Optimal frame designs for multitasking devices with weight restrictions

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Abstract

Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in \mathbb{N}^m$ be a finite sequence (of dimensions) and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n}$ be a sequence of positive numbers (of weights), where $\mathbb{I}_k = \{1, \ldots, k\}$ for $k \in \mathbb{N}$. We introduce the $(\alpha, \mathbf{d})$-designs i.e., m-tuples $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ such that $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n}$ is a finite sequence in $\mathbb{C}^{d_j}$, $j \in \mathbb{I}_m$, and such that the sequence of non-negative numbers $(\|f_{ij}\|^2)_{j \in \mathbb{I}_m}$ forms a partition of α_i, $i \in \mathbb{I}_n$. We characterize the existence of $(\alpha, \mathbf{d})$-designs with prescribed properties in terms of majorization relations. We show, by means of a finite-step algorithm, that there exist $(\alpha, \mathbf{d})$-designs $\Phi^{\mathrm{op}} = (\mathcal{F}_j^{\mathrm{op}})_{j \in \mathbb{I}_m}$ that are universally optimal; that is, for every convex function $\varphi : [0, \infty) \to [0, \infty)$ then Φ^{op} minimizes the joint convex potential induced by φ among $(\alpha, \mathbf{d})$-designs, namely

$$\sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j^{\mathrm{op}}) \leq \sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j)$$

for every $(\alpha, \mathbf{d})$-design $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$, where $\mathrm{P}_\varphi(\mathcal{F}) = \mathrm{tr}(\varphi(S_\mathcal{F}))$; in particular, Φ^{op} minimizes both the joint frame potential and the joint mean square error among $(\alpha, \mathbf{d})$-designs. We show that in this case $\mathcal{F}_j^{\mathrm{op}}$ is a frame for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$. This corresponds to the existence of optimal encoding-decoding schemes for multitasking devices with energy restrictions.

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Contents

1 Introduction

A finite sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ of vectors in $\mathbb{C}^d$ is a frame for $\mathbb{C}^d$ if $\mathcal{F}$ is a (possibly redundant) system of generators for $\mathbb{C}^d$. In this case, it is well known that there exist finite sequences $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ in $\mathbb{C}^d$ - the so called duals of $\mathcal{F}$ - such that

$$f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle \, g_i = \sum_{i \in \mathbb{I}_n} \langle f, g_i \rangle \, f_i \quad \text{for} \quad f \in \mathbb{C}^d. \tag{1}$$

Thus, we can encode/decode the vector f in terms of the inner products $(\langle f, f_i \rangle)_{i \in \mathbb{I}_n} \in \mathbb{C}^n$: (see [7, 12, 13] and the references therein). The frame operator $S_{\mathcal{F}} \in \mathcal{M}_d(\mathbb{C})^+$ is given by

$$S_{\mathcal{F}} f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle \, f_i \quad \text{for} \quad f \in \mathbb{C}^d. \tag{2}$$

If $S_{\mathcal{F}}$ is invertible (i.e. if $\mathcal{F}$ is a frame) the canonical dual of $\mathcal{F}$ is given by $g_i = S_{\mathcal{F}}^{-1} f_i$ for $i \in \mathbb{I}_n$; this dual plays a central role in applications since it has several optimal (minimal) properties within the set of duals of $\mathcal{F}$. Unfortunately, the computation of the canonical dual depends on finding $S_{\mathcal{F}}^{-1}$, which is a challenging task from the numerical point of view. A way out of this problem is to consider those frames $\mathcal{F}$ for which $S_{\mathcal{F}}^{-1}$ is easy to compute (e.g. tight frames). In general, the numerical stability of the computation of $S_{\mathcal{F}}^{-1}$ depends on the spread of the eigenvalues of $S_{\mathcal{F}}$. In [4] Benedetto and Fickus introduced a convex functional called the frame potential of a sequence $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ given by

$$\text{FP}(\mathcal{F}) = \sum_{i, j \in \mathbb{I}_n} |\langle f_i, f_j \rangle|^2 \geq 0. \tag{3}$$

In [4] the authors showed that under some normalization conditions, $\text{FP}(\mathcal{F})$ provides an scalar measure of the spread of the eigenvalues of $\mathcal{F}$. More explicitly, the authors showed that the minimizers of FP among sequences $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ for which $\|f_i\| = 1$, $i \in \mathbb{I}_n$, are exactly the n/d-tight frames. It is worth pointing out that these minimizers are also optimal for transmission through noisy channels (in which erasures of the frame coefficients may occur, see [6, 20]).

In some applications of frame theory, we are drawn to consider frames $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ such that $\|f_i\|^2 = \alpha_i$, $i \in \mathbb{I}_n$, for some prescribed sequence $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{>0})^n$; this is known as the (classical) frame design problem. In practice, we can think of frames with prescribed norms as designs for encoding-decoding schemes to be applied by a device with some sort of energy restrictions (e.g. a device with limited access to energy power): in this case, control of the norms of the frame elements amounts to control the energy needed to apply the linear scheme.

It is then natural to wonder whether there are tight frames with norms prescribed by α. This question has motivated the study of the frame design problem (see [1, 8, 10, 11, 14, 15, 16, 21] and [17, 18, 23, 22, 25, 26, 27] for the more general frame completion problem with prescribed norms). It is well known that in some cases there are no tight frames in the class of sequences in $\mathbb{C}^d$ with norms prescribed by α; in these cases, it is natural to consider minimizers of the frame potential within this class, since the eigenvalues of the frame operator of such minimizers have minimal spread (thus, inducing more stable linear reconstruction processes). These considerations lead to the study of optimal designs with prescribed structure. In [9], the authors compute the structure of such minimizers and show it resembles that of tight frames.

It is worth pointing out that there are other measures of the spread of the spectra of frame operators (e.g. the mean squared error (MSE)). It turns out that both the MSE and the FP lie within the class of convex potentials introduced in [24]. It is shown in [24] that there are solutions $\mathcal{F}^{\text{op}}$ to the frame design problem which are **structural** in the sense that they are minimizers of every convex potential (e.g. MSE and FP) among frames with squared norms prescribed by α. A fundamental

tool to show the existence of such structural optimal frame designs is the so-called majorization in $\mathbb{R}^n$, which is a partial order used in matrix analysis (see [5]).

Motivated originally in the study of optimal finitely generated shift invariant systems with norm restrictions, for a finitely generated shift invariant subspace of $L^2(\mathbb{R}^d)$ (see [2, Section 4.2.] and also [3]), in the present paper we consider extensions of the (classical) frame design problems as follows: given a finite sequence (of dimensions) $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in \mathbb{N}^m$ and a sequence (of weights) $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in \mathbb{R}^n_{>0}$, we consider the set $\mathcal{D}(\alpha, \mathbf{d})$ of $(\alpha, \mathbf{d})$-designs. i.e. m-tuples $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ such that each $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n}$ is a finite sequence in $\mathbb{C}^{d_j}$, and

$$\sum_{j \in \mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n. \tag{4}$$

Notice that the restrictions on the norms above involve vectors in the (possibly different) spaces $f_{ij} \in \mathbb{C}^{d_j}$ for $j \in \mathbb{I}_m$. The $(\alpha, \mathbf{d})$-designs appear as the discretizations in the context of finitely generated shift invariant systems (see [2]). On the other hand, as in the case of frames with prescribed norms, $(\alpha, \mathbf{d})$-designs can be considered as encoding-decoding schemes to be applied by a multitasking device with some sort of energy restriction (e.g. due to isolation, or devices that are far from energy networks); in case $\mathcal{F}_j$ is a frame for $\mathbb{C}^{d_j}$ for $j \in \mathbb{I}_m$, then $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ induces linear schemes in the spaces $(\mathbb{C}^{d_j})_{j \in \mathbb{I}_m}$ that run in parallel. In this case, we want to control the overall energy needed (in each step of the encoding-decoding scheme) to apply simultaneously the m linear schemes, through the restrictions in Eq.(4).

It is natural to consider those $(\alpha, \mathbf{d})$-designs that give rise to the more stable multitasking processes. In order the measure the overall stability of the family $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ we can consider the joint frame potential of Φ or the joint MSE of Φ given by

$$\mathrm{FP}\,(\Phi) = \sum_{j \in \mathbb{I}_m} \mathrm{FP}\,(\mathcal{F}_j) \quad , \quad \mathrm{MSE}(\Phi) = \sum_{j \in \mathbb{I}_m} \mathrm{MSE}(\mathcal{F}_j) \quad \text{respectively .}$$

More generally, given a convex function $\varphi : [0, \infty) \to [0, \infty)$ we introduce the joint convex potential $\mathrm{P}_\varphi(\Phi)$ induced by φ (see Section 3.1 for details); this family of convex potentials (that contains the joint frame potential and joint MSE) provides natural measures of numerical stability of the family $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$. We remark that they are the same potentials considered in the previously mentioned context of finitely generated shift invariant systems in [2].

Given $(\alpha, \mathbf{d})$ as above, in this work we characterize the sequences of positive operators $S_j \in \mathcal{M}_{d_j}(\mathbb{C})^+$ for $j \in \mathbb{I}_m$, for which there exist $(\alpha, \mathbf{d})$-designs $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ such that $S_{\mathcal{F}_j} = S_j$, for $j \in \mathbb{I}_m$. Our characterization is obtained in terms of the spectra of the operators S_j and majorization relations, and it extends the well known solution of the classical frame design problem.

Then, we construct $(\alpha, \mathbf{d})$-designs Φ^{op} that are optimal in $\mathcal{D}(\alpha, \mathbf{d})$; in this setting, optimality is measured in terms of joint convex potentials, as discussed above. The kernel of this problem is the computation of the optimal spectral structure among sequences in $\mathcal{D}(\alpha, \mathbf{d})$.

We point out that our approach to these problems is constructive; indeed, we describe a finite step algorithm that produces designs $\Phi \in \mathcal{D}(\alpha, \mathbf{d})$ with prescribed spectral structure and optimal designs $\Phi^{\mathrm{op}} \in \mathcal{D}(\alpha, \mathbf{d})$ as above. Moreover, we include several numerical examples of optimal $(\alpha, \mathbf{d})$-designs obtained with the implementation of our algorithm in MATLAB. We further obtain the uniqueness of the spectral structure of optimal $(\alpha, \mathbf{d})$-designs. Moreover, we show that the optimal spectral structure does not depend on the particular choice of the convex potential. As a consequence, our results generalize the results in [4, 9, 24].

The existence of optimal $(\alpha, \mathbf{d})$-designs as above settles in the affirmative a conjecture in [2, Section 4.2.] regarding the existence of optimal finitely generated shift invariant systems (for a finitely

generated shift invariant subspace of $L^2(\mathbb{R}^d)$) with norm restrictions, with respect to convex potentials (see also [3]). On the other hand, our results have potential applications in comunication theory, e.g. in the study of the capacity of Multiple Input Multiple Output (MIMO) Additive White Gaussian Noise (AWGN) channels (see [19]).

