Physics of Fluids

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Cite as: Phys. Fluids **31**, 083108 (2019); https://doi.org/10.1063/1.5110480
Submitted: 17 May 2019 . Accepted: 07 August 2019 . Published Online: 27 August 2019

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Physics of Fluids ARTICLE scitation.org/journal/phf

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ABSTRACT

This work focused on obtaining an improved and expanded general theoretical analysis of a two-dimensional film draining on a quasivertical plate, solving rigorous mass, momentum, and energy balances. A dimensional analysis and scaling was used to simplify the mathematical description, and a generalized Newtonian fluid was assumed as the film-forming material. A new quantity that governs the draining flow and film characteristics, called viscous dissipation, was proposed as part of the novel analytical expressions obtained in this work. Velocity profile, average velocity, flow rate, and local and average film thickness expressions can be obtained, allowing to simplify the overall calculation complexity and to find new potential analytical expressions using more complex rheological models.

I. INTRODUCTION

Draining flow is common in many industrial processes. For example, it is considered a critical step in self-metered free coating techniques, such as dip coating, because it is assumed that draining will determine the thickness and global characteristics of the film.[1] If the draining flow is considered as an isothermal and nonevaporative phenomenon, as occurs in many practical situations, the analytical solutions of balance equations can be obtained. In this sense, efforts in previous works have been made in order to obtain analytical expressions for the main outputs (velocity, flow rate, and film thickness) that describe the film draining flow on regular geometries, such as vertical plates.[2–5]

During free draining, the shear flow of the film-forming material is due mainly to gravity forces that generate the movement of the film. The energy that must be supplied to maintain the relative motion of a fluid under simple shear, and that is often considered to be a dissipated power, is usually referred to as viscous dissipation.[6,7]

To the author's knowledge, the influence of viscous dissipation on the draining flow behavior has not yet been analyzed and discussed in the literature. For this reason, the objective of this work was to obtain an improved and expanded general theoretical analysis of a two-dimensional draining film on a quasivertical plate, solving rigorous mass, momentum, and energy balances. The mathematical description was simplified using a dimensional analysis and scaling, while a generalized Newtonian fluid was assumed as the film-forming material. A new quantity that governs the draining flow and film characteristics, called viscous dissipation, was proposed as part of the novel analytical expressions obtained here. This study proposes to increase the number of ways of calculation and the interconnection of the main variables helping to obtain their analytical expressions for a given rheological model.

II. THEORETICAL APPROACH

The problem described in this work is derived from a simplified description of the film draining process shown in Fig. 1. Here, a film is drained from the surface of a quasivertical plate under the effect of gravity. Basically, at the beginning ($t = 0$), a plate is submerged into a vessel containing a film-forming fluid. The surface plate is perfectly in contact with the fluid. Then, the plate is withdrawn from the fluid letting gravity to act onto the film. This produces deformations in the film that lead to draining. The flow that arises from this process can be modeled from mass, momentum, and energy balances. In the following sections, a general but simple model to describe the main variables in a film draining process will be presented.

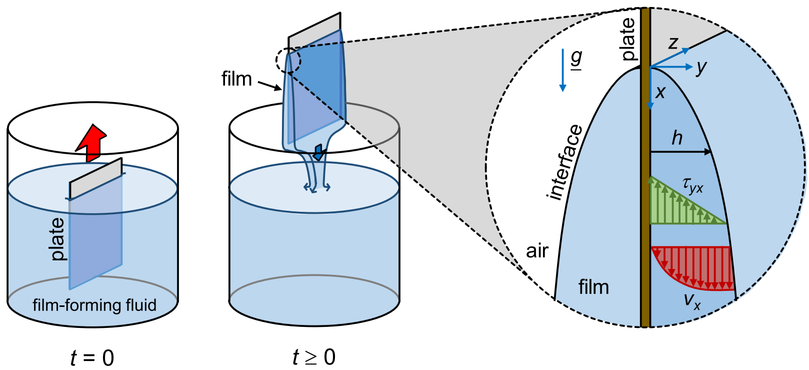

FIG. 1. Film draining process representation.

A. Equations of change

The main equations that describes the phenomena taken place in the system shown in Fig. 1 are as follows:[8]

- mass balance (continuity),

$$\frac{\partial \rho}{\partial t} + \underline{\nabla} \cdot (\rho \underline{v}) = 0, \tag{1}$$

- momentum balance (Cauchy),

$$\rho\left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \underline{\nabla}\, \underline{v}\right) = -\underline{\nabla} p - \underline{\nabla} \cdot \underline{\underline{\tau}} + \underline{F}_e, \tag{2}$$

- kinetic energy balance,

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho v^2\right) = -\underline{\nabla} \cdot \tfrac{1}{2}\rho v^2 \underline{v} - \underline{\nabla} \cdot p\underline{v} + p\left(\underline{\nabla} \cdot \underline{v}\right) - \underline{\nabla} \cdot \left(\underline{\underline{\tau}} \cdot \underline{v}\right) + \underline{\underline{\tau}} : \underline{\nabla}\, \underline{v} + \underline{v} \cdot \underline{F}_e, \tag{3}$$

where ρ is the density, t is the time, $\underline{v}$ is the velocity vector, p is the thermodynamic pressure, $\underline{\underline{\tau}}$ is the shear stress tensor, and $\underline{F}_e$ is the volumetric forces vector. Now, due to the complexity of the system expressed by Eqs. (1)–(3), analytical solutions are extremely difficult or impossible to obtain. Several methods can be used to tackle this issue and obtain useful mathematical models. Here, a combination of assumptions and a basic dimensional analysis will be used.

