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# On the geometry of generalized inverses

**E. Andruchow**[*1], **G. Corach**[**2], and **M. Mbekhta**[***3]

[1] Instituto de Ciencias, Universidad Nacional de Gral. Sarmiento, J. M. Gutierrez entre J. L. Suarez y Verdi, (1613) Los Polvorines, Argentina

[2] Departamento de Matemática, Facultad de Ingeniería, Universidad de Buenos Aires, Paseo Colón 850 (1063) Buenos Aires, Argentina

[3] UFR de Mathematiques, F – 59655 Villeneuve d'Ascq, France

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We study the set $S = \{(a, b) \in A \times A : aba = a,\ bab = b\}$ which pairs the relatively regular elements of a Banach algebra $A$ with their pseudoinverses, and prove that it is an analytic submanifold of $A \times A$. If $A$ is a C$^*$-algebra, inside $S$ lies a copy the set $\mathcal{I}$ of partial isometries, we prove that this set is a $C^\infty$ submanifold of $S$ (as well as a submanifold of $A$). These manifolds carry actions from, respectively, $G_A \times G_A$ and $U_A \times U_A$, where $G_A$ is the group of invertibles of $A$ and $U_A$ is the subgroup of unitary elements. These actions define homogeneous reductive structures for $S$ and $\mathcal{I}$ (in the differential geometric sense). Certain topological and homotopical properties of these sets are derived. In particular, it is shown that if $A$ is a von Neumann algebra and $p$ is a purely infinite projection of $A$, then the connected component $\mathcal{I}_p$ of $p$ in $\mathcal{I}$ is simply connected. If $1 - p$ is also purely infinite, then $\mathcal{I}_p$ is contractible.



This paper contains a study of the differential geometry of the set $S$ of all pairs $(a, b)$ of elements of a Banach algebra $A$ such that $b$ (resp. $a$) is a generalized inverse of $a$ (resp. $b$), and of the set $\mathcal{I}$ of all partial isometries of a C*-algebra $A$. It can be seen as a continuation of [12] for the Banach algebra setting, and of [25] and [26] for the C*-algebra setting. The first appearences of generalized inverse methods in analysis go back to Fredholm, Hilbert and Hurwitz. In an explicit form, it was E. H. Moore who first considered what today is called de Moore-Penrose inverse. It was only after 1950, when Bjerhammar [10], and Penrose [29]–[30] rediscovered the Moore-Penrose pseudoinverse of a matrix, that the subject took great impulse (see the recent paper [9] by A. Ben Israel for an account of this story). Its popularity came from the extremely vast and diversified field of its applications in many scientific disciplines. The reader is referred to the book [27], edited by M. Z. Nashed, which contains many developments, applications and a list of 1776 references related to generalized inverses.

We shall concentrate in some local and global topological-geometrical properties of $S$ and $\mathcal{I}$. The first description of the connected components of $\mathcal{I}$ for the algebra $L(H)$ of all linear bounded operators on a Hilbert space $H$ was done by Halmos and McLaughlin [18]: they proved that two partial isometries $a,\ c$ in $L(H)$ belong to the same connected component of $\mathcal{I}$ if and only if they have the same rank, nullity and co-rank. A simplification of their proof, found by R. G. Douglas (see [17], solution to problem 131) shows that $a,\ c$ belong to the same component if and only if there exist unitaries $u,\ v$ in $L(H)$ such that $uav^* = c$. Mbekhta and Skhiri [25] extended these results to the Calkin algebra, and Mbekhta and Stratila [26] proved them for von Neumann algebras. The topology of the set $S$ has been studied in part by J. L. Taylor [32]. B. Gramsch [16] studied the set of Fredholm operators with fixed dimension of the kernel, and proved that it is an analytic homogeneous manifold. He also extended some of these results to the context of Frechet algebras with open group of invertible elements. In [12] there is a geometrical study of certain parts of $S$. Observe that $(a, b)$ belongs to $S$ if and only if $aba = a$ and $bab = b$; in particular $ab$ and $ba$ belong to $Q$, the set of idempotents of $A$. For a fixed $r$ in $Q$, consider the set $S_r = \{(a, b) \in S : ar = a,\ rb = b,\ ba = r\}$. There is a natural action of $G_A$ over $S_r$ given by

* Corresponding author: e-mail: eandruch@ungs.edu.ar, Phone: ? , Fax: ?
** e-mail: gcorach@fi.uba.ar, Phone: ? , Fax: ?
*** e-mail: mostafa.mbekhta@agat.univ-lille1.fr, Phone: ? , Fax: ?

$u\,.\,(a,b)=(ua,bu^{-1})$ and this action describes $S_r$ as a discrete union of homogeneous spaces of $G_A$. Of course, there are many natural actions of $G_A$ over $S$ but what Halmos-McLaughlin's result says is that it is an action of $G_A \times G_A$ over $S$ what seems to be needed, if one wants that the orbits contain the connected components. This is exactly the strategy we shall follow. Of course, for the case of $\mathcal{I}$ we shall look for an action of $U_A \times U_A$ over $\mathcal{I}$, where $U_A$ denotes the group of unitaries of $A$.

It turns out that $S$ is what geometers call a reductive homogeneous space. The reader is referred to the classical book of Kobayashi and Nomizu [21] for finite dimensional manifolds, and to [22] and [5] for a general description of these reductive spaces, in the case of Banach algebras and algebras of operators. As examples of reductive homogeneous spaces in the context of Banach and C$^*$-algebras we mention sets of projections [13], spheres of C*-Hilbert modules [2], projective spaces of C*-algebras and Hilbert modules [3], [4], and so on. The geometrical methods used here may have some relevance in the study of generalized resolvent problems as those considered by Apostol and Clancey [6]–[7] and Mbekhta [23]–[24]. We intend to proceed with these matters elsewhere. Let us mention that in the C$^*$-algebra case the subset $\{(a,b)\in S : a\geq 0,\ b\geq 0\}$ plays a relevant role in the so called "fidelity theory" of quantum physics, studied by Josza [19], Uhlmann [33] and Alberti [1].

The contents of the paper are as follows. In Section 2 we define the action of $G_A \times G_A$ on $S$ by

$$(u,v)\,.\,(a,b) \;=\; \big(uav^{-1}, vbu^{-1}\big)$$

and determine its orbits. This allows one to determine also the connected components of $S$. Moreover, it is shown that for a fixed $(a,b)$ in $S$ the map

$$\pi_{(a,b)} \;:\; G_A \times G_A \;\longrightarrow\; S\,, \quad \pi_{(a,b)}(u,v) \;=\; (u,v)\,.\,(a,b)$$

admits continuous local cross sections, so that, using results of Raeburn [31], the orbits of the action (i.e. the images of the maps $\pi_{(a,b)}$) are submanifolds of $A \times A$ and $\pi_{(a,b)}$ is a submersion. There is also a description of the properties of the map $\varphi(a,b)=(ab,ba)$ from $S$ to $Q\times Q$, where $Q$ is the set of all idempotents of $A$. The structure of the orbits of $Q$ under the similarity action by $G_A$ is relevant for the results in this section. The reader is referred to [34] for a nice study of the components of $Q$.

Section 3 is devoted to define and study a natural connection on $S$ and a reductive structure on the homogeneous space determined by $\pi_{(a,b)}$. We explicitely describe the geodesics of the connection by means of the tangent maps of the local sections defined in Section 2.

The C$^*$-algebra case is studied at Section 4. If $A$ is a unital C$^*$-algebra then $\mathcal{I}$ can be identified with the subset of $S$ of all pairs $(u,u^*)$, for $u$ in $\mathcal{I}$ and restrict the action of $G_A \times G_A$ to $U_A \times U_A$. This action has similar properties so we get the description of all connected components of $\mathcal{I}$ for an arbitrary C$^*$-algebra, extending the results mentioned before. In particular, the connected component in $\mathcal{I}$ of $u$ is the set $U_0uU_0$, where $U_0$ denotes the connected component in $U_A$ of the identity. We prove that $\mathcal{I}$ is a $C^\infty$ submanifold of $A$, or, under the above identification, of $S$. The map

$$\varphi \;:\; \mathcal{I} \;\longrightarrow\; P\times P\,, \quad \varphi(u) \;=\; (uu^*, u^*u)$$

is a fibration, which enables one to prove that, for example, if $A$ is a von Neumann algebra, and $p$ and $1-p$ are purely infinite projections, then the connected component of $p$ in the set $\mathcal{I}$ is contractible.