The paper is organized as follows. In Section 2 we recall the notion of majorization together with some fundamental results about this pre-order. We also include some notions and results related with finite frame theory and convex potentials. In Section 3 we formalize the notion of $(\alpha, \mathbf{d})$-designs and describe in detail our main goals. In Section 3.2 we state and prove our main results, that include an effective characterization of the existence of $(\alpha, \mathbf{d})$-designs with prescribed spectral structure as well as the existence of (universal) optimal designs. The paper ends with Section 4, in which we present some general comments about the problems we studied, and several numerical examples that exhibit the properties of the optimal $(\alpha, \mathbf{d})$-designs computed with a finite step algorithm.

2 Preliminaries

In this section we introduce the notation, terminology and results from matrix analysis and frame theory that we will use throughout the paper. General references for these results are the texts [5] and [7, 12, 13]. In what follows we adopt the following

Notation and terminology. We let $\mathcal{M}_{k,d}(\mathcal{S})$ be the set of $k \times d$ matrices with coefficients in $\mathcal{S} \subset \mathbb{C}$ and write $\mathcal{M}_{d,d}(\mathbb{C}) = \mathcal{M}_d(\mathbb{C})$ for the algebra of $d \times d$ complex matrices. We denote by $\mathcal{H}(d) \subset \mathcal{M}_d(\mathbb{C})$ the real subspace of selfadjoint matrices and by $\mathcal{M}_d(\mathbb{C})^+ \subset \mathcal{H}(d)$ the cone of positive semidefinite matrices. We let $\mathcal{U}(d) \subset \mathcal{M}_d(\mathbb{C})$ denote the group of unitary matrices. For $d \in \mathbb{N}$, let $\mathbb{I}_d = \{1, \ldots, d\}$ and let $\mathbb{1}_d = (1)_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ be the vector with all its entries equal to 1.

Given $x = (x_i)_{i \in \mathbb{I}_d} \in \mathbb{R}^d$ we denote by $x^\downarrow = (x_i^\downarrow)_{i \in \mathbb{I}_d}$ (respectively $x^\uparrow = (x_i^\uparrow)_{i \in \mathbb{I}_d}$) the vector obtained by rearranging the entries of x in non-increasing (respectively non-decreasing) order. We denote by $(\mathbb{R}^d)^\downarrow = \{x^\downarrow : x \in \mathbb{R}^d\}$, $(\mathbb{R}_{\geq 0}^d)^\downarrow = \{x^\downarrow : x \in \mathbb{R}_{\geq 0}^d\}$ and analogously for $(\mathbb{R}^d)^\uparrow$ and $(\mathbb{R}_{\geq 0}^d)^\uparrow$.

Given a matrix $A \in \mathcal{H}(d)$ we denote by $\lambda(A) = \lambda^\downarrow(A) = (\lambda_i(A))_{i \in \mathbb{I}_d} \in (\mathbb{R}^d)^\downarrow$ the eigenvalues of A counting multiplicities and arranged in non-increasing order, and by $\lambda^\uparrow(A)$ the same vector but ordered in non-decreasing order. If $x, y \in \mathbb{C}^d$ we denote by $x \otimes y \in \mathcal{M}_d(\mathbb{C})$ the rank-one matrix given by $(x \otimes y) z = \langle z, y \rangle\, x$, for $z \in \mathbb{C}^d$.

2.1 Majorization

Next we recall the notion of majorization between vectors, that will play a central role throughout our work.

Definition 2.1.   1. Let $x, y \in \mathbb{R}^d$. We say that x is *submajorized* by y, and write $x \prec_w y$, if

$$\sum_{i \in \mathbb{I}_j} x_i^\downarrow \leq \sum_{i \in \mathbb{I}_j} y_i^\downarrow \quad \text{for every} \quad 1 \leq j \leq d\,.$$

If $x \prec_w y$ and $\operatorname{tr} x = \sum_{i \in \mathbb{I}_d} x_i = \sum_{i \in \mathbb{I}_d} y_i = \operatorname{tr} y$, then x is *majorized* by y, and write $x \prec y$.

2. Let $x \in \mathbb{R}_{\geq 0}^n$ and $y \in \mathbb{R}_{\geq 0}^d$ with $n > d$. Then we define the notions of $\prec$ and $\prec_w$ between the vectors x and y (of different size) by changing y by $y \oplus 0_{n-d} := (y, 0, \ldots, 0) \in \mathbb{R}^n$. Then

$$x \prec y \quad \text{if} \quad \sum_{i \in \mathbb{I}_n} x_i = \sum_{i \in \mathbb{I}_d} y_i \quad \text{and} \quad \sum_{i \in \mathbb{I}_j} x_i^\downarrow \leq \sum_{i \in \mathbb{I}_j} y_i^\downarrow \quad \text{for} \quad 1 \leq j \leq d\,. \tag{5}$$

and similarly one defines $y \prec x$.   △

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It is well known that majorization is related with the class $\mathcal{DS}(d)$ of doubly stochastic matrices i.e., formed by $D \in \mathcal{M}_d(\mathbb{C})$ with real non-negative entries such that each row sum and column sum equals one.

Theorem 2.2 (See [5]). Let $x, y \in \mathbb{R}^d$. Then

$$x \prec y \quad \Longleftrightarrow \quad \text{there exists} \quad D \in \mathcal{DS}(d) \quad \text{such that} \quad x = Dy \ . \qquad \square$$

Remark 2.3. Let $x, y \in \mathbb{R}^d$ be such that $x \prec y$. Using [5, Theorem II.1.10] we get a finite step algorithm (based on the so-called T-transformations) that constructs $D \in \mathcal{DS}(d)$ such that $x = Dy$.

Majorization is intimately related with tracial inequalities of convex functions. The following result summarizes these relations (see for example [5]): $\triangle$

Theorem 2.4. Let $x, y \in \mathbb{R}^d$. If $\varphi : I \to \mathbb{R}$ is a convex function defined on an interval $I \subseteq \mathbb{R}$ such that $x, y \in I^d$ then:

1. If $x \prec y$, then $\operatorname{tr} \varphi(x) \overset{\text{def}}{=} \sum_{i \in \mathbb{I}_d} \varphi(x_i) \leq \sum_{i \in \mathbb{I}_d} \varphi(y_i) = \operatorname{tr} \varphi(y)$.

2. If only $x \prec_w y$, but φ is an increasing convex function, then still $\operatorname{tr} \varphi(x) \leq \operatorname{tr} \varphi(y)$.

3. If $x \prec y$ and φ is a strictly convex function such that $\operatorname{tr} \varphi(x) = \operatorname{tr} \varphi(y)$ then, $x^{\downarrow} = y^{\downarrow}$. $\square$

2.2 Frames and convex potentials

In what follows we adopt the following

Notation and terminology: let $\mathcal{F} = \{f_i\}_{i \in \mathbb{I}_n}$ be a finite sequence in $\mathbb{C}^d$. Then,

1. $T_{\mathcal{F}} \in \mathcal{M}_{d,n}(\mathbb{C})$ is the synthesis operator given by $T_{\mathcal{F}} x = \sum_{i \in \mathbb{I}_n} x_i f_i$, for $x = (x_i)_{i \in \mathbb{I}_n} \in \mathbb{C}^n$.

2. $T_{\mathcal{F}}^* \in \mathcal{M}_{n,d}(\mathbb{C})$ is the analysis operator and it is given by $T_{\mathcal{F}}^* f = (\langle f, f_i \rangle)_{i \in \mathbb{I}_n}$, for $f \in \mathbb{C}^d$.

3. $S_{\mathcal{F}} \in \mathcal{M}_d(\mathbb{C})^+$ denotes the frame operator of $\mathcal{F}$ and it is given by $S_{\mathcal{F}} = T_{\mathcal{F}} T_{\mathcal{F}}^*$. Hence,

$$S_{\mathcal{F}} f = \sum_{i \in \mathbb{I}_n} \langle f, f_i \rangle f_i = \sum_{i \in \mathbb{I}_n} (f_i \otimes f_i) f \quad \text{for} \quad f \in \mathbb{C}^d \ .$$

4. We say that $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if it spans $\mathbb{C}^d$; equivalently, $\mathcal{F}$ is a frame for $\mathbb{C}^d$ if $S_{\mathcal{F}}$ is a positive invertible operator acting on $\mathbb{C}^d$.

In several applied situations it is desired to construct a finite sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in (\mathbb{C}^d)^n$, in such a way that the spectra of the frame operator of $\mathcal{G}$ is given by some $\lambda \in (\mathbb{R}_{\geq 0}^d)^{\downarrow}$ and the squared norms of the frame elements are prescribed by a sequence of positive numbers $\alpha = (\alpha_i)_{i \in \mathbb{I}_n}$. This is known as the (classical) frame design problem and it has been studied by several research groups (see for example [1, 8, 10, 11, 14, 15, 16, 21]). The following result characterizes the existence of such frame designs in terms of majorization relations.

Theorem 2.5 ([1, 23]). Let $\lambda \in \mathbb{R}_{\geq 0}^d$ and consider $a = (a_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{>0}^n)^{\downarrow}$. Then there exists a sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n}$ in $\mathbb{C}^d$ such that $\lambda(S_{\mathcal{G}}) = \lambda^{\downarrow}$ and $\|g_i\|^2 = a_i$ for $i \in \mathbb{I}_n$ if and only if $a \prec \lambda$. $\square$

The previous result shows the flexibility of structured frame designs, which is important in applied situations. Also, numerical stability of the encoding-decoding scheme induced by a frame plays a role in applications; hence, a central problem in this area is to described the structured frame designs that maximize the stability of their encoding-decoding scheme. One of the most important

(scalar) measures of stability is the so-called frame potential introduced by Benedetto and Fickus in [4] given by

$$\mathrm{FP}\,(\mathcal{F}) = \sum_{i,\,j\in\mathbb{I}_n} |\langle f_i\,,\,f_j\rangle|^2 = \mathrm{tr}(S_{\mathcal{F}}^2) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n\,.$$

Benedetto and Fickus have shown that (under certain normalization conditions) minimizers of the frame potential induce the most stable encoding-decoding schemes. More generally, we can measure the stability of the scheme induced by the sequence $\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n$ in terms of convex potentials. In order to introduce these potentials we consider the sets

$$\mathrm{Conv}(\mathbb{R}_{\geq 0}) = \{\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\,:\,\varphi \text{ is a convex function }\}$$

and $\mathrm{Conv}_{\mathrm{s}}(\mathbb{R}_{\geq 0}) = \{\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0}) : \varphi \text{ is strictly convex }\}$.

Definition 2.6. Following [24] we consider the convex potential P_φ associated to $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$, given by

$$\mathrm{P}_\varphi(\mathcal{F}) \;= \mathrm{tr}\,\varphi(S_{\mathcal{F}}) = \sum_{i\in\mathbb{I}_d} \varphi(\lambda_i(S_{\mathcal{F}})) \quad \text{for} \quad \mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n\,,$$

where the matrix $\varphi(S_{\mathcal{F}})$ is defined by means of the usual functional calculus. $\triangle$

Convex potentials allow us to model several well known measures of stability considered in frame theory. For example, in case $\varphi(x) = x^2$ for $x \in \mathbb{R}_{\geq 0}$ then P_φ is the Benedetto-Fickus frame potential; in case $\varphi(x) = x^{-1}$ for $x \in \mathbb{R}_{>0}$ then P_φ is known as the mean squared error (MSE).