The main assumptions and preferences used in this work are as follows: (1) the film-forming fluid is incompressible ($\rho \neq f(\underline{x}, t)$), (2) the external forces are mainly gravitational ($\underline{F}_e = \rho \underline{g}$), (3) the surface interactions are negligible (capillary number Ca $\gg 1$), (4) the system is open ($\underline{\nabla} p \approx 0$), (5) the system can be represented in Cartesian coordinates ($\underline{x} = x\underline{i} + y\underline{j} + z\underline{k}$), (6) the problem is mainly two dimensional (i.e., $v_z \approx 0$ and changes in the z-direction are negligible), and (7) gravity acts in the x-direction ($\underline{g} = g_x \underline{i}$). Some additional minor assumptions will be made throughout the manuscript when needed.

Consequently, a dimensional analysis is useful in order to obtain simpler expressions of those equations that are also representative of the phenomena taking place in the studied process. The following dimensionless variables are defined:

$$\begin{aligned}
&\tilde{v}_x = \frac{v_x}{U}, \quad \tilde{v}_y = \frac{v_y}{V}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{y} = \frac{y}{h_L}, \\
&\tilde{v} = \frac{v}{\sqrt{V^2 + U^2}} = \frac{v}{\sqrt{\varepsilon^2 U^2 + U^2}} = \frac{v}{U\sqrt{\varepsilon^2 + 1}}, \\
&\tilde{\tau}_{xy} = \tilde{\tau}_{yx} = \frac{\tau_{xy}}{\eta_R(U/h_L + V/L)} = \frac{\tau_{xy}}{\eta_R(U/h_L)(1 + \varepsilon^2)}, \\
&\tilde{\tau}_{xx} = \frac{\tau_{xx}}{\eta_R(U/L)}, \quad \tilde{\tau}_{yy} = \frac{\tau_{yy}}{\eta_R(V/h_L)}, \\
&\tilde{p} = \frac{p}{\eta_R(U/h_L)}, \quad \tilde{t} = \frac{t}{(L/U)}, \quad \varepsilon = \frac{h_L}{L},
\end{aligned} \tag{4}$$

where U and V are the reference velocities for the x-direction and y-direction, respectively (m/s), L is the length of the plate (m), h_L is the local thickness of the film at L (m), η_R is an apparent steady state viscosity at a reference condition, and ε is the slenderness of the system.

The dimensionless forms of the shear stress tensor components were chosen taking into account that:[9] $\tau_{yx} = -\eta(\partial v_y/\partial x + \partial v_x/\partial y)$, $\tau_{xx} = -2\eta(\partial v_x/\partial x)$, and $\tau_{yy} = -2\eta(\partial v_y/\partial y)$. Based on the previous assumptions and a nondimensionalization process, the significant dimensionless components of the momentum balance [Eq. (2)] can be written as

$$\mathrm{Re}\,\varepsilon\left(\frac{\partial \tilde{v}_x}{\partial \tilde{t}} + \tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right) = -\varepsilon \frac{\partial \tilde{\tau}_{xx}}{\partial \tilde{x}} - \left(\varepsilon^2 + 1\right) \frac{\partial \tilde{\tau}_{yx}}{\partial \tilde{y}} + \mathrm{St}, \tag{5}$$

$$\mathrm{Re}\,\varepsilon\left(\frac{\partial \tilde{v}_y}{\partial \tilde{t}} + \tilde{v}_x \frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right) = -\left(\varepsilon^2 + 1\right) \frac{\partial \tilde{\tau}_{xy}}{\partial \tilde{x}} - \frac{\partial \tilde{\tau}_{yy}}{\partial \tilde{y}}, \tag{6}$$

where St = Re/Fr is the Stokes number,[10] $\mathrm{Re} = \rho U h_L/\eta_R$ is the Reynolds number, and $\mathrm{Fr} = U^2/(g_x h_L)$ is the Froude number.

In a slender system, such as a film draining, the length of the film on the plate is much larger than the average thickness of the film. That is, $\varepsilon \ll 1$. Also, if the fluid is highly viscous and/or the film is thin, the flow tends to be laminar and $\mathrm{Re}\,\varepsilon \ll 1$. Then, the minimum set of dimensionless equations that can be used to describe the flow of a coating film during draining becomes

$$\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0, \tag{7}$$

$$\frac{\partial \tilde{\tau}_{yx}}{\partial \tilde{y}} = \mathrm{St}, \tag{8}$$

$$\frac{\partial \tilde{\tau}_{xy}}{\partial \tilde{x}} + \frac{\partial \tilde{\tau}_{yy}}{\partial \tilde{y}} = 0. \tag{9}$$

Here, the order of magnitude of Re and Fr should be similar in order to obtain an analytical solution different from a constant. Equations (7)–(9) are similar to the ones obtained through a lubrication approximation.