In Section 5, we study the subset $S_\Delta=\{(a,b)\in S : ab=ba\}$, which can be regarded as the set of elements of $A$ which are invertible in some corner $qAq$ of $A$ (here $q$ is an idempotent whose range equals the range of $a$). This set is a submanifold of $A\times A$, and the base space of a fibre bundle. If $A$ is a von Neumann algebra, this fact enables one to show that for every projection $q$ in $A$, it holds $\pi_1(S_\Delta,(q,q))\simeq \pi_1(G_{qAq},q)$, where $G_{qAq}$ denotes the group of invertibles of $qAq$.

## 1 Generalized invertible elements

Let $A$ be a Banach algebra with identity and define

$$S \;=\; \{(a,b) :\ aba=a,\ bab=b\}\,.$$

If $G_A$ denotes the group of invertible elements of $A$, consider the left action

$$L \,:\, G_A \times G_A \times S \;\longrightarrow\; S\,, \quad L_{(w,z)}(a,b) \;=\; (w,z)\,.\,(a,b) \;=\; \big(waz^{-1}, zbw^{-1}\big)\,.$$

The action of $G_A \times G_A$ over $S$ induces a partition of $S$ in orbits: the orbit of $(a,b) \in S$ is $O_{(a,b)} = \{(waz^{-1}, zbw^{-1}) : w, z \in G_A\}$. Denote by $C_{(a,b)}$ the connected component of $(a,b)$ in $S$. In order to study the orbits of the action $L$, recall the similarity action of $G_A$ over the set $Q$ of all idempotents of $A$, namely $L' : G_A \times Q \to Q$, $L'(w,q) = wqw^{-1}$ for $w \in G_A$, $q \in Q$. It is well-known that the map $\pi_q : G_A \to Q$, $\pi_q(w) = wqw^{-1}$ admits local cross sections at any point of its image. More precisely, given $q_0$ in the image of $\pi_q$ there exist an open neighborhood $U$ of $q_0$ in $Q$ and an analytic map $\sigma' : U \to G_A$ such that $\pi_q(\sigma'(r)) = r$ for all $r \in U$. In fact, it suffices to define, for a convenient $\alpha$ between 0 and 1, $U = \{r \in Q : \|r - q_0\| < \alpha\}$ and $\sigma'(r) = rq + (1-r)(1-q)$. Observe that, for $\alpha$ small enough, the image of $\sigma'$ is contained in the connected component $G_0$ of 1 in $G_A$. For each $q \in Q$ denote by $O_q$ the orbit of $q$ by the action $L'$, i.e. $O_q = \{wqw^{-1} : w \in G_A\}$ and by $C_q$ the connected component of $q$ in $Q$. By the remarks above, each orbit is open and therefore closed, so that, in particular, $C_q = \{wqw^{-1} : w \in G_0\}$. The following result establishes the existence of continuous local cross sections for the action (cf. 4.2 of [16]).

**Proposition 1.1** *For every $(a,b) \in S$ there exist an open neighborhood $W$ of $(a,b)$ in $S$ and a continuous mapping $\sigma : W \to G_0 \times G_0$ such that $L_{\sigma(a',b')} = (a',b')$ for all $(a',b') \in W$.*

Proof. As mentioned before, if $(a,b) \in S$ then $q_1 = ab$ and $q_2 = ba$ belong to $Q$. By the remarks above, there exist two analytic maps $\sigma_1' : W_1 \to G_A$, $\sigma_2' : W_2 \to G_A$ where $W_1$ (resp. $W_2$) is an open neighborhood of $q_1$ (resp. $q_2$) in $Q$ such that $\pi_{q_k}(\sigma_k'(r)) = r$ for all $r \in W_k$, $k = 1,2$. Define $W = \{(a',b') \in S : a'b' \in W_1, b'a' \in W_2\}$. Then $W$ is an open subset of $S$ because the map $\phi : S \to Q \times Q$, $\phi(c,d) = (cd, dc)$ is obviously continuous. Define

$$\sigma \;:\; W \;\longrightarrow\; G_A \times G_A\,, \quad \sigma(a',b') \;=\; (a'\sigma_2'(b'a')b + (1-a'b')\sigma_1'(a'b'), \sigma_2'(b'a'))$$

for $(a',b') \in W$. Clearly $\sigma$ is analytic. Observe that

$$[a'\sigma_2'(b'a')b + (1-a'b')\sigma_1'(a'b')]^{-1} \;=\; a\sigma_2'(b'a')^{-1}b' + \sigma_1'(a'b')^{-1}(1-a'b')\,. \tag{1.1}$$

Indeed, note that $\sigma_1'(a'b')ab = a'b'\sigma_1'(a'b')$ and $\sigma_2'(b'a')ba = b'a'\sigma_2'(b'a')$. Then

$$\begin{aligned}
&\big[a\sigma_2'(b'a')^{-1}b' + \sigma_1'(a'b')^{-1}(1-a'b')\big]\big[a'\sigma_2'(b'a')b + (1-a'b')\sigma_1'(a'b')\big] \\
&\quad=\; a\sigma_2'(b'a')^{-1}b'a'\sigma_2'(b'a')b + \sigma_1'(a'b')^{-1}(1-a'b')\sigma_1'(a'b') \\
&\quad=\; ab + 1 - ab \;=\; 1\,.
\end{aligned}$$

The product on the reverse order is dealt analogously. Let us show that

$$L_{\sigma(a',b')}(a,b) \;=\; (a',b')\,, \tag{1.2}$$

for all $(a',b') \in W$. First compute

$$[a'\sigma_2'(b'a')b + (1-a'b')\sigma_1'(a'b')]a\sigma_2'(b'a')^{-1}\,.$$

Note that $a'\sigma_2'(b'a')ba\sigma_2'(b'a')^{-1} = a'b'a' = a'$ and the other summand equals zero because

$$(1-a'b')\sigma_1'(a'b')a \;=\; (1-a'b')\sigma_1'(a'b')aba \;=\; (1-a'b')a'b'\sigma_1'(a'b')a \;=\; 0\,.$$

Next compute

$$\sigma_2'(b'a')b[a'\sigma_2'(b'a')b + (1-a'b')\sigma_1'(a'b')]^{-1}\,,$$

which by (1.1) equals

$$\sigma_2'(b'a')b\big[a\sigma_2'(b'a')^{-1}b' + \sigma_1'(a'b')^{-1}(1-a'b')\big]\,.$$

The first summand gives $\sigma_2'(b'a')ba\sigma_2'(b'a')^{-1}b' = b'a'b' = b'$, and the second equals zero, because

$$b\sigma_1'(a'b')^{-1}(1-a'b') \;=\; bab\sigma_1'(a'b')^{-1}(1-a'b') \;=\; b\sigma_1'(a'b')a'b'(1-a'b') \;=\; 0\,.$$

Therefore (1.2) holds. □

**Remark 1.2** If $\sigma_k(W_k)$ is contained in $G_0$ for $k = 1,\ 2$, then $\sigma(W)$ is contained in the connected component of the identity in $G_A \times G_A$, which is $G_0 \times G_0$.

**Corollary 1.3** *The orbits of the action of $G_A \times G_A$ are open and closed. In particular, the connected component of $(a, b)$ in $S$ is $(G_A \times G_A)_0 \cdot (a, b)$.*

The last result proves that

$$C_{(a,b)} \ = \ \big\{ \big(uav^{-1}, vbu^{-1}\big) : u, v \in G_0 \big\}$$

and

$$O_{(a,b)} \ = \ \big\{ \big(uav^{-1}, vbu^{-1}\big) : u, v \in G_A \big\}\,.$$

**Example 1.4** 1. If $a = b = 0$, then $(a, b) \in S$ and $C_{(a,b)} = O_{(a,b)} = \{(0, 0)\}$.

2. If $a \in G$ and $b = a^{-1}$, then $(a, b) \in S$, $C_{(a,b)} = \{(u, u^{-1}) : u \in G_0\}$ and $O_{(a,b)} = \{u, u^{-1} : u \in G\}$.