Going back to the problem of stable designs, it is worth pointing out the existence of structured designs that are optimal with respect to every convex potential. Indeed, given $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}_{\geq 0}^n$ and $d \in \mathbb{N}$ with $d \leq n$, the α-torus is the set:

$$\mathcal{B}_{\alpha,\,d} = \{\mathcal{F} = \{f_i\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^d)^n\,:\,\|f_i\|^2 = \alpha_i\,,\,\,i \in \mathbb{I}_n\}\,. \tag{6}$$

We endow $\mathcal{B}_{\alpha,\,d}$ (which is a product space) with the product metric. The structure of (local) minimizers of convex potentials in $\mathcal{B}_{\alpha,\,d}$ has been extensively studied. The first results were obtained for the frame potential in [4] and in a more general context in [9]. The case of general convex potentials was studied in [17, 18, 22, 23, 24, 25, 26, 27] (in some cases in the more general setting of frame completion problems with prescribed norms).

3 On $(\alpha,\,\mathbf{d})$-design problems

We begin this section by introducing notation and terminology that allow us to model the $(\alpha,\,\mathbf{d})$-design problems, including the optimal design problem with prescribed weights. Then, we state and prove our main results for $(\alpha,\,\mathbf{d})$-designs.

3.1 Modeling the problem

Now we generalize the α-torus to the multi-frames:

Definition 3.1. Let $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in \mathbb{R}_{>0}^n$ and $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ be such that $d_1 \leq n$.

1. An $(\alpha,\,\mathbf{d})$-design is an m-tuple

$$\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}\,,\quad \text{where} \quad \mathcal{F}_j = \{f_{ij}\}_{i\in\mathbb{I}_n} \in (\mathbb{C}^{d_j})^n \quad \text{for} \quad j \in \mathbb{I}_m$$

and such that $\sum\limits_{j\in\mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i$, for $i \in \mathbb{I}_n$.

2. We denote by $\mathcal{D}(\alpha\,,\mathbf{d})$ the set of all $(\alpha\,,\mathbf{d})$-designs. We point out that (in order to simplify our description of the model) we consider $(\alpha\,,\mathbf{d})$-designs in a broad sense; namely, if $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\mathbf{d})$ then $\mathcal{F}_j$ is not necessarily a frame for $\mathbb{C}^{d_j}$, for $j \in \mathbb{I}_m$.

3. In order to compare the overall stability of the linear encoding-decoding schemes induced by an $(\alpha\,,\mathbf{d})$-design we introduce the following potentials: Given $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ we consider the joint potential induced by φ on $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\mathbf{d})$ given by

$$\mathrm{P}_\varphi(\Phi) = \sum_{j\in\mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \sum_{j\in\mathbb{I}_m} \mathrm{tr}\ \varphi(S_{\mathcal{F}_j}) = \sum_{j\in\mathbb{I}_m} \sum_{i\in\mathbb{I}_{d_j}} \varphi(\lambda_i(S_{\mathcal{F}_j}))\,. \qquad \triangle$$

Consider the notation and terminology of Definition 3.1. We can now describe the main problems that we consider in this work as follows:

P1. Determine necessary and sufficient conditions for the existence of $(\alpha\,,\mathbf{d})$-designs with prescribed spectral structure and describe algorithmic procedures to construct such designs, in case they exist.

P2. Given $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ determine the existence and structure of those $\Phi_\varphi \in \mathcal{D}(\alpha\,,\mathbf{d})$ that minimize the joint convex potential P_φ in $\mathcal{D}(\alpha\,,\mathbf{d})$, that is

$$\mathrm{P}_\varphi(\Phi_\varphi) = \min\{\mathrm{P}_\varphi(\Phi):\ \Phi \in \mathcal{D}(\alpha\,,\mathbf{d})\}\,. \qquad (7)$$

In this case we say that Φ_φ is an P_φ-optimal $(\alpha\,,\mathbf{d})$-design. Determine whether these P_φ-optimal $(\alpha\,,\mathbf{d})$-designs depend on the particular choice of P_φ, for strictly convex functions $\varphi \in \mathrm{Conv}_\mathrm{s}(\mathbb{R}_{\geq 0})$.

P3. Describe an algorithmic procedure that computes P_φ-optimal $(\alpha\,,\mathbf{d})$-designs.

P4. Characterize the P_φ-optimal $(\alpha\,,\mathbf{d})$-designs in terms of some structural properties.

We will solve problems P1.-P4. In particular, we will show that if $\Phi_\varphi = (\mathcal{F}_j)_{j\in\mathbb{I}_m}$ is an P_φ-optimal $(\alpha\,,\mathbf{d})$-design for $\varphi \in \mathrm{Conv}_\mathrm{s}(\mathbb{R}_{\geq 0})$ then, $\mathcal{F}_j$ is a frame for $\mathbb{C}^{d_j}$ for each $j \in \mathbb{I}_m$ (see Section 3.2). Moreover, we will show that P_φ-optimal $(\alpha\,,\mathbf{d})$-designs do not depend on the particular choice of P_φ, for strictly convex functions $\varphi \in \mathrm{Conv}_\mathrm{s}(\mathbb{R}_{\geq 0})$.

3.2 Main results

In this section we state and prove our main results; these include the existence of $(\alpha\,,\mathbf{d})$-designs with prescribed spectral structure, and designs with some special structure which turn out to be optimal designs in the sense of Problem (P2). We further show the uniqueness of the spectral structure of optimal $(\alpha\,,\mathbf{d})$-designs.

Our first main result characterizes the existence of $(\alpha\,,\mathbf{d})$-designs with prescribed spectral structure. We formalize problem P1. in terms of the following

Definition 3.2. Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ be such that $d_1 \leq n$. Let

$$\mu_j = (\mu_{i,j})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}_{\geq 0}^{d_j})^\downarrow \quad \text{for} \quad j \in \mathbb{I}_m \quad \text{and set} \quad \mathcal{M} := \{\mu_j\}_{j\in\mathbb{I}_m} \in \prod_{j\in\mathbb{I}_m} (\mathbb{R}_{\geq 0}^{d_j})^\downarrow\,.$$

We say that the pair $(\alpha\,,\mathcal{M})$ is **admissible** if there exists $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\mathbf{d})$ such that

$$\lambda(S_{\mathcal{F}_j}) = \mu_j \quad \text{for every} \quad j \in \mathbb{I}_m\,.$$

In this case we denote $\mathcal{M} = \mathcal{M}_\Phi$. $\hfill \triangle$

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In order to obtain an effective characterization of admissibility, we introduce the notion of (α, m)-weight partition matrix.

Remark 3.3. Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ be such that $d_1 \leq n$.

1. We consider the set of (α, m)-weight partitions given by

$$P_{\alpha, m} = \{A = (a_{ij})_{i \in \mathbb{I}_n, j \in \mathbb{I}_m} \in \mathcal{M}_{n,m}(\mathbb{C}) : a_{ij} \geq 0 \quad \text{and} \quad \sum_{j \in \mathbb{I}_m} a_{ij} = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n \}.$$

2. A sequence $(\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d}) \iff$ its matrix of weights

$$A = \left\{ \|f_{ij}\|^2 \right\}_{i \in \mathbb{I}_n \ j \in \mathbb{I}_m} \in P_{\alpha, m}.$$

Lemma 3.4 (A first characterization of admissible pairs). *Consider a pair $(\alpha, \mathcal{M})$ as in Definition 3.2 above. Then the following conditions are equivalent.*

1. *The pair $(\alpha, \mathcal{M})$ is admissible.*

2. *There exists a matrix $A \in P_{\alpha, m}$ such that*

$$c_j(A) \prec \mu_j \quad \text{for every} \quad j \in \mathbb{I}_m, \tag{8}$$

where $c_j(A) \in \mathbb{R}^n_{\geq 0}$ denotes the j-th column of A.

*In particular, the set of sequences $\mathcal{M}$ such that $(\alpha, \mathcal{M})$ is admissible is **convex** in $\prod_{j \in \mathbb{I}_m} \mathbb{R}^{d_j}$.*

Proof. If the pair $(\alpha, \mathcal{M})$ is admissible, let $(\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ be such that $\lambda(S_{\mathcal{F}_j}) = \mu_j$, for $j \in \mathbb{I}_m$. Let $A \in P_{\alpha, m}$ be given by $c_j(A) = (\|f_{ij}\|^2)_{i \in \mathbb{I}_n}$, where $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n}$, for $j \in \mathbb{I}_m$. Then, by the Theorem 2.5, $c_j(A) \prec \mu_j$ for $j \in \mathbb{I}_m$.

Conversely, assume that there exists $A \in P_{\alpha,m}$ with $c_j(A) \prec \mu_j$, for $j \in \mathbb{I}_m$. Then, again by Theorem 2.5, for each $j \in \mathbb{I}_m$ there exists $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ such that $c_j(A) = (\|f_{ij}\|^2)_{i \in \mathbb{I}_n}$ and $\lambda(S_{\mathcal{F}_j}) = \mu_j$. Hence, $(\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ which shows that the pair $(\alpha, \mathcal{M})$ is admissible. $\square$

The following result provides an effective method to determine whether a given pair is admissible or not.

Theorem 3.5. *Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}^n_{\geq 0})^\downarrow$ be such that $d_1 \leq n$. Given a sequence $\mathcal{M} = \{\mu_j\}_{j \in \mathbb{I}_m} \in \prod_{j \in \mathbb{I}_m} (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$, set*

$$\sigma_{\mathcal{M}} := \sum_{j \in \mathbb{I}_m} (\mu_j \oplus 0_{d_1 - d_j}) \in (\mathbb{R}^{d_1}_{\geq 0})^\downarrow. \tag{9}$$

Then, the pair $(\alpha, \mathcal{M})$ is admissible if and only if $\alpha \prec \sigma_{\mathcal{M}}$.

Proof. Assume first that the pair $(\alpha, \mathcal{M})$ is admissible. Then, by Lemma 3.4, there exists $A \in P_{\alpha, m}$ such that $c_j(A) = (a_{ij})_{i \in \mathbb{I}_n} \prec \mu_j = (\mu_{i,j})_{i \in \mathbb{I}_{d_j}} \in (\mathbb{R}^{d_j}_{\geq 0})^\downarrow$, for $j \in \mathbb{I}_m$. Fix an index $j \in \mathbb{I}_m$. Note that $n \geq d_1 = \min\{n, d_1\}$. Hence, by hypothesis, for $k \in \mathbb{I}_{d_1}$ we have that

$$\sum_{i=1}^{k} a_{ij} \leq \sum_{i=1}^{k} (c_j(A)^\downarrow)_i \leq \sum_{i=1}^{\min\{k, d_j\}} \mu_{i,j} \implies \sum_{i=1}^{k} \alpha_i = \sum_{j \in \mathbb{I}_m} \sum_{i=1}^{k} a_{ij} \leq \sum_{j \in \mathbb{I}_m} \sum_{i=1}^{\min\{k, d_j\}} \mu_{i,j}. \tag{10}$$

Notice that

$$\sigma_{\mathcal{M}} = (\sigma_i)_{i \in \mathbb{I}_{d_1}} = \sum_{j \in \mathbb{I}_m} (\mu_j \oplus 0_{d_1 - d_j}) \implies \sum_{j \in \mathbb{I}_m} \sum_{i=1}^{\min\{k, d_j\}} \mu_{i,j} = \sum_{i=1}^{k} \sigma_i \ , \quad k \in \mathbb{I}_{d_1} \ .$$

Then, Eq. (10) shows that $\alpha \prec \sigma_{\mathcal{M}}$ (the equality $\operatorname{tr} \sigma_{\mathcal{M}} = \operatorname{tr} \alpha$ is clear).