Equation (3) will be analyzed in Sec. II F.

Even though Eqs. (7)–(9) were obtained assuming that the gravitational effect on the y-direction was negligible, small tilt angles of the substrate can be accepted as long as the conditions presented in Sec. II B are fulfilled. So, the quasivertical plates can be described simply by using the following expression for g_x:[2]

$$g_x = |\underline{g}| \cos\theta, \tag{10}$$

where $|\underline{g}|$ is the magnitude of $\underline{g}$ and θ is the angle between the x-axis and the direction of $\underline{g}$.

B. Range of theoretical validity of the approach

An important feature of the approach presented in Sec. II A is the capability to estimate the expected range of validity of Eqs. (7)–(9). The following set of conditions was assumed to be true:

$$\varepsilon \ll 1, \quad \mathrm{Re}\,\varepsilon \ll 1, \quad \mathrm{St} = \mathcal{O}(1), \tag{11}$$

where $\mathcal{O}(1)$ means "in the order of one."

Here, as stated by Peralta *et al.*,[2] it is necessary to define two parameters in order to evaluate those conditions: (1) η_R and (2) U.

As $\mathrm{St} = \mathcal{O}(1)$, a valid assumption would be that $U = \mathcal{O}\left(\rho g_x h_L^2/\eta_R\right)$. The calculation of η_R will depend on the constitutive (rheological) model adopted to estimate η.

C. Constitutive equation

To solve the system of Eqs. (7)–(9), an equation that relates the film tension with the film deformation rates is needed. The simplest, but general, model to describe this relationship is the Generalized Newtonian Fluid (GNF).[8] Many important fluids are described as a GNF. In these materials, the relationship between the shear stress tensor and the shear rate tensor ($\underline{\underline{\dot{\gamma}}}$) is given by[9]

$$\underline{\underline{\tau}} = -\eta(\tau, \dot{\gamma}, p, T, C)\underline{\underline{\dot{\gamma}}}, \tag{12}$$

where η is the apparent steady-state viscosity, τ is the second invariant or norm of $\underline{\underline{\tau}}$, $\dot{\gamma}$ is the second invariant or norm of $\underline{\underline{\dot{\gamma}}}$, T is the temperature, and C is the concentration. The minus sign in Eq. (12) is adopted mainly for consistency purposes between momentum, heat, and mass transfer.[8]

Several models have been used to successfully estimate η for a wide range of fluids and operative conditions,[2,4,5] for example, Ofoli *et al.*, extended Quemada, and Carreau-Yasuda, among other models. These particular models are versatile because they include multiple simpler but important rheological models such as Carreau, Cross, Heinz-Casson, Casson, Herschel-Bulkey, Sisko, Ellis, Meter-Bird, Reiner-Phillipoff, and Bingham, among other models.

The values of τ and $\dot{\gamma}$ can be obtained from their definitions,[8]

$$\tau = \sqrt{\frac{1}{2}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}}, \tag{13}$$

$$\dot{\gamma} = \sqrt{\frac{1}{2}\,\underline{\underline{\dot{\gamma}}} : \underline{\underline{\dot{\gamma}}}}, \tag{14}$$

where $\underline{\underline{\dot{\gamma}}} = \underline{\nabla}\,\underline{v} + \left(\underline{\nabla}\,\underline{v}\right)^T$.

On one hand, assuming that $\underline{\underline{\tau}}$ is symmetric, and using two-dimensional Cartesian coordinates and the dimensionless parameters defined in Eq. (4), Eqs. (13) and (14) can be written as

$$\tilde{\tau} = \sqrt{\frac{\varepsilon^2}{2}\left(\tilde{\tau}_{xx}^2 + \tilde{\tau}_{yy}^2\right) + (1+\varepsilon^2)^2\tilde{\tau}_{yx}^2}, \tag{15}$$

$$\tilde{\dot{\gamma}} = \sqrt{2\varepsilon^2\left[\left(\frac{\partial \tilde{v}_x}{\partial \tilde{x}}\right)^2 + \left(\frac{\partial \tilde{v}_y}{\partial \tilde{y}}\right)^2\right] + \left(\frac{\partial \tilde{v}_x}{\partial \tilde{y}} + \varepsilon^2\frac{\partial \tilde{v}_y}{\partial \tilde{x}}\right)^2}, \tag{16}$$

where $\tilde{\tau} = \tau/[\eta_R(U/h_L)]$ and $\tilde{\dot{\gamma}} = \dot{\gamma}/(U/h_L)$.

On the other hand, and based on Eq. (8), the only component of $\underline{\underline{\tau}}$ needed to be computed is τ_{yx}. Then, using Eq. (4), the component $\tilde{\tau}_{yx}$ yields

$$\tilde{\tau}_{yx} = -\tilde{\eta}\left(\varepsilon^2\frac{\partial \tilde{v}_y}{\partial \tilde{x}} + \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right), \tag{17}$$

where $\tilde{\eta} = \eta/\eta_R$.