3. If $a = p \in Q$ and $b = p$ then $(a, b) \in S$, here $C_{(a,b)} = \{(upv^{-1}, vpu^{-1}) : u, v \in G_0\}$ and, analogously, $O_{(a,b)} = \{(upv^{-1}, vpu^{-1}) : u, v \in G\}$

We begin now the geometrical study of the orbit of $S$

**Theorem 1.5** *The orbit $O_{(a,b)}$ is an analytical submanifold of $A \times A$, and the map*

$$\pi_{(a,b)} \ : \ G_A \times G_A \ \longrightarrow \ O_{(a,b)}\,, \quad \pi_{(a,b)}(w, z) \ = \ (w, z)\,.\,(a, b)$$

*is an analitic submersion.*

P r o o f. Consider the map $s : A \times A \to A \times A$ given by

$$s(x, y) \ = \ \big(xb + (1 - xy)(1 - ab),\ yxba + (1 - yx)(1 - ba)\big)\,.$$

Note that $s$ restricted to the open neighbourhood $W \subset O_{(a,b)}$ of (1.1) above coincides with $\sigma$. It is clearly an analytic map. Now the map $\pi_{(a_0,b_0)}$ is open, because it has continuous local cross sections. Let us regard it as a map from $G_A \times G_A$ to $A \times A$ in order to differentiate it. Denote by $\delta = d\big(\pi_{(a,b)}\big)_{(1,1)}$. We claim that the kernel and the image of $\delta : T(G_A \times G_A)_{(1,1)} \simeq A \times A \to A \times A$ are complemented subspaces of $A \times A$. Indeed, the identity

$$\pi_{(a,b)} \circ s \circ \pi_{(a,b)} \ = \ \pi_{(a,b)}$$

holds on a neighbourhood of $(1, 1)$ in $G_A \times G_A$ (which is open in $A \times A$), because $s$ coincides with $\sigma$ on a neighbourhood of $(a, b) = \pi_{(a,b)}(1, 1)$. By differentiating this identity one gets

$$\delta \circ ds_{(a,b)} \circ \delta \ = \ \delta\,.$$

This implies that $\delta$ is relatively regular in $L(A \times A)$, and therefore has complemented kernel and image. We now use Proposition 1.5 of [31], which states that in this case $O_{(a,b)}$ is a submanifold and $\pi_{(a,b)}$ is a submersion.

It follows that each orbit $O_{(a,b)}$ (which is a union of connected components of $S$) is an analytic homogeneous space of $G_A \times G_A$. □

The tangent space $TS_{(a,b)}$ can be computed in two ways. First, a tangent vector at $(a, b)$ is the derivative at $t = 0$ of a curve $(a(t), b(t)) \in S$ such that $(a(0), b(0)) = (a, b)$, thus

$$TS_{(a,b)} \ = \ \{(x, y) \in A \times A : xba + abx + aya = x,\ yab + bay + bxb = y\}\,.$$

Second, using the fact that $\pi_{(a,b)}$ is a submersion, it holds

$$TS_{(a,b)} \ = \ \operatorname{Im}\delta \ = \ \{(xa - ay, yb - bx) : x, y \in A\}\,.$$

Note that this subspace is complemented in $A \times A$.

We shall need the following results.

**Proposition 1.6** *Let $p, q \in Q$. Then the following properties are equivalent:*
1. *there exist $x, y \in A$ such that $xp = qx$, $py = yq$, $yxp = p$, $xyq = q$;*
2. *there exist $a, b \in$ such that $aba = a$, $bab = b$, $ap = qa$, $pb = bq$, $bap = p$, $abq = q$;*
3. *there exist $w, z \in A$ such that $wz = q$, $zw = p$;*
4. *there exist $c, d \in A$ such that $cdc = c$, $dcd = d$, $cd = q$, $dc = p$.*

P r o o f. The proof is straightforward. To prove that 1 implies 2, put $a = xp$ and $b = py$. To prove that 2 implies 3 put $w = a$ and $z = b$. To prove that 3 implies 4 put $c = qw$ and $d = zq$. To prove that 4 implies 1 put $x = c$ and $y = d$. □

**Proposition 1.7** *Each of the above properties defines an equivalence relation.*

P r o o f. If suffices to prove that 1. defines an equivalence relation. In fact, if $xp = qx$, $py = yq$, $yxp = p$, $xyq = q$ and $wq = rw$, $qz = zr$, $zwq = q$, $wzr = r$ then $wxp = rwx$, $pyz = yzr$, $yzwxp = p$, $wxyzr = r$. □

**Remark 1.8** If $(a, b) \in S$ and $b' \in A$, then $(a, b') \in S$ if and only if there exist $x, y \in A$ such that $ya = a$, $ax = a$, $b' = xby$ if and only if there exist $x, y \in G_A$ such that $ya = a$, $ax = a$, $b' = xby$.

P r o o f. $x = 1 - ba + b'a$, $y = 1 - ab + ab'$. □

Consider now the analytic map

$$\varphi \;:\; S \;\longrightarrow\; Q \times Q\,, \quad (a, b) \;\longmapsto\; (ab, ba)$$

which plays a relevant role in what follows. First, we observe that the last proposition characterizes the image of $\varphi$.

**Corollary 1.9** *Let $p, q \in Q$. The pair $(p, q)$ belongs to the image of $\varphi$ if and only if there exist $x, y \in A$ such that $xy = q$, $yx = p$.*

P r o o f. It suffices to prove that $xy = q$ and $yx = p$ then there exists $(a, b) \in S$ such that $ab = p$, $ba = q$. Take $b = qx$, $a = yq$. □

Observe first that $\varphi$ is compatible with the actions of $G_A \times G_A$ on $S$ and of $G_A$ on $Q$. In other terms, that $\varphi((w, z) \cdot (a, b)) = (w \cdot ab, z \cdot ba) = (wabw^{-1}, zbaz^{-1})$, for all $w, z \in G_A$, $(a, b) \in S$. The proof of the following two results is straightforward.

**Proposition 1.10** *For all $(a, b) \in S$ it holds $\varphi\big(O_{(a,b)}\big) = O_{ab} \times O_{ba}$.*

**Corollary 1.11** *If $(q, r) \in \operatorname{im} \varphi$ then $(q, uru^{-1}) \in \operatorname{im} \varphi$ for all $u \in G_A$.*

**Proposition 1.12** *The map $\varphi : O_{(a,b)} \to O_{ab} \times O_{ba}$ is an analytic submersion.*

P r o o f. Consider the following diagram

$$\begin{tikzcd} G_A \times G_A \arrow[r, "\pi_{(a,b)}"] \arrow[dr, "\pi_{ab}\times\pi_{ba}"'] & O_{(a,b)} \arrow[d, "\varphi"] \\ & O_{ab} \times O_{ba} \end{tikzcd}$$

where $\pi_{ab} : G_A \to O_{ab} = \{gabg^{-1} : g \in G_A\}$, $\pi_{ab}(g) = gabg^{-1}$ is the analytic submersion (i.e. homogeneous space) induced by the similarity action of $G_A$ in $Q$, $g\,.\,(ab) = gabg^{-1}$, and analogously for $ba$. It is apparent that this diagram is commutative. Denote by $\sigma_{ab}$ (resp. $\sigma_{ba}$) the analytic local cross section for $\pi_{ab}$ (resp. $\pi_{ba}$) as in (1.1). Then

$$\pi_{(a,b)} \circ (\sigma_{ab} \times \sigma_{ba})$$

is an analytical local cross section for $\varphi$. Therefore $\varphi$ is a submersion. □

Denote by $S_\Delta = \{(a, b) \in S : ab = ba\}$. Then $(a, b), (a, c) \in S_\Delta$ implies that $b = c$. The map

$$\phi \;:\; S_\Delta \;\longrightarrow\; Q$$

is surjective.