For the converse, in order to show that $(\alpha, \mathcal{M})$ is admissible, we prove that there exists $A \in P_{\alpha,m}$ such that $c_j(A) \prec \mu_j$, for $j \in \mathbb{I}_m$ (see Lemma 3.4). Indeed, since $\alpha \prec \sigma_{\mathcal{M}}$, by Theorem 2.2 there exists a doubly stochastic matrix $D \in \mathcal{DS}(n)$, such that $D(\sigma_{\mathcal{M}} \oplus 0_{n-d_1}) = \alpha$. Consider $A \in \mathcal{M}_{n,m}(\mathbb{R}_{\geq 0})$ determined by $c_j(A) = D(\mu_j \oplus 0_{n-d_j})$, for $j \in \mathbb{I}_m$. Notice that in this case by construction, $c_j(A) \prec \mu_j \oplus 0_{n-d_j} \implies c_j(A) \prec \mu_j$ for $j \in \mathbb{I}_m$ and

$$A \, \mathbb{1}_m = \sum_{j \in \mathbb{I}_m} c_j(A) = \sum_{j \in \mathbb{I}_m} D(\mu_j \oplus 0_{n-d_j}) = D(\sigma_{\mathcal{M}} \oplus 0_{n-d_1}) = \alpha \ .$$

Thus, $A \in P_{\alpha,m}$ and the pair $(\alpha, \mathcal{M})$ is admissible. □

Remark 3.6 (Finite-step algorithm for constructing $(\alpha, \mathbf{d})$-designs with prescribed spectral structure). With the notation of Theorem 3.5, assume that $\alpha \prec \sigma_{\mathcal{M}}$. Hence, in this case the pair $(\alpha, \mathcal{M})$ is admissible. By Remark 2.3, there is a finite step algorithm that constructs $D \in \mathcal{DS}(n)$ such that $\alpha = D(\sigma_{\mathcal{M}} \oplus 0_{n-d_1})$. From the previous proof we see that if we consider $A \in \mathcal{M}_{n,m}(\mathbb{R}_{\geq 0})$ determined by $c_j(A) = D(\mu_j \oplus 0_{n-d_j})$, for $j \in \mathbb{I}_m$, then $A \in P_{\alpha,m}$. Moreover, by construction we have that $c_j(A) \prec \mu_j \oplus 0_{n-d_j}$ ($\implies c_j(A) \prec \mu_j$) for $j \in \mathbb{I}_m$.

We can now apply finite step algorithms (such as the one-sided Bendel-Mickey algorithm, see [10, 11, 14, 16]) and obtain $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n} \in (\mathbb{C}^{d_j})^n$ such that $(\|f_{ij}\|^2)_{i \in \mathbb{I}_n} = c_j(A)$ and such that $\lambda(S_{\mathcal{F}_j}) = \mu_j$ for $j \in \mathbb{I}_m$. Therefore, we get $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ such that $\mathcal{M} = \mathcal{M}_\Phi$ in a constructive way. △

The following definition introduces an m-tuple of vectors (of eigenvalues), associated to every $(\alpha, \mathbf{d})$-design, and a large vector constructed from the juxtaposition of the elements of this set. These are going to be useful in proving the existence of optimal designs in terms of majorization relations, related with problem P2 above. (see Theorem 3.14 below).

Definition 3.7. Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^\downarrow$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^\downarrow$ be such that $d_1 \leq n$. Let $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ and let $S_j = S_{\mathcal{F}_j} \in \mathcal{M}_{d_j}(\mathbb{C})^+$ denote the frame operators of $\mathcal{F}_j$, for $j \in \mathbb{I}_m$. We define

$$\mathcal{M}_\Phi = \{\lambda(S_j)\}_{j \in \mathbb{I}_m} \in \prod_{j \in \mathbb{I}_m} (\mathbb{R}_{\geq 0}^{d_j})^\downarrow \quad \text{and} \quad \Lambda_\Phi = \big(\lambda(S_1), \ldots, \lambda(S_m)\big) \in \mathbb{R}_{\geq 0}^d \ , \qquad (11)$$

where $d = \operatorname{tr} \mathbf{d} = \sum_{j \in \mathbb{I}_m} d_j$ and each $\lambda(S_j) \in (\mathbb{R}_{\geq 0}^{d_j})^\downarrow$ is the vector of eigenvalues of S_j, for $j \in \mathbb{I}_m$.

Recall that given a generic sequence $\mathcal{M} \in \prod_{j \in \mathbb{I}_m} (\mathbb{R}_{\geq 0}^{d_j})^\downarrow$, we say that the pair $(\alpha, \mathcal{M})$ is admissible if there exists $\Phi \in \mathcal{D}(\alpha, \mathbf{d})$ such that $\mathcal{M} = \mathcal{M}_\Phi$, which in turns is equivalent to $\sigma_{\mathcal{M}} \prec \alpha$. We also remark that Λ_Φ is not an ordered vector. We shall use the specific order of its entries given in Eq.(11) in order to preserve the convexity properties given by Lemma 3.4. △

Remark 3.8. Consider the notation in Definition 3.7. If $\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$ and P_φ denotes the joint convex potential induced by φ (see Definition 2.6) then,

$$\mathrm{P}_\varphi(\Phi) = \sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \sum_{j \in \mathbb{I}_m} \operatorname{tr}(\varphi(\lambda(S_j))) = \sum_{\ell \in \mathbb{I}_{|d|}} \varphi((\Lambda_\Phi)_\ell) =: \operatorname{tr}(\varphi(\Lambda_\Phi)) \ . \qquad (12)$$

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Therefore, by Theorem 2.4 and Eq. (12), the existence of an (optimal) $(\alpha, \mathbf{d})$-design satisfying Eq. (7) for every $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ is equivalent to the existence of $\Psi = (\mathcal{G}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ such that

$$\Lambda_\Psi \prec \Lambda_\Phi \quad \text{for every} \quad \Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d}). \qquad \triangle$$

Remark 3.9. Consider the notation in Definition 3.7. In the rest of this section we shall show the existence of $(\alpha, \mathbf{d})$-designs $\Phi^{\mathrm{op}} = (\mathcal{F}_j^{\mathrm{op}})_{j \in \mathbb{I}_m}$ that are optimal with respect to every joint convex potential (see Theorem 3.15). It turns out that these optimal designs have some special features.

In this remark we describe the special structure of the associated sequence $\mathcal{M}_{\Phi^{\mathrm{op}}}$ (and introduce the necessary notation to describe this structure) in order to make more intelligible the next statements, which are intended to construct admissible pairs with this (optimal) structure. Let

$$\mu_j^{\mathrm{op}} = (\mu_{ij}^{\mathrm{op}})_{i \in \mathbb{I}_{d_j}} = \lambda(S_{\mathcal{F}_j^{\mathrm{op}}}) \in (\mathbb{R}_{\geq 0}^{d_j})^{\downarrow} \quad \text{for every} \quad j \in \mathbb{I}_m.$$

denote the eigenvalues of the frame operators of $\mathcal{F}_j^{\mathrm{op}}$. Then they must have the following structure: Each vector $\mu_j^{\mathrm{op}} \in (\mathbb{R}_{\geq 0}^{d_j})^{\downarrow}$ is a (truncated) copy of the first vector $\mu_1^{\mathrm{op}} \in (\mathbb{R}_{\geq 0}^{d_1})^{\downarrow}$, i.e.

$$\mu_{ij}^{\mathrm{op}} = \mu_{i1}^{\mathrm{op}} \quad \text{for every} \quad i \in \mathbb{I}_j \quad \text{and every} \quad j \in \mathbb{I}_m. \qquad (13)$$

In detail, let $\sigma(S_{\mathcal{F}_1^{\mathrm{op}}}) = \{\gamma_1, \ldots, \gamma_p\}$, with $\gamma_1 > \ldots > \gamma_p \geq 0$. Then there exist indexes $g_0 = 0 < g_1 < \ldots < g_p = d_1$ (that we shall construct looking for admissibility) such that

$$\{i \in \mathbb{I}_{d_1} : \mu_{i1} = \gamma_\ell\} = \{i : g_{\ell-1} + 1 \leq i \leq g_\ell\} \quad \text{for} \quad \ell \in \mathbb{I}_p.$$

We define the following constants, which only depend on the data $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$:

$$h_i := \#\{j \in \mathbb{I}_m : d_j \geq i\} \quad \text{for} \quad i \in \mathbb{I}_{d_1}. \qquad (14)$$

Notice that,

$$h := (h_i)_{i \in \mathbb{I}_{d_1}} = \sum_{i=1}^{m} \mathbb{1}_{d_i} \oplus 0_{d_1 - d_i}. \qquad (15)$$

Using the relations in Eq. (13) we get that

$$\Lambda_{\Phi^{\mathrm{op}}}^{\downarrow} = (\gamma_\ell \mathbb{1}_{r_\ell})_{\ell \in \mathbb{I}_p} \quad \text{where} \quad r_\ell = \sum_{i=g_{\ell-1}+1}^{g_\ell} h_i, \quad \ell \in \mathbb{I}_p. \qquad (16)$$

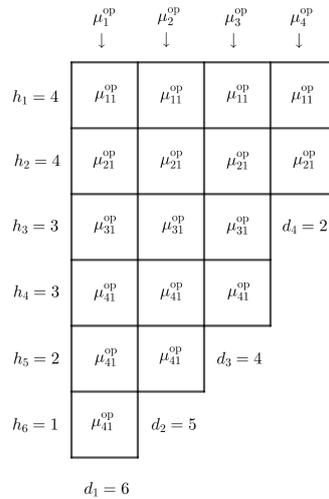We give an example of this situation for $m = 4$ and $\mathbf{d} = (6, 5, 4, 2)$ in Figure 1.

Figure 1: A graphic example of the structure of $(\mu_j^{\mathrm{op}})_{j \in \mathbb{I}_4}$.