As $\varepsilon^2 \ll 1$, Eqs. (15)–(17) can be written in a dimensional form as

$$\tau = |\tau_{yx}|, \tag{18}$$

$$\dot{\gamma} = \left|\frac{\partial v_x}{\partial y}\right|, \tag{19}$$

$$\tau_{yx} = -\eta\frac{\partial v_x}{\partial y}. \tag{20}$$

Additionally, the shear stress acting on the film can be calculated from Eqs. (8), (19), and (20). In the dimensional form,

$$\tau_{yx} = -\rho g_x(h - y). \tag{21}$$

This expression, which agrees with the literature for film draining systems,[11] states that the shear stress profile in the y-direction is linear with a slope of ρg_x (Fig. 2). Moreover, the maximum shear stress (as a negative value due to the adopted sign convention) is estimated at the wall (i.e., $y = 0$) as $\tau_m = -\rho g_x h$ and the minimum is zero at the fluid-air interface (i.e. $y = h$). The profile of τ_{yx} is linear in the y-direction, regardless of the fluid nature.

D. Viscous dissipation

Viscous dissipation (P) can be defined as the rate of irreversible conversion from kinetic to internal energy and expressed as[8]

$$P = -\underline{\underline{\tau}} : \underline{\nabla}\,\underline{v}. \tag{22}$$

Taking into account that the components and variations of parameters in the z-direction are negligible, the symmetry of $\underline{\underline{\tau}}$, and

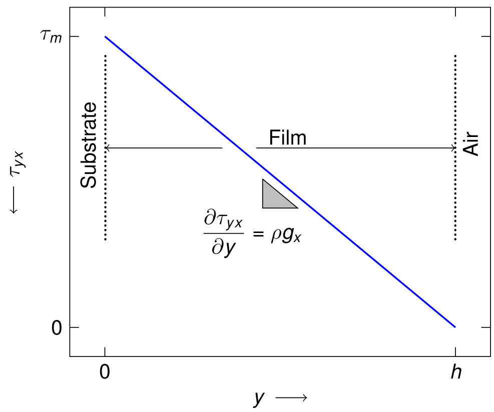

FIG. 2. General representation of the shear stress (τ_{yx}) film profile as a function of y.

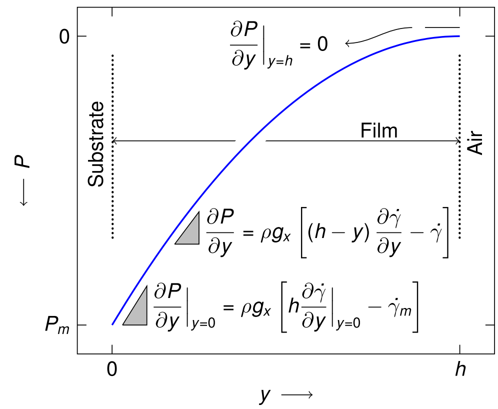

FIG. 3. General representation of the viscous dissipation (P) profile as a function of y.

Eq. (4), the dimensionless viscous dissipation ($\tilde{P}$) can be expressed as

$$\tilde{P} = -\varepsilon^2 \tilde{\tau}_{xx} \frac{\partial \tilde{v}_x}{\partial \tilde{x}} - \tilde{\tau}_{yx}\left(\frac{\partial \tilde{v}_x}{\partial \tilde{y}} + \varepsilon^2 \frac{\partial \tilde{v}_y}{\partial \tilde{x}}\right) - \varepsilon^2 \tilde{\tau}_{yy} \frac{\partial \tilde{v}_y}{\partial \tilde{y}}, \tag{23}$$

where $\tilde{P} = P/[\eta_R(U/h_L)]$. Then, considering that $\varepsilon^2 \ll 1$, Eq. (23) yields (in dimensional form)

$$P = -\tau_{yx} \frac{\partial v_x}{\partial y}. \tag{24}$$

Taking into account Eqs. (18) and (19), then

$$P = \tau \dot{\gamma}. \tag{25}$$

This expression shows that the rate at which the energy is lost irreversibly can be calculated by multiplying the norms of the shear stress tensor and the shear rate tensor and agrees with the result found in literature.[6] In practice, this energy must effectively be supplied continuously to maintain the relative motion under simple shear, despite the friction between fluid layers. Thermodynamically, this energy contributes to the increase in the internal energy of the system, generating the rise of its temperature. But the heat exchange with the surroundings, in particular, with the solid walls, may allow the system to remain at the same temperature. Although these effects are usually negligible for gases, they become significant for viscous liquids under high shear rates.[6]

A general viscous dissipation profile as a function of y, for a certain value of x, is presented in Fig. 3. The film dissipation shows a maximum value $P_m = -\tau_m \dot{\gamma}_m$ (maximum degradation of energy) at the substrate-fluid interface (i.e., $y = 0$). Conversely, a minimum value of zero is obtained at the air-fluid interface (i.e. $y = h$). The profile shows a convex shape with a maximum slope ($\rho g_x [h(\partial\dot{\gamma}/\partial y)_0 - \dot{\gamma}_m]$) at $y = 0$ and a slope of zero (minimum) at $y = h$.