## 2 A connection in $S$

In this section we obtain a differential equation to lift smooth curves in $S$. This procedure leads to a notion of parallel transport, and a covariant derivative in $S$. In the previous section we used the local cross section $\sigma = \sigma_{(a,b)}$, which locally has the property

$$\sigma_{(a,b)}(a', b') \,.\, (a, b) \;=\; (a', b') \,.$$

Suppose that $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is a curve in $S$, for $t \in [0, 1]$. For a fixed $t \in [0, 1]$, let $\mathcal{P}$ be a partition $\{0 = t_0 < t_1 < \ldots < t_n = t\}$ of $[0, t]$ such that for each $i = 1, \ldots, n$, $\gamma(t_i)$ lies in the domain of $\sigma_{\gamma(t_{i-1})}$. It follows that the element

$$g_t \;=\; \sigma_{\gamma(t_k)}(\gamma(t))\sigma_{\gamma(t_{k-1})}(\gamma(t_k)) \ldots \sigma_{\gamma(0)}(\gamma(t_1))$$

is invertible and verifies $g_t \,.\, \gamma(0) = \gamma(t)$ (where $k$ is defined by $t_k < t \leq t_{k+1}$). Now we would like to take limit when the norms of the partitions $\mathcal{P}$ tend to zero, in order to obtain an invertible element $\Gamma(t) = \lim_{|\pi| \to 0} g_t$, which is independent of the partitions, and lifts the curve $\gamma$, i.e. $\Gamma(t) \,.\, \gamma(0) = \gamma(t)$ for all $t$. Instead of this, we shall obtain a linear differential equation, whose solution is $\Gamma$. If $h$ is small enough, then $t_k < t + h$ and therefore

$$\frac{g_{t+h} - g_t}{h} \;=\; \frac{\sigma_{\gamma(t)}(\gamma(t+h)) - 1}{h}\, g_t \,.$$

Taking limits here, with respect to partitions and with respect to $h$, one gets the equation

$$\Gamma'(t) \;=\; \frac{d}{dh}\, \sigma_{\gamma(t)}(\gamma(t+h))\Big|_{h=0} \Gamma(t) \,.$$

A straightforward computation shows that

$$\frac{d}{dh}\, \sigma_{\gamma(t)}(\gamma(t+h))\Big|_{h=0} \;=\; \Delta(t) \;=\; (\Delta_1(t), \Delta_2(t)) \,,$$

where

$$\Delta_1 \;=\; \gamma_1'\gamma_2 - \gamma_1\gamma_2' + \gamma_1\gamma_2'\gamma_1\gamma_2 \quad \text{and} \quad \Delta_2 \;=\; \gamma_2'\gamma_1 - \gamma_2\gamma_1' + 2\gamma_2\gamma_1'\gamma_2\gamma_1 \,. \tag{2.1}$$

The only fact used to obtain these formulae is that $\gamma_1\gamma_2\gamma_1 = \gamma_1$ implies that $\gamma_1\gamma_2'\gamma_1\gamma_2 + \gamma_1\gamma_2\gamma_1'\gamma_2 = 0$, and an analogous identity in the reverse order of the $\gamma_i$.

**Proposition 2.1** *Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a smooth curve in $S$ with $\gamma(0) = (a, b)$. Let $\Delta$ be as in* (2.1)*. The unique solution of the linear equation*

$$\begin{aligned} \Gamma'(t) \;&=\; \Delta(t)\Gamma(t) \,, \\ \Gamma(0) \;&=\; (1, 1) \end{aligned} \tag{2.2}$$

*is a curve $\Gamma : [0, 1] \to A \times A$ with values in $G_A \times G_A$ which satisfies*

$$\Gamma(t) \,.\, (a, b) \;=\; \gamma(t) \,, \quad t \in [0, 1] \,.$$

P r o o f. It is a standard fact that a solution of a linear equation with invertible initial condition stays invertible [14].

Let us see that $\Gamma = (\Gamma_1, \Gamma_2)$ lifts $\gamma$. Compute

$$\begin{aligned} \left(\Gamma_1^{-1}\gamma_1\Gamma_2\right)' \;&=\; -\,\Gamma_1^{-1}\Gamma_1'\Gamma_1^{-1}\gamma_1\Gamma_2 + \Gamma_1^{-1}\gamma_1'\Gamma_2 + \Gamma_1'\gamma_1\Gamma_2' \\ &=\; -\,\Gamma_1^{-1}(\gamma_1'\gamma_2 - \gamma_1\gamma_2' + \gamma_1\gamma_2'\gamma_1\gamma_2)\gamma_1\Gamma_2 + \Gamma_1^{-1}\gamma_1\Gamma_2 \\ &\quad +\, \Gamma_1^{-1}\gamma_1(\gamma_2'\gamma_1 - \gamma_2\gamma_1' + 2\gamma_2\gamma_1'\gamma_2\gamma_1)\Gamma_2 \,. \end{aligned}$$

We claim that this expression vanishes. It suffices to show that

$$0 \;=\; -\gamma_1'\gamma_2\gamma_1 + \gamma_1\gamma_2'\gamma_1 - \gamma_1\gamma_2'\gamma_1\gamma_2\gamma_2 + \gamma_1' + \gamma_1\gamma_2'\gamma_1 - \gamma_1\gamma_2\gamma_1' + 2\gamma_1\gamma_2\gamma_1'\gamma_2\gamma_1\,.$$

This is clearly the case, because

$$\gamma_1' \;=\; \gamma_1'\gamma_2\gamma_1 + \gamma_1\gamma_2'\gamma_1 + \gamma_1\gamma_2\gamma_1'\,.$$

Therefore $\Gamma_1^{-1}(t)\gamma_1(t)\Gamma_2(t)$ is constant. Since it equals $a$ for $t = 0$, it follows that $\gamma_1(t) = \Gamma_1(t)a\Gamma_2^{-1}(t)$. Analogously one sees that $\gamma_2(t) = \Gamma_2(t)b\Gamma_2^{-1}(t)$, i.e.

$$(\gamma_1(t), \gamma_2(t)) \;=\; (\Gamma_1(t), \Gamma_2(t))\,.\,(a, b)\,. \qquad \square$$

Let us introduce now a reductive structure on the homogeneous space

$$\pi_{(a,b)} \;:\; G_A \times G_A \;\longrightarrow\; O_{(a,b)} \subset S\,.$$

The isotropy group is $F = F_{(a,b)} = \{(k, h) \in G_A \times G_A : (k, h)\,.\,(a, b) = (a, b)\}$, i.e. $(k, h) \in F$ if $ka = ah$ and $hb = bk$. A reductive structure on the homogeneous space $(G_A \times G_A)/F$ is a decomposition $\mathcal{F} \oplus \mathcal{H}$ of the Lie algebra of $G_A \times G_A$, where $\mathcal{F}$ is the Lie algebra of $F$, and the supplement $\mathcal{H}$ is $Ad(F)$-invariant. Let us make these things precise in our particular context. First, the Lie algebra of $G_A \times G_A$ identifies with $A \times A$, with the usual conmutator as the Lie bracket. With this identification, $\mathcal{F}$ is apparently given by

$$\mathcal{F} \;=\; \{(x, y) \in A \times A : xa = ay,\; yb = bx\}\,.$$

For $(k, h) \in F$, $ad(k, h) : A \times A \to A \times A$ is the inner automorphism given by the invertible $(k, h)$, that is, $ad(k, h)(x, y) = (kxk^{-1}, hyh^{-1})$. This map is clearly linear, therefore $Ad(k, h) = ad(k, h)$. We must exhibit a complement for $\mathcal{F} \subset A \times A$ which is invariant for these inner automorphisms. To do so, differentiate

$$\sigma_{(a,b)} \;:\; \mathcal{V} \subset S \;\longrightarrow\; G_A \times G_A\,,$$
$$\Sigma \;=\; d\big(\sigma_{(a,b)}\big)_{(a,b)} : T(S)_{(a,b)} \;\longrightarrow\; A \times A\,, \quad \Sigma(x, y) \;=\; (xb - ay + ayab, ya - bx + 2bxba)\,.$$

As in (1.5), $\delta = d\big(\pi_{(a,b)}\big)_{(1,1)}$ and $\Sigma$ verify $\delta \circ \Sigma \circ \delta = \delta$. It follows that $\Sigma \circ \delta$ is an idempotent of $L(A \times A)$ whose kernel equals the kernel of $\delta$, which in turn coincides with $\mathcal{F}$. It follows that the range of $\Sigma \circ \delta$ is a supplement for $\mathcal{F}$. Explicitely,

$$\Sigma \circ \delta(x, y) \;=\; (xab - ayb + abx - abxab, yba + bay - 2bayba)\,.$$

We claim that $\mathcal{H} := R(\Sigma \circ \delta)$ is $Ad(F)$-invariant. Indeed, if $(k, h) \in F$, then

$$\begin{aligned}
&\Sigma \circ \delta\big(kxk^{-1}, hyh^{-1}\big)\\
&\quad= \big(kxk^{-1}ab - ahyh^{-1}b + abkxk^{-1} - abkxk^{-1}ab, hyh^{-1}ba + bayhyh^{-1} - 2bahyh^{-1}ba\big)\\
&\quad= \big(kxabak^{-1} - kaybk^{-1} + kabxk^{-1} - kabxabk^{-1}, hybah^{-1} + hbayh^{-1} - 2hbaybah^{-1}\big)\\
&\quad= (k, h)\big(\Sigma \circ \delta(x, y)\big)\big(k^{-1}, h^{-1}\big)\,.
\end{aligned}$$