For example, if we assume that

$$\mu_{11}^{\mathrm{op}} = \mu_{21}^{\mathrm{op}} = \mu_{31}^{\mathrm{op}} = \gamma_1 \ , \quad \mu_{41}^{\mathrm{op}} = \mu_{51}^{\mathrm{op}} = \gamma_2 \quad \text{and} \quad \mu_{61}^{\mathrm{op}} = \gamma_3 \quad \text{with} \quad \gamma_1 > \gamma_2 > \gamma_3$$

then we have: $g_0 = 0$, $g_1 = 3$, $g_2 = 5$ and hence, $r_1 = 11$, $r_2 = 5$, $r_3 = 1$; therefore, we compute $\Lambda_{\Phi^{\mathrm{op}}}^{\downarrow} = (\gamma_1 \, \mathbb{1}_{11} \, , \, \gamma_2 \, \mathbb{1}_5 \, , \, \gamma_3 \, \mathbb{1}_1) \in \mathbb{R}_{>0}^{17}$ in this case. $\triangle$

In order to obtain the our next main result, we consider the following

Notation 3.10. Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow}$ be such that $d_1 \leq n$. Let $h = (h_i)_{i \in \mathbb{I}_{d_1}}$ be defined as in Eq. (14). For $1 \leq s \leq t \leq d_1$, denote by $P_{s,t}$ and Q_t the ratios

$$P_{s,t} = \frac{\sum_{i=s}^{t} \alpha_i}{\sum_{i=s}^{t} h_i} \quad \text{and} \quad Q_t = \frac{\sum_{i=t}^{n} \alpha_i}{\sum_{i=t}^{d_1} h_i} \, . \tag{17}$$

The following result is a technical construction that we will use to build optimal $(\alpha \, , \mathbf{d})$-designs with a spectral picture as in Remark 3.9.

Theorem 3.11. *Let* $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ *and* $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow}$ *be such that* $d_1 \leq n$. *Consider the vector* $h = (h_i)_{i \in \mathbb{I}_{d_1}}$ *as in Eq. (15). Then, there exist*

$$p \in \mathbb{I}_{d_1} \quad \text{and} \quad g_1, \ldots, g_p \in \mathbb{N} \quad \text{with} \quad 0 = g_0 < g_1 < \cdots < g_p = d_1$$

such that, if we define $\gamma_i = P_{g_{i-1}+1, g_i}$, *for* $i \in \mathbb{I}_{p-1}$ *and* $\gamma_p = Q_{g_{p-1}+1}$ *according with Eq. (17), then*

1. $\gamma_1 > \ldots > \gamma_p > 0$;

2. *They satisfy the following "block" majorizations:*

$$(\gamma_i \, h_k)_{k=g_{i-1}+1}^{g_i} \succ (\alpha_i)_{k=g_{i-1}+1}^{g_i} \quad \text{for} \quad i \in \mathbb{I}_{p-1} \quad \text{and}$$

$$(\gamma_p \, h_k)_{k=g_{p-1}+1}^{d_1} \succ (\alpha_i)_{k=g_{p-1}+1}^{n} \, . \tag{18}$$

In particular,

$$(\gamma_1 \, \mathbb{1}_{g_1-g_0} \, , \, \gamma_2 \, \mathbb{1}_{g_2-g_1} \, , \, \ldots \, , \, \gamma_p \, \mathbb{1}_{g_p-g_{p-1}}) \circ h \succ \alpha \, , \tag{19}$$

where $\circ$ *denotes the entry-wise product.*

Proof. First note that d_1 clearly satisfies $Q_{d_1} \geq P_{d_1, d_1}$. Then we can define the index

$$s^* = \min\{j \in \mathbb{I}_{d_1} : Q_j \geq P_{j,k} \quad \text{for every} \quad j \leq k \leq d_1\} \tag{20}$$

We denote $c = Q_{s^*}$. Therefore, by Eq. (17),

$$c \sum_{i=s^*}^{k} h_i \geq \sum_{i=s^*}^{k} \alpha_i \quad \text{for every} \quad s^* \leq k \leq d_1 \, .$$

In other words, $(\alpha_k)_{k=s^*}^{n} \prec (h_k \, c)_{k=s^*}^{d_1}$. If $s^* = 1$ then we set $p = 1$, $g_0 = 0$, $g_1 = d_1$ and $\gamma_1 = c > 0$. Then items 1. and 2. of the statement are satisfied in this case.

Otherwise, $s^* > 1$ and we proceed to find the step g_1. First, we define γ_1:

$$\gamma_1 = \max\{P_{1,k} \, : \, 1 \leq k \leq s^* - 1\} \, ,$$

and then we define g_1 as:
$$g_1 = \max\{j \in \mathbb{I}_{s^*-1} : P_{1,j} = \gamma_1\}.$$

By construction, we obtain that
$$(\gamma_1 h_i)_{i=1}^{g_1} \succ (\alpha_i)_{i=1}^{g_1}.$$

Now, if $g_1 = s^* - 1$ then we set $p = 2$, $g_0 = 0$, $g_2 = d_1$ and $\gamma_2 = c > 0$.

Otherwise, $g_1 < s^* - 1$ (and having the index s^* fixed), we define g_2 in a similar way:

$$\gamma_2 = \max\{P_{g_1+1,k} : g_1 < k \leq s^* - 1\}$$

and then,
$$g_2 = \max\{g_1 + 1 \leq j \leq s^* - 1 : P_{g_1+1,j} = \gamma_2\}.$$

Again, by construction we have that

$$(\gamma_2 h_i)_{i=g_1+1}^{g_2} \succ (\alpha_i)_{i=g_1+1}^{g_2}.$$

We claim that $\gamma_1 > \gamma_2$. Indeed, suppose that $P_{g_1+1,g_2} = \gamma_2 \geq \gamma_1 = P_{1,g_1}$. Then,

$$
\begin{aligned}
P_{1,g_2} - P_{1,g_1} &= \frac{\sum_{i=1}^{g_1} \alpha_i + \sum_{i=g_1+1}^{g_2} \alpha_i}{\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i} - \frac{\sum_{i=1}^{g_1} \alpha_i}{\sum_{i=1}^{g_1} h_i} = \\
&= \frac{\left(\sum_{i=1}^{g_1} h_i\right)\left(\sum_{i=g_1+1}^{g_2} \alpha_i\right) - \left(\sum_{i=g_1+1}^{g_2} h_i\right)\left(\sum_{i=1}^{g_1} \alpha_i\right)}{\sum_{i=1}^{g_1} h_i \left(\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i\right)} \\
&= \frac{\left(\sum_{i=1}^{g_1} h_i\right)\left(\sum_{i=g_1+1}^{g_2} h_i\right)(\gamma_2 - \gamma_1)}{\sum_{i=1}^{g_1} h_i \left(\sum_{i=1}^{g_1} h_i + \sum_{i=g_1+1}^{g_2} h_i\right)} \geq 0.
\end{aligned}
$$

Hence, $P_{1,g_2} = P_{1,g_1} = \gamma_1$ which contradicts the definition of g_1, so the claim is proved.

We can continue inductively with this process, that is, once we find $g_{k-1} < s^* - 1$ we compute first γ_k as the maximum among $P_{g_{k-1}+1,l}$, with $g_{k-1} + 1 \leq l \leq s^* - 1$ and then define $g_k \leq s^* - 1$ as the maximum index $g_{k-1} + 1 \leq l \leq s^* - 1$ such that $P_{g_{k-1}+1,l} = \gamma_k$. As before, this construction guarantees the corresponding block majorization.

Notice that in the last step, corresponding to the $p - 1$ iteration of the process, we necessarily have $g_{p-1} = s^* - 1$. Define $\gamma_p = Q_{g_{p-1}+1} = c > 0$ and $g_p = d_1$.

By construction, and the previous remarks we have that $\gamma_1 > \gamma_2 > \ldots > \gamma_{p-1}$ and item 18 is satisfied. It remains to prove that $\gamma_{p-1} > \gamma_p$.

Suppose, on the contrary, that $\gamma_p \geq \gamma_{p-1}$. Consider $\bar{c} = Q_{g_{p-2}+1}$. Clearly, $\bar{c}$ is a convex combination of γ_p and γ_{p-1}, say $\bar{c} = t\gamma_{p-1} + (1-t)\gamma_p$. In particular, $\gamma_{p-1} \leq \bar{c} \leq \gamma_p$. Therefore, we have that

$$P_{g_{p-2}+1,l} \leq \gamma_{p-1} \leq \bar{c}, \quad \text{for } g_{p-2} + 1 \leq l \leq g_{p-1} \tag{21}$$

Let $g_{p-1} + 1 \leq l < d_1$, and denote by $A = \sum_{i=g_{p-2}+1}^{g_{p-1}} h_i$, $B = \sum_{i=g_{p-1}+1}^{l} h_i$ and $C = \sum_{i=l+1}^{d_1} h_i$.

Notice that, with this notation, in the convex combination that generates $\bar{c}$, we have $t = \frac{A}{A+B+C}$. Then, since $\gamma_{p-1} \leq \gamma_p$,

$$\frac{A}{A+B}\gamma_{p-1} + \frac{B}{A+B}\gamma_p \leq \frac{A}{A+B+C}\gamma_{p-1} + \frac{B+C}{A+B+C}\gamma_p$$

Hence, since by definition of γ_p we have $P_{g_{p-1}+1,\,l} \leq \gamma_p$, we obtain

$$\frac{A}{A+B}\,\gamma_{p-1} + \frac{B}{A+B}\,P_{g_{p-1}+1,\,l} \leq \frac{A}{A+B+C}\,\gamma_{p-1} + \frac{B+C}{A+B+C}\,\gamma_p \tag{22}$$

since $\frac{A}{A+B}\,\gamma_{p-1} + \frac{B}{A+B}\,P_{g_{p-1}+1,\,l} = P_{g_{p-2}+1,\,l}$ and $\frac{A}{A+B+C}\,\gamma_{p-1} + \frac{B+C}{A+B+C}\,\gamma_p = \overline{c}$, we deduce

$$P_{g_{p-2}+1,\,l} \leq \overline{c} \;, \qquad \text{for} \quad g_{p-1} + 1 \leq l \leq d_1 \tag{23}$$

Therefore, Eqs. (21), (23) imply that, for $j = g_{p-2} + 1 < s^* = g_{p-1} + 1$,

$$P_{j,\,l} \leq Q_j \;, \qquad \text{for} \quad l = j,\ldots,d_1$$

which contradicts the construction of s^*. So we can conclude that $\gamma_{p-1} > \gamma_p$ and the theorem is proved. $\qquad\square$

Next, we introduce the following vectors associated to a pair $(\alpha\,,\mathbf{d})$.

Definition 3.12. Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow}$ be such that $d_1 \leq n$. Let $0 = g_0 < \cdots < g_p = d_1$ and $\gamma_1 > \cdots > \gamma_p > 0$ be as in Theorem 3.11. Then, we set:

1. $\mu_1^{\mathrm{op}} = (\mu_{i\,1}^{\mathrm{op}})_{i\in\mathbb{I}_{d_1}} = (\gamma_k \mathbb{1}_{g_k - g_{k-1}})_{k=1}^p \in (\mathbb{R}_{>0}^{d_1})^{\downarrow}$;

2. For $j > 1$ set $\mu_j^{\mathrm{op}} = (\mu_{i\,j}^{\mathrm{op}})_{i\in\mathbb{I}_{d_j}} \in (\mathbb{R}_{>0}^{d_j})^{\downarrow}$ such that $\mu_{i\,j}^{\mathrm{op}} = \mu_{i\,1}^{\mathrm{op}}$ for $i \in \mathbb{I}_{d_j}$.

3. Denote $\mathcal{M}^{\mathrm{op}} = \{\mu_j^{\mathrm{op}}\}_{j\in\mathbb{I}_m} \in \prod_{j\in\mathbb{I}_m} (\mathbb{R}_{>0}^{d_j})^{\downarrow}$.