E. Velocity profile

The velocity profile in the film can be calculated by integrating Eq. (19),

$$v_x = \int_0^y \dot{\gamma}\, dy. \tag{26}$$

Here, Eq. (26) needs to be rearranged accordingly based on the dependency of η on either τ or $\dot{\gamma}$. Taking into account Eqs. (18)–(20), the v_x can be calculated as

$$v_x = \frac{1}{\rho g_x} \int_\tau^{\tau_m} \frac{\tau}{\eta(\tau)} d\tau, \tag{27}$$

$$v_x = \frac{1}{\rho g_x} \int_{\dot{\gamma}}^{\dot{\gamma}_m} \left[\dot{\gamma}^2 \frac{\partial \eta(\dot{\gamma})}{\partial \dot{\gamma}} + \dot{\gamma}\, \eta(\dot{\gamma})\right] d\dot{\gamma}, \tag{28}$$

where τ_m is the maximum shear stress at position x and $\dot{\gamma}_m$ is the maximum shear stress at the same position x.

A general velocity profile as a function of y is shown in Fig. 4. The profile is concave with a maximum value v_m at the air-fluid interface and a minimum value of zero at $y = 0$. The profile slope is the local shear rate $\dot{\gamma}$ [Eq. (19)], so a maximum $\dot{\gamma}_m$ is expected at $y = 0$ and a minimum of zero at $y = h$.

F. Average velocity and flow rate

The area-averaged velocity ($\langle v_x \rangle_A$) can be calculated by

$$\langle v_x \rangle_A = \frac{1}{A} \int_A v_x\, dA, \tag{29}$$

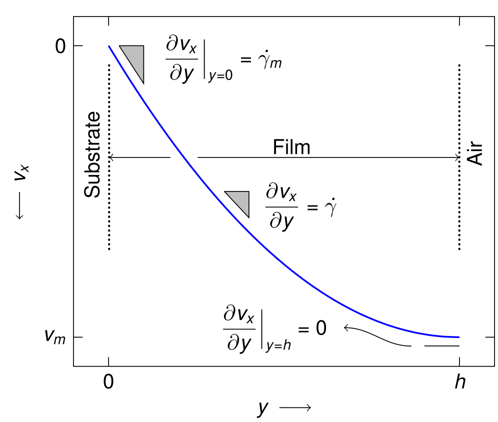

FIG. 4. General representation of the velocity profile (v_x) as a function of y.

where A is the perpendicular area to the flow at a position x. Here, $dA = dydz$ and $A = Wh$, where W is the plate width. So, Eq. (29) can be written as the average (in y-direction) velocity

$$\langle v_x \rangle_y = \frac{1}{h}\int_0^h v_x\, dy. \tag{30}$$

Also, defining the flow rate as $Q = \int_A v_x\, dA$, the average velocity can also be calculated as

$$\langle v_x \rangle_y = \frac{Q_W}{h}, \tag{31}$$

where Q_W is the flow rate per unit of the plate width.

An additional expression that can help to calculate the average velocity is obtained from the kinetic energy balance [Eq. (3)]. Taking into account Eq. (4), the symmetry of $\underline{\underline{\tau}}$, and that the components and variations in the z-direction are negligible, Eq. (3) yields

$$\begin{aligned} \mathrm{Re}\,\varepsilon\left[\frac{\partial\left(\frac{1}{2}\tilde{v}^2\right)}{\partial\tilde{t}} + \frac{\partial\left(\frac{1}{2}\tilde{v}^2\tilde{v}_x\right)}{\partial\tilde{x}} + \frac{\partial\left(\frac{1}{2}\tilde{v}^2\tilde{v}_y\right)}{\partial\tilde{y}}\right] &= -\varepsilon\left[\frac{\partial(\tilde{p}\tilde{v}_x)}{\partial\tilde{x}} + \frac{\partial(\tilde{p}\tilde{v}_y)}{\partial\tilde{y}}\right] + \varepsilon\tilde{p}\left(\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \frac{\partial\tilde{v}_y}{\partial\tilde{y}}\right) \\ &\quad - \varepsilon^2\left(\frac{\partial\tilde{\tau}_{xx}\tilde{v}_x}{\partial\tilde{x}} + \frac{\partial\tilde{\tau}_{xy}\tilde{v}_y}{\partial\tilde{x}}\right) + \left(\frac{\partial\tilde{\tau}_{yx}\tilde{v}_x}{\partial\tilde{y}} + \varepsilon^2\frac{\partial\tilde{\tau}_{yy}\tilde{v}_y}{\partial\tilde{y}}\right) \\ &\quad + \left(\varepsilon^2\tilde{\tau}_{xx}\frac{\partial\tilde{v}_x}{\partial\tilde{x}} + \tilde{\tau}_{xy}\frac{\partial\tilde{v}_x}{\partial\tilde{y}}\right) + \varepsilon^2\left(\tilde{\tau}_{yx}\frac{\partial\tilde{v}_y}{\partial\tilde{x}} + \tilde{\tau}_{yy}\frac{\partial\tilde{v}_y}{\partial\tilde{y}}\right) + \mathrm{St}\,\tilde{v}_x. \end{aligned} \tag{32}$$

