

Schur complements of selfadjoint Krein space operators

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Abstract

Given a bounded selfadjoint operator W on a Krein space \mathcal{H} and a closed subspace \mathcal{S} of \mathcal{H} , the Schur complement of W to \mathcal{S} is defined under the hypothesis of weak complementability. A variational characterization of the Schur complement is given and the set of selfadjoint operators W admitting a Schur complement with these variational properties is shown to coincide with the set of \mathcal{S} -weakly complementable selfadjoint operators.

Keywords: Schur complements, Krein spaces, oblique projections
47A58, 47B50, 47A64

1. Introduction

The notion of Schur complement (or shorted operator) of B to \mathcal{S} for a positive operator B on a Hilbert space \mathcal{H} and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace, was introduced by M.G. Krein [16]. When $\leq_{\mathcal{H}}$ is the usual order in $L(\mathcal{H})$, he proved that the set $\{X \in L(\mathcal{H}) : 0 \leq_{\mathcal{H}} X \leq_{\mathcal{H}} B \text{ and } R(X) \subseteq \mathcal{S}^{\perp}\}$ has a maximum element, which he defined as the Schur complement $B_{/\mathcal{S}}$ of B to \mathcal{S} . This notion was later rediscovered by Anderson and Trapp [1]. If B is represented as the 2×2 block matrix $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ with respect to the decomposition of $\mathcal{H} = \mathcal{S} \oplus \mathcal{S}^{\perp}$, they established the formula

$$B_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & c - y^*y \end{pmatrix}$$

where y is the unique solution of the equation $b = a^{1/2}x$ such that the range inclusion $R(y) \subseteq \overline{R(a)}$ holds. The solution always exists because B is positive, in which case a is also positive and the range inclusion $R(b) \subseteq R(a^{1/2})$ holds.

In [4] Antezana et al., extended the Schur complement to any bounded operator B satisfying a weak complementability condition with respect to a given pair of closed subspaces \mathcal{S} and \mathcal{T} , by giving an Anderson-Trapp type formula. In particular, if B is a bounded selfadjoint operator, $\mathcal{S} = \mathcal{T}$ and $B = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$, this condition reads $R(b) \subseteq R(|a|^{1/2})$, which as noted, is automatic for positive operators.

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Later, Massey and Stojanoff [20] studied many properties of the Schur complement of an \mathcal{S} -weakly complementable selfadjoint operator B when \mathcal{S} is B -positive.

In this paper we show that if B is a bounded selfadjoint operator which is \mathcal{S} -weakly complementable then $B_{/\mathcal{S}}$ can be characterized as the solution of a *min – max* problem, extending the original approach of Krein. But, more importantly, the converse is true, in the sense that if the solution of this *min – max* problem exists then B has to be \mathcal{S} -weakly complementable. In other words, the \mathcal{S} -weakly complementable operators are exactly those selfadjoint operators admitting a Schur complement that satisfies these variational properties.

A closed-form expression for the Schur complement $B_{/\mathcal{S}}$ of B to \mathcal{S} is also established, in terms of a family of densely defined projections with prescribed nullspace \mathcal{S}^\perp (Theorem 3.14). This formula is new even in the case of positive B .

We then turn to the consideration of a bounded selfadjoint operator W on a Krein space $(\mathcal{H}, [\ , \])$. For a fixed signature operator J on \mathcal{H} , JW is selfadjoint in the Hilbert space inner product $\langle \ , \ \rangle$ associated with J . If \mathcal{S} is a given closed subspace of \mathcal{H} , JW is assumed to be \mathcal{S} -weakly complementable and J_α is any other signature operator on \mathcal{H} then two key results are established: $J_\alpha W$ is \mathcal{S} -weakly complementable (Theorem 4.4) and $J(JW)_{/\mathcal{S}} = J_\alpha(J_\alpha W)_{/\mathcal{S}}$ (Theorem 4.5).

Based on these results we extend the notions of \mathcal{S} -weak complementability and Schur complement to the Krein space setting. A bounded selfadjoint operator W on a Krein space \mathcal{H} is \mathcal{S} -weakly complementable if, for some (and, hence, any) signature operator J , JW is \mathcal{S} -weakly complementable in the corresponding Hilbert space. If this is the case then the Schur complement of W to \mathcal{S} is $W_{/[\mathcal{S}]} := J(JW)_{/\mathcal{S}}$.

In this fashion we obtain a simple way of computing the Schur complement of \mathcal{S} -weakly complementable selfadjoint operators in Krein spaces. This definition allows us to “translate” the properties obtained in Hilbert spaces to the Krein space setting in a straightforward way.

If \mathcal{S} is a regular subspace of \mathcal{H} (meaning that $\mathcal{H} = \mathcal{S} [\dot{+}] \mathcal{S}^{\perp[\dot{+}]}$) then it is possible to give a characterization of the \mathcal{S} -weak complementability of W in terms of the entries of the first row of the 2×2 block matrix representation of W with respect to $\mathcal{S} [\dot{+}] \mathcal{S}^{\perp[\dot{+}]}$. Indeed if $W = (w_{ij})_{i,j=1,2}$ and $w_{11} = dd^\#$ is a Bognár-Krámli factorization of w_{11} obtained as in [14, Theorem 1.1], then W is \mathcal{S} -weakly complementable if, and only if, $R(w_{12}) \subseteq R(d)$. In this case, $W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^\#y \end{pmatrix}$ with y the only solution of the equation $w_{12} = dx$. The result may be viewed as a neat Krein space counterpart of the Hilbert space results in [4].

Based on a formula given by Pekarev [22], Maestripieri and Martínez Pería [18] extended the concept of the Schur complement to bounded selfadjoint operators in Krein spaces with the so-called “unique factorization property”. Another approach was given by Mary [19]. He defined the Schur complement of a bounded operator $W = (w_{ij})_{i,j=1,2}$ when the range $R(w_{11})$ and the nullspace $N(w_{11})$ of w_{11} are regular subspaces. The approach we adopt has greater scope and is less restrictive.

The paper has three additional sections. Section 2 is a brief expository introduction to Krein spaces and operators on them and serves to fix the notation and give some results that are needed in the following sections. Section 3 is entirely devoted to the study of complementability and the Schur complement of a selfadjoint operator on a Hilbert space. In Section 4 we present our main results concerning the Schur complement of a Krein space operator. This section includes three subsections: the first deals with the notion of weak complementability on Krein spaces; the second presents an application inspired on some completion problems previously considered in Hilbert and Krein spaces by Baidiuk and Hassi in [6] and [7]; in the last subsection our notion of Schur complement in the Krein space setting is compared to those in [18] and [19].

2. Preliminaries

We assume that all Hilbert spaces are complex and separable. If \mathcal{H} and \mathcal{K} are Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ stands for the space of all the bounded linear operators from \mathcal{H} to \mathcal{K} . When $\mathcal{H} = \mathcal{K}$ we write, for short, $L(\mathcal{H})$. The domain, range and nullspace of any given $A \in L(\mathcal{H}, \mathcal{K})$ are denoted by $Dom(A)$, $R(A)$ and $N(A)$, respectively. Given a subset $\mathcal{T} \subseteq \mathcal{K}$, the preimage of \mathcal{T} under A is denoted by $A^{-1}(\mathcal{T})$ so $A^{-1}(\mathcal{T}) = \{x \in \mathcal{H} : Ax \in \mathcal{T}\}$.

The direct sum of two closed subspaces \mathcal{M} and \mathcal{N} of \mathcal{H} is represented by $\mathcal{M} \dot{+} \mathcal{N}$. If \mathcal{H} is decomposed as $\mathcal{H} = \mathcal{M} \dot{+} \mathcal{N}$, the projection onto \mathcal{M} with nullspace \mathcal{N} is denoted by $P_{\mathcal{M}/\mathcal{N}}$ and abbreviated $P_{\mathcal{M}}$ when $\mathcal{N} = \mathcal{M}^\perp$. In general, \mathcal{Q} indicates the subset of all the oblique projections in $L(\mathcal{H})$, namely, $\mathcal{Q} := \{Q \in L(\mathcal{H}) : Q^2 = Q\}$.

$L(\mathcal{H})^s$ stands for the set of selfadjoint operators in $L(\mathcal{H})$. Denote by $GL(\mathcal{H})$ the group of invertible operators in $L(\mathcal{H})$, $L(\mathcal{H})^+$ the cone of positive semidefinite operators in $L(\mathcal{H})$ and $GL(\mathcal{H})^+ := GL(\mathcal{H}) \cap L(\mathcal{H})^+$. Given two operators $S, T \in L(\mathcal{H})$, the notation $T \leq_{\mathcal{H}} S$ signifies that $S - T \in L(\mathcal{H})^+$. Given any $T \in L(\mathcal{H})$, $|T| := (T^*T)^{1/2}$ is the modulus of T and $T = U|T|$ is the polar decomposition of T , with U the partial isometry such that $N(U) = N(T)$.

The following is a well-known result about range inclusion and factorizations of operators.

Lemma 2.1 (Douglas' Lemma [12]). *Let $Y \in L(\mathcal{K}_1, \mathcal{H})$ and $Z \in L(\mathcal{K}_2, \mathcal{H})$. Then $R(Z) \subseteq R(Y)$ if and only if there exists $D \in L(\mathcal{K}_2, \mathcal{K}_1)$ such that $Z = YD$.*

Amongst the solutions of the equation $Z = YX$, there exists a unique operator $D_0 \in L(\mathcal{H})$ such that $N(Z) = N(D_0)$ and $R(D_0) \subseteq \overline{R(Y^)}$.*

The operator D_0 is called the *reduced solution* of $Z = YX$.

Given $B \in L(\mathcal{H})$ selfadjoint and \mathcal{S} a closed subspace of \mathcal{H} , we say that \mathcal{S} is *B-positive* if $\langle Bs, s \rangle > 0$ for every $s \in \mathcal{S}$, $s \neq 0$. *B-nonnegative*, *B-neutral*, *B-negative* and *B-nonpositive* subspaces are defined analogously. If \mathcal{S} and \mathcal{T} are two closed subspaces of \mathcal{H} , the notation $\mathcal{S} \oplus_B \mathcal{T}$ is used to indicate the orthogonal direct sum of \mathcal{S} and \mathcal{T} when, in addition, $\langle Bs, t \rangle = 0$ for every $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

The following is a consequence of the spectral theorem for Hilbert space selfadjoint operators.

Lemma 2.2. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Then \mathcal{S} can be represented as*

$$\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_- \quad (2.1)$$

where \mathcal{S}_+ and \mathcal{S}_- are closed, \mathcal{S}_+ is *B-nonnegative*, \mathcal{S}_- is *B-nonpositive*.

Let

$$B = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \quad \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \quad (2.2)$$

be the matrix decomposition of B induced by \mathcal{S} and consider $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ as in (2.1). Then, the matrix representations of $a, |a|, |a|^{1/2} \in L(\mathcal{S})^s$ induced by \mathcal{S}_+ are: $a = \begin{bmatrix} a_+ & 0 \\ 0 & -a_- \end{bmatrix}$, $|a| = \begin{bmatrix} a_+ & 0 \\ 0 & a_- \end{bmatrix}$,

$|a|^{1/2} = \begin{bmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{bmatrix}$, respectively.

Let us write $b := \begin{bmatrix} b_+ \\ b_- \end{bmatrix} : \mathcal{S}^\perp \rightarrow \begin{bmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{bmatrix}$, where $b_\pm = P_{\mathcal{S}_\pm} b$. Then B can be written as

$$B = \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & -a_- & b_- \\ b_+^* & b_-^* & c \end{bmatrix} \quad \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix} . \quad (2.3)$$

In [1, Theorem 3], the Schur complement $B_{/\mathcal{S}}$ of an operator $B \in L(\mathcal{H})^+$ was characterized in the following fashion: if the matrix representation of B is given by (2.2), then

$$B_{/\mathcal{S}} = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f \end{bmatrix},$$

where f is the reduced solution of $a^{1/2}x = b$ (which always exists for positive operators). The next lemma characterizes the positive operators in terms of its matrix decomposition. It follows easily from the fact that $B - B_{/\mathcal{S}} \geq_{\mathcal{H}} 0$.

Lemma 2.3. *Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and $B \in L(\mathcal{H})^s$ with matrix decomposition induced by \mathcal{S} as in (2.2). Then $B \in L(\mathcal{H})^+$ if and only if*

$$a \geq_{\mathcal{H}} 0, \quad b = b^*, \quad R(b) \subseteq R(a^{1/2}), \quad \text{and } c = f^*f + t,$$

with f the reduced solution of the equation $b = a^{1/2}x$ and $t \geq_{\mathcal{H}} 0$.

2.1. Krein Spaces

Although familiarity with operator theory on Krein spaces is presumed, we include some basic notions. Standard references on Krein spaces and operators on them are [3], [5] and [8]. We also refer to [13] and [14] as authoritative accounts of the subject.