Since this supplement $\mathcal{H}$ is obtained by means of $\sigma_{(a,b)}$, it follows that the Equation (2.2) lifts curves horizontally, i.e. if $\gamma = (\gamma_1, \gamma_2)$ is a smooth curve in $S$ with $\gamma(0) = (a, b)$, and $\Gamma$ is the solution of (2.2) with $\Gamma(0) = (1, 1)$, then $\Gamma'(t) \in \Gamma(t)\,.\,\mathcal{H}$

The reductive structure induces a linear connection in $S$. For example, if $\gamma \subset S$ is a smooth curve and $(x, y)$ is a tangent vector at $\gamma(0)$, then the parallel transport of $(x, y)$ along $\gamma$ is given by

$$\Gamma(t)\,.\,(x, y)\,,$$

where $\Gamma$ is the solution of (2.2) with $\Gamma(0) = (1, 1)$. In particular, given $(a, b) \in S$ and $(x, y) \in T(S)_{(a,b)}$, the unique geodesic $\gamma$ in $S$ satisfying $\gamma(0) = (a, b)$ and $\gamma'(0) = (x, y)$ is the curve

$$\gamma(t) \;=\; \big(e^{t\Sigma_1(x,y)}, e^{t\Sigma_2(x,y)}\big)\,.\,(a, b)\,.$$

These are standard facts from the theory of homogeneous reductive spaces. For a survey on the theory in the context of operator algebras, see [22].

## 3 Partial isometries

In this section we consider the case when $A$ is a C$^*$-algebra, and focus on pairs $(a,b) \in S$ such that $a^* = b$, i.e. $a = u$ is a partial isometry.

**Proposition 3.1** *Let $u, v$ be two partial isometries of $A$ such that $\|u - v\| < 1$. Then there exist unitaries $\nu$ and $\gamma$ in $A$ such that*

$$\gamma u \nu^* \; = \; v \, .$$

*Moreover, $\gamma$ and $\nu$ are $C^\infty$ formulas in terms of $u$ and $v$.*

This result has been obtained in the case of the algebra $L(H)$ by Halmos and Mc Laughlin in [18] (see also a proof in [25]) and extended by Mbekhta and Strătilă for von Neumann algebras [26]. The following proof is valid for any unital C$^*$-algebra.

Proof. Denote $p = u^*u$ and $q = v^*v$. We claim that if $\|u - v\| < 1$ then $p$ and $q$ are unitarily equivalent. Indeed, as in (3.1) of [26]

$$\|p - pq\| \; = \; \|u^*u - u^*uv^*v\| \; \leq \; \|u(1 - v^*v)\| \; = \; \|(u - v)(1 - v^*v)\| \; < \; 1 \, .$$

It is a folklore fact that two such projections $p$, $q$ are unitarily equivalent in $A$. We give a proof. Clearly this inequality implies that $\|p - pqp\| < 1$, and therefore the element $pqp$ is invertible in $pAp$. Let $w_1 = q(pqp)^{-1/2}$, where the inverse of $pqp$ is taken in $pAp$. Note that $qp = w_1|qp|$ is the polar decomposition of $qp$. Indeed, $w_1$ is a partial isometry:

$$w_1^* w_1 \; = \; (pqp)^{-1/2} q (qpq)^{-1/2} \; = \; (pqp)^{-1/2} pqp ((qpq)^{-1/2} \; = \; p \, ,$$

and

$$w_1 w_1^* \; = \; qp(pqp)^{-1} pq \, .$$

Note that this term is a projection $r$, with $r \leq q$. Reasoning analogously one obtains also that $qpq$ is invertible in $qAq$, therefore the range of $qp$ equals the range of $q$. It follows that $r$ and $q$ have the same range, and then $r = q$. Thus we have proved that $p$ and $q$ are equivalent in $A$, and that this equivalence is implemented by $w_1$. Now $\|(1 - p) - (1 - p)(1 - q)\| = \|q - pq\| < 1$ for identical reasons. It follows that there is another partial isometry $w_2$ implementing equivalence between $1 - p$ and $1 - q$, which is also an explicit formula in terms of $p = u^*u$ and $q = v^*v$. Then $\nu = w_1 + w_2$ is a unitary element of $A$ such that

$$\nu u^* u \nu^* \; = \; v^* v \, .$$

Analogously, one constructs an explicit $\sigma \in U_A$ such that

$$\sigma u u^* \sigma^* \; = \; v v^* \, .$$

Let us emphasize the fact that these unitaries can be obtained as real analytic functions (in fact sums of powers of the elements $u^*u$, $uu^*$, $v^*v$ and $vv^*$).

It follows that the partial isometries $\sigma u \nu^*$ and $v$ have the same initial and final spaces. Therefore the element

$$\omega \; = \; v(\sigma u \nu^*)^* + (1 - vv^*)$$

is a unitary of $A$. Moreover

$$\omega \sigma u \nu^* \; = \; v(\sigma u \nu^*)^* (\sigma u \nu^*) + (1 - vv^*) \sigma u \nu^* \, .$$

Note that the second term equals zero, and that in the first term $(\sigma u \nu^*)^*(\sigma u \nu^*) = v^* v$. Therefore

$$\omega \sigma u \nu^* \; = \; v v^* v \; = \; v \, .$$

Choose $\gamma = \omega \sigma$. □

The unitary $\gamma$ above can be simplified,

$$\gamma \;=\; \omega\sigma \;=\; v\nu u^* + \sigma(1 - uu^*)\,.$$

These formulas define a cross section for the action of the group $U_A \times U_A$ on partial isometries, $(\omega_1, \omega_2)\,.\,u = \omega_1 u \omega_2^*$, which is the restriction of the former action of $G_A \times G_A$ on general pairs (partial isometries are identified with pairs $(a, a^*)$ in $S$). From the above proposition one obtains local cross section for this action, namely

$$\mu_u \;:\; \{v : vv^*v = v,\ \|u - v\| < 1\} \;\longrightarrow\; U_A \times U_A\,, \quad \mu_u(v) \;=\; (\gamma, \nu)\,. \tag{3.1}$$

The following result is a generalization of Theorem 3.1 of [26].

**Corollary 3.2** *Let $u$, $v$ be two partial isometries in $A$. Then $u$ and $v$ are homotopic (i.e. connected by a continuous path of partial isometries) if and only if there exist unitaries $\gamma$ and $\nu$ in the conected component of $1$ in $U_A$ such that $\gamma u \nu^* = v$.*

P r o o f. The only if part is clear. Suppose that $u(t)$ is a continuous path of partial isometries such that $u(0) = u$ and $u(1) = v$. Let $\{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a partition of the unit interval such that for consecutive $t_i$ it holds that $\|u(t_i) - u(t_{i+1})\| < 1$. It follows that $u(t_i)$ and $u(t_{i+1})$ are conjugate by a pair of unitaries $\gamma_i$, $\nu_i$ which are of the form

$$\gamma_i \;=\; e^{ix_i} \quad \text{and} \quad \nu_i \;=\; e^{iy_i}\,,$$

for $x_i$, $y_i$ selfadjoint elements in $A$. Therefore

$$v \;=\; \gamma_{n-1} \ldots \gamma_0 u \nu_0^* \ldots \nu_{n-1}^* \;=\; e^{ix_{n-1}} \ldots e^{ix_1} u e^{-iy_0} \ldots e^{-iy_{n-1}}\,.$$

Clearly these unitaries belong to the connected component of $1$. □

**Proposition 3.3** *The set $\mathcal{I}$ of partial isometries is a $C^\infty$ submanifold of $A$. If $\mathcal{I}$ is identified with the set of pairs $\{(a, b) \in S : b = a^*\}$, it is a $C^\infty$ submanifold of $S$.*

P r o o f. The proof proceeds as in the analogous fact of the previous section. The cross section (3.1) can be extended to an open neighbourhood of $u$ in $A$. The same argument holds if one regards partial isometries as pairs. This proves that the unitary orbits of $u$ are submanifolds, both of $A$ and $S$. Next note that the set of partial isometries is a discrete union of unitary orbits, because partial isometries at distance less than $1$ are unitarily equivalent. Therefore the whole set of partial isometries is a submanifold. □

As with $S$, the tangent space $T\mathcal{I}_u$ can be computed in two ways,

$$T\mathcal{I}_u \;=\; \{x \in A : x = xu^*u + uu^*x + ux^*u\}\,,$$

or equivalently

$$T\mathcal{I}_u \;=\; \{xu - uy : x, y \in A \text{ with } x^* = -x,\ y^* = -y\}\,.$$

This space is complemented in $A$.