Now, considering that $\mathrm{Re}\,\varepsilon \ll 1$, $\varepsilon \ll 1$, and $\varepsilon^2 \ll 1$, the fluid is incompressible, and Eq. (25), the dimensional form of Eq. (32) is

$$\frac{\partial \tau_{yx} v_x}{\partial y} + P = \rho g_x v_x. \tag{33}$$

This result means that the rate of work done by external forces on the fluid per unit volume transforms into (i) the rate of irreversible conversion of kinetic energy into internal energy per unit volume (i.e., viscous dissipation) and (ii) the rate of work done by viscous forces on the fluid per unit volume.[8] In other words, the energy is transformed to a form of energy used to change the volume and the shape of the film, respectively.[12,13] Integrating Eq. (33) with respect to y (between 0 and h), dividing each term by h, and considering that $(\tau_{yx}v_x)_h = (\tau_{yx}v_x)_0 = 0$, Eq. (33) can be combined with Eq. (31) to yield

$$\langle P \rangle_y = \rho g_x \langle v_x \rangle_y = \rho g_x \frac{Q_W}{h}, \tag{34}$$

where

$$\langle P \rangle_y = \frac{1}{h}\int_0^h P dy. \tag{35}$$

Equation (34) shows a proportional relationship between the average viscous dissipation, the flow rate, and the average velocity.

G. Local film thickness

A mass balance on an incompressible film with no evaporation can be written as[2]

$$\frac{\partial h}{\partial t} + \frac{\partial Q_W}{\partial h}\frac{\partial h}{\partial x} = 0. \tag{36}$$

Taking into account that $h(t, 0) = 0$ (the contact line is pinned), the solution of Eq. (36) is[2]

$$\frac{x}{t} = \frac{\partial Q_W}{\partial h} = \frac{\partial}{\partial h}(h\langle v_x \rangle_y). \tag{37}$$

Using Eq. (34) in Eq. (37),

$$\frac{x}{t} = \frac{1}{\rho g_x}\frac{\partial}{\partial h}\int_0^h P dy. \tag{38}$$

Now, using the Leibniz integral rule and the triple product rule [applied to Eq. (21)], Eq. (38) can be transformed as

$$\frac{x}{t} = \frac{1}{\rho g_x}\int_0^h \frac{\partial P}{\partial h}\frac{\partial y}{\partial \tau_{yx}}\frac{\partial \tau_{yx}}{\partial h} dh = \frac{1}{\rho g_x}\int_0^{P_m} dP. \tag{39}$$

Then, the local film thickness can be estimated by

$$\frac{x}{t} = \frac{P_m}{\rho g_x} = -\frac{\tau_m \dot{\gamma}_m}{\rho g_x} = h\dot{\gamma}_m, \tag{40}$$

where $P_m = -\tau_m\dot{\gamma}_m$ is the viscous dissipation on the substrate surface (i.e., maximum viscous dissipation).

This result is consistent with the ones found by Pendergrass[14] and Keeley *et al.*[15] for apparent viscosities depending on τ and $\dot{\gamma}$, respectively. It is worth mentioning that Eq. (40) was found by considering only $P = f(h, y)$ and independently of taking into account a functionality of the apparent viscosity on τ or $\dot{\gamma}$. Therefore, Eq. (40) might be taken as a more general result than the one obtained by Pendergrass[14] and by Keeley *et al.*[15]

As $\partial Q_W/\partial h = h\dot{\gamma}_m$, the term $h\dot{\gamma}_m$ could be regarded as a characteristic velocity at which singularities of the initial or boundary data propagate inside the domain where the equation is posed.[16]

The general form of the h profile as a function of the space-time variable x/t is shown in Fig. 5. The profile starts at $x/t = 0$ from zero and increases continuously with a concave shape. At $x/t = 0$ (i.e., $x = 0$ or $y \to \infty$), the profile shows an infinite slope. As x/t progress toward infinity (i.e., $x \to \infty$ or $t = 0$), the slope $[1 - \partial \ln \dot{\gamma}_m/\partial \ln(x/t)]/\dot{\gamma}_m$ decreases continuously to zero.