Consider a linear space \mathcal{H} with an indefinite metric; i.e., a sesquilinear Hermitian form $[\cdot, \cdot]$. A vector $x \in \mathcal{H}$ is said to be *positive* if $[x, x] > 0$. A subspace \mathcal{S} of \mathcal{H} is *positive* if every $x \in \mathcal{S}$, $x \neq 0$, is a positive vector. *Negative, nonnegative, nonpositive* and *neutral* vectors and subspaces are defined likewise.

We say that two closed subspaces \mathcal{M} and \mathcal{N} are *orthogonal*, and write $\mathcal{M} \perp \mathcal{N}$, if $[m, n] = 0$ for every $m \in \mathcal{M}$ and $n \in \mathcal{N}$. Denote the orthogonal direct sum of two closed subspaces \mathcal{M} and \mathcal{N} by $\mathcal{M} \dot{+} \mathcal{N}$.

Given any subspace \mathcal{S} of \mathcal{H} , the *orthogonal companion* of \mathcal{S} in \mathcal{H} is defined as

$$\mathcal{S}^{\perp} := \{x \in \mathcal{H} : [x, s] = 0 \text{ for every } s \in \mathcal{S}\}.$$

An indefinite metric space $(\mathcal{H}, [\cdot, \cdot])$ is a *Krein space* if it admits a decomposition as an orthogonal direct sum

$$\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-, \quad (2.4)$$

where $(\mathcal{H}_+, [\cdot, \cdot])$ and $(\mathcal{H}_-, -[\cdot, \cdot])$ are Hilbert spaces. Any decomposition with these properties is called a *fundamental decomposition* of \mathcal{H} .

Given a Krein space $(\mathcal{H}, [\cdot, \cdot])$ with a fundamental decomposition $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$, the (orthogonal) direct sum of the Hilbert spaces $(\mathcal{H}_+, [\cdot, \cdot])$ and $(\mathcal{H}_-, -[\cdot, \cdot])$ is a Hilbert space, $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Notice that the inner product $\langle \cdot, \cdot \rangle$ and the corresponding quadratic norm $\|\cdot\|$ depend on the fundamental decomposition.

Every fundamental decomposition of \mathcal{H} has an associated *signature operator*: $J := P_+ - P_-$ with $P_{\pm} := P_{\mathcal{H}_{\pm}/\mathcal{H}_{\mp}}$. The indefinite metric and the inner product corresponding to a fundamental decomposition of \mathcal{H} with signature operator J are related to each other by

$$\langle x, y \rangle = [Jx, y] \quad (x, y \in \mathcal{H}).$$

If \mathcal{H} is a Krein space, $L(\mathcal{H})$ stands for the vector space of all the linear operators on \mathcal{H} which are bounded in an associated Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$. Since the norms generated by different fundamental decompositions of a Krein space \mathcal{H} are equivalent (see, for instance, [5, Theorem 7.19]), $L(\mathcal{H})$ does not depend on the chosen underlying Hilbert space, all of which are equivalent.

Given $T \in L(\mathcal{H})$, $T^{\#}$ is the unique operator satisfying

$$[Tx, y] = [x, T^{\#}y] \text{ for every } x, y \in \mathcal{H}.$$

$L(\mathcal{H})^{[s]}$ denotes the set of the operators $T \in L(\mathcal{H})$ such that $T = T^{\#}$. The selfadjoint operator $T \in L(\mathcal{H})$ is *positive* if $[Tx, x] \geq 0$ for every $x \in \mathcal{H}$. The notation $S \leq T$ signifies that $T - S$ is positive.

A (closed) subspace \mathcal{S} of a Krein space \mathcal{H} is *regular* if it is itself a Krein space in the indefinite metric of \mathcal{H} . A subspace \mathcal{S} is regular if and only if $\mathcal{H} = \mathcal{S} \dot{+} \mathcal{S}^{\perp}$ or, equivalently, if it is the range of a selfadjoint projection, i.e., there exists $Q \in \mathcal{Q}$ such that $Q = Q^{\#}$ and $R(Q) = \mathcal{S}$ (see [5, Proposition 1.4.19]). Clearly, \mathcal{S} is regular if and only if \mathcal{S}^{\perp} is regular.

Suppose that \mathcal{S} is a regular subspace with fundamental decomposition $\mathcal{S} = \mathcal{S}_+ \dot{+} \mathcal{S}_-$. Then, by [14, Theorem 1.6], there exists a fundamental decomposition of $\mathcal{H} = \mathcal{H}_+ \dot{+} \mathcal{H}_-$ such that $\mathcal{S}_{\pm} \subseteq \mathcal{H}_{\pm}$. In this case,

$$\mathcal{H}_{\pm} = \mathcal{S}_{\pm} \dot{+} \mathcal{N}_{\pm},$$

where $\mathcal{S}^{[\perp]} = \mathcal{N}_+ [\perp] \mathcal{N}_-$ is a fundamental decomposition of $\mathcal{S}^{[\perp]}$. Now, consider $J_1 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{matrix}$ and $J_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{matrix} \mathcal{N}_+ \\ \mathcal{N}_- \end{matrix}$, signature operators of \mathcal{S} and $\mathcal{S}^{[\perp]}$, respectively. Then

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix} \quad (2.5)$$

is a signature operator for \mathcal{H} .

Given $W \in L(\mathcal{H})^s$ and \mathcal{S} a closed subspace of \mathcal{H} , we say that \mathcal{S} is W -positive if $[Ws, s] > 0$ for every $s \in \mathcal{S}$, $s \neq 0$. W -nonnegative, W -neutral, W -negative and W -nonpositive subspaces are defined likewise. If \mathcal{S} and \mathcal{T} are two closed subspaces of \mathcal{H} , the notation $\mathcal{S} [\perp]_W \mathcal{T}$ is used to indicate the direct sum of \mathcal{S} and \mathcal{T} when, additionally, $[Ws, t] = 0$ for every $s \in \mathcal{S}$ and $t \in \mathcal{T}$.

3. Complementability and Schur complement for selfadjoint operators in Hilbert spaces

The notion of complementability of an operator $B \in L(\mathcal{H})$ with respect to two given closed subspaces \mathcal{S} and \mathcal{T} of \mathcal{H} was studied for matrices by Ando [2] and extended to operators in Hilbert spaces by Carlson and Haynsworth [10]. In [4] Antezana et al. defined a weaker notion, that of *weak complementability*, and extended the notion of the Schur complement to this context. We use these ideas when $\mathcal{S} = \mathcal{T}$ and $B \in L(\mathcal{H})^s$. In what follows, we recall both definitions for this particular case:

Definition. Let $B \in L(\mathcal{H})^s$ and let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace. Then B is \mathcal{S} -complementable if

$$\mathcal{H} = \mathcal{S} + B^{-1}(\mathcal{S}^\perp).$$

In [11] it was shown that B is \mathcal{S} -complementable if and only if there exists a B -selfadjoint projection onto \mathcal{S} ; i.e., the set

$$\mathcal{P}(B, \mathcal{S}) := \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, BQ = Q^*B\}$$

is not empty. It was also proven that, if

$$B = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}, \quad (3.1)$$

then B is \mathcal{S} -complementable if and only if $R(b) \subseteq R(a)$.

This naturally leads to the following definition.

Definition. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace, and $B \in L(\mathcal{H})^s$ with representation as in (3.1). Then B is \mathcal{S} -weakly complementable if

$$R(b) \subseteq R(|a|^{1/2}).$$

When $R(a)$ is closed both notions coincide and therefore the notion of weak complementability is distinct only in the infinite dimensional setting. Every positive operator B is \mathcal{S} -weakly complementable.

Proposition 3.1. *Let $B \in L(\mathcal{H})^s$. Then B is \mathcal{S} -weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$ if and only if B is semidefinite.*

Proof. If B is semidefinite then B is \mathcal{S} -weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$, because in this case, if $B \in L(\mathcal{H})$ is represented as in (3.1) for any \mathcal{S} , by Lemma 2.3, $R(b) \subseteq R((\pm a)^{1/2})$.

Conversely, suppose that B is \mathcal{S} -weakly complementable for every closed subspace $\mathcal{S} \subseteq \mathcal{H}$ and that B is not definite. Then there exists $x_0 \in \mathcal{H} \setminus \{0\}$ such that x_0 is B -neutral and $x_0 \notin N(B)$. Let $\mathcal{S} = \text{span} \{x_0\}$ and suppose that $B \in L(\mathcal{H})^s$ is represented as in (3.1). Then $\langle By, y \rangle = \langle ay, y \rangle = 0$ for every $y \in \mathcal{S}$. Hence $a = 0$ and $b = 0 = b^*$, because B is \mathcal{S} -weakly complementable. Then, $\mathcal{S} \subseteq N(B)$ which is a contradiction. Therefore, B is semidefinite. \square

Also, B is \mathcal{S} -weakly complementable and \mathcal{S} is B -nonnegative if and only if $a \in L(\mathcal{S})^+$ and $R(b) \subseteq R(a^{1/2})$. In fact, if \mathcal{S} is B -nonnegative then, for every $s \in \mathcal{S}$,

$$0 \leq \langle Bs, s \rangle = \langle as, s \rangle,$$

whence $a \in L(\mathcal{S})^+$ and, since B is \mathcal{S} -weakly complementable, $R(b) \subseteq R(a^{1/2})$. The converse is similar. Analogously, B is \mathcal{S} -weakly complementable and \mathcal{S} is B -nonpositive if and only if $-a \in L(\mathcal{S})^+$ and $R(b) \subseteq R((-a)^{1/2})$.

We recall the definition of Schur complement for an \mathcal{S} -weakly complementable selfadjoint operator.

Definition. Let $\mathcal{S} \subseteq \mathcal{H}$ be a closed subspace and let $B \in L(\mathcal{H})^s$ be \mathcal{S} -weakly complementable. When B is as in (3.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a . The *Schur complement* of B to \mathcal{S} is defined as

$$B/\mathcal{S} := \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix}.$$

$B_{\mathcal{S}} := B - B/\mathcal{S}$ is the \mathcal{S} -compression of B .

If B is positive, B/\mathcal{S} coincides with the usual Schur complement of B to \mathcal{S} .

3.1. Variational characterization of the Schur complement

For $B \in L(\mathcal{H})^s$ and \mathcal{S} a closed subspace of \mathcal{H} , define

$$\mathcal{M}^-(B, \mathcal{S}^\perp) := \{X \in L(\mathcal{H})^s, X \leq_{\mathcal{H}} B, R(X) \subseteq \mathcal{S}^\perp\},$$

$$\mathcal{M}^+(B, \mathcal{S}^\perp) := \{X \in L(\mathcal{H})^s, B \leq_{\mathcal{H}} X, R(X) \subseteq \mathcal{S}^\perp\}.$$

The next proposition shows that B is \mathcal{S} -weakly complementable if and only if $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_+^\perp)$ are non-empty, where $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1).

Proposition 3.2. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then, the following statements are equivalent:*

- i) B is \mathcal{S} -weakly complementable;
- ii) there exist $B_1, B_2, B_3 \in L(\mathcal{H})^s$, $B_2, B_3 \geq_{\mathcal{H}} 0$ such that $B = B_1 + B_2 - B_3$ and $\mathcal{S} \subseteq N(B_1)$, $\mathcal{S}_- \subseteq N(B_2)$, $\mathcal{S}_+ \subseteq N(B_3)$;
- iii) the sets $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_+^\perp)$ are non-empty;
- iv) B is \mathcal{S}_\pm -weakly complementable.

Proof. $i) \Rightarrow ii)$: Let $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ be any decomposition as in (2.1). Suppose that the matrix representation of B induced by \mathcal{S}_+ is as in (2.3) and B is \mathcal{S} -weakly complementable. Then $R(b_+) \subseteq R(a_+^{1/2})$ and $R(b_-) \subseteq R(a_-^{1/2})$. In fact, since $R(b) \subseteq R(|a|^{1/2})$, for every $y \in \mathcal{S}^\perp$, there exists $s \in \mathcal{S}$ such that $by = |a|^{1/2}s$. Therefore, for every $y \in \mathcal{S}^\perp$, $b_\pm y = P_{\mathcal{S}_\pm} by = P_{\mathcal{S}_\pm} |a|^{1/2}s = a_\pm^{1/2}s$. Let f be the reduced solution of $b_+ = a_+^{1/2}x$ and g the reduced solution of $b_- = -a_-^{1/2}x$. Set $B_1 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c - f^*f + g^*g \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$,

$B_2 := \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & 0 & 0 \\ b_+^* & 0 & f^*f \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$ and $B_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & a_- & -b_- \\ 0 & -b_-^* & g^*g \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$. Then $B = B_1 + B_2 - B_3$, $\mathcal{S} \subseteq N(B_1)$, $\mathcal{S}_- \subseteq N(B_2)$, $\mathcal{S}_+ \subseteq N(B_3)$ and, by Lemma 2.3, $B_2, B_3 \geq_{\mathcal{H}} 0$.