The next result shows that the restriction of $\varphi : S \to Q \times Q$ to $\mathcal{I}$ has nice geometric properties.

**Proposition 3.4** *The map*

$$\varphi \;:\; \mathcal{I} \;\longrightarrow\; P \times P\,, \quad \varphi(u) \;=\; (uu^*, u^*u)$$

*is a $C^\infty$ submersion.*

P r o o f. The proof proceeds analogously as in (1.12). □

Recall the differential equation (2.2), based on the local cross section for the action of $G_A \times G_A$. Our next result shows that the analogous equation lifts curves of partial isometries.

**Proposition 3.5** *Let $u : [0,1] \to \mathcal{I}$ be a smooth curve of partial isometries with $u(0) = u$. Then the unique solution $U$ of the equation in $A \times A$*

$$U' = \Lambda U, \quad U(0) = (1,1),$$

*where*

$$\Lambda = (\Lambda_1, \Lambda_2) = \big(u'u^* - uu^{*\prime} + uu^{*\prime}uu^*, u^{*\prime}u - u^*u' + 2u^*u'u^*u\big),$$

*is a curve $U(t) = (\nu_1(t), \nu_2(t))$ in $U_A \times U_A$ which satisfies*

$$U(t) . u = u(t), \quad t \in [0,1].$$

P r o o f. Note that

$$(\nu_1\nu_1^*)' = \Lambda_1\nu_1\nu_1^* + \nu_1\nu_1^*\Lambda_1^*,$$

i.e. $\nu_1\nu_1^*$ is a solution of the linear equation

$$X' = \Lambda_1 X + X\Lambda_1^*, \quad X(0) = 1.$$

The map $X \equiv 1$ is also a solution, indeed,

$$\begin{aligned}\Lambda_1 + \Lambda_1^* &= u'u^* - uu^{*\prime} + uu^{*\prime}uu^* + \big(u'u^* - uu^{*\prime} + uu^{*\prime}uu^*\big)^* \\ &= uu^{*\prime}uu^* + uu^*u'u^* = 0.\end{aligned}$$

This last fact is obtained as in (2.1), differentiating $u^*uu^*u = u^*u$. It follows that $\nu_1\nu_1^* \equiv 1$. In order to prove that $\nu_1^*\nu_1 \equiv 1$, observe that

$$(\nu_1^*\nu_1)' = \nu_1^*\Lambda_1^*\nu_1 + \nu_1^*\Lambda_1\nu_1 = 0.$$

Therefore $\nu_1$ is a curve of unitaries. Analogously, one proves that $\nu_2$ is a curve of unitaries. To finish the proof, note that the curve $\gamma = (u, u^*)$ is a curve in $S$, and that for this $\gamma$, the function $\Delta$ of (2.2) coincides with $\Lambda$. This implies, by (2.1), that the solution $U$ lifts $\gamma$, and $u$. □

As in the previous section, one can prove that this equation is in fact the transport equation of a linear connection in the space $\mathcal{I}$ of partial isometries. We shall study this connection elsewhere.

In [2] and [4] there is a study of the structure of the set of partial isometries of a $C^*$-algebra with initial space $p \in A$:

$$\mathcal{S}_p(A) = \{x \in A : x^*x = p\}.$$

Clearly $\mathcal{S}_p(A) \subset \mathcal{I}$. Moreover, one has the following:

**Remark 3.6** Let $\varphi_2 : \mathcal{I} \to P$, $\varphi_2(u) = u^*u$ (i.e. the second coordinate of $\varphi$ above). Then $\varphi_2$ is a submersion (in fact, a retraction), and $\mathcal{S}_p(A) = \varphi_2^{-1}(p)$. Therefore $\mathcal{S}_p(A)$ is a $C^\infty$ submanifold of $\mathcal{I}$.

Let us finish this section with an example, which illustrates how the maps here defined can be used to determine properties of the spaces. Suppose that $A$ is a von Neumann algebra, and let $u \in \mathcal{I}$ such that $uu^*$ and $u^*u$ are unitarily equivalent. Then there exists a projection $p \in A$ in the orbit of $u$ under $U_A \times U_A$. Since the unitary group is connected in this case, the orbit of $u$ coincides with the connected component $\mathcal{I}_p$ of $p$ in $\mathcal{I}$. On the other hand, the orbit of $p$ under the (inner automorphism) action of $U_A$, coincides with the connected component of $p$ in $P$. The submersion

$$\varphi : \mathcal{I}_p \longrightarrow P_p \times P_p, \quad \varphi(u) = (uu^*, u^*u)$$

is in particular a fibre bundle, with fibre (over $p$) equal to the unitary group $U_{pAp}$.

**Proposition 3.7** *In the above situation, if $p$ is purely infinite, then $\mathcal{I}_p$ is simply connected. If both $p$ and $1-p$ are purely infinite, then $\mathcal{I}_p$ is contractible.*

P r o o f. The connected components of the space of projections of a von Neumann algebra are simply connected (see [4]). On the other hand, if $p$ is purely infinite, then the unitary group $U_{pAp}$ is contractible in the norm topology [11]. The tail of the homotopy exact sequence of $\varphi$ yields

$$0 \;=\; \pi_1(U_{pAp},1) \;\longrightarrow\; \pi_1(\mathcal{I}_p,p) \;\xrightarrow{\varphi_*}\; \pi_1(P_p \times P_p,(p,p)) \;=\; 0\,.$$

If also $1-p$ is purely infinite, consider the fibre bundle

$$U_A \;\longrightarrow\; P_p \quad \text{defined by} \quad u \;\longmapsto\; upu^*$$

with fibre $\{p\}' \cap U_A \;\simeq\; U_{pAp} \times U_{(1-p)A(1-p)}$ [13]. It follows that $U_A$, as well as $U_{pAp}$ and $U_{(1-p)A(1-p)}$, are contractible. Therefore $P_p$ has trivial homotopy groups of all orders. Using this fact in the homotopy exact sequence of $\varphi$, we get

$$0 \;=\; \pi_n(U_{pAp},1) \;\longrightarrow\; \pi_n(\mathcal{I}_p,p) \;\xrightarrow{\varphi_*}\; \pi_n(P_p \times P_p,(p,p)) \;=\; 0$$

for all $n$. Therefore $\mathcal{I}_p$ is a differentiable manifold with trivial homotopy groups. Then, by Palais' results [28] $\mathcal{I}_p$ is contractible. □

If $p$ is purely infinite but $1-p$ is finite, then $\pi_2(\mathcal{I}_p,p) = \pi_2(P_p \times P_p,(p,p))$ is non trivial. For example, if $(1-p)A(1-p)$ is a $II_1$-factor, this group equals $\mathbb{R} \times \mathbb{R}$ [4].

## 4 The bundle $S_\Delta$ of invertible elements

In this section we focus on the properties of the set

$$S_\Delta \;=\; \{(a,b) \in S : ab = ba\}\,.$$

First note that this set can be regarded as the bundle of the invertible groups $G_{qAq}$ of all corner subalgebras $qAq$ of $A$. Explicitely, the map

$$S_\Delta \;\xrightarrow{\sim}\; \bigcup_{q\in Q} G_{qAq} \;=\; \{(x,q) : x \in G_{qAq}\}\,, \quad (a,b) \;\longrightarrow\; (a,ab)$$

is a homeomorphism with inverse $(x,q) \mapsto (x,x^{-1})$ (inverse taken in $qAq$). The map $\varphi : S_\Delta \to Q$, $\varphi(a,b) = ab$, translates as the fibration of units

$$\bigcup_{q\in Q} G_{qAq} \;\longrightarrow\; Q\,, \quad (g,q) \;\longrightarrow\; q\,.$$

Let us establish the next result concerning the smooth structure of $S_\Delta$.