At this point, the vector μ_1^{op} (and so the sequence $\mathcal{M}^{\mathrm{op}}$) depends on the choice of the indexes $0 = g_0 < \cdots < g_p = d_1$ from Theorem 3.11. Nevertheless, we shall see now that, when rearranged, $\mathcal{M}^{\mathrm{op}}$ has minimality properties for majorization, so that they are univocally determined. $\qquad\triangle$

Remark 3.13. Let $\gamma_1 \geq \ldots \geq \gamma_p \in \mathbb{R}$ and consider $\lambda = (\gamma_1 \mathbb{1}_{r_1}, \ldots, \gamma_p \mathbb{1}_{r_p}) = (\lambda_i)_{i\in\mathbb{I}_r} \in (\mathbb{R}^r)^{\downarrow}$, where $r \stackrel{\text{def}}{=} \sum_{i\in\mathbb{I}_p} r_i$. Set $s_k = \sum_{j\in\mathbb{I}_k} r_j$, for $k \in \mathbb{I}_p$. Given $\beta \in (\mathbb{R}^r)^{\downarrow}$ such that $\mathrm{tr}(\lambda) = \mathrm{tr}(\beta)$ then

$$\lambda \prec \beta \iff \sum_{i\in\mathbb{I}_k} \gamma_i\,r_i \leq \sum_{j\in\mathbb{I}_{s_k}} \beta_j \;, \qquad \text{for} \quad k \in \mathbb{I}_{p-1} \;. \tag{24}$$

Indeed, if the right conditions hold and there exists $0 \leq k \leq p-1$ with $s_k < t < s_{k+1}$ ($s_0 = 0$) and such that $\sum_{j\in\mathbb{I}_t} \lambda_j > \sum_{j\in\mathbb{I}_t} \beta_j$, it is easy to see that

$$\sum_{j=s_k+1}^{t} \beta_j < \sum_{j=s_k+1}^{t} \lambda_j = (t - s_k)\,\gamma_{k+1} \implies \beta_t < \gamma_{k+1} \implies \sum_{j\in\mathbb{I}_{s_{k+1}}} \beta_j < \sum_{i\in\mathbb{I}_{k+1}} \gamma_i\,r_i \;,$$

which contradicts our assumption (24). Therefore $\lambda \prec \beta$. $\qquad\triangle$

We can now state our second main result.

Theorem 3.14. Let $\mathbf{d} = (d_j)_{j\in\mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ and $\alpha = (\alpha_i)_{i\in\mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow}$ be such that $d_1 \leq n$.

1. If we let $\mathcal{M}^{\mathrm{op}}$ be as in Definition 3.12 then, $(\alpha\,,\mathcal{M}^{\mathrm{op}})$ is admissible. In particular, there exists $\Phi^{\mathrm{op}} = (\mathcal{F}_j^{\mathrm{op}})_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\mathbf{d})$ such that $\mathcal{M}^{\mathrm{op}} = \mathcal{M}_{\Phi^{\mathrm{op}}}$.

2. If $\Phi = (\mathcal{F}_j)_{j\in\mathbb{I}_m} \in \mathcal{D}(\alpha\,,\mathbf{d})$, then

$$\Lambda_{\Phi^{\mathrm{op}}} \prec \Lambda_{\Phi}\;,$$

where $\Lambda_{\Phi^{\mathrm{op}}}\,,\,\Lambda_{\Phi} \in \mathbb{R}_{\geq 0}^d$ are as in Definition 3.7.

Proof. 1. By Definition 3.12, each vector $\mu_i^{\mathrm{op}} \oplus 0_{d_1 - d_i} = \mu_1^{\mathrm{op}} \circ \left(\mathbb{1}_{d_i} \oplus 0_{d_1 - d_i} \right)$. Then

$$\sigma_{\mathcal{M}^{\mathrm{op}}} = \sum_{i \in \mathbb{I}_m} (\mu_i^{\mathrm{op}} \oplus 0_{d_1 - d_i}) = \mu_1^{\mathrm{op}} \circ \sum_{i=1}^{m} \left(\mathbb{1}_{d_i} \oplus 0_{d_1 - d_i} \right) \overset{(15)}{=} \mu_1^{\mathrm{op}} \circ h \ ,$$

using the vector $h = (h_i)_{i \in \mathbb{I}_{d_1}}$ defined from $\mathbf{d}$ as in Eqs. (14) and (15). Therefore, by Eq. (19) in Theorem 3.11, $\sigma_{\mathcal{M}^{\mathrm{op}}} \succ \alpha$, so the statement follows from Theorem 3.5.

2. Let $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ be such that $\mathcal{F}_j = \{f_{ij}\}_{i \in \mathbb{I}_n}$, for $j \in \mathbb{I}_m$. Then

$$\sum_{j \in \mathbb{I}_m} \|f_{ij}\|^2 = \alpha_i \quad \text{for} \quad i \in \mathbb{I}_n \,. \tag{25}$$

On the other hand, if we denote $\lambda_{ij} = \lambda_i(S_{\mathcal{F}_j})$ for $j \in \mathbb{I}_m$ and $i \in \mathbb{I}_{d_j}$, we also have that

$$(\|f_{ij}\|^2)_{i \in \mathbb{I}_n} \prec \lambda(S_{\mathcal{F}_j}) = (\lambda_{ij})_{i \in \mathbb{I}_{d_j}} \quad \text{for} \quad j \in \mathbb{I}_m \,.$$

Hence, we conclude that

$$\sum_{i \in \mathbb{I}_s} \|f_{ij}\|^2 \leq \sum_{i=1}^{\min\{s, d_j\}} \lambda_{ij} \quad \text{for} \quad s \in \mathbb{I}_n \quad \text{and} \quad j \in \mathbb{I}_m \,. \tag{26}$$

Let $\Phi^{\mathrm{op}} \in \mathcal{D}(\alpha, \mathbf{d})$ be as in item 1. We also consider $p \in \mathbb{N}$, $g_0 = 0 < g_1 < \ldots < g_p = d_1$ and $\gamma_1 > \ldots > \gamma_p > 0$ as in Theorem 3.11. Let $r_1, \ldots, r_p \in \mathbb{N}$ such that $\Lambda_{\Phi^{\mathrm{op}}}^{\downarrow} = (\gamma_\ell \mathbb{1}_{r_\ell})_{\ell \in \mathbb{I}_p}$ as in Eq. (16). By Remark 3.13, in order to prove that $\Lambda_{\Phi^{\mathrm{op}}} \prec \Lambda_\Phi$ it is sufficient to check that

$$\sum_{\ell \in \mathbb{I}_q} r_\ell \gamma_\ell \leq \sum_{i \in \mathbb{I}_{s_q}} (\Lambda_\Phi)_i^{\downarrow} \ , \quad \text{where} \quad s_q = \sum_{\ell \in \mathbb{I}_q} r_\ell \ , \quad \text{for every} \quad q \in \mathbb{I}_{p-1} \ , \tag{27}$$

because $\operatorname{tr} \Lambda_{\Phi^{\mathrm{op}}} = \operatorname{tr} \alpha = \operatorname{tr} \Lambda_\Phi$. Fix $q \in \mathbb{I}_{p-1}$ and consider the set

$$S_q = \{(i, j): \ 1 \leq i \leq \min\{g_q, d_j\} \ , \ j \in \mathbb{I}_m\} \,.$$

It is easy to see, using Eq. (15) (or looking at the rows and columns of the Figure 1), that

$$\#S_q = \sum_{j \in \mathbb{I}_m} \min\{g_q, d_j\} \overset{(15)}{=} \sum_{i \in \mathbb{I}_{g_q}} h_i = \sum_{\ell \in \mathbb{I}_q} \left(\sum_{i = g_{\ell-1}+1}^{g_\ell} h_i \right) \overset{(16)}{=} \sum_{\ell \in \mathbb{I}_q} r_\ell = s_q \,. \tag{28}$$

Therefore we can show Eq. (27) as follows: For every $q \in \mathbb{I}_{p-1}$,

$$\begin{aligned}
\sum_{(i, j) \in S_q} \lambda_{ij} &= \sum_{j \in \mathbb{I}_m} \sum_{i=1}^{\min\{g_q, d_j\}} \lambda_{ij} \overset{(26)}{\geq} \sum_{j \in \mathbb{I}_m} \sum_{i \in \mathbb{I}_{g_q}} \|f_{ij}\|^2 \overset{(25)}{=} \sum_{i \in \mathbb{I}_{g_q}} \alpha_i \\
&= \sum_{\ell \in \mathbb{I}_q} \left(\sum_{i = g_{\ell-1}+1}^{g_\ell} \alpha_i \right) \overset{(18)}{=} \sum_{\ell \in \mathbb{I}_q} \left(\sum_{i = g_{\ell-1}+1}^{g_\ell} \gamma_\ell h_i \right) \overset{(16)}{=} \sum_{\ell \in \mathbb{I}_q} r_\ell \gamma_\ell \,.
\end{aligned}$$

Since $\sum_{i \in \mathbb{I}_{s_q}} (\Lambda_\Phi)_i^{\downarrow} \overset{(28)}{\geq} \sum_{(i, j) \in S_q} \lambda_{ij}$, then Eq. (27) follows, and $\Lambda_{\Phi^{\mathrm{op}}} \prec \Lambda_\Phi$. $\qquad\square$

Theorem 3.14 together with the argument in Remark 3.8 allow us to obtain our third main result.

Theorem 3.15. *Let $\mathbf{d} = (d_j)_{j \in \mathbb{I}_m} \in (\mathbb{N}^m)^{\downarrow}$ and $\alpha = (\alpha_i)_{i \in \mathbb{I}_n} \in (\mathbb{R}_{\geq 0}^n)^{\downarrow}$ be such that $d_1 \leq n$.*

1. *Let $\Phi^{\mathrm{op}} = (\mathcal{F}_j^{\mathrm{op}})_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ be as in Theorem 3.14. If $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$ then we have that*

$$P_\varphi(\Phi^{\mathrm{op}}) \leq \mathrm{P}_\varphi(\Phi) \quad \textit{for every} \quad \Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d}). \qquad (29)$$

2. *Moreover, if $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ is such that there exists $\varphi \in \mathrm{Conv}_{\mathrm{s}}(\mathbb{R}_{\geq 0})$ for which Φ is a global minimum for P_φ (i.e., if equality holds in Eq. (29)), then*

$$\Lambda_\Phi = \Lambda_{\Phi^{\mathrm{op}}} \quad \textit{and} \quad \mathcal{M}_\Phi = \mathcal{M}^{\mathrm{op}}, \quad \textit{so that} \quad \lambda(S_{\mathcal{F}_j}) = \mu_j^{\mathrm{op}} \in (\mathbb{R}_{>0}^{d_j})^{\downarrow}, \qquad (30)$$

and, in particular, $\mathcal{F}_j$ is a frame for $\mathbb{C}^{d_j}$ for every $j \in \mathbb{I}_m$.

