REVISTA DE LA
UNIÓN MATEMÁTICA ARGENTINA
Volumen 46, Número 1, 2005, Páginas 53–66

SOME OPERATOR INEQUALITIES FOR UNITARILY INVARIANT NORMS*†

CRISTINA CANO, IRENE MOSCONI‡ AND DEMETRIO STOJANOFF§

ABSTRACT. Let $L(\mathcal{H})$ be the algebra of bounded operators on a complex separable Hilbert space $\mathcal{H}$. Let N be a unitarily invariant norm defined on a norm ideal $\mathcal{I} \subseteq L(\mathcal{H})$. Given two positive invertible operators $P, Q \in L(\mathcal{H})$ and $k \in (-2, 2]$, we show that $N\big(PTQ^{-1} + P^{-1}TQ + kT\big) \geq (2 + k)N(T)$, $T \in \mathcal{I}$. This extends Zhang's inequality for matrices. We prove that this inequality is equivalent to two particular cases of itself, namely $P = Q$ and $Q = P^{-1}$. We also characterize those numbers k such that the map $\Upsilon : L(\mathcal{H}) \to L(\mathcal{H})$ given by $\Upsilon(T) = PTQ^{-1} + P^{-1}TQ + kT$ is invertible, and we estimate the induced norm of Υ^{-1} acting on the norm ideal $\mathcal{I}$. We compute sharp constants for the involved inequalities in several particular cases.

1. INTRODUCTION

Let $\mathcal{H}$ be a Hilbert space and denote by $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. In 1990, Corach-Porta-Recht [6] show that, for every invertible selfadjoint operator $S \in L(\mathcal{H})$ and for every $T \in L(\mathcal{H})$, it holds that

$$\big\|STS^{-1} + S^{-1}TS\big\| \geq 2\big\|T\big\|. \tag{1}$$

Several authors have proved generalizations and alternative proofs of inequality (1). For example, Bhatia and Davis [3], Kittaneh (in two different ways, see [11] and [12]), and Andruchow, Corach and Stojanoff [2]. On the other hand, in 1993, Livshits and Ong [14] studied the invertibility of the map $T \longmapsto STS^{-1} + S^{-1}TS$ for not necessarily selfadjoint $S \in L(\mathcal{H})$. In 2001, Seddik, [15] proved that, for $S \in L(\mathcal{H})$ invertible and positive, $T \in L(\mathcal{H})$, and $k = 0, 1, 2$,

$$\big\|kT + STS^{-1} + S^{-1}TS\big\| \geq (k + 2)\big\|T\big\|. \tag{2}$$

In 1999 Zhan [18] showed that, given two positive invertible matrices $P, Q \in \mathcal{M}_n(\mathbb{C})$, $T \in \mathcal{M}_n(\mathbb{C})$, and $k \in (-2, 2]$, then

$$|||PTQ^{-1} + P^{-1}TQ + kT||| \geq (2 + k)\, |||\, T|||, \tag{3}$$

for every unitarily invariant norm $||| \cdot |||$ on $\mathcal{M}_n(\mathbb{C})$.

*Keywords: positive matrices, inequalities, unitarily invariant norm.
†AMS Subject Classifications: Primary 47A30, 47B15.
‡Partially supported by Universidad Nac. del Comahue
§Partially supported CONICET (PIP 4463/96), UNLP 11 X350 and ANPCYT (PICT03-09521)

In this paper we work with unitarily invariant norms defined in some ideal $\mathcal{I}$ of $L(\mathcal{H})$ (see Remark 2.1 or Simon's book [17]). We show that, for every unitarily invariant norm $|||\cdot|||$, the following inequalities are equivalent, for every $k \in (-2, 2]$:

$$|||PTQ^{-1} + P^{-1}TQ + kT||| \geq (2+k)\, |||\, T\, ||| \quad \text{for} \quad P, Q \in Gl(\mathcal{H})^+,\ T \in \mathcal{I}\ . \quad (4)$$

$$|||STS + S^{-1}TS^{-1} + kT||| \geq (2+k)\, |||\, T\, ||| \quad \text{for} \quad S \in Gl(\mathcal{H})^+,\ T \in \mathcal{I}\ . \quad (5)$$

$$|||kT + STS^{-1} + S^{-1}TS||| \geq (k+2)\, |||\, T\, ||| \quad \text{for} \quad S \in Gl(\mathcal{H})^+,\ T \in \mathcal{I}\ . \quad (6)$$