Now, taking into account the implicitness of h in Eq. (40), depending on the complexity of η, the local thickness profile can be

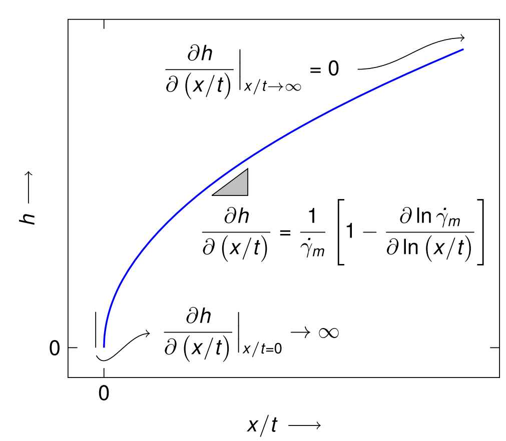

FIG. 5. General representation of the local film profile (h) as a function of y.

difficult to calculate. The simplest calculation strategy is to find h by iteration (for example, fixed-point iteration[17]) using the following rearrangement of Eq. (40):

$$h = \sqrt{\frac{\eta_m}{\rho g_x}\frac{x}{t}}. \tag{41}$$

This form of Eq. (41) has proven to converge quickly to roots that are real and with physical meaning.[3–5]

Based on Eqs. (37) and (40), a useful additional and convenient way to calculate Q_W is

$$Q_W = h\langle v_x\rangle_y = \frac{1}{\rho g_x}\int_0^h P_m\, dh = \int_0^h h\dot{\gamma}_m\, dh. \tag{42}$$

The relative convenience of using Eq. (42) to obtain Q_W will depend on the complexity of $h\dot{\gamma}_m$.

H. Average film thickness

As stated by Peralta *et al.*,[2] the uniformity of the film is one of the main properties to be evaluated. This quantity can be estimated by the ratio of the average thickness to the local thickness.[18] The area-averaged film thickness is defined by

$$\langle h\rangle_A = \frac{1}{A}\int_A h\, dA, \tag{43}$$

where A is the surface area of the plate. Defining $dA = dzdx$, then $A = Wx$, where W is the total width of the plate. So, Eq. (43) yields

$$\langle h\rangle_x = \frac{1}{x}\int_0^x h\, dx, \tag{44}$$

where $\langle h\rangle_x$ is the average film thickness in the x direction (main flow). Equation (44) can be rearranged by using integration by parts

$$\frac{\langle h\rangle_x}{h} = 1 - \frac{1}{xh}\int_0^h x\, dh. \tag{45}$$

Using Eq. (37) into Eq. (45),

$$\frac{\langle h\rangle_x}{h} = 1 - \frac{1}{h(\partial Q_W/\partial h)}\int_0^h \frac{\partial Q_W}{\partial h} dh. \tag{46}$$

Then,

$$\frac{\langle h\rangle_x}{h} = 1 - \frac{Q_W/h}{\partial Q_W/\partial h}. \tag{47}$$

Taking into account Eqs. (37), (38), and (40),

$$\frac{\langle h\rangle_x}{h} = 1 - \frac{\langle P\rangle_y}{P_m}. \tag{48}$$

This expression indicates that there is a simple and proportional relationship between the ratio of the average viscous dissipation to the maximum viscous dissipation (i.e., on the substrate surface) and the ratio of the average film thickness to the local film thickness. That is, the more homogeneously the energy is dissipated through the film thickness ($\langle P\rangle_y$ is approaching to P_m) the less homogeneous will be the film ($\langle h\rangle_x/h \to 1/2$). Equation (48) is a useful mathematical expression to see that, for a generalized Newtonian fluid (shear thinning or thickening), the film thickness homogeneity ranges $1/2 \le \langle h\rangle_x/h \le 1$. This result is useful to see that $0 \le \langle P\rangle_y/P_m \le 1/2$.

An important expression that relates viscous dissipation, flow rate, and the characteristic velocity of the system can be obtained by combining Eqs. (34), (40), (47), and (48),

$$\frac{\langle P\rangle_y}{P_m} = \frac{\partial \ln h}{\partial \ln Q_W} = \frac{\langle v_x\rangle_y}{h\dot{\gamma}_m} = \zeta. \tag{49}$$

This equation states that the homogeneity of energy dissipation (ζ) is proportional to the rate of variation of $\ln h$ with respect to $\ln Q_W$. Also, ζ is proportional to the homogeneity of velocities for a certain x. That is, the higher ζ the higher the energy is dissipated into the film and the quicker the information is transferred within the film to produce flow.

Based on Eqs. (48) and (49), the general functionality of a dimensionless average film thickness $\langle h\rangle_x^*$ as a function of ζ is shown in Fig. 6. The profile is linear with a maximum value of 1 (i.e., $\langle h\rangle_x = h$) at $\zeta = 0$ and a minimum value of zero at $\zeta = 1$. As h increases constantly with x (Fig. 5), a value of $\langle h\rangle_x^* = 1$ would mean that the profile of h is constant. Consequently, the film viscous energy would be totally dissipated for $y > 0$. For example, this situation can be obtained by a solid film.

I. Calculation of parameters

One of the main features of this work can be seen in Fig. 7. This figure shows the main parameters used to describe the system schematized in Fig. 1 and how they are connected. Originally, one could argue that 5 parameters were needed ($\dot{\gamma}$, v_x, $\langle v_x\rangle_y$, h, and $\langle h\rangle_x$). These parameters were sequentially calculated using a combination of steps involving integrals and derivatives represented (Fig. 7) by red sinuous and blue-green arrows, respectively. That is, $\langle h\rangle_x$ could be calculated from h, h from $\langle v_x\rangle_y$, and so on. Here, as each step represents an additional challenge depending on the model complexity, the overall calculation process could become hard or impossible very quickly. Pendergrass[14] and Keeley *et al.*[15] presented a way to calculate h directly (using an algebraic equation) from $\dot{\gamma}$ for $\eta = f(\tau)$ and $\eta = f(\dot{\gamma})$, respectively. This represented a shortcut in the previous calculation sequence reducing the number of steps needed to calculate $\langle h\rangle_x$ to only one integration. However, there is no reduction in the calculation complexity for the rest of the parameters.