$ii) \Rightarrow iii)$: Since $B_1 + B_2 = B + B_3 \geq_{\mathcal{H}} B$ and $R(B_1 + B_2) \subseteq \mathcal{S}_+^\perp$, $B_1 + B_2 \in \mathcal{M}^+(B, \mathcal{S}_+^\perp)$. Similarly, $B_1 - B_3 \in \mathcal{M}^-(B, \mathcal{S}_+^\perp)$.

$iii) \Rightarrow iv)$: Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}_+^\perp)$. Since $R(X_0) \subseteq \mathcal{S}_+^\perp$, the matrix representation of X_0 induced by \mathcal{S}_+ is $X_0 = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_+^\perp \end{matrix}$, for some $d \in L(\mathcal{S}_+^\perp)$. Suppose that the matrix representation of B induced by \mathcal{S}_+ is $B = \begin{bmatrix} a' & b' \\ b'^* & c' \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_+^\perp \end{matrix}$, with $a' \in L(\mathcal{S}_+)^+$. Since

$$\begin{bmatrix} a' & b' \\ b'^* & c' - d \end{bmatrix} = B - X_0 \geq_{\mathcal{H}} 0,$$

by Lemma 2.3, $R(b') \subseteq R(a^{1/2})$ and B is \mathcal{S}_+ -weakly complementable. In a similar way, B is \mathcal{S}_- -weakly complementable.

$iv) \Rightarrow i)$: Suppose that the matrix representation of B induced by \mathcal{S}_+ is as in (2.3), since B is \mathcal{S}_\pm -weakly complementable, $R(b_\pm) \subseteq R(a_\pm^{1/2})$. Thus,

$$R(b) \subseteq R(b_+) + R(b_-) \subseteq R(a_+^{1/2}) \oplus R(a_-^{1/2}) = R(|a|^{1/2}),$$

and B is \mathcal{S} -weakly complementable. \square

The following result characterizes the weak \mathcal{S} -complementability of B when \mathcal{S} is B -nonnegative. A similar result holds in the B -nonpositive case. Several of the equivalences were also proven in [20, Proposition 3.3]. Nonetheless, we include the proofs for the sake of completeness.

Proposition 3.3. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Then the following statements are equivalent:*

- i) \mathcal{S} is B -nonnegative and B is \mathcal{S} -weakly complementable;*
- ii) there exist $B_1, B_2 \in L(\mathcal{H})^s$, $B_2 \geq_{\mathcal{H}} 0$ such that $B = B_1 + B_2$ and $\mathcal{S} \subseteq N(B_1)$;*
- iii) the set $\mathcal{M}^-(B, \mathcal{S}^\perp)$ is non-empty;*
- iv) the set $\mathcal{M}^-(B, \mathcal{S}^\perp)$ has a maximum element, namely,*

$$B_{/\mathcal{S}} = \max \mathcal{M}^-(B, \mathcal{S}^\perp).$$

Proof. If \mathcal{S} is B -nonnegative then, in the decomposition of \mathcal{S} as in (2.1), $\mathcal{S}_+ = \mathcal{S}$ and $\mathcal{S}_- = \{0\}$. Applying Proposition 3.2, the equivalence $i) \Leftrightarrow ii)$ and the implication $ii) \Rightarrow iii)$ follow.

$iii) \Rightarrow i)$: Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}^\perp)$. For all $s \in \mathcal{S}$, $\langle Bs, s \rangle \geq \langle X_0s, s \rangle = 0$, because $X_0 \leq_{\mathcal{H}} B$ and $R(X_0) \subseteq \mathcal{S}^\perp$. Then \mathcal{S} is B -nonnegative. In this case, applying Proposition 3.2, we also have that B is \mathcal{S} -weakly complementable.

$iii) \Leftrightarrow iv)$: Suppose that $X_0 \in \mathcal{M}^-(B, \mathcal{S}^\perp)$, then by $iii) \Rightarrow i)$, \mathcal{S} is B -nonnegative and B is \mathcal{S} -weakly complementable. Decompose X_0 as $X_0 = X_{0+} - X_{0-}$, with $X_{0\pm} \in L(\mathcal{S}^\perp)^+$. Since $X_{0+} - X_{0-} \leq_{\mathcal{H}} B$, it follows that $0 \leq_{\mathcal{H}} X_{0+} \leq_{\mathcal{H}} B + X_{0-}$. Thus, by [1, Theorem 1],

$$0 \leq_{\mathcal{H}} X_{0+} \leq (B + X_{0-})_{/\mathcal{S}} = B_{/\mathcal{S}} + X_{0-},$$

where the last equality is a result of the fact that if $Z \in L(\mathcal{H})$ is selfadjoint and $R(Z) \subseteq \mathcal{S}^\perp$ then $B + Z$ is \mathcal{S} -weakly complementable and $(B + Z)_{/\mathcal{S}} = B_{/\mathcal{S}} + Z$. Therefore $X_0 \leq_{\mathcal{H}} B_{/\mathcal{S}}$. Finally, as $B_{/\mathcal{S}} \in L(\mathcal{H})^s$, $R(B_{/\mathcal{S}}) \subseteq \mathcal{S}^\perp$ and, by Lemma 2.3, $B_{/\mathcal{S}} \leq_{\mathcal{H}} B$. Hence $B_{/\mathcal{S}} \in \mathcal{M}^-(B, \mathcal{S}^\perp)$. Thus $B_{/\mathcal{S}} = \max \mathcal{M}^-(B, \mathcal{S}^\perp)$.

The converse is straightforward. \square

The Schur complement of B to \mathcal{S} satisfies a variational characterization as a min-max if and only if B is \mathcal{S} -weakly complementable, as the following theorem shows.

Theorem 3.4. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then B is \mathcal{S} -weakly complementable if and only if there exist $\min \mathcal{M}^+(\max \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp)$ and $\max \mathcal{M}^-(\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp)$. In this case,*

$$B_{/\mathcal{S}} = \min \mathcal{M}^+(\max \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = \max \mathcal{M}^-(\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp).$$

Proof. Suppose that B is \mathcal{S} -weakly complementable. If the matrix representation of B induced by \mathcal{S}_+ is as in (2.3), then $R(b_\pm) \subseteq R(a_\pm^{1/2})$. Let f be the reduced solution of $b_+ = a_+^{1/2}x$ and g the reduced solution of $b_- = -a_-^{1/2}x$. Then, by Proposition 3.3,

$$B_{/\mathcal{S}_+} = \max \mathcal{M}^-(B, \mathcal{S}_+^\perp) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_- & b_- \\ 0 & b_-^* & c - f^*f \end{bmatrix} \begin{array}{l} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{array}.$$

Thus $B_{/S_+}$ is \mathcal{S}_- -weakly complementable and \mathcal{S}_- is $B_{/S_+}$ -nonpositive. Again by Proposition 3.3,

$$(B_{/S_+})_{/S_-} = \min \mathcal{M}^+ (\max \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f + g^*g \end{bmatrix}.$$

In a similar way,

$$(B_{/S_-})_{/S_+} = \max \mathcal{M}^- (\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp) = \begin{bmatrix} 0 & 0 \\ 0 & c - f^*f + g^*g \end{bmatrix}.$$

Conversely, if there exist $\min \mathcal{M}^+(\max \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp)$ and $\max \mathcal{M}^-(\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp)$, then the sets $\mathcal{M}^-(B, \mathcal{S}_+^\perp)$ and $\mathcal{M}^+(B, \mathcal{S}_-^\perp)$ are non-empty. So by Proposition 3.2, B is \mathcal{S} -weakly complementable.

In this case, notice that

$$b = \begin{bmatrix} b_+ \\ b_- \end{bmatrix} = \begin{bmatrix} a_+^{1/2} & 0 \\ 0 & a_-^{1/2} \end{bmatrix} \begin{bmatrix} f \\ -g \end{bmatrix} = |a|^{1/2}(f - g),$$

and since

$$R(f - g) \subseteq \overline{R(a_+^{1/2})} \oplus \overline{R(a_-^{1/2})} = \overline{R(|a|^{1/2})},$$

$y := f - g$ is the reduced solution of the equation $b = |a|^{1/2}x$. Also, if $u = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \end{matrix}$, then

$a = u|a| = |a|u$ is the polar decomposition of a . Therefore $y^*uy = [f^* - g^*] \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} f \\ -g \end{bmatrix} = f^*f - g^*g$

and

$$B_{/S} = \min \mathcal{M}^+(\max \mathcal{M}^-(B, \mathcal{S}_+^\perp), \mathcal{S}_-^\perp) = \max \mathcal{M}^-(\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp).$$

□

Corollary 3.5. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). If B is \mathcal{S} -weakly complementable then*

$$B_{/S} = (B_{/S_+})_{/S_-} = (B_{/S_-})_{/S_+}.$$

In [1, Theorem 5], Anderson and Trapp proved that if $B \in L(\mathcal{H})^+$ and \mathcal{S} is a closed subspace of \mathcal{H} then

$$B_{/S} = \inf \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}.$$

More generally:

Theorem 3.6. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then B is \mathcal{S} -weakly complementable if and only if there exist*

$$\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right)$$

and

$$\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right).$$

In this case,

$$\begin{aligned} B_{/S} &= \sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right) \\ &= \inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right). \end{aligned}$$

In order to prove Theorem 3.6, we require the following lemmas.

Lemma 3.7. *Let $B \in L(\mathcal{H})^*$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). If B is \mathcal{S} -weakly complementable and $B = B_1 + B_2 - B_3$ is any decomposition as in Proposition 3.2, then*

$$B/\mathcal{S} = B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_-.$$

Proof. Since B is \mathcal{S} -weakly complementable, by Proposition 3.2, B is \mathcal{S}_\pm -weakly complementable. Then, proceeding as in the proof of Proposition 3.3, it can be checked that $B_1 + B_2 - B_3/\mathcal{S}_- = \min \mathcal{M}^+(B, \mathcal{S}_-^\perp)$.

On the other hand,

$$B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_- = \max \mathcal{M}^-(B_1 + B_2 - B_3/\mathcal{S}_-, \mathcal{S}_+^\perp).$$

In fact, $B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_- \in \mathcal{M}^-(B_1 + B_2 - B_3/\mathcal{S}_-, \mathcal{S}_+^\perp)$. This follows since $B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_-$ is selfadjoint, $B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_- \leq_{\mathcal{H}} B_1 + B_2 - B_3/\mathcal{S}_-$ and, by [1, Corollary 4], $R(B_3/\mathcal{S}_-) \subseteq \overline{R(B_3)} \subseteq \mathcal{S}_+^\perp$. Let $Y \in \mathcal{M}^-(B_1 + B_2 - B_3/\mathcal{S}_-, \mathcal{S}_+^\perp)$, and decompose Y as $Y = Y_+ - Y_-$, with $Y_\pm \in L(\mathcal{S}_+^\perp)^+$. Since $Y_+ - Y_- \leq_{\mathcal{H}} B_1 + B_2 - B_3/\mathcal{S}_-$,

$$0 \leq_{\mathcal{H}} Y_+ + B_3/\mathcal{S}_- \leq_{\mathcal{H}} B_1 + B_2 + Y_-.$$

Now, since $R(B_1 + Y_-) \subseteq \mathcal{S}_+^\perp$, [1, Theorem 1] gives that

$$0 \leq_{\mathcal{H}} Y_+ + B_3/\mathcal{S}_- \leq_{\mathcal{H}} (B_1 + B_2 + Y_-)/\mathcal{S}_+ = B_1 + B_2/\mathcal{S}_+ + Y_-;$$

i.e., $Y \leq_{\mathcal{H}} B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_-$. Therefore,

$$B_1 + B_2/\mathcal{S}_+ - B_3/\mathcal{S}_- = \max \mathcal{M}^-(\min \mathcal{M}^+(B, \mathcal{S}_-^\perp), \mathcal{S}_+^\perp) = B/\mathcal{S}.$$

□

Lemma 3.8. *Let $B \in L(\mathcal{H})^+$ and let \mathcal{S} be a closed subspace of \mathcal{H} decomposed as in (2.1). For any $Q_- \in \mathcal{Q}$, $N(Q_-) = \mathcal{S}_-$,*

$$\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- = Q_-^* B/\mathcal{S}_+ Q_-.$$

Proof. Let $Q_- \in \mathcal{Q}$, $N(Q_-) = \mathcal{S}_-$. By [1, Theorem 5], $B/\mathcal{S}_+ \leq Q_+^* B Q_+$, for every $Q_+ \in \mathcal{Q}$, $N(Q_+) = \mathcal{S}_+$. Therefore $Q_-^* B/\mathcal{S}_+ Q_-$ is a lower bound of the set $\{Q_-^* Q_+^* B Q_+ Q_- : Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+\}$.

If B is invertible then, by [11, Section 4], B is \mathcal{S}_+ -complementable. So, by [11, Proposition 4.2], there exists $Q_+ \in \mathcal{Q}$, $N(Q_+) = \mathcal{S}_+$ such that $B/\mathcal{S}_+ = Q_+^* B Q_+$. Then clearly in this case, the infimum is actually a minimum.