We give a proof of inequality (6), using a technical result about unitarily invariant norms, which allows us to obtain a reduction to the matricial case. In this case, we use a result of Bhatia and Parthasarathy [4], and some properties of the Hadamard product of matrices. This result was previously proved for $k = 0$ in [2], for not necessarily positive S, P and Q. We study the operators associated to the three mentioned inequalities, and their restriction as operators on the norm ideal $\mathcal{I}$. We compute their spectra and, in some cases, their reduced minimum moduli (also called *conorms*). The rest of the paper deals with the estimation of sharp constants for inequality (5), with respect to the usual norm of $L(\mathcal{H})$. We get the optimal constant, if one restricts to operators $T \in L(\mathcal{H})^+$. Using the notion of Hadamard index for positive matrices, studied in [7], we compute, for a fixed $S \in Gl(\mathcal{H})^+$, the constant

$$M(S,k) = \max\{C \geq 0 : \|STS + S^{-1}TS^{-1} + kT\| \geq C\|T\| \text{ for every } T \in L(\mathcal{H})^+\},$$

for $k \geq 0$ (see Proposition 5.6). Finally, we give some partial results for $T \in \mathcal{M}_n(\mathbb{C})$, in lower dimensions, showing numerical estimates of sharp constants. For $n = 2,\ 3$ and 4, we characterize the best intervals J_n such that the inequality (6) holds in $\mathcal{M}_n(\mathbb{C})$ for every $k \in J_n$.

In section 2, we fix several notations and state some preliminary results. We expose with some detail the theory of unitarily invariant norms defined on norm ideals of $L(\mathcal{H})$, proving some technical results in this area. In section 3, we show the equivalence of the mentioned inequalities and we give the proof of (6). In section 4, we study the associated operators. In section 5, we describe the theory of Hadamard index, and we use it to obtain a description of the constant $M(S,k)$. In section 6 we study the case of matrices of lower dimensions.

We wish to acknowledge Prof. G. Corach who shared with us fruitful discussions concerning these matters.

2. PRELIMINARIES

Let $\mathcal{H}$ be a separable Hilbert space, and $L(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. We denote $L_0(\mathcal{H})$ the ideal of compact operators, $\mathcal{G}l(\mathcal{H})$ the group of invertible operators, $L(\mathcal{H})_h$ the set of hermitian operators, $L(\mathcal{H})^+$ the set of positive definite operators, $\mathcal{U}(\mathcal{H})$ the unitary group, and $Gl(\mathcal{H})^+$ the set of invertible positive definite operators.

Given an operator $A \in L(\mathcal{H})$, $R(A)$ denotes the range of A, $\ker A$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $|A| = (A^*A)^{1/2}$ the modulus of A, $\rho(A)$ the spectral radius of A, and $\|A\|$ the spectral norm of A. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, we denote by $P_{\mathcal{S}}$ the orthogonal projection onto $\mathcal{S}$.

When $\dim \mathcal{H} = n < \infty$, we shall identify $\mathcal{H}$ with $\mathbb{C}^n$, $L(\mathcal{H})$ with $\mathcal{M}_n(\mathbb{C})$, and we use the following notations: $\mathcal{H}(n)$ for $L(\mathcal{H})_h$, $\mathcal{M}_n(\mathbb{C})^+$ for $L(\mathcal{H})^+$, $\mathcal{U}(n)$ for $\mathcal{U}(\mathcal{H})$, and $\mathcal{G}l(n)$ for $\mathcal{G}l(\mathcal{H})$. A norm $|||\cdot|||$ in $\mathcal{M}_n(\mathbb{C})$ is called *unitarily invariant* if $|||UAV||| = |||A|||$ for every $A \in \mathcal{M}_n(\mathbb{C})$ and $U, V \in \mathcal{U}(n)$.

Remark 2.1. The notion of unitarily invariant norms can be defined also for operators on Hilbert spaces. We give some basic definitions (see Simon's book [17]): Let $A \in L_0(\mathcal{H})$. Then also $|A| \in L_0(\mathcal{H})$. We denote by $s(A) = (s_k(A))_{k\in\mathbb{N}}$, the sequence of eigenvalues of $|A|$, taken in non increasing order and with multiplicity. If $\dim R(A) = n < \infty$, we take $s_k(A) = 0$ for $k > n$. The numbers $s_k(A)$ are called *the singular values* of A.

Denote by $\mathcal{C}_0$ the set of complex sequences which converge to zero. Consider $\mathcal{C}_F \subseteq \mathcal{C}_0$ the set of sequences with finite non zero entries. For $a \in \mathcal{C}_0$, denote $|a| = (|a_n|)_{n\in\mathbb{N}} \in \mathcal{C}_0$. A *gauge symmetric function (or symmetric norm)* is a map $g : \mathcal{C}_F \to \mathbb{R}$ which satisfy the following properties:

- g is a norm on $\mathcal{C}_F$,
- $g(a) = g(|a|)$ for every $a \in \mathcal{C}_F$, and
- g is invariant under permutations.