**Proposition 4.1** *$S_\Delta$ is an analytic submanifold of $A \times A$.*

P r o o f. Let $(a,b) \in S_\Delta$, with $ab = ba = q$. It is clear that the diagonal set $D \subset Q\times Q$, $D = \{(r,r) : r \in Q\}$ is an analytic submanifold. It follows that the connected component of $(a,b)$ in $S_\Delta$, which coincides with the connected component of $(a,b)$ in $\varphi^{-1}(D)$, is a submanifold. □

**Remark 4.2** 1. Let $(a,b) \in S$. Then $O_{(a,b)} \cap S_\Delta \neq \emptyset$ if and only if the idempotents $ab$ and $ba$ are similar. Indeed, if $(c,d) \in S_\Delta$ lies in the orbit of $(a,b)$, then there exist invertibles $g$, $k$ such that $c = gak^{-1}$ and $d = kbg^{-1}$, and therefore $gabg^{-1} = cd = da = kbak^{-1}$. On the other hand, if $ab = gbag^{-1}$ for some $g \in G_A$, then $(1,g)\,.\,(a,b) \in S_\Delta \cap O_{(a,b)}$.

2. It has been noticed that the connected components of $S$ coincide with the orbits under the action of $G_0$. The facts above imply that the connected component of $(a,b)$ in $S_\Delta$ is the intersection of $S_\Delta$ with the orbit of $(a,b)$ under the action of $G_0$.

3. If $q \in Q$, then $(q,q) \in S_\Delta$. Moreover, if $(a,b) \in S_\Delta$, then $O_{(a,b)} \cap S_\Delta$ contains an element $(q,q)$. Indeed, if $(a,b) \in S_\Delta$, the elements

$$\alpha \;=\; b^2+1-ab \quad \text{and} \quad \beta \;=\; b+1-ab$$

are invertible, with inverses $\alpha^{-1} = a^2+1-ab$ and $\beta^{-1} = a+1-ab$. Observe that

$$\alpha a\beta^{-1} \;=\; (b^2+1-ab)a(a+1-ab) \;=\; a^2b^2 \;=\; ab\,,$$

and analogously $\beta b\alpha^{-1} = ab$. This shows that $(ab,ba) \in O_{(a,b)} \cap S_\Delta$.

If $q \in Q$, consider the subgroup $G_q = \{g \in G_A : g(1-q) = 1-q\}$ of $G_A$. Its elements are of the form $g = x+1-q$, with $x \in G_{qAq}$, i.e. $G_q$ is the image of natural imbedding of $G_{qAq}$ in $G_A$. Note that $G_q \subset G_{qAq} \times G_{(1-q)A(1-q)} := H_q$. Apparently, $H_q = \{q\}' \cap G_A$. Consider the analytic map

$$\mu_q \;:\; G_A \times H_q \;\longrightarrow\; S_\Delta\,, \quad \mu_q(g,h) \;=\; \big(ghqg^{-1}, gq(gh)^{-1}\big)\,.$$

The image of $\mu_q$ equals $S_\Delta \cap O_{(q,q)}$ (or equivalently, the subbundle of the invertible groups $G_{rAr}$ for all idempotents $r$ which are similar to $q$). As remarked above, this intersection is a union of connected components of $S_\Delta$. Indeed, an element in $S_\Delta \cap O_{(q,q)}$ is of the form $(\alpha q\beta^{-1}, \beta q\alpha^{-1})$, with $\alpha q\alpha^{-1} = \alpha q\beta^{-1}\beta q\alpha^{-1} = \beta q\alpha^{-1}\alpha q\beta^{-1} = \beta q\beta^{-1}$. In other words, $\alpha^{-1}\beta \in H_q$. Therefore $(\alpha q\beta^{-1}, \beta q\alpha^{-1}) = \mu_q(\alpha, \alpha^{-1}\beta)$.

Let us compute now the fibre $F = \mu_q^{-1}(q,q)$. A pair $(g,h) \in F$ if $ghqg^{-1} = q$ and $gqh^{-1}g^{-1} = q$. This clearly implies that $g$ commutes with $q$ and $hq = q$. It is apparent that the converse is also true. It follows that

$$F \;=\; H_q \times G_{1-q}\,,$$

which is a subgroup of $G_A \times G_A$. Note though that $\mu_q$ is not defined by an action on $S_\Delta$ (we found no natural action on $S_\Delta$ insofar). However, we have the following result:

**Proposition 4.3** *The map*

$$\mu_q \;:\; G_A \times H_q \;\longrightarrow\; S_\Delta \cap O_{(q,q)}\,, \quad \mu_q(g,h) \;=\; \big(ghqg^{-1}, gqh^{-1}g^{-1}\big)$$

*is a locally trivial fibre bundle with fibre $F = H_q \times G_{1-q}$.*

P r o o f. Let us construct smooth local cross sections for $\mu_q$. First on a neighbourhood of $(q,q)$ in $S_\Delta$. Let $(a,b) \in S_\Delta$ be close enough to $(q,q)$ so that the element $\gamma = abq + (1-ab)(1-q)$ is invertible. This condition defines a neighbourhood $\mathcal{U}_{(q,q)}$ of $(q,q)$ in $S_\Delta$. Put

$$g \;=\; a^2\gamma + \gamma(1-q) \quad \text{and} \quad h \;=\; \gamma^{-1}a\gamma + 1 - q\,.$$

The element $g$ is invertible, with inverse $g^{-1} = \gamma^{-1}b^2 + (1-q)\gamma^{-1}$. The inverse of $h$ is $\gamma^{-1}b\gamma + 1 - q$. Note also that $h$ commutes with $q$, i.e. $h \in H_q$ (in fact $h \in G_q$). Finally, note that

$$\begin{aligned} ghqg^{-1} \;&=\; \big(a^2\gamma + \gamma(1-q)\big)\big(\gamma^{-1}a\gamma + 1 - q\big)q\big(\gamma^{-1}b^2 + (1-q)\gamma^{-1}\big) \\ &=\; \big(a^2\gamma + \gamma(1-q)\big)\gamma^{-1}a\gamma q\gamma^{-1}b^2 \\ &=\; a^3(ab)b^2 + \gamma(1-q)\gamma^{-1}a\gamma q\gamma^{-1}b^2\,. \end{aligned}$$

Note that since $ab = ba$, $a^3(ab)b^2 = a^4b^3 = a$. The other term vanishes, $\gamma(1-q)\gamma^{-1}a = (1-ab)a = 0$. Therefore $ghqg^{-1} = a$. Analogously, $gqh^{-1}g^{-1} = b$. It follows that

$$(a,b) \;\longmapsto\; (g,h) \;=\; s_{(q,q)}(a,b)\,, \quad (a,b) \in \mathcal{U}_{(q,q)}$$

is a local cross section for $\mu_q$ near $(q,q)$.

If now $(a_0,b_0)$ is an arbitrary point in $S_\Delta \cap O_{(q,q)}$, there exist $r,s \in G_A$ such that $(a_0,b_0) = (r,s)\,.\,(q,q)$. If $(a,b) \in S_\Delta$ is close enough to $(a_0,b_0)$ so that $(r^{-1},s^{-1})\,.\,(a,b)$ lies in the domain of $s_{(q,q)}$, then $s_{(a_0,b_0)}(a,b) = (rg, r^{-1}sh)$ $\big($where $(g,h) = s_{(q,q)}((r^{-1},s^{-1}).(a,b))\big)$ defines a cross section near $(a_0,b_0)$.