For a non invertible B , consider $\varepsilon > 0$. If F is any lower bound of the set $\{Q_-^* Q_+^* B Q_+ Q_- : Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+\}$ then, for any $Q_+ \in \mathcal{Q}$, $N(Q_+) = \mathcal{S}_+$,

$$F \leq Q_-^* Q_+^* B Q_+ Q_- \leq Q_-^* Q_+^* (B + \varepsilon I) Q_+ Q_-.$$

Since $B + \varepsilon I$ is invertible, it follows that $F \leq Q_-^* (B + \varepsilon I)/\mathcal{S}_+ Q_-$. As ε is arbitrary, [1, Corollary 2] yields $F \leq Q_-^* B/\mathcal{S}_+ Q_-$.

□

Proof of Theorem 3.6. Suppose that B is \mathcal{S} -weakly complementable and write $B = B_1 + B_2 - B_3$, with $\mathcal{S} \subseteq N(B_1)$, $\mathcal{S}_- \subseteq N(B_2)$, $\mathcal{S}_+ \subseteq N(B_3)$ and $B_2, B_3 \geq_{\mathcal{H}} 0$ (see Proposition 3.2). Let $Q_- \in \mathcal{Q}$, $N(Q_-) = \mathcal{S}_-$. Then, for any $Q_+ \in \mathcal{Q}$, $N(Q_+) = \mathcal{S}_+$,

$$Q_-^* Q_+^* B Q_+ Q_- = B_1 + Q_-^* Q_+^* B_2 Q_+ Q_- - Q_-^* B_3 Q_-.$$

By Lemma 3.8, $\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B_2 Q_+ Q_- = Q_-^* B_2 /_{\mathcal{S}_+} Q_-$. Therefore,

$$\begin{aligned} \inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- &= B_1 + Q_-^* B_2 /_{\mathcal{S}_+} Q_- - Q_-^* B_3 Q_- \\ &= B_1 + B_2 /_{\mathcal{S}_+} - Q_-^* B_3 Q_-, \end{aligned}$$

where we used the fact that $R(B_2 /_{\mathcal{S}_+}) \subseteq \overline{R(B_2)} \subseteq \mathcal{S}_-^\perp$ (see [1, Corollary 4]). Finally, by [20, Proposition 3.7] and Lemma 3.7,

$$\begin{aligned} &\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right) = \\ &= \sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} (B_1 + B_2 /_{\mathcal{S}_+} - Q_-^* B_3 Q_-) = B_1 + B_2 /_{\mathcal{S}_+} - B_3 /_{\mathcal{S}_-} = B /_{\mathcal{S}}. \end{aligned}$$

The second equality follows in a similar way.
Conversely, suppose that

$$\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right)$$

and

$$\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right).$$

exist. Then, for every $Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-$, there exists

$$\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_-.$$

In particular, for $Q_- = P_{\mathcal{S}_-^\perp}, T_0 := \inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp}$. Then $P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} - T_0 \geq_{\mathcal{H}} 0$ for every $Q_+ \in \mathcal{Q}$ such that $N(Q_+) = \mathcal{S}_+$. Thus $T_0 = P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} - (P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} - T_0)$ is selfadjoint.

Since $R(Q_+^* B Q_+) \subseteq \mathcal{S}_+^\perp$, then $P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} = P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp}$ and

$$T_0 = \inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp}.$$

Suppose that the matrix decomposition of T_0 is given by $T_0 = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12}^* & t_{22} & t_{23} \\ t_{13}^* & t_{23}^* & t_{33} \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$. Then

$$P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} - T_0 = \begin{bmatrix} -t_{11} & -t_{12} & -t_{13} \\ -t_{12}^* & -t_{22} & -t_{23} \\ -t_{13}^* & t_{23}^* & P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp} - t_{33} \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then $t_{11} \leq_{\mathcal{H}} 0$ and $t_{22} \leq_{\mathcal{H}} 0$. Also, since $P_{\mathcal{S}_-^\perp} T_0 P_{\mathcal{S}_-^\perp} \leq_{\mathcal{H}} P_{\mathcal{S}_-^\perp} Q_+^* B Q_+ P_{\mathcal{S}_-^\perp}$ for every $Q_+ \in \mathcal{Q}$ such that $N(Q_+) = \mathcal{S}_+$, then $P_{\mathcal{S}_-^\perp} T_0 P_{\mathcal{S}_-^\perp} \leq_{\mathcal{H}} T_0$. Therefore

$$T_0 - P_{\mathcal{S}_-^\perp} T_0 P_{\mathcal{S}_-^\perp} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{12}^* & t_{22} & t_{23} \\ t_{13}^* & t_{23}^* & 0 \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then, by Lemma 2.3, $t_{11} \geq_{\mathcal{H}} 0$ so $t_{11} = 0$, $t_{22} \geq_{\mathcal{H}} 0$ so $t_{22} = 0$, and also $t_{12} = t_{12}^* = t_{13} = t_{13}^* = t_{23} = t_{23}^* = 0$. Hence, $R(T_0) \subseteq \mathcal{S}^\perp$. Therefore

$$P_{\mathcal{S}_-^\perp} Q_+^* (B - T_0) Q_+ P_{\mathcal{S}_-^\perp} \geq_{\mathcal{H}} 0 \text{ for every } Q_+ \in \mathcal{Q} \text{ such that } N(Q_+) = \mathcal{S}_+.$$

Let us show that $\langle (B - T_0)x, x \rangle \geq 0$ for every $x \in \mathcal{S}^\perp = \mathcal{S}_+ \oplus \mathcal{S}^\perp$. Fix $x \in \mathcal{S}^\perp$; if $x \in \mathcal{S}_+$ then $\langle (B - T_0)x, x \rangle = \langle Bx, x \rangle \geq 0$, because $\mathcal{S}_+ \subseteq N(T_0)$ and \mathcal{S}_+ is B -nonnegative. If $x \notin \mathcal{S}_+$ then $P_{\mathcal{S}^\perp}x \neq 0$ and there exists a subspace \mathcal{M} such that $x \in \mathcal{M}$ and $\mathcal{M} \dot{+} \mathcal{S}_+ = \mathcal{H}$. Take $Q_+ = P_{\mathcal{M}} // \mathcal{S}_+$; then $x = Q_+x = Q_+P_{\mathcal{S}^\perp}x$. Thus $\langle (B - T_0)x, x \rangle = \langle P_{\mathcal{S}^\perp}Q_+^*(B - T_0)Q_+P_{\mathcal{S}^\perp}x, x \rangle \geq 0$. Since $x \in \mathcal{S}^\perp$ is arbitrary, $\langle (B - T_0)x, x \rangle \geq 0$ for every $x \in \mathcal{S}^\perp$. If the matrix decomposition of B is as in (2.3),

$$P_{\mathcal{S}^\perp}(B - T_0)P_{\mathcal{S}^\perp} = \begin{bmatrix} a_+ & 0 & b_+ \\ 0 & 0 & 0 \\ b_+^* & 0 & c - t_{33} \end{bmatrix} \geq_{\mathcal{H}} 0.$$

Then, by Lemma 2.3, $R(b_+) \subseteq R(a_+^{1/2})$.

Analogously, since $\inf_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\sup_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}^-} Q_+^*Q_-^*BQ_-Q_+ \right)$ exists, it follows that $R(b_-) \subseteq R(a_-^{1/2})$. Therefore $R(b) \subseteq R(b_+) + R(b_-) \subseteq R(a_+^{1/2}) \oplus R(a_-^{1/2}) = R(|a|^{1/2})$ and B is \mathcal{S} -weakly complementable. \square

Corollary 3.9. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that \mathcal{S} is B -nonnegative. Then B is \mathcal{S} -weakly complementable if and only if there exists $\inf \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$. In this case,*

$$B_{/\mathcal{S}} = \inf \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}. \quad (3.2)$$

A similar result holds when \mathcal{S} is B -nonpositive, replacing \inf by \sup .

The following proposition shows that the infimum in (3.2) is indeed a minimum if and only if B is \mathcal{S} -complementable and \mathcal{S} is B -nonnegative. Recall that when B is \mathcal{S} -complementable, the set

$$\mathcal{P}(B, \mathcal{S}) := \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}, BQ = Q^*B\}$$

is not empty. See definition on page 5.

Proposition 3.10. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that B is \mathcal{S} -weakly complementable. Then*

$$B_{/\mathcal{S}} = \min \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$$

if and only if B is \mathcal{S} -complementable and \mathcal{S} is B -nonnegative. In this case,

$$B_{/\mathcal{S}} = B(I - Q),$$

for any $Q \in \mathcal{P}(B, \mathcal{S})$.

A similar result holds when \mathcal{S} is B -nonpositive, replacing \min by \max .

Proof. Suppose that B is \mathcal{S} -complementable and \mathcal{S} is B -nonnegative. Then, by [20, Proposition 4.6], $B_{/\mathcal{S}} = \min \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$. In this case $B_{/\mathcal{S}} = B(I - Q)$, for any $Q \in \mathcal{P}(B, \mathcal{S})$.

Conversely, suppose that $B_{/\mathcal{S}} = \min \{Q^*BQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$. Let B be as in (3.1) and $Q := \begin{pmatrix} 0 & e \\ 0 & I \end{pmatrix}$ with $e \in L(\mathcal{S}^\perp, \mathcal{S})$. Then $Q \in \mathcal{Q}$ and $N(Q) = \mathcal{S}$. Let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a . From $B_{/\mathcal{S}} \leq_{\mathcal{H}} Q^*BQ$, it is easy to check that

$$0 \leq \left\langle u(f + |a|^{1/2}ue)y, (f + |a|^{1/2}ue)y \right\rangle \text{ for every } y \in \mathcal{S}^\perp \text{ and } e \in L(\mathcal{S}^\perp, \mathcal{S}).$$

Since $R(f) \subseteq \overline{R(|a|^{1/2})}$ then $\overline{R(|a|^{1/2})} = \overline{R(f)} \oplus \overline{R(|a|^{1/2})} \cap R(f)^\perp$. In particular, for every $s \in \mathcal{S}$, $|a|^{1/2}s = t + v$, with $t \in \overline{R(f)}$ and $v \in \overline{R(|a|^{1/2})} \cap R(f)^\perp$. If $s \in \mathcal{S}$ and $\varepsilon > 0$, then there exist $y_\varepsilon \in \mathcal{S}^\perp$ and $e_\varepsilon \in L(\mathcal{S}^\perp, \mathcal{S})$ such that $\| |a|^{1/2}s - (fy_\varepsilon + |a|^{1/2}ue_\varepsilon y_\varepsilon) \| < \varepsilon$. Therefore

$$\begin{aligned} \langle as, s \rangle &= \left\langle u|a|^{1/2}s, |a|^{1/2}s \right\rangle \\ &= \left\langle u \lim_{\varepsilon \rightarrow 0} [(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon], \lim_{\varepsilon \rightarrow 0} [(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon] \right\rangle \\ &= \lim_{\varepsilon \rightarrow 0} \left\langle u(f + |a|^{1/2}ue_\varepsilon)y_\varepsilon, (f + |a|^{1/2}ue_\varepsilon)y_\varepsilon \right\rangle \geq 0 \end{aligned}$$

and \mathcal{S} is B -nonnegative.

In this case, by Proposition 3.3, $B_{/\mathcal{S}} \in \mathcal{M}^-(B, \mathcal{S}^\perp)$, so that $B_{/\mathcal{S}} \leq_{\mathcal{H}} B$. Let $Q_0 \in \mathcal{Q}$ with $N(Q_0) = \mathcal{S}$ such that $B_{/\mathcal{S}} = Q_0^* B Q_0$. Then, $B - Q_0^* B Q_0 \geq_{\mathcal{H}} 0$. From $Q_0^*(B - Q_0^* B Q_0)Q_0 = 0$, it follows that $(B - Q_0^* B Q_0)Q_0 = 0$, which implies that $B Q_0 = Q_0^* B Q_0$. Thus, $E_0 := I - Q_0 \in \mathcal{P}(B, \mathcal{S})$ and B is \mathcal{S} -complementable. \square

If $B \in L(\mathcal{H})^s$ and \mathcal{S} is a closed subspace of \mathcal{H} such that B is \mathcal{S} -complementable, then there exists a B -selfadjoint projection Q into \mathcal{S} that can be decomposed as the sum of two B -selfadjoint projections Q_+, Q_- with B -nonnegative and B -nonpositive ranges, respectively.

Lemma 3.11. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then, the following statements are equivalent:*

- i) B is \mathcal{S} -complementable;*
- ii) B is \mathcal{S}_\pm -complementable;*
- iii) B is \mathcal{S} -weakly complementable and $B_{/\mathcal{S}_\pm}$ is \mathcal{S}_\mp -complementable.*

In this case, there exists $Q \in \mathcal{P}(B, \mathcal{S})$ that can be decomposed as $Q = Q_+ + Q_-$, where $Q_\pm \in \mathcal{P}(B, \mathcal{S}_\pm)$. Moreover, $R(Q_+) \perp R(Q_-)$ and $Q_+ Q_- = Q_- Q_+ = 0$.