Proof. Let $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$, then by Theorem 3.14 we know that $\Lambda_{\Phi^{\mathrm{op}}} \prec \Lambda_\Phi$. Therefore, by Remark 3.8 we get that for every $\varphi \in \mathrm{Conv}(\mathbb{R}_{\geq 0})$,

$$\mathrm{P}_\varphi(\Phi^{\mathrm{op}}) = \sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j^{\mathrm{op}}) = \mathrm{tr}(\varphi(\Lambda_{\Phi^{\mathrm{op}}})) \leq \mathrm{tr}(\varphi(\Lambda_\Phi)) = \sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \mathrm{P}_\varphi(\Phi).$$

If $\varphi \in \mathrm{Conv}_{\mathrm{s}}(\mathbb{R}_{\geq 0})$ and $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})$ is such that equality holds in Eq. (29), let

$$\mathcal{C} = \{\Lambda_\Psi : \ \Psi = (\mathcal{G}_j)_{j \in \mathbb{I}_m} \in \mathcal{D}(\alpha, \mathbf{d})\} \ \subseteq \ \mathbb{R}_{\geq 0}^d.$$

By Lemma 3.4 it follows that $\mathcal{C}$ is a convex set. We finally introduce $F_\varphi : \mathcal{C} \to \mathbb{R}_{\geq 0}$, $F_\varphi(\Lambda) = \mathrm{tr}(\varphi(\Lambda))$ for $\Lambda \in \mathcal{C}$. Since φ is strictly convex we immediately see that F - which is defined on the convex set $\mathcal{C}$ - is strictly convex as well. Hence, there exists a *unique* $\Lambda_\varphi \in \mathcal{C}$ such that

$$F(\Lambda_\varphi) = \min\{F(\Lambda) : \ \Lambda \in \mathcal{C}\}.$$

Notice that by hypothesis, we have that $F(\Lambda_\Phi) = F(\Lambda_{\Phi^{\mathrm{op}}}) = \min\{F(\Lambda) : \ \Lambda \in \mathcal{C}\}$ so then

$$(\lambda(S_{\mathcal{F}_j}))_{j \in \mathbb{I}_m} = \Lambda_\Phi = \Lambda_\varphi = \Lambda_{\Phi^{\mathrm{op}}} = (\lambda(S_{\mathcal{F}_j^{\mathrm{op}}}))_{j \in \mathbb{I}_m}. \qquad \square$$

Remark 3.16. As a consequence of Eq.'s (29) and (30), the sequence $\mathcal{M}^{\mathrm{op}}$ of Definition 3.12 and the indexes $(g_i)_{i \in \mathbb{I}_p}$ of Theorem 3.11 are univocally determined.

On the other hand, with the notation of Theorem 3.15, if we assume that $m = 1$, then the previous theorem recovers the main results from [9, 24, 25, 26]. In this case, the optimal spectra μ^{op} is obtained in terms of the water-filling construction. Hence, our results can be considered as a multivariated extension of the water-filling construction (see [26]). $\triangle$

4 Final comments and examples

4.1 On the weight partitions

By Theorem 3.15 and Remark 3.16, the spectral structure of all $(\alpha, \mathbf{d})$-designs that minimize a strictly convex potential on $\mathcal{D}(\alpha, \mathbf{d})$ is unique. It is natural to wonder whether the (α, m)-weight partitions corresponding to such minimizers also coincide. It turns out that this is not the case, as we shall see in the following example:

Let $\alpha = \mathbb{1}_6 \in (\mathbb{R}_{>0}^6)^{\downarrow}$, $m = 2$ and let $\mathbf{d} = (4, 2) \in \mathbb{N}^2$. In this case $\mathcal{M}^{\mathrm{op}}$ is given by

$$\mu_1^{\mathrm{op}} = \mathbb{1}_4 \quad \text{and} \quad \mu_2^{\mathrm{op}} = \mathbb{1}_2,$$

since $\alpha = \mathbb{1}_6 \prec (2, 2, 1, 1) = \sigma_{\mathcal{M}^{\mathrm{op}}}$ (so that $\mathcal{M}^{\mathrm{op}}$ is $(\alpha, \mathbf{d})$-admissible), and because the associated vector $\Lambda^{\mathrm{op}} = (\mu_1^{\mathrm{op}}, \mu_2^{\mathrm{op}}) = \mathbb{1}_6$ is minimal for majorization. Let

$$A_1 = \left(\frac{4}{6} \mathbb{1}_6, \frac{2}{6} \mathbb{1}_6 \right) \in P_{\alpha, 2} \subseteq \mathcal{M}_{6,2}(\mathbb{C}) \quad \text{and} \quad A_2 = \left(c_1(A_2), c_2(A_2) \right) \in P_{\alpha, 2}$$

where $c_1(A_2) = (\mathbb{1}_4 \, , 0_2)$ and $c_2(A_2) = (0_4 \, , \mathbb{1}_2)$. It is easy to see that both matrices satisfy Eq. (8) in Lemma 3.4. Thus, we can construct $\Phi^1 = (\mathcal{F}_1^1 \, , \mathcal{F}_2^1) \in \mathcal{D}(\alpha \, , \mathbf{d})$ with weight partition A_1, in such a way that $\mathcal{F}_1^1$ is a Parseval frame for $\mathbb{C}^4$ and $\mathcal{F}_2^1$ is a Parseval frame for $\mathbb{C}^2$ (both of 6 vectors). On the other hand, if we let $\{e_\ell^{(k)}\}_{\ell \in \mathbb{I}_k}$ denote the canonical basis of $\mathbb{C}^k$ for $k \in \mathbb{N}$ and let $\Phi^2 = (\mathcal{F}_1^2 \, , \mathcal{F}_2^2) \in \mathcal{D}(\alpha \, , \mathbf{d})$ with

$$\mathcal{F}_1^2 = \{e_1^{(4)}, \ldots, e_4^{(4)}, 0, 0\} \in (\mathbb{C}^4)^6 \quad \text{and} \quad \mathcal{F}_2^2 = \{0, 0, 0, 0, e_1^{(2)}, e_2^{(2)}\} \in (\mathbb{C}^2)^6 \, ,$$

then Φ^2 has weight partition A_2. Clearly $\mathcal{M}_{\Phi_1} = \mathcal{M}_{\Phi_2} = \mathcal{M}^{\mathrm{op}}$. That is, A_1 and A_2 are both associated to optimal $(\alpha \, , \mathbf{d})$-designs. Thus, weight partitions inducing optimal $(\alpha \, , \mathbf{d})$-designs are not unique. Note that the $(\alpha \, , \mathbf{d})$-designs Φ^1 and Φ^2 are qualitatively different.

Remark 4.1. The fact that there are many $(\alpha \, , m)$-partition matrices $A \in P_{\alpha \, , m}$ that are associated to minimizers, as in the previous example, shows that the construction of optimal $(\alpha \, , \mathbf{d})$-designs can not be reduced to a convex optimization problem in the space $P_{\alpha \, , m}$ of $(\alpha \, , m)$-weight partitions.

We remark that in a previous version of this paper we constructed an algorithm which produced a particular matrix $A^{\mathrm{op}} \in P_{\alpha \, , m}$ associated to a minimizer. That is, once A^{op} was constructed we considered the so-called water-filling of the columns $c_j(A^{\mathrm{op}})$ in dimension d_j (see [27]), which lead to the optimal spectra μ_j^{op}, for $j \in \mathbb{I}_m$.

Our new strategy, based on Theorem 3.5, allow us to compute directly the optimal spectra μ_j^{op} for $j \in \mathbb{I}_m$ (and to show the existence of $(\alpha \, , \mathbf{d})$-designs with these spectra). Once the optimal spectra are computed then, using Remark 3.6, we can compute several associated weight partitions that in turn allow us to compute optimal $(\alpha \, , \mathbf{d})$-designs in an effective way (see Section 4.3 below). This new approach has decreased considerably the length of the exposition of our results herein. $\triangle$

4.2   A compact description of the problem

There is a reformulation of the problems of this paper in a more concise model. Let α and $\mathbf{d}$ be as in Definition 3.1. Set $d = \operatorname{tr} \mathbf{d}$ and assume that $\mathcal{H} = \mathbb{C}^d = \bigoplus_{j \in \mathbb{I}_m} \mathcal{H}_j$ for some subspaces with $\dim \mathcal{H}_j = d_j$, for $j \in \mathbb{I}_m$. Let us denote by $P_j : \mathcal{H} \to \mathcal{H}_j \subseteq \mathcal{H}$ the corresponding projections.

Notice that a sequence $\mathcal{G} = \{g_i\}_{i \in \mathbb{I}_n} \in \mathcal{B}_{\alpha \, , d} \subseteq \mathcal{H}^n \iff$ the sequence $\Phi = (\mathcal{F}_j)_{j \in \mathbb{I}_m}$ determined by $\mathcal{F}_j = P_j(\mathcal{G})$ (i.e. $f_{ij} = P_j(g_i) \in \mathcal{H}_j \cong \mathbb{C}^{d_j}$, $i \in \mathbb{I}_n$) for $j \in \mathbb{I}_m$, satisfies that $\Phi \in \mathcal{D}(\alpha \, , \mathbf{d})$.

Consider the pinching map $\mathcal{C}_{\mathbf{d}} : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ given by $\mathcal{C}_{\mathbf{d}}(A) = \sum_{j \in \mathbb{I}_m} P_j \, A \, P_j$, for every $A \in \mathcal{M}_d(\mathbb{C})$. Then, for each $\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$ we can define a $\mathbf{d}$-pinched potential

$$\mathrm{P}_{\varphi \, , \mathbf{d}}(\mathcal{G}) \stackrel{\mathrm{def}}{=} \operatorname{tr} \varphi(\mathcal{C}_{\mathbf{d}}(S_{\mathcal{G}})) \quad \text{for every} \quad \mathcal{G} \in \mathcal{H}^n \, ,$$

which describes simultaneously the behavior of the projections of $\mathcal{G}$ to each subspace $\mathcal{H}_j$. Actually, with the previous notations,

$$\mathrm{P}_{\varphi \, , \mathbf{d}}(\mathcal{G}) = \sum_{j \in \mathbb{I}_m} \operatorname{tr} \varphi(P_j \, S_{\mathcal{G}} \, P_j) = \sum_{j \in \mathbb{I}_m} \mathrm{P}_\varphi(\mathcal{F}_j) = \mathrm{P}_\varphi(\Phi) \, .$$

Therefore the problem of finding optimal $(\alpha \, , \mathbf{d})$-designs (and studying their properties) translates to the study of sequences $\mathcal{G} \in \mathcal{B}_{\alpha \, , d}$ which minimize the $\mathbf{d}$-pinched potentials $\mathrm{P}_{\varphi \, , \mathbf{d}}$.

We point out that for $\varphi \in \operatorname{Conv}(\mathbb{R}_{\geq 0})$ and $\mathcal{G} \in \mathcal{H}^n$

$$\mathrm{P}_{\varphi \, , \mathbf{d}}(\mathcal{G}) \leq \operatorname{tr} \varphi(S_{\mathcal{G}}) \quad \text{but} \quad \mathrm{P}_{\varphi \, , \mathbf{d}}(\mathcal{G}) \neq \operatorname{tr} \varphi(S_{\mathcal{G}}) \quad \text{in general}$$

(see Definition 2.6). Therefore, previous results related with the structure of minimizers of convex potentials in $\mathcal{B}_{\alpha \, , d}$ (e.g. [24]) do not apply to the $\mathbf{d}$-pinched potential and we require a new approach to study this problem, as shown in Example 4.3. In the paper we use the more complicated notation of sequences because it has proved to be more useful for all the computations detailed before.

