Q)$ is proper.

If neither of these two happen, it is clear that neither $R(P) + N(Q)$ nor (equivalently) the sum of the orthogonals $N(P) + R(Q)$ is closed. \square

We have the following:

Theorem 4.2. *Let $I, J \subset \mathbb{R}^n$ with finite Lebesgue measure. Then*

1. $R(P_I) + R(Q_J)$ is a closed proper subset of $L^2(\mathbb{R}^n)$, with infinite codimension. The sum is direct ($R(P_I) \cap R(Q_J) = \{0\}$).
2. $N(P_I) + N(Q_J) = L^2(\mathbb{R}^n)$, and the sum is not direct ($N(P_I) \cap N(Q_J)$ is infinite dimensional).
3. $R(P_I) + N(Q_J)$ and $N(P_I) + R(Q_J)$ are proper dense subspaces of $L^2(\mathbb{R}^n)$, and $R(P_I) \cap N(Q_J) = N(P_I) \cap R(Q_J) = \{0\}$.

Proof. By the cited result [9], two projections P, Q , satisfy that $R(P) + R(Q)$ is closed and $R(P) \cap R(Q) = \{0\}$ if and only if $\|PQ\| < 1$. It is also known (see above, [13]) that $\|P_I Q_J\| < 1$. The intersection of these spaces is, in our case (using the notation of the Halmos decomposition)

$$R(P_I) \cap R(Q_J) = \mathcal{H}_{11} = \{0\}.$$

As remarked above, Lenard proved that $\mathcal{H}_{11} = \mathcal{H}_{10} = \mathcal{H}_{01} = \{0\}$, and \mathcal{H}_{00} is infinite dimensional. The orthogonal complement of this sum is

$$(R(P_I) + R(Q_J))^\perp = N(P_I) \cap N(Q_J) = \mathcal{H}_{00}.$$

Thus the first assertion follows.

In our case $\|P_I - Q_J\| = 1$ ([13], [22]) thus we may apply the above Lemma.

The first condition cannot happen:

$$(N(P_I) + R(Q_J))^\perp = R(P_I) \cap N(Q_J) = \mathcal{H}_{10} = \{0\}.$$

By a similar argument, neither the second condition can happen. Thus $R(P_I) + R(Q_J)$ is non closed, and its orthogonal complement is trivial. Thus the second and third assertions follow. \square

Remark 4.3. It is known (see for instance [12]), that if P, Q are projections with PQ compact and $R(P) \cap R(Q) = \{0\}$, then

$$\|PQ\| < 1.$$

In [6], the second named author and A. Maestriperieri studied the set of operators $T \in \mathcal{B}(\mathcal{H})$ which are of the form $T = PQ$. Among other properties, they proved that T may have many factorizations, but there is a minimal factorization (called *canonical factorization* of T), namely

$$T = P_{\overline{R(T)}} P_{N(T)^\perp},$$

which satisfies that if $T = PQ$, then $R(T) \subset R(P)$ and $N(T)^\perp \subset R(Q)$ (or equivalently $N(Q) \subset N(T)$). Following this notation,

Proposition 4.4. *The factorization $P_I Q_J$ is canonical.*

Proof. Put $T = P_I Q_J$. Using Halmos decomposition in this particular case ($\mathcal{H} = \mathcal{H}_{00} \oplus (\mathcal{L} \times \mathcal{L})$), apparently

$$P_I Q_J P_I = 0 \oplus \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix},$$

and thus $R(P_I Q_J P_I) = 0 \oplus (R(C) \times 0)$. Recall that $C^2 > 0$, and thus C^2 has dense range. It follows that

$$\overline{R(T)} = \overline{R(P_I Q_J)} = \overline{R(P_I Q_J P_I)} = 0 \oplus (\mathcal{L} \times 0),$$

which is precisely the range of P_I : $\overline{R(T)} = R(P_I)$. Note the following elementary fact:

$$N(PQ) = N(Q) \oplus (R(Q) \cap N(P)).$$

For the factorization $T = P_I Q_J$ it is known ([22]) that $R(Q_J) \cap N(P_I) = 0$. Thus

$$N(T) = N(P_I Q_J) = N(Q_J)$$

and the proof follows. \square

In [6] it is proven that if $T = PQ = P_0 Q_0$, and the latter is the canonical factorization, then

$$\|P_0 f - Q_0 f\| \leq \|P f - Q f\|$$

for any $f \in L^2(\mathbb{R}^n)$. In particular $\|P_0 - Q_0\| \leq \|P - Q\|$. In our case we get the following result

Corollary 4.5. *Let P, Q projections in $L^2(\mathbb{R}^n)$ such that $PQ = P_I Q_J$. Then for any $f \in L^2(\mathbb{R}^n)$ one has*

$$\|P_I f - Q_J f\|_2 \leq \|P f - Q f\|_2.$$

In particular, $\|P_I - Q_J\| \leq \|P - Q\|$.