Let us exhibit a trivialization for $\mu_q$ near $(q,q)$. Trivializations near other points of $S_\Delta$ are obtained analogously. Consider the map

$$\phi \;=\; \phi_{(q,q)} \;:\; \mathcal{U}_{(q,q)} \times F \;\longrightarrow\; \mu_q^{-1}\big(\mathcal{U}_{(q,q)}\big)\,, \quad \phi((a,b),(f_1,f_2)) \;=\; \big(g, f_1^{-1}hf_1\big)(f_1,f_2)\,,$$

where $(g,h) = s_{(q,q)}(a,b)$. Clearly this map is smooth, and is well defined, because $f_1, f_2$ and $h$ lie in $H_q$. Note that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{U}_{(q,q)} \times F \arrow[r, "\phi"] \arrow[dr, "P_1"'] & \mu_{(q,q)}^{-1}(\mathcal{U}_{(q,q)}) \arrow[d, "\mu_q"] \\
& \mathcal{U}_{(q,q)}
\end{tikzcd}$$

where $P_1((a,b),(f_1,f_2)) = (a,b)$. Indeed,

$$\begin{aligned}
\mu_q(\phi((a,b),(f_1,f_2)) \;&=\; \mu_q\big(gf_1, f_1^{-1}hf_1f_2\big) \\
&=\; \big(gf_1f_1^{-1}hf_1f_2qf_1^{-1}g^{-1}, gf_1qf_2^{-1}f_1^{-1}h^{-1}f_1f_1^{-1}g^{-1}\big) \\
&=\; \big(ghf_1f_2qf_1^{-1}g^{-1}, gf_1qf_2^{-1}f_1^{-1}h^{-1}g^{-1}\big)\,.
\end{aligned}$$

Note that $(f_1,f_2) \in F$ means that $f_1$ commutes with $q$ and $f_2q = qf_2 = q$ $\big($and $f_2^{-1}q = qf_1^{-1} = q\big)$. Therefore this last expression equals

$$\big(ghqg^{-1}, gqh^{-1}g^{-1}\big) \;=\; \mu_q\big(s_{(q,q)}(a,b)\big) \;=\; (a,b)\,.$$

Finally, $\phi$ is a diffeomorphism. Let us exhibit its inverse

$$\psi \;:\; \mu_q^{-1}\big(\mathcal{U}_{(q,q)}\big) \;\longrightarrow\; \mathcal{U}_{(q,q)} \times F\,, \quad \psi(g,h) \;=\; (\mu_q(g,h),(f_1,f_2))\,,$$

where $f_1$ and $f_2$ are constructed as follows. Denote by $(g',h') = s_{(q,q)}(\mu_q(g,h))$, and put

$$f_1 \;=\; (g')^{-1}g \quad \text{and} \quad f_2 \;=\; g^{-1}g'(h')^{-1}(g')^{-1}gh\,.$$

Clearly $\psi$ is smooth. Let us show that it is well defined, i.e. that $f_1 \in H_q$ and $f_2 \in G_{1-q}$. Note that $\mu_q(g,h) = \mu_q(g',h')$. This implies that

$$(gh,g)\,.\,(q,q) \;=\; (g'h',g')\,.\,(q,q)$$

or equivalently

$$\big((h')^{-1}(g')^{-1}gh, (g')^{-1}g\big)\,.\,(q,q) \;=\; (q,q)\,.$$

This implies $f_0 := (h')^{-1}(g')^{-1}gh$ and $f_1 = (g')^{-1}g$ satisfy

$$f_0q \;=\; qf_1 \quad \text{and} \quad f_1q \;=\; qf_0\,.$$

These relations imply that $f_0$ and $f_1$ commute with $q$ and $f_0q = f_1q$. In particular, $f_1 \in H_q$ as desired. On the other hand, note that $f_2 = f_1^{-1}f_0$, and therefore $f_2q = f_1^{-1}f_0q = q$, i.e. $f_2 \in G_{1-q}$. It is a routine verification to show that $\psi$ is the inverse of $\phi$. Trivializations near other points of $S_\Delta$ are obtained in an analogous manner. □

Let us consider now the particular case when $A$ is a von Neumann algebra. Here $G_A$ is connected, and the range of $\mu_q$ equals the connected component of $(q,q)$ in $S_\Delta$. Moreover, since any idempotent is similar to a selfadjoint projection, we may assume $q^* = q$. Now, the $\pi_1$ group of the unitary orbit of a projection in a von Neumann algebra is trivial (see [4], 4.5). This is equivalent to saying that in the fibre bundle

$$U_A \;\longrightarrow\; \{uqu^* : u \in U_A\}\,, \quad u \;\longmapsto\; uqu^*\,,$$

studied in [13], the inclusion map from the fibre $\{q\}' \cap U_A$ into $U_A$ is surjective at the $\pi_1$-level. We need to adapt this result to the similarity action.

**Lemma 4.4** *Let $q$ be a projection in $A$. Then the similarity orbit of $q$ is simply connected and the inclusion map $H_q \hookrightarrow G_A$ induces a surjection $\pi_1(H_q, 1) \to \pi_1(G_A, 1) \to 0$.*

Proof. The similarity action $G_A \times Q \to Q$ defined by $g\,.\,q = gqg^{-1}$, gives for every $q \in Q$ a fibre bundle

$$G_A \longrightarrow Q\,, \quad g \longmapsto gqg^{-1}$$

whose image is the connected component (which is also the similarity orbit because $A$ is a von Neumann algebra) of $q$ in $Q$, and whose fibre is $H_q$. Then, considering the homotopy exact sequence of this bundle, it suffices to prove that the similarity orbit of $q$ has trivial $\pi_1$ group. As noted above, we may choose $q = q^*$. Let $q(t) \in Q$ be a continuous curve with $q(0) = q(1) = q$. There is a lifting curve $g(t) \in G_A$ satisfying $q(t) = g(t)qg(t)^{-1}$. Let $g(t) = u(t)|g(t)|$ be the polar decomposition of $g(t)$. The set of invertible positive elements is convex, therefore one can perform the following deformation of the curve $q(t)$:

$$F(s,t) \;=\; u(t)(1-s+s|g(t)|)q(1-s+s|g(t)|)^{-1}u^*(t)\,, \quad (s,t) \in [0,1]\times[0,1]\,.$$

Note that for each fixed $s$, $F(s,t)$ is a curve in $Q$, with $F(1,t) = q(t)$ and $F(0,t)$ in the unitary orbit of $q$. Also note that $g(0)$ and $g(1)$ commute with $q$, and therefore also $u(0)$, $u(1)$, $|g(0)|$ and $|g(1)|$ commute with $q$. This implies that for each fixed $s$, $F(s,0) = F(s,1) = q$. In other words, $q(t)$ is homotopic to a curve in the unitary orbit of $q$, with a homotpy which fixes endpoints. By the result from [4] cited above, this curve $q(t)$ can be further deformed to a constant curve. □

**Theorem 4.5** *If $A$ is a von Neumann algebra and $q$ is a projection in $A$, then*

$$\pi_1(S_\Delta, (q,q)) \;=\; \pi_1(G_{qAq}, q)\,.$$

*In particular, if $q$ is a purely infinite projection, then the connected component of $(q,q)$ in $S_\Delta$ is contractible.*

Proof. Consider the tail of the homotopy exact sequence of the fibre bundle $\mu_q$,

$$\ldots \longrightarrow \pi_1(H_q,1)\oplus\pi_1(G_{1-q},1) \xrightarrow{i_*} \pi_1(G_A,1)\oplus\pi_1(H_q,1) \xrightarrow{(\mu_q)_*} \pi_1(S_\Delta,(q,q)) \longrightarrow 0\,.$$

Clearly the morphism $i_*$ induced by the inclusion $i : H_q \times G_{1-q} \hookrightarrow G_A \times H_q$ splits as the sum of

$$(i_1)_* \;:\; \pi_1(H_q,1) \longrightarrow \pi_1(G_A,1)\,,$$

which is onto by the lemma above, and the morphism

$$(i_2)_* \;:\; \pi_1(G_{1-q},1) \longrightarrow \pi_1(H_q,1)\,.$$

Note that $H_q = G_q \oplus G_{1-q}$, therefore the image of $(i_2)_*$ equals $\{0\}\oplus\pi_1(G_{1-q},1)$. Now

$$\begin{aligned}
\pi_1(S_\Delta,(q,q)) &= \big(\pi_1(G_A,1)\oplus\pi_1(H_q,1)\big)/\ker(\mu_q)_* \\
&= \big(\pi_1(G_A,1)\oplus\pi_1(H_q,1)\big)/\mathrm{Im}\, i_* \\
&= \big(\pi_1(G_A,1)\oplus\pi_1(H_q,1)\big)/\big(\pi_1(G_A,1)\oplus\pi_1(G_{1-q})\big) \\
&= \pi_1(H_q,1)/\pi_1(G_{1-q},1) \\
&= \pi_1(G_q,1)\,.
\end{aligned}$$

Observe that $G_q \simeq G_{qAq}$.

If $q$ is a purely infinite projection, then $G_{qAq}$ is contractible [11]. The connected component of $(q,q)$ is therefore a differentiable manifold with trivial homotopy groups of all orders, it follows, again by Palais' results [28], that it is contractible. □

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