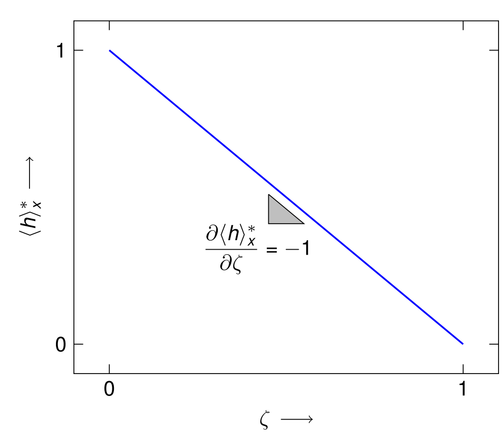

FIG. 6. General representation of the dimensionless average film profile ($\langle h\rangle_x^* = \langle h\rangle_x/h$) as a function of ζ.

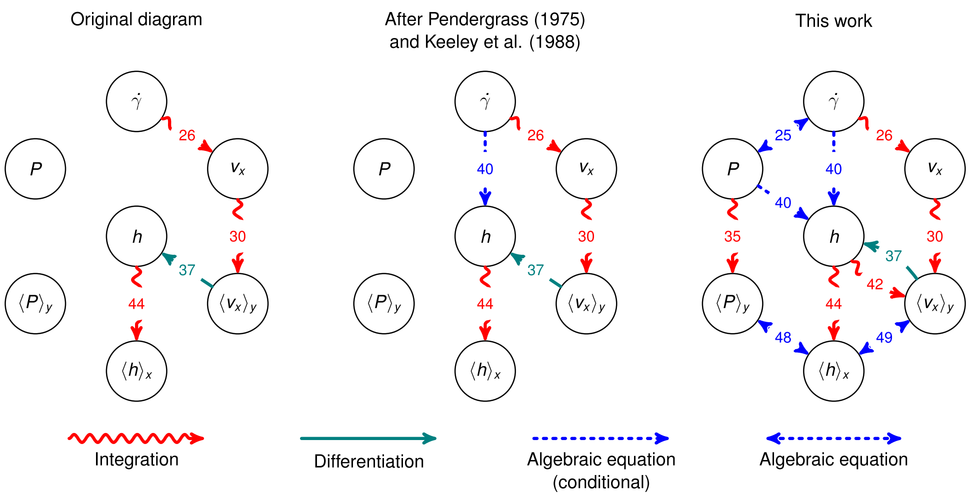

FIG. 7. Diagrams of the steps required to calculate the main quantities of the system. Numbers on the arrows correspond to the respective equations in the text.

This work presents a leap forward in the calculation process and description capabilities of the system described in Fig. 1 and its main parameters. The incorporation of P and $\langle P\rangle_y$ to the list of quantities that are related to the flow description gives new expressions and paths to calculate the main parameters of the system. This increment in the interconnection and ways of calculation of the main variables that can be seen from left to right in Fig. 7. Now, h, P, and $\dot{\gamma}$ can be interchangeable by using Eqs. (25) and (40) instead of using complex integrals. The same is valid for the averaged quantities $\langle P\rangle_y$, $\langle h\rangle_x$ and $\langle v_x\rangle_y$ [Eqs. (48) and (49)]. This significantly reduces the overall calculation complexity and gives a deeper insight of the described problem (i.e., how the viscous dissipation is governing the film thickness). For example, at least three independent paths can be taken simultaneously (through P, h, or v_x) to calculate $\langle h\rangle_x$ from $\dot{\gamma}$.

III. CONCLUSIONS

An improved general and expanded theoretical analysis of a two-dimensional film draining from a quasivertical plate was presented, solving rigorous mass, momentum, and energy balances. A scaling and dimensional analysis was used to simplify the mathematical description. A generalized Newtonian fluid was assumed as the film material. New analytical expressions for the main parameters that describe the flow were found. Also, this work proposes a new parameter (viscous dissipation) that governs the flow and film characteristics. The combination of these two features helps us to simplify the overall calculation complexity, allowing us to find new analytical expressions using more complex rheological models. As a result, the ways of calculation and the interconnection between the main variables is increased, helping us to obtain their analytical expressions for a given rheological model.

ACKNOWLEDGMENTS

This research was supported partially by Universidad Nacional del Litoral (Project CAI+D 2016-50420150100002 LI) (Santa Fe, Argentina), Consejo Nacional de Investigaciones Científicas y Técnicas (Project PIP 2015-11220150100185CO) (Argentina), and Agencia Nacional de Promoción Científica y Tecnológica (Project ANPCyT 2015-365) (Argentina).

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