Proof. *i) \Leftrightarrow ii) :* Let $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ be any decomposition as in (2.1). Suppose that the matrix representation of B is as in (2.3) and B is \mathcal{S} -complementable. Then $R(b_+) \subseteq R(a_+)$ and $R(b_-) \subseteq R(a_-)$. In fact, since $R(b) \subseteq R(a)$, for every $y \in \mathcal{S}^\perp$, there exists $s \in \mathcal{S}$ such that $by = as$. Therefore, for every $y \in \mathcal{S}^\perp$, $b_\pm y = P_{\mathcal{S}_\pm} by = P_{\mathcal{S}_\pm} as = a_\pm s$ and B is \mathcal{S}_\pm -complementable. The converse follows in a similar way using that $R(a) = R(a_+) \oplus R(a_-)$.

i) \Leftrightarrow iii) : It can be proven in a similar way as in *i) \Leftrightarrow ii)* using the decomposition of $B_{/\mathcal{S}_\pm}$ given in the proof of Theorem 3.4. In this case, let f be the reduced solution of $b_+ = a_+ x$ and g the reduced solution

of $b_- = -a_- x$. Set $Q_+ := \begin{bmatrix} I & 0 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$ and $Q_- := \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & -g \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}$. Then $Q_\pm \in \mathcal{P}(B, \mathcal{S}_\pm)$,

$R(Q_+) \perp R(Q_-)$ and $Q_+ Q_- = Q_- Q_+ = 0$. Finally, since

$$R(f - g) \subseteq \overline{R(a_+)} \oplus \overline{R(a_-)} = \overline{R(a)},$$

$y := f - g$ is the reduced solution of the equation $b = ax$. Therefore $Q := Q_+ + Q_- = \begin{bmatrix} I & 0 & f \\ 0 & I & -g \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix} =$

$$\begin{bmatrix} I & y \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} \in \mathcal{P}(B, \mathcal{S}). \quad \square$$

Corollary 3.12. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} such that B is \mathcal{S} -complementable. Then*

$$B_{/\mathcal{S}} = B(I - Q) = (I - Q)^* B$$

for any $Q \in \mathcal{P}(B, \mathcal{S})$.

Proof. Let $Q^0 \in \mathcal{P}(B, \mathcal{S})$ such that there exist $Q_\pm^0 \in \mathcal{P}(B, \mathcal{S}_\pm)$ with $Q^0 = Q_+^0 + Q_-^0$, $R(Q_+^0) \perp R(Q_-^0)$ and $Q_+^0 Q_-^0 = Q_-^0 Q_+^0 = 0$, as in Lemma 3.11. Set $\mathcal{S}_\pm := R(Q_\pm^0)$, then $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$, as in Lemma 2.2. By Lemma 3.11, $B_{/\mathcal{S}_+}$ is \mathcal{S}_- -complementable and since \mathcal{S}_- is $B_{/\mathcal{S}_+}$ -nonpositive, Corollary 3.5 together with Proposition 3.10 give

$$B_{/\mathcal{S}} = (B_{/\mathcal{S}_+})_{/\mathcal{S}_-} = B_{/\mathcal{S}_+}(I - Q_-^0).$$

Then, once again by Lemma 3.11, B is \mathcal{S}_+ -complementable and by Proposition 3.10,

$$B_{/\mathcal{S}} = B_{/\mathcal{S}_+}(I - Q_-^0) = B(I - Q_+^0)(I - Q_-^0) = B(I - Q_+^0 - Q_-^0) = B(I - Q^0).$$

Now take any $Q \in \mathcal{P}(B, \mathcal{S})$, then by [11, Theorem 3.5] and [17, Proposition 3.2], $Q = Q^0 + T$, for some $T \in L(\mathcal{H})$ with $R(T) \subseteq N(B) \cap \mathcal{S}$ and $\mathcal{S} \subseteq N(T)$. Therefore

$$B_{/\mathcal{S}} = B(I - Q^0) = B(I - (Q - T)) = B(I - Q).$$

□

Corollary 3.12 shows that $B_{/\mathcal{S}}$ coincides with the Schur complement defined in [15] for a bounded selfadjoint operator B and $\mathcal{S} \subseteq \mathcal{H}$ a closed subspace such that B is \mathcal{S} -complementable.

Corollary 3.13. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} such that B is \mathcal{S} -weakly complementable. Suppose that $\mathcal{S} = \mathcal{S}_+ \oplus_B \mathcal{S}_-$ is any decomposition as in (2.1). Then, B is \mathcal{S} -complementable if and only if*

$$\begin{aligned} B_{/\mathcal{S}} &= \max_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\min_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right) \\ &= \min_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\max_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right). \end{aligned}$$

Proof. Suppose that B is \mathcal{S} -complementable. Then, by Theorem 3.6, Lemma 3.11 and Corollary 3.12, the result follows.

Conversely, suppose that

$$\begin{aligned} B_{/\mathcal{S}} &= \max_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} \left(\min_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- \right) \\ &= \min_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} \left(\max_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_+^* Q_-^* B Q_- Q_+ \right). \end{aligned}$$

Suppose that the matrix representation of B is as in (2.3). Since B is \mathcal{S} -weakly complementable, take B_1, B_2 and B_3 as in the proof of Proposition 3.2. Let $Q_- \in \mathcal{Q}$ with $N(Q_-) = \mathcal{S}_-$. Then, by the proof of Theorem 3.6,

$$\min_{Q_+ \in \mathcal{Q}, N(Q_+) = \mathcal{S}_+} Q_-^* Q_+^* B Q_+ Q_- = Q_-^* (B_1 + B_{2/\mathcal{S}_+} - B_3) Q_-.$$

Observe that

$$B_1 + B_{2/\mathcal{S}_+} - B_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -a_- & b_- \\ 0 & b_-^* & c - f^* f \end{bmatrix} \begin{matrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{S}^\perp \end{matrix}.$$

Since there exists $\max_{Q_- \in \mathcal{Q}, N(Q_-) = \mathcal{S}_-} Q_-^* (B_1 + B_{2/\mathcal{S}_+} - B_3) Q_-$, by Proposition 3.10, $B_1 + B_{2/\mathcal{S}_+} - B_3$ is \mathcal{S}_- -complementable and then $R(b_-) \subseteq R(a_-)$. In a similar fashion, $B_1 + B_2 - B_{3/\mathcal{S}_-}$ is \mathcal{S}_+ -complementable and $R(b_+) \subseteq R(a_+)$. Therefore, $R(b) \subseteq R(b_-) + R(b_+) \subseteq R(a_-) \oplus R(a_+) = R(a)$. Hence B is \mathcal{S} -complementable. □

3.2. A formula for the Schur complement

When the operator B is \mathcal{S} -complementable, the Schur complement can be written as $B_{/\mathcal{S}} = (I - F)B$, for any bounded projection with $N(F) = \mathcal{S}^\perp$ such that $(FB)^* = FB$. In fact, from Corollary 3.12, it suffices to take $F = Q^*$, for any $Q \in \mathcal{P}(B, \mathcal{S})$.

In this section, we show that a similar formula for $B_{/\mathcal{S}}$ can be given when B is \mathcal{S} -weakly complementable. In this case the projection need not be bounded, but it is densely defined.

Theorem 3.14. *Let $B \in L(\mathcal{H})^s$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Then B is \mathcal{S} -weakly complementable if and only if there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $E|P_{\mathcal{S}} B P_{\mathcal{S}}|^{1/2} \in L(\mathcal{H})$ and $EB \in L(\mathcal{H})^s$. In this case,*

$$B_{/\mathcal{S}} = (I - E)B.$$

Remark. The densely defined projection E is closed if and only if the pair (B, \mathcal{S}) is quasi-compatible; i.e., $\mathcal{H} = \mathcal{S} + B^{-1}(\mathcal{S}^\perp)$. Moreover, $E \in L(\mathcal{H})$ if and only if B is \mathcal{S} -complementable.

Proof. Suppose that B is \mathcal{S} -weakly complementable. If the matrix decomposition of B induced by \mathcal{S} is as in (3.1), let f be the reduced solution of $b = |a|^{1/2}x$ and $a = u|a|$ the polar decomposition of a . Write $(|a|^{1/2})^\dagger$ for the Moore-Penrose inverse of $|a|^{1/2}$ and set

$$E = \begin{bmatrix} I & 0 \\ f^*u(|a|^{1/2})^\dagger & 0 \end{bmatrix}.$$

Then $Dom(E) = Dom(|a|^{1/2})^\dagger \oplus \mathcal{S}^\perp$ and E is a densely defined projection with $N(E) = \mathcal{S}^\perp$. On the other hand, since $R(B) \subseteq R(|a|^{1/2}) \oplus \mathcal{S}^\perp$, the product $(I - E)B$ is well defined. Moreover

$$\begin{aligned} (I - E)B &= \begin{bmatrix} 0 & 0 \\ -f^*u(|a|^{1/2})^\dagger & I \end{bmatrix} \begin{bmatrix} |a|^{1/2}u|a|^{1/2} & |a|^{1/2}f \\ f^*|a|^{1/2} & c \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & c - f^*uf \end{bmatrix} = B/\mathcal{S} \end{aligned}$$

is bounded and selfadjoint. Finally,

$$E|P_{\mathcal{S}}BP_{\mathcal{S}}|^{1/2} = \begin{bmatrix} |a|^{1/2} & 0 \\ f^*u & 0 \end{bmatrix} \in L(\mathcal{H}).$$

Conversely, suppose that there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $E|P_{\mathcal{S}}BP_{\mathcal{S}}|^{1/2}$ and $EB \in L(\mathcal{H})^s$. Then the matrix decomposition of $I - E$ is

$$I - E = \begin{bmatrix} 0 & 0 \\ y & I \end{bmatrix},$$

with $y : Dom(y) \subseteq \mathcal{S} \rightarrow \mathcal{S}^\perp$ and $\overline{Dom(y)} = \mathcal{S}$. If the matrix decomposition of B is as in (3.1), since $(I - E)B$ is selfadjoint, it follows that, $ya = -b^*$ and yb is bounded and selfadjoint. From the fact that $E|P_{\mathcal{S}}BP_{\mathcal{S}}|^{1/2} \in L(\mathcal{H})$, we have that $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$ and, since $ya = -b^*$, we also have that $y|a|^{1/2}u|a|^{1/2} = -b^*$. Then $b = |a|^{1/2}(-y|a|^{1/2}u)^*$, $R(b) \subseteq R(|a|^{1/2})$ and B is \mathcal{S} -weakly complementable. \square

4. Schur complement in Krein spaces

In this section we adapt the definitions of complementability and weak complementability given in Section 3 to a bounded selfadjoint operator W acting on a Krein space $(\mathcal{H}, [\ , \])$. From now on all spaces are assumed to be Krein spaces unless otherwise stated.

Definition. Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . The operator W is called \mathcal{S} -complementable if

$$\mathcal{H} = \mathcal{S} + W^{-1}(\mathcal{S}^{[\perp]}).$$

If W is \mathcal{S} -complementable then, for any fundamental decomposition $\mathcal{H} = \mathcal{H}_+ [+] \mathcal{H}_-$ with signature operator J , we get that $\mathcal{H} = \mathcal{S} + (JW)^{-1}(\mathcal{S}^\perp)$. Therefore, W is \mathcal{S} -complementable if and only if the selfadjoint operator JW is \mathcal{S} -complementable in (the Hilbert space) $(\mathcal{H}, \langle \ , \ \rangle)$ for any (and then for every) signature operator J . From this, it follows that W is \mathcal{S} -complementable if and only if there exists a projection Q onto \mathcal{S} such that $WQ = Q^\#W$.

In this case, if the matrix representation of JW induced by \mathcal{S} is

$$JW = \begin{bmatrix} a & b \\ b^* & c \end{bmatrix}, \tag{4.1}$$

the \mathcal{S} -complementability of W is equivalent to $R(b) \subseteq R(a)$ (see [11, Proposition 3.3]).

In a similar fashion we define the \mathcal{S} -weak complementability in Krein spaces, with respect to a fixed signature operator J .

Definition. Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . The operator W is \mathcal{S} -weakly complementable with respect to a signature operator J if JW is \mathcal{S} -weakly complementable in $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.

Next, we show that the \mathcal{S} -weak complementability of W does not depend on the signature operator. In order to do so, we need to establish some technical lemmas. Some additional notation is also required: consider J and J_α two signature operators and set $\alpha = J_\alpha J$. Denote by $\mathcal{H} = (\mathcal{H}, \langle \cdot, \cdot \rangle)$ the Hilbert space associated to J and by $\mathcal{H}_\alpha = (\mathcal{H}, \langle \cdot, \cdot \rangle_{J_\alpha})$, where $\langle x, y \rangle_{J_\alpha} = [J_\alpha x, y] = \langle \alpha^{-1}x, y \rangle$, the Hilbert space associated to J_α . Then $\alpha \geq_{\mathcal{H}} 0$ and $\alpha \geq_{\mathcal{H}_\alpha} 0$. Notice that $\mathcal{S}^{\perp_\alpha} = \alpha(\mathcal{S}^\perp)$ is the orthogonal complement of \mathcal{S} in \mathcal{H}_α and, for $T \in L(\mathcal{H})$, $T^{*\alpha} = \alpha T^* \alpha^{-1}$ is the adjoint of T in \mathcal{H}_α . Also, $T^{*\alpha} = T$ if and only if $\alpha T^* = T\alpha$.