4.3 Examples

Theorems 3.5, 3.14 and 3.15, combined with Remark 3.6 allow us to describe a finite step algorithmic process for the effective construction of optimal designs $\Phi^{\mathrm{op}} \in \mathcal{D}(\alpha\,,\mathbf{d})$ from the initial data $(\alpha, \mathbf{d})$.

A possible scheme for the algorithmic procedure would be as follows:

Algorithm 4.2. IMPUT DATA: $(\alpha\,,\mathbf{d})$.

STEP 1. Along with the computation of the vector h, by means of an iterative process, we compute the value of s^* defined in Theorem 3.11. If $s^* = 1$, we set $p = 1$, $g_1 = 1$ and $\gamma_1 = Q_1$. In this case, $\gamma_1 \mathbb{1}_{d_1} \circ h \succ \alpha$ and go to STEP 3. Otherwise, set $j = 1$ and continue with

STEP 2. By computing the maximum among the means $P_{j,k}$, $j \leq k \leq s^* - 1$ we compute g_j and γ_j as it is indicated in Theorem 3.11. If $g_j = s^* - 1$, rename $j = p - 1$, and $\gamma_p = Q_{g_{p-1}+1}$ and go to STEP 3. Otherwise, set $j = g_j + 1$ and repeat STEP 2. This step produces the set of optimal spectra $\mathcal{M}^{\mathrm{op}} = \{\mu_j^{\mathrm{op}}\}_{j \in \mathbb{I}_m}$, such that $(\alpha\,,\mathcal{M})$ is admissible.

STEP 3. Apply the finite step algorithm described in Remark 3.6 and obtain $\Phi^{\mathrm{op}} = (\mathcal{F}_j^{\mathrm{op}})_{j \in \mathbb{I}_m}$ (as described in Theorem 3.14) as output.

The following examples were obtained via an implementation of Algorithm 4.2 using MATLAB.

Example 4.3. Consider the family of weights given by $\alpha = \{9, 8, 7, 5, 4, 2.5, 2, 2, 1.5, 0.6, 0.5\}$ and suppose that the dimensions to be considered are $\mathbf{d} = (7\,,5\,,3)$. In this case, the optimal spectra $\mathcal{M}^{\mathrm{op}}$ are determined, as in Definition 3.12, by

$$\mu_1^{\mathrm{op}} = (3, 2.7583, 2.7583, 2.7583, 2.7583, 2.7583, 2.7583)\,.$$

If $\sigma_{\mathcal{M}^{\mathrm{op}}} \in \mathbb{R}^7$ is defined as in Theorem 3.5, then, $\alpha \prec \sigma_{\mathcal{M}^{\mathrm{op}}}$ by Theorem 3.14. Using Remark 3.6 we construct $D \in \mathcal{DS}(11)$ such that $D(\sigma_{\mathcal{M}^{\mathrm{op}}} \oplus 0_4) = \alpha$. Setting $A \in \mathcal{M}_{11,3}(\mathbb{R}_{\geq 0})$, such that $c_j(A) = D\left(\mu_j^{\mathrm{op}} \oplus 0_{11-d_j}\right) \in \mathbb{R}_{\geq 0}^{11}$, for $j = 1, 2, 3$, we get (for example) the following partition of α:

$$A = \begin{bmatrix} 3 & 3 & 3 \\ 2.7583 & 2.7583 & 2.4833 \\ 2.7583 & 2.7583 & 1.4833 \\ 2.7583 & 1.8135 & 0.4282 \\ 2.5267 & 1.1307 & 0.3425 \\ 1.5792 & 0.7067 & 0.2141 \\ 1.2634 & 0.5654 & 0.1713 \\ 1.2634 & 0.5654 & 0.1713 \\ 0.9475 & 0.4240 & 0.1285 \\ 0.3790 & 0.1696 & 0.0514 \\ 0.3158 & 0.1413 & 0.0428 \end{bmatrix} \in P_{\alpha\,,3}\,.$$

Once we have the partitions and optimal spectra, we can construct examples of frames using these data, applying known algorithms like one-sided Bendel-Mickey algorithm (see [10, 11, 14, 16]):

$$\mathcal{F}_1 = \begin{bmatrix} 0.0705 & 0.1956 & -0.0616 & -0.6865 & -0.6865 & 0.3994 & -0.0845 & -0.3230 & -1.1553 & 0.2649 & -0.3180 \\ 0.2804 & -0.2311 & -0.2142 & 0.2434 & 0.2434 & -0.4716 & 1.2808 & 0.2534 & -0.4197 & 0.3309 & -0.5206 \\ 0.0380 & -0.1106 & -0.5728 & -0.8134 & -0.8134 & -0.2257 & 0.3005 & 0.0342 & 0.9482 & 0.1009 & -0.2125 \\ -0.0004 & -0.1760 & -0.3643 & -0.2125 & -0.2125 & -0.3592 & 0.2804 & 0.2989 & -0.4753 & -1.0345 & 0.9956 \\ -0.4655 & -0.4260 & -0.5815 & 0.0796 & 0.0796 & -0.8695 & -0.8294 & 0.3134 & -0.3310 & 0.0127 & -0.6235 \\ 0.1120 & 0.2501 & 0.3128 & -0.1034 & -0.1034 & 0.5106 & -0.0368 & 1.2019 & 0.0419 & -0.6107 & -0.7246 \\ 0.0391 & -0.0112 & 0.0316 & 0.0949 & 0.0949 & -0.0229 & 0.1448 & -0.9781 & 0.1061 & -1.0607 & -0.8232 \end{bmatrix}$$

$$\mathcal{F}_2 = \begin{bmatrix} 0.1841 & 0.2017 & 0.3189 & 0.0682 & -0.2093 & -0.2340 & -0.2960 & 0.6595 & 0.5432 & 0.0340 & 1.3437 \\ 0.0249 & 0.0273 & 0.0432 & 0.6049 & 0.6893 & 0.7707 & 0.9748 & 0.0893 & 0.4598 & 0.3451 & 0.1842 \\ -0.1947 & -0.2132 & -0.3372 & -0.3744 & -0.1430 & -0.1599 & -0.2022 & -0.6973 & 1.2517 & 0.5169 & -0.1238 \\ -0.2625 & -0.2876 & -0.4547 & -0.2253 & 0.1440 & 0.1610 & 0.2037 & -0.9404 & -0.4997 & -0.3351 & 1.0506 \\ -0.0015 & -0.0016 & -0.0025 & -0.0619 & -0.0723 & -0.0808 & -0.1022 & -0.0053 & -0.6598 & 1.5029 & 0.2041 \end{bmatrix}$$

$$\mathcal{F}_3 = \begin{bmatrix} -0.0342 & -0.0375 & -0.0593 & -0.3714 & -0.3952 & -0.4419 & -0.5590 & -0.6249 & -1.1632 & -0.2605 & -0.3888 \\ -0.1953 & -0.2139 & -0.3383 & -0.1805 & 0.0479 & 0.0536 & 0.0678 & 0.0758 & 0.1410 & -1.4873 & 0.5519 \\ 0.0592 & 0.0649 & 0.1026 & -0.0281 & -0.1129 & -0.1263 & -0.1597 & -0.1786 & -0.3324 & 0.4511 & 1.5950 \end{bmatrix}$$

Let $\Phi^{\mathrm{op}} = (\mathcal{F}_1\,,\mathcal{F}_2\,,\mathcal{F}_3) \in \mathcal{D}(\alpha\,,\mathbf{d})$. Then, by Theorem 3.15 we have that

$$\min\{\mathrm{P}_\varphi(\Phi)\ :\ \ \Phi \in \mathcal{D}(\alpha\,,\mathbf{d})\} = \mathrm{P}_\varphi(\Phi^{\mathrm{op}}) = 3\,\varphi(3) + 12\,\varphi(2.7583)\,.$$

Now, with the notation and terminology of Section 4.2, we get the following lower bound for the **d**-pinched potential (notice that $d = d_1 + d_2 + d_3 = 15$)

$$\min\{\mathrm{tr}\,\varphi(S_\mathcal{G})\ :\ \ \mathcal{G} \in \mathcal{B}_{\alpha\,,15}\} \geq \min\{\mathrm{P}_{\varphi\,,\mathbf{d}}(\mathcal{G})\ :\ \ \mathcal{G} \in \mathcal{B}_{\alpha\,,15}\} = 3\,\varphi(3) + 12\,\varphi(2.7583)\,. \qquad (31)$$

Indeed, since $d = 15 > n = 11$, we have that (see [24])

$$\min\{\mathrm{tr}\,\varphi(S_\mathcal{G})\ :\ \ \mathcal{G} \in \mathcal{B}_{\alpha\,,15}\} = \sum_{j\in\mathbb{I}_{11}} \varphi(\alpha_j) + 4\,\varphi(0)\,. \qquad (32)$$

Moreover, in case $\varphi \in \mathrm{Conv}_\mathrm{s}(\mathbb{R}_{\geq 0})$ then the minimizers of the potential $\mathcal{G} \mapsto \mathrm{tr}\,\varphi(S_\mathcal{G})$ in $\mathcal{B}_{\alpha\,,15}$ are sequences $\mathcal{G} = \{g_i\}_{i\in\mathbb{I}_{11}} \in \mathcal{H}^{11}$ of mutually orthogonal vectors. If we further choose $\varphi(x) = x^2$, $x \geq 0$, then the reader can check that nor the minimal value, nor the geometric structure of minimizers of Eqs. (31) and (32) coincide. $\triangle$

Example 4.4. When $\alpha = \{20, 19.5, 10, 5, 4.5, 3, 2.4, 2\}$ and $\mathbf{d} = \{5, 4, 4, 3, 2\}$, Algorithm 4.2 constructs the optimal spectra given by

$$\mu_1^{\mathrm{op}} = (4, 3.9, 3.3625, 3.3625, 3.3625)$$
$$\mu_2^{\mathrm{op}} = (4, 3.9, 3.3625, 3.3625)$$
$$\mu_3^{\mathrm{op}} = (4, 3.9, 3.3625, 3.3625)$$
$$\mu_4^{\mathrm{op}} = (4, 3.9, 3.3625)$$
$$\mu_5^{\mathrm{op}} = (4, 3.9)$$

where the smaller spectrum does not have the constant 3.3625. As before, using Remark 3.6, we obtain the following partition:

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 3.9 & 3.9 & 3.9 & 3.9 & 3.9 \\ 3.3625 & 2.8875 & 2.5 & 1.25 & 0 \\ 1.9896 & 1.1354 & 1.25 & 0.625 & 0 \\ 1.7907 & 1.0218 & 1.125 & 0.5625 & 0 \\ 1.1938 & 0.6812 & 0.75 & 0.375 & 0 \\ 0.955 & 0.545 & 0.6 & 0.3 & 0 \\ 0.7959 & 0.4541 & 0.5 & 0.25 & 0 \end{bmatrix} \in P_{\alpha\,,5}\,.$$

$\triangle$

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