References

- [1] Amrein, W. O.; Sinha, K. B. On pairs of projections in a Hilbert space. *Linear Algebra Appl.* 208/209 (1994), 425–435.
- [2] Andruchow, E.; Operators which are the difference of two projections. *J. Math. Anal. Appl.* 420 (2014), no. 2, 1634–1653.
- [3] Arias, A.; Gudder, S. Almost sharp quantum effects. *J. Math. Phys.* 45 (2004), no. 11, 4196–4206.
- [4] Berthier, A. M.; Jauch, J. M. A theorem on the support of functions in $L^2(\mathbf{R})$ and of their Fourier transforms. *Lett. Math. Phys.* 1 (1975/76), no. 2, 93–97.
- [5] Böttcher, A.; Spitkovsky, I. M. A gentle guide to the basics of two projections theory. *Linear Algebra Appl.* 432 (2010), no. 6, 1412–1459.
- [6] Corach, G.; Maestriperi, A. Products of orthogonal projections and polar decompositions. *Linear Algebra Appl.* 434 (2011), no. 6, 1594–1609.
- [7] Corach, G.; Porta, H.; Recht, L. The geometry of spaces of projections in C^* -algebras. *Adv. Math.* 101 (1993), no. 1, 59–77.
- [8] Davis, C. Separation of two linear subspaces. *Acta Sci. Math. Szeged* 19 (1958) 172–187.
- [9] Deutsch, F. *Best approximation in inner product spaces*, Springer-Verlag, New York, 2001.
- [10] Dixmier, J. Position relative de deux variétés linéaires fermées dans un espace de Hilbert. (French) *Revue Sci.* 86, (1948). 387–399.
- [11] Donoho, D. L.; Stark, P. B. Uncertainty principles and signal recovery. *SIAM J. Appl. Math.* 49 (1989), no. 3, 906–931.
- [12] Feshchenko, I. S. On closedness of the sum of n subspaces of a Hilbert space. *Ukrainian Math. J.* 63 (2012), no. 10, 1566–1622.
- [13] Folland, G. B.; Sitaram, A. The uncertainty principle: a mathematical survey. *J. Fourier Anal. Appl.* 3 (1997), no. 3, 207–238.
- [14] Fuchs, W. H. J., On the magnitude of Fourier transforms. *Proc. Intern. Math. Congr.*, Amsterdam, North-Holland, Amsterdam (1954).
- [15] Gröchenig, K. *Foundations of time-frequency analysis. Applied and Numerical Harmonic Analysis.* Birkhuser Boston, Inc., Boston, MA, 2001.
- [16] Halmos, P. R. Two subspaces. *Trans. Amer. Math. Soc.* 144 (1969) 381–389.
- [17] Havin, V.; Jöricke, B. *The uncertainty principle in harmonic analysis*, Springer-Verlag, Berlin, 1994.
- [18] Hogan, J. A.; Lakey, J. D. *Time-frequency and time-scale methods. Adaptive decompositions, uncertainty principles, and sampling*, Birkhäuser, Boston, 2005.

- [19] Krein, M. G. The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II. (Russian) *Mat. Sbornik N.S.* 21(63), (1947). 365–404.
- [20] Landau, H. J.; Pollak, H. O. Prolate spheroidal wave functions, Fourier analysis and uncertainty. II. *Bell System Tech. J.* 40 (1961), 65–84.
- [21] Landau, H. J.; Pollak, H. O. Prolate spheroidal wave functions, Fourier analysis and uncertainty. III. The dimension of the space of essentially time- and band-limited signals. *Bell System Tech. J.* 41 (1962), 1295–1336.
- [22] Lenard, A. The numerical range of a pair of projections. *J. Functional Analysis* 10 (1972), 410–423.
- [23] Nees, M. Products of orthogonal projections as Carleman operators. *Integral Equations Operator Theory* 35 (1999), no. 1, 85–92.
- [24] Porta, H.; Recht, L. Minimality of geodesics in Grassmann manifolds. *Proc. Amer. Math. Soc.* 100 (1987), no. 3, 464–466.
- [25] Slepian, D.; Pollak, H. O. Prolate spheroidal wave functions, Fourier analysis and uncertainty. I. *Bell System Tech. J.* 40 (1961), 43–63.
- [26] Smith, K. T., The uncertainty principle on groups, *SIAM J. Appl. Math.* 50 (1990), 876–882.

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