Denote by $|T|_\alpha$ the modulus of T in \mathcal{H}_α and by $P_S^\alpha = P_{\mathcal{S}/\mathcal{S}^{\perp_\alpha}}$ the orthogonal projection onto \mathcal{S} in \mathcal{H}_α . If $T \geq_{\mathcal{H}_\alpha} 0$, we indicate by $T^{1/2_\alpha}$ the square root of T in \mathcal{H}_α . Frequently, we will use that if $T \geq_{\mathcal{H}} 0$ or $T = T^*$ then $\alpha T \geq_{\mathcal{H}_\alpha} 0$ or $\alpha T = (\alpha T)^{*\alpha}$, respectively.

Lemma 4.1. *Let \mathcal{S} be a closed subspace of \mathcal{H} , J and J_α two signature operators and $\alpha = J_\alpha J$. Then*

$$\tilde{\alpha} := P_S^\alpha \alpha|_{\mathcal{S}} = (P_S \alpha^{-1}|_{\mathcal{S}})^{-1} \in GL(\mathcal{S})^+.$$

Proof. The projection P_S^α can be expressed in terms of P_S and α as

$$P_S^\alpha = P_S(P_S \alpha^{-1} P_S + (I - P_S) \alpha^{-1} (I - P_S))^{-1} \alpha^{-1};$$

see [11, Section 4]. Therefore,

$$\begin{aligned} P_S^\alpha \alpha|_{\mathcal{S}} &= P_S(P_S \alpha^{-1} P_S + (I - P_S) \alpha^{-1} (I - P_S))^{-1}|_{\mathcal{S}} = \\ &= (P_S \alpha^{-1}|_{\mathcal{S}})^{-1} \in GL(\mathcal{S})^+. \end{aligned}$$

□

By Lemma 4.1, $\mathcal{S}_{\tilde{\alpha}} := (\mathcal{S}, \langle \cdot, \cdot \rangle_{\tilde{\alpha}})$ is a Hilbert space. Also, from the discussion before Lemma 4.1, if $a \in L(\mathcal{S})^s$ then $(\tilde{\alpha}a)^{*\tilde{\alpha}} = \tilde{\alpha}a$.

Lemma 4.2. *Let \mathcal{S} be a closed subspace of \mathcal{H} , J and J_α two signature operators, $\alpha = J_\alpha J$ and $a \in L(\mathcal{S})^s$. Let $a' = P_S^\alpha \alpha a$. Then*

$$R(|a'|_{\tilde{\alpha}}^{1/2_{\tilde{\alpha}}}) = R((P_S^\alpha \alpha|_{\mathcal{S}})^{1/2_\alpha}) = \tilde{\alpha} R(|a|^{1/2}). \quad (4.2)$$

Proof. First observe that

$$(P_S^\alpha \alpha|_{\mathcal{S}})^{1/2_\alpha} = (\tilde{\alpha}|_{\mathcal{S}})^{1/2_\alpha} = P_S^\alpha \alpha X_0 P_S^\alpha,$$

with $X_0 = \tilde{\alpha}^{-1/2} (\tilde{\alpha}^{1/2} |a| \tilde{\alpha}^{1/2})^{1/2} \tilde{\alpha}^{-1/2}$. In fact, $X_0 \in L(\mathcal{S})^+$, because $\tilde{\alpha}^{1/2} \in GL(\mathcal{S})^+$ by Lemma 4.1. Clearly, $X_0 \tilde{\alpha} X_0 = |a|$. Therefore, $(\tilde{\alpha} X_0)^2 = \tilde{\alpha} |a|$. Also, $\tilde{\alpha} X_0 P_S^\alpha = P_S^\alpha \alpha X_0 P_S^\alpha \geq_{\mathcal{H}_\alpha} 0$.

Then $(\tilde{\alpha} X_0 P_S^\alpha)^2 = \tilde{\alpha} X_0 P_S^\alpha \tilde{\alpha} X_0 P_S^\alpha = \tilde{\alpha} X_0 \tilde{\alpha} X_0 P_S^\alpha = \tilde{\alpha} |a| P_S^\alpha$. Thus,

$$(\tilde{\alpha} |a| P_S^\alpha)^{1/2_\alpha} = \tilde{\alpha} X_0 P_S^\alpha = P_S^\alpha \alpha X_0 P_S.$$

Now, since $X_0 \tilde{\alpha} X_0 = |a|$, Douglas' Lemma yields $R(X_0 \tilde{\alpha}^{1/2}) = R(|a|^{1/2})$. Therefore, because $\tilde{\alpha}^{1/2} \in GL(\mathcal{S})^+$, $R(X_0) = R(|a|^{1/2})$ (see Lemma 4.1). Then

$$R((P_S^\alpha \alpha|_{\mathcal{S}})^{1/2_\alpha}) = R(P_S^\alpha \alpha X_0 P_S) = R(\tilde{\alpha} X_0 P_S^\alpha) = \tilde{\alpha} R(X_0) = \tilde{\alpha} R(|a|^{1/2})$$

and the second equality in (4.2) follows. Using that $(\tilde{\alpha}a)^{*\tilde{\alpha}} = \tilde{\alpha}a$, $R(|a'|_{\tilde{\alpha}}) = R(a') = R(\tilde{\alpha}a) = R(\tilde{\alpha}|a|)$. Then, applying Douglas' Lemma and the operator monotonicity of the square root in $\mathcal{S}_{\tilde{\alpha}}$ (see [21]), we get that

$$R(|a'|_{\tilde{\alpha}}^{1/2_{\tilde{\alpha}}}) = R((\tilde{\alpha}|a|)^{1/2_{\tilde{\alpha}}}).$$

Finally, from $R((\tilde{\alpha}|a|)^{1/2_{\tilde{\alpha}}}) = R((P_S^\alpha \alpha|_{\mathcal{S}})^{1/2_\alpha})$, we get the first equality. □

Lemma 4.3. *Let \mathcal{S} be a closed subspace of \mathcal{H} , J and J_α two signature operators, $\alpha = J_\alpha J$ and $a \in L(\mathcal{S})^s$. Let $a' = P_S^\alpha \alpha a$, $\Gamma := \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, $\Gamma_\alpha := \begin{bmatrix} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp\alpha} \end{matrix}$ and E a densely defined projection with $N(E) = \mathcal{S}^\perp$ such that $E\Gamma \in L(\mathcal{H})$. Then $\alpha E \alpha^{-1}$ is a densely defined projection with $N(\alpha E \alpha^{-1}) = \mathcal{S}^{\perp\alpha}$ such that $\alpha E \alpha^{-1} \Gamma_\alpha \in L(\mathcal{H})$.*

Proof. Clearly, $\alpha E \alpha^{-1}$ is a densely defined projection with $N(\alpha E \alpha^{-1}) = \mathcal{S}^{\perp\alpha}$. Let us see that $\alpha E \alpha^{-1} \Gamma_\alpha \in L(\mathcal{H})$. Since E is a densely defined projection with $N(E) = \mathcal{S}^\perp$ such that $E\Gamma \in L(\mathcal{H})$, the matrix decomposition of E is

$$E = \begin{bmatrix} I & 0 \\ y & 0 \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix} = (I + y)P_S,$$

with $y : \text{Dom}(y) \subseteq \mathcal{S} \rightarrow \mathcal{S}^\perp$, $\overline{\text{Dom}(y)} = \mathcal{S}$ and $y|a|^{1/2} \in L(\mathcal{S}, \mathcal{S}^\perp)$. Also, $\Gamma_\alpha = |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} P_S^\alpha + (I - P_S^\alpha)$. Since, by Lemma 4.2, $R(|a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}}) = R(\tilde{\alpha}|a|^{1/2})$, $|a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} \geq_{\mathcal{S}^\perp} 0$ and $\tilde{\alpha}|a|^{1/2} \geq_{\mathcal{S}^\perp} 0$, there exists $g \in GL(\mathcal{S})$ such that

$$|a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} = \tilde{\alpha}|a|^{1/2}g.$$

Then $P_S \alpha^{-1} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} = P_S \alpha^{-1} \tilde{\alpha} |a|^{1/2} g = P_S \alpha^{-1} P_S \tilde{\alpha} |a|^{1/2} g = \tilde{\alpha}^{-1} \tilde{\alpha} |a|^{1/2} g = |a|^{1/2} g$.

Therefore, $R(P_S \alpha^{-1} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}}) = R(|a|^{1/2}) \subseteq \text{Dom}(y)$ and

$$y P_S \alpha^{-1} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} = y |a|^{1/2} g \in L(\mathcal{S}, \mathcal{S}^\perp).$$

Thus

$$\alpha E \alpha^{-1} \Gamma_\alpha = \alpha P_S \alpha^{-1} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} P_S^\alpha + \alpha y P_S \alpha^{-1} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} P_S^\alpha \in L(\mathcal{H}).$$

□

Now we are ready to show that the weak \mathcal{S} -complementability of W does not depend on the fundamental decomposition of \mathcal{H} .

Theorem 4.4. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that W is \mathcal{S} -weakly complementable for some signature operator J . Then W is \mathcal{S} -weakly complementable for any other signature operator J_α .*

Proof. Suppose that W is \mathcal{S} -weakly complementable for some signature operator J and the matrix decomposition of JW is as in (4.1). Observe that $a \in L(\mathcal{S})^s$.

By Theorem 3.14, there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $EJW \in L(\mathcal{H})^s$. Also, if $\Gamma := \begin{bmatrix} |a|^{1/2} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^\perp \end{matrix}$, then $E\Gamma \in L(\mathcal{H})$. Let J_α be another signature operator and $\alpha = J_\alpha J$. If $J_\alpha W = \alpha JW = \begin{bmatrix} a' & b' \\ b'^* & c' \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp\alpha} \end{matrix}$ then

$$a' = P_S^\alpha \alpha JW P_S^\alpha = P_S^\alpha \alpha a P_S^\alpha.$$

Consider $\alpha E \alpha^{-1}$ and $\Gamma_\alpha := \begin{bmatrix} |a'|_{\tilde{\alpha}}^{1/2\tilde{\alpha}} & 0 \\ 0 & I \end{bmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp\alpha} \end{matrix}$. By Lemma 4.3, $\alpha E \alpha^{-1}$ is a densely defined projection with $N(\alpha E \alpha^{-1}) = \mathcal{S}^{\perp\alpha}$ such that $\alpha E \alpha^{-1} \Gamma_\alpha \in L(\mathcal{H})$. Also, $(\alpha E \alpha^{-1})(J_\alpha W) = \alpha E JW$ is selfadjoint in \mathcal{H}_α . Therefore, again by Theorem 3.14, $J_\alpha W$ is \mathcal{S} -weakly complementable. □

From now on, since the \mathcal{S} -weak complementability does not depend on the fundamental decomposition of \mathcal{H} , we simply say that W is \mathcal{S} -weakly complementable, whenever W is \mathcal{S} -weakly complementable with respect to a signature operator J . In particular, if $W \geq 0$ then W is \mathcal{S} -weakly complementable.

Following the ideas of [4], we extend the notion of Schur complement to selfadjoint operators in Krein spaces:

Definition. Let $W \in L(\mathcal{H})^{[s]}$, \mathcal{S} a closed subspace of \mathcal{H} and J a signature operator. Suppose that W is \mathcal{S} -weakly complementable. The *Schur complement* of W to \mathcal{S} corresponding to J is

$$W_{/[\mathcal{S}]}^J = J(JW)_{/\mathcal{S}},$$

and the \mathcal{S} -compression of W is $W_{[\mathcal{S}]}^J = W - W_{/[\mathcal{S}]}^J = J(JW)_{\mathcal{S}}$.

Theorem 4.5. Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that W is \mathcal{S} -weakly complementable and J is a signature operator, then

$$W_{/[\mathcal{S}]}^J = W_{/[\mathcal{S}]}^{J_\alpha},$$

for any other signature operator J_α ; i.e., the Schur complement does not depend on the fundamental decomposition of \mathcal{H} .

Henceforth we write $W_{/[\mathcal{S}]}$ for this operator.

Proof. Suppose that W is \mathcal{S} -weakly complementable. Then, by Lemma 4.3, JW is \mathcal{S} -weakly complementable for any signature operator J . By Theorem 3.14, there exists a densely defined projection E with $N(E) = \mathcal{S}^\perp$ such that $EJW \in L(\mathcal{H})^s$, $E\Gamma \in L(\mathcal{H})$ and $W_{/[\mathcal{S}]}^J = J(I - E)JW$. Let J_α be another signature operator, $\alpha = J_\alpha J$ and consider $\alpha E \alpha^{-1}$. Then, by Theorem 4.4, $\alpha E \alpha^{-1}$ is a densely defined projection with $N(\alpha E \alpha^{-1}) = \mathcal{S}^{\perp\alpha}$ such that $\alpha E \alpha^{-1} \Gamma_\alpha \in L(\mathcal{H})$. Therefore, by Theorem 3.14,

$$W_{/[\mathcal{S}]}^{J_\alpha} = J_\alpha \alpha (I - E) \alpha^{-1} J_\alpha W = J(I - E)JW = W_{/[\mathcal{S}]}^J.$$

□

Corollary 4.6. Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that W is \mathcal{S} -weakly complementable. Then there exists a densely defined projection E with $N(E) = \mathcal{S}^{\perp\perp}$ such that

$$W_{/[\mathcal{S}]} = (I - E)W.$$

Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} , define

$$\begin{aligned} \mathcal{N}^-(W, \mathcal{S}^{[\perp]}) &:= \{X \in L(\mathcal{H})^{[s]} : X \leq W, R(X) \subseteq \mathcal{S}^{[\perp]}\}, \\ \mathcal{N}^+(W, \mathcal{S}^{[\perp]}) &:= \{X \in L(\mathcal{H})^{[s]} : W \leq X, R(X) \subseteq \mathcal{S}^{[\perp]}\}. \end{aligned}$$

If J is any signature operator,

$$\mathcal{N}^\pm(W, \mathcal{S}^{[\perp]}) = J\mathcal{M}^\pm(JW, \mathcal{S}^\perp).$$

Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Then, applying Lemma 2.2 to $B = JW$, with J any signature operator, \mathcal{S} can be decomposed as

$$\mathcal{S} = \mathcal{S}_+ [\dot{+}]_W \mathcal{S}_-, \tag{4.3}$$

where \mathcal{S}_+ and \mathcal{S}_- are closed, \mathcal{S}_+ is W -nonnegative, \mathcal{S}_- is W -nonpositive and, moreover, $\mathcal{S}_+ \perp \mathcal{S}_-$.

Proposition 4.7. Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ [\dot{+}]_W \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J . Then the following statements are equivalent:

- i) W is \mathcal{S} -weakly complementable;
- ii) there exist $W_1, W_2, W_3 \in L(\mathcal{H})^{[s]}$, $W_2, W_3 \geq 0$ such that $W = W_1 + W_2 - W_3$ and $\mathcal{S} \subseteq N(W_1)$, $\mathcal{S}_- \subseteq N(W_2)$, $\mathcal{S}_+ \subseteq N(W_3)$;
- iii) the sets $\mathcal{N}^-(W, \mathcal{S}_+^{[\perp]})$ and $\mathcal{N}^+(W, \mathcal{S}_-^{[\perp]})$ are non-empty;
- iv) W is \mathcal{S}_\pm -weakly complementable.

Proof. This follows from Proposition 3.2. \square

The following theorem proves that the set $\mathcal{N}^-(W, \mathcal{S}^{[\perp]})$ has a maximum element if and only if \mathcal{S} is W -nonnegative and W is \mathcal{S} -weakly complementable. A similar result can be proven if \mathcal{S} is W -nonpositive and W is \mathcal{S} -weakly complementable.

Proposition 4.8. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Then \mathcal{S} is W -nonnegative and W is \mathcal{S} -weakly complementable if and only if the set $\mathcal{N}^-(W, \mathcal{S}^{[\perp]})$ has a maximum element.*

In this case,

$$W_{/[\mathcal{S}]} = \max \mathcal{N}^-(W, \mathcal{S}^{[\perp]}).$$

Proof. This follows from Proposition 3.3. \square

Theorem 4.9. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ [\dot{+}]_W \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J . Then W is \mathcal{S} -weakly complementable if and only if there exists $\min \mathcal{N}^+(\max \mathcal{N}^-(W, \mathcal{S}_+^{[\perp]}), \mathcal{S}_-^{[\perp]})$ and $\max \mathcal{N}^-(\min \mathcal{N}^+(B, \mathcal{S}_-^{[\perp]}), \mathcal{S}_+^{[\perp]})$.*

In this case,

$$\begin{aligned} W_{/[\mathcal{S}]} &= \min \mathcal{N}^+(\max \mathcal{N}^-(W, \mathcal{S}_+^{[\perp]}), \mathcal{S}_-^{[\perp]}) \\ &= \max \mathcal{N}^-(\min \mathcal{N}^+(B, \mathcal{S}_-^{[\perp]}), \mathcal{S}_+^{[\perp]}). \end{aligned}$$

Proof. This follows from Theorem 3.4. \square

Corollary 4.10. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ [\dot{+}]_W \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J . If W is \mathcal{S} -weakly complementable, then*

$$W_{/[\mathcal{S}]} = (W_{/[\mathcal{S}_+]})_{/[\mathcal{S}_-]} = (W_{/[\mathcal{S}_-]})_{/[\mathcal{S}_+]}$$

Theorem 4.11. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $\mathcal{S} = \mathcal{S}_+ [\dot{+}]_W \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J . W is \mathcal{S} -weakly complementable if and only if there exist*

$$\sup_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} \left(\inf_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} E_-^\# E_+^\# W E_+ E_- \right)$$

and

$$\inf_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} \left(\sup_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} E_+^\# E_-^\# W E_- E_+ \right).$$

In this case,

$$\begin{aligned} W_{/[\mathcal{S}]} &= \sup_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} \left(\inf_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} E_-^\# E_+^\# W E_+ E_- \right) \\ &= \inf_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} \left(\sup_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} E_+^\# E_-^\# W E_- E_+ \right). \end{aligned}$$

Proof. For any signature operator J , if $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is the associated Hilbert space,

$$J\{Q^* J W Q : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\} = \{E^\# W E : E \in \mathcal{Q}, N(E) = \mathcal{S}\}. \quad (4.4)$$

Also, there exists $\inf_{\leq} \{E^\# W E : E = E^2, N(E) = \mathcal{S}\}$ if and only if there exists $\inf_{\leq \mathcal{H}} \{Q^* J W Q : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$. Moreover

$$\inf_{\leq} \{E^\# W E : E = E^2, N(E) = \mathcal{S}\} = J \inf_{\leq \mathcal{H}} \{Q^* J W Q : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}. \quad (4.5)$$

Analogously, there exists $\sup_{\leq} \{E^\# W E : E = E^2, N(E) = \mathcal{S}\}$ if and only if there exists $\sup_{\leq \mathcal{H}} \{Q^* J W Q : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}$. Moreover

$$\sup_{\leq} \{E^\# W E : E = E^2, N(E) = \mathcal{S}\} = J \sup_{\leq \mathcal{H}} \{Q^* J W Q : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\}. \quad (4.6)$$

The result follows from (4.4), (4.5), (4.6) and Theorem 3.6. \square

Corollary 4.12. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that \mathcal{S} is W -nonnegative. Then W is \mathcal{S} -weakly complementable if and only if there exists inf $\{E^\#WE : E = E^2, N(E) = \mathcal{S}\}$.*

In this case,

$$W_{/[\mathcal{S}]} = \inf \{E^\#WE : E = E^2, N(E) = \mathcal{S}\}.$$

A similar result holds when \mathcal{S} is W -nonpositive, replacing inf by sup.

Proposition 4.13. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that W is \mathcal{S} -weakly complementable. Then,*

$$W_{/[\mathcal{S}]} = \min \{E^\#WE : E \in \mathcal{Q}, N(E) = \mathcal{S}\}$$

if and only if W is \mathcal{S} -complementable and \mathcal{S} is W -nonnegative.

In this case,

$$W_{/[\mathcal{S}]} = W(I - Q),$$

with Q any projection onto \mathcal{S} such that $WQ = Q^\#W$.

A similar result holds when \mathcal{S} is W -nonpositive, replacing min by max.

Proof. For any signature operator J , if $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is the associated Hilbert space, by (4.4) and Proposition 3.10,

$$\begin{aligned} W_{/[\mathcal{S}]} &= J(JW)_{/[\mathcal{S}]} = J \min_{\leq \mathcal{H}} \{Q^*JWQ : Q \in \mathcal{Q}, N(Q) = \mathcal{S}\} \\ &= \min \{E^\#WE : E \in \mathcal{Q}, N(E) = \mathcal{S}\} \end{aligned}$$

if and only if $\mathcal{H} = \mathcal{S} + (JW)^{-1}(\mathcal{S}^\perp)$ and \mathcal{S} is JW -nonnegative (in the Hilbert space \mathcal{H}) if and only if W is \mathcal{S} -complementable and \mathcal{S} is W -nonnegative.

The operator Q is any projection onto \mathcal{S} such that $WQ = Q^\#W$ if and only if $Q \in \mathcal{P}(JW, \mathcal{S})$ for any signature operator J . Therefore, in these cases, by Proposition 3.10, $W_{/[\mathcal{S}]} = J(JW)_{/[\mathcal{S}]} = J(JW)(I - Q) = W(I - Q)$, for any of these projections. \square

Corollary 4.14. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} such that W is \mathcal{S} -complementable. Then*

$$W_{/[\mathcal{S}]} = W(I - Q),$$

for Q any projection onto \mathcal{S} such that $WQ = Q^\#W$.

Proof. This follows proceeding as in Corollary 3.12 and by Proposition 4.13. \square

Theorem 4.15. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} such that W is \mathcal{S} -weakly complementable. Suppose that $\mathcal{S} = \mathcal{S}_+ \dot{+}_W \mathcal{S}_-$ is any decomposition as in (4.3) for some signature operator J . Then, W is \mathcal{S} -complementable if and only if*

$$\begin{aligned} W_{/[\mathcal{S}]} &= \min_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} \left(\max_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} E_-^\# E_+^\# W E_+ E_- \right) \\ &= \max_{E_- \in \mathcal{Q}, N(E_-) = \mathcal{S}_-} \left(\min_{E_+ \in \mathcal{Q}, N(E_+) = \mathcal{S}_+} E_+^\# E_-^\# W E_- E_+ \right). \end{aligned}$$

Proof. This follows by (4.4) and Corollary 3.13. \square

4.1. Weak complementability for regular subspaces

Any $W \in L(\mathcal{H})^{[s]}$ can be written in the form

$$W = DD^\#$$

where $D \in L(\mathcal{K}, \mathcal{H})$ for some Krein space \mathcal{K} and $N(D) = \{0\}$. This factorization, in general, is not unique. Such factorizations are known as *Bognár-Krámlí factorizations*, see [9].

Let J be any signature operator of \mathcal{H} . Then, JW is selfadjoint in the corresponding Hilbert space. If $JW = U|JW| = |JW|U$ is the polar factorization of JW , then $\mathcal{K} := \overline{R(|JW|)}$ is a Krein space with signature operator $J_{\mathcal{K}} := U|_{\mathcal{K}}$. Define $D : \mathcal{K} \rightarrow \mathcal{H}$ by

$$Dk := J|JW|^{1/2}k, \quad k \in \mathcal{K}. \quad (4.7)$$

Then, $N(D) = \{0\}$, $D^{\#} = J_{\mathcal{K}}|JW|^{1/2} = U|JW|^{1/2}$ and $DD^{\#} = W$ (cf. [14, Theorem 1.1]).

Definition. A Bognár-Krámli factorization of an operator $W \in L(\mathcal{H})^{[s]}$ which is constructed by the method described above is called a *polar factorization* of W (see [14, Lecture 6]).

Lemma 4.16. *Let $W \in L(\mathcal{H})^{[s]}$ have polar factorizations $W = DD^{\#} = EE^{\#}$ with $D \in L(\mathcal{K}, \mathcal{H})$ and $E \in L(\mathcal{K}', \mathcal{H})$. Then*

$$R(D) = R(E).$$

Proof. In this case, following similar arguments as in [14, Theorem 6.1] and [14, Theorem 6.2], it can be shown that there exists a unique $L \in L(\mathcal{K}', \mathcal{K})$ such that $E = DL$ and $D = EL^{\#}$. Clearly, $R(D) = R(E)$. \square

Let \mathcal{S} be a regular subspace of \mathcal{H} , then $W \in L(\mathcal{H})^{[s]}$ can be represented as a 2×2 block matrix in the form

$$W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^{\#} & w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}. \quad (4.8)$$

Theorem 4.17. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a regular subspace of \mathcal{H} . Suppose that W is represented as in (4.8) and $w_{11} = dd^{\#}$ is any polar factorization of w_{11} . Then W is \mathcal{S} -weakly complementable if and only if $R(w_{12}) \subseteq R(d)$.*

In this case,

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^{\#}y \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix},$$

with $y \in L(\mathcal{S}^{\perp}, \mathcal{K})$ the only solution of the equation $w_{12} = dx$.

Proof. Take $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$ a signature operator for \mathcal{H} as in (2.5).

Then, $JW = \begin{pmatrix} J_1 w_{11} & J_1 w_{12} \\ J_2 w_{12}^{\#} & J_2 w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$ and W is \mathcal{S} -weakly complementable if and only if $R(J_1 w_{12}) \subseteq R(|J_1 w_{11}|^{1/2})$ or, equivalently, $R(w_{12}) \subseteq R(J_1 |J_1 w_{11}|^{1/2}) = R(d)$. Indeed, if $e := J_1 |J_1 w_{11}|^{1/2}$ then, by (4.7), $w_{11} = ee^{\#}$ is a polar factorization of w_{11} and, by Lemma 4.16, $R(e) = R(d)$.

In this case, let $y \in L(\mathcal{S}^{\perp}, \mathcal{K})$ be the only solution of the equation $w_{12} = dx$. Observe that

$$(JW)_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & J_2 w_{22} - f^* u f \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix},$$

where f is the reduced solution of $J_1 w_{12} = |J_1 w_{11}|^{1/2} x$ and u is the partial isometry corresponding to the polar decomposition of $J_1 w_{11}$. Clearly, $w_{12} = J_1 |J_1 w_{11}|^{1/2} f = ef$, so that $ef = dy$. As in the proof of Lemma 4.16, a unique bounded operator l can be found such that $e = dl$ and $d = el^{\#}$. Therefore, $y^{\#} d^{\#} = f^{\#} e^{\#} = f^{\#} l^{\#} d^{\#}$ and, since $d^{\#}$ has a dense range, $y^{\#} = f^{\#} l^{\#}$. In a similar way, $l^{\#} y = f$. Therefore, $y^{\#} y = f^{\#} l^{\#} y = f^{\#} f$. Finally, since $J_2 f^* u = f^{\#}$, it follows that

$$W_{/[\mathcal{S}]} = J(JW)_{/\mathcal{S}} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - J_2 f^* u f \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - y^{\#} y \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}.$$

\square

4.2. An application to a completion problem

Let \mathcal{S} be a regular subspace of \mathcal{H} and consider a bounded incomplete block operator

$$W^0 = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^\# & * \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}, \quad (4.9)$$

with $w_{11} \in L(\mathcal{S})^{[s]}$.

Following the ideas of Baidiuk in [6, Theorem 2.1], the next theorem solves a completion problem for any bounded incomplete operator W^0 of the form (4.9).

Proposition 4.18. *Let \mathcal{S} be a regular subspace of \mathcal{H} and W^0 be an incomplete block operator of the form (4.9). Assume that the number of negative squares $\nu_-[w_{11}]$ of the quadratic form $[w_{11}f, f]$, $f \in \mathcal{S}$, is finite. Let $w_{11} = dd^\#$ be any polar factorization of w_{11} . Then, there exists a completion W of W^0 with some operator $w_{22} \in L(\mathcal{S}^{\perp})^{[s]}$ such that $\nu_-[W] = \nu_-[w_{11}]$ if and only if $R(w_{12}) \subseteq R(d)$.*

In this case, if y is the unique bounded solution of the equation $w_{12} = dx$, the operator $y^\#y \in L(\mathcal{S}^{\perp})$ is the minimum in the solution set

$$\mathcal{W} = \{w_{22} \in L(\mathcal{S}^{\perp})^{[s]} : W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^\# & w_{22} \end{pmatrix} : \nu_-[W] = \nu_-[w_{11}]\},$$

and this solution set admits the description

$$\mathcal{W} = \{w_{22} \in L(\mathcal{S}^{\perp})^{[s]} : w_{22} = y^\#y + z, \text{ where } z = z^\# \geq 0\}.$$

Proof. Take $J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$ a signature operator for \mathcal{H} as in (2.5).

Then, $JW = \begin{pmatrix} J_1w_{11} & J_1w_{12} \\ J_2w_{12}^\# & * \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix}$. Then, by [6, Theorem 2.1], there exists a completion W of W^0 if and only if $R(J_1w_{12}) \subseteq R(|J_1w_{11}|^{1/2})$ or equivalently, proceeding as in Theorem 4.17, $R(w_{12}) \subseteq R(d)$.

In this case, by [6, Theorem 2.1], any selfadjoint operator completion of JW^0 admits the representation

$$JW = \begin{pmatrix} J_1w_{11} & J_1w_{12} \\ J_2w_{12}^\# & J_2w_{22} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{\perp} \end{matrix},$$

with $J_2w_{22} = f^*uf + z$ and f the reduced solution of $J_1w_{12} = |J_1w_{11}|^{1/2}x$, u the partial isometry corresponding to the polar decomposition of J_1w_{11} , and $z \in L(\mathcal{S}^{\perp})^+$. Then, as in the proof of Theorem 4.17, if y is the unique bounded solution of the equation $w_{12} = dx$, the set of completions of W^0 has the form $W = \begin{pmatrix} w_{11} & w_{12} \\ w_{12}^\# & w_{22} \end{pmatrix}$ with $w_{22} = y^\#y + z$, where $z = z^\# \geq 0$. \square

Remark. Any completion W of an incomplete block operator W^0 of the form (4.9) has the same \mathcal{S} -compression: $W_{[\mathcal{S}]}$. Moreover, for any completion W , $W_{[\mathcal{S}]} \leq W$.

4.3. Comparison with other notions of Schur complement in Krein spaces

In [19], Mary proved that any weakly regular operator $B \in L(\mathcal{K}, \mathcal{H})$ (i.e., any operator such that $\overline{R(B)}$ and $N(B)$ are regular subspaces) admits a (unique) closed Moore-Penrose inverse. That is, there exists an operator $B^\dagger : \text{Dom}(B^\dagger) = R(B) \dot{+} R(B)^{\perp} \subseteq \mathcal{H} \rightarrow \mathcal{K}$ such that BB^\dagger is a symmetric projection from $\text{Dom}(B^\dagger)$ onto $R(B)$ with nullspace $N(B)$ and $B^\dagger B$ is a symmetric projection from \mathcal{K} onto $R(B^\dagger) = N(B)^{\perp}$ with nullspace $N(B)$ (see [19, Corollary 2.9 and Lemma 2.2]).

Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a regular subspace of \mathcal{H} . Suppose that W is represented as in (4.8) and $w_{11} = dd^\#$ is a polar factorization of w_{11} . If $\overline{R(w_{11})}$ is regular, then $d, d^\#$ and w_{11} are weakly regular, therefore, there exist $d^\dagger, (d^\#)^\dagger$ and w_{11}^\dagger , which are weakly regular and $(d^\#)^\dagger = (d^\dagger)^\#$ (see [19, Theorem 2.8 and Theorem 2.15]).

Suppose that W is \mathcal{S} -weakly complementable. Then $R(w_{12}) \subseteq R(d)$ (see Theorem 4.17) and $d(d^\dagger w_{12}) = w_{12}$. Therefore, $d^\dagger w_{12} \in L(\mathcal{S}^{[\perp]}, \mathcal{K})$ is the unique solution of the equation $dx = w_{12}$. Thus, by Theorem 4.17,

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - (d^\dagger w_{12})^\# d^\dagger w_{12} \end{pmatrix} \begin{matrix} \mathcal{S} \\ \mathcal{S}^{[\perp]} \end{matrix}.$$

If in addition $R(w_{11})$ is closed, then $R(d)$ is regular. Thus, $(d^\dagger)^\# d^\dagger w_{12}$ is well defined, $w_{11}^\dagger = (d^\#)^\dagger d^\dagger \in L(\mathcal{S})$ and

$$W_{/[\mathcal{S}]} = \begin{pmatrix} 0 & 0 \\ 0 & w_{22} - w_{12}^\# w_{11}^\dagger w_{12} \end{pmatrix} = W_{/[\mathcal{S}]}^{XM},$$

where $W_{/[\mathcal{S}]}^{XM}$ is the Schur complement of W to \mathcal{S} as defined by Mary. See [19, Theorem 2.20].

For a positive operator W in a Hilbert space \mathcal{H} and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, Pekarev [22] showed that the Schur complement $W_{/\mathcal{S}}$ of W to \mathcal{S} can be expressed as $W_{/\mathcal{S}} = W^{1/2}(I - P_{\mathcal{M}})W^{1/2}$ where $\mathcal{M} = \overline{W^{1/2}(\mathcal{S})}$. In [18], Pekarev's result was taken as an inspiration to extend the concept to the more general Krein space setting. In that paper a (bounded) selfadjoint operator W is said to have the unique factorization property (UFP) if for any two Bognár-Krámlí factorizations of $W = D_1 D_1^\# = D_2 D_2^\#$, there is an isomorphism U such that $D_1 = D_2 U$.

For $W \in L(\mathcal{H})^{[s]}$ with the UFP and a closed subspace $\mathcal{S} \subseteq \mathcal{H}$, consider $\mathcal{M} = \overline{D^\#(\mathcal{S})}$ and suppose that \mathcal{M} is a regular subspace of \mathcal{K} . The Schur complement of W to \mathcal{S} is then defined in that paper as

$$W_{/[\mathcal{S}]}^{M-MP} = D(I - Q)D^\#,$$

where Q is the selfadjoint projection onto \mathcal{M} .

The next result shows that, when W is \mathcal{S} -complementable the regularity of \mathcal{M} can be omitted. If $P_{\mathcal{M}/\mathcal{M}^{[\perp]}}$ is the *projection-like* operator with domain $\mathcal{M} [\dot{+}] \mathcal{M}^{[\perp]}$, range \mathcal{M} and nullspace $\mathcal{M}^{[\perp]}$, then the operator $D(I - P_{\mathcal{M}/\mathcal{M}^{[\perp]}})D^\#$ is well defined and bounded. In this case,

$$D(I - P_{\mathcal{M}/\mathcal{M}^{[\perp]}})D^\# = W_{/[\mathcal{S}]}.$$

Since any bounded selfadjoint operator W can be written in the form $W = DD^\#$ with $D : \mathcal{K} \rightarrow \mathcal{H}$ injective, \mathcal{K} a Krein space, it follows that W need not have the UFP.

Proposition 4.19. *Let $W \in L(\mathcal{H})^{[s]}$ and let \mathcal{S} be a closed subspace of \mathcal{H} . Suppose that $W = DD^\#$ with $D \in L(\mathcal{K}, \mathcal{H})$, $N(D) = \{0\}$ and W is \mathcal{S} -complementable. Let $\mathcal{M} = \overline{D^\#(\mathcal{S})}$. Then*

$$W_{/[\mathcal{S}]} = D(I - P_{\mathcal{M}/\mathcal{M}^{[\perp]}})D^\#.$$

Proof. Since W is \mathcal{S} -complementable, we have that $\mathcal{H} = \mathcal{S} + W^{-1}(\mathcal{S}^{[\perp]})$. Suppose that $W = DD^\#$ with $D \in L(\mathcal{K}, \mathcal{H})$ and $N(D) = \{0\}$. Then

$$R(D^\#) = D^\#(\mathcal{S}) [\dot{+}] R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]}.$$

Furthermore, the sum is direct because $\{0\} = R(D^\#)^{[\perp]} \supseteq D^\#(\mathcal{S})^{[\perp]} \cap \overline{D^\#(\mathcal{S})}$. Therefore,

$$R(D^\#) = D^\#(\mathcal{S}) [\dot{+}] R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]} \subseteq \mathcal{M} [\dot{+}] \mathcal{M}^{[\perp]}.$$

Let $T := P_{\mathcal{M}/\mathcal{M}^{[\perp]}} D^\#$; since $R(D^\#) \subseteq \text{Dom}(P_{\mathcal{M}/\mathcal{M}^{[\perp]}}) = \mathcal{M} [\dot{+}] \mathcal{M}^{[\perp]}$, T is well defined. Let Q be any projection onto \mathcal{S} such that $WQ = Q^\#W$. Then, for every $x \in \mathcal{H}$,

$$Tx = TQx + T(I - Q)x.$$

Since $Qx \in \mathcal{S}$, $TQx = P_{\mathcal{M}/\mathcal{M}^{[\perp]}} D^\# Qx = D^\# Qx$. Also, $T(I - Q)x = 0$ because $D^\#(I - Q)x \in R(D^\#(I - Q)) = D^\#N(Q) \subseteq D^\#(W^{-1}(\mathcal{S}^{[\perp]})) = R(D^\#) \cap D^\#(\mathcal{S})^{[\perp]} \subseteq \mathcal{M}^{[\perp]} = N(T)$. Therefore,

$$T = P_{\mathcal{M}/\mathcal{M}^{[\perp]}} D^\# = D^\# Q \in L(\mathcal{H}).$$

Thus, by Corollary 4.14,

$$W_{/[\mathcal{S}]} = W(I - Q) = DD^\#(I - Q) = D(I - P_{\mathcal{M}/\mathcal{M}^{[\perp]}})D^\#.$$

□

Acknowledgements

We thank Professor M.A. Dritschel who read the original version of this paper and gave us useful advise.

We thank the anonymous referee for carefully reading our manuscript and helping us to improve this paper with several useful comments.

Maximiliano Contino and Alejandra Maestripieri were supported by CONICET PIP 0168. The work of Stefania Marcantognini was done during her stay at the Instituto Argentino de Matemática with an appointment funded by the CONICET. She is greatly grateful to the institute for its hospitality and to the CONICET for financing her post.

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