We say that g is *normalized* if $g(e_1) = 1$. For $a \in \mathcal{C}_0$, define

$$g(a) = \sup_{n\in\mathbb{N}} g(a_1, \dots, a_n, 0, \dots) \in \mathbb{R} \cup \{+\infty\} .$$

A unitarily invariant norm in $L_0(\mathcal{H})$ is a map $|||\cdot||| : L_0(\mathcal{H}) \to \mathbb{R}^+ \cup \{\infty\}$ given by $|||A||| = g(s(A))$, $A \in L_0(\mathcal{H})$, where g is a symmetric norm. The set

$$\mathcal{I} = \mathcal{I}_g = \{A \in L_0(\mathcal{H}) : |||A||| < \infty\}$$

is a selfadjoint ideal of $L(\mathcal{H})$, called the norm ideal associated to $|||\cdot|||$. Then $(\mathcal{I}, |||\cdot|||)$ is a Banach space, and the following properties hold:

1. If $B \in \mathcal{I}$, then $|||B||| = |||B^*|||$, and $|||AB||| \le \|A\| \; |||B|||$ for every $A \in L(\mathcal{H})$.
2. If $A \in L(\mathcal{H})$ has finite rank, then $A \in \mathcal{I}$, because $s(A) \in \mathcal{C}_F$.
3. If $\dim R(A) = 1$, then $|||A||| = s_1(A)g(e_1) = g(e_1)\|A\|$.
4. Given $A \in L_0(\mathcal{H})$ and $B \in \mathcal{I}$ such that
$$\|A\|_{(k)} := \sum_{j=1}^{k} s_j(A) \le \sum_{j=1}^{k} s_j(B) = \|B\|_{(k)} \qquad \text{for every} \qquad k \in \mathbb{N},$$
then $A \in \mathcal{I}$ and $|||A||| \le |||B|||$.
5. For every $\varepsilon > 0$ and $T \in \mathcal{I}$, there exists a finite rank operator S such that $|||T - S||| < \varepsilon$.

Some well known examples of unitarily invariant norms are the Schatten p-norms $|||A|||_p = \operatorname{tr}(|A|^p)^{1/p}$, for $1 \le p \le \infty$, and the Ky-Fan norms $\|\cdot\|_{(k)}$, $k \in \mathbb{N}$. The usual norm, which coincides with $\|\cdot\|_{(1)}$ and $|||\cdot|||_\infty$ when restricted to $L_0(\mathcal{H})$, is also unitarily invariant. △

Proposition 2.2. *Let N be an unitarily invariant norm on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$. Let $\{P_F\}_{F\in\mathcal{F}}$ be a increasing net of projections in $L(\mathcal{H})^+$ which converges strongly* *to the identity (i.e., $P_F\,x \xrightarrow[F\in\mathcal{F}]{} x$ for every $x \in \mathcal{H}$). Then*

$$P_F T \xrightarrow[F\in\mathcal{F}]{N} T \quad \text{and} \quad P_F T P_F \xrightarrow[F\in\mathcal{F}]{N} T \quad \text{for every} \quad T \in \mathcal{I}\ .$$

Proof. By Remark 2.1, for every $\varepsilon > 0$, there exists a finite rank operator S such that $N(T-S) < \varepsilon$. For every $A \in \mathcal{I}$ and every projection $P \in L(\mathcal{H})$, it holds that $N(PAP) \le N(PA) \le N(A)$. In particular, $N(P_F(T-S)P_F) \le N(P_F(T-S)\,) < \varepsilon$ for every $F \in \mathcal{F}$. Hence, we can assume that $\dim R(T) = n < \infty$. Given $F \in \mathcal{F}$, denote $Q_F = I - P_F$. Since $N(\,Q_F T Q_F\,) \le N(\,Q_F T)$ and

$$T - P_F T P_F = Q_F T + T Q_F - Q_F T Q_F, \quad F \in \mathcal{F},$$

it suffices to prove that $N(T - P_F T) = N(Q_F T) \xrightarrow[F\in\mathcal{F}]{} 0$. Fix $x \in \mathcal{H}$. Note that $\|(T^* Q_F T)x\| \le \|T\|\,\|Q_F(Tx)\| \xrightarrow[F\in\mathcal{F}]{} 0$. Therefore

$$\|(T^* Q_F T)^{1/2} x\|^2 = \langle (T^* Q_F T)x, x\rangle \le \|(T^* Q_F T)x\|\,\|x\| \xrightarrow[F\in\mathcal{F}]{} 0\ .$$

This implies that $(T^* Q_F T)^{1/2} = |Q_F T| \xrightarrow[F\in\mathcal{F}]{SOT} 0$, and all these operators act on the fixed finite dimensional subspace $\ker T^{\perp}$, where the convergence of operators in every norm (included N) is equivalent to the SOT (or strong) convergence. □

Remark 2.3. Let $|||\cdot|||$ be a unitarily invariant norm defined on a norm ideal $\mathcal{I} \subseteq L(\mathcal{H})$. The space $L(\mathcal{H} \oplus \mathcal{H})$ can be identified with the algebra of block 2×2 matrices with entries in $L(\mathcal{H})$, denoted by $L(\mathcal{H})^{2\times 2}$. Denote by $\mathcal{I}_2$ the ideal of $L(\mathcal{H}\oplus\mathcal{H})$ associated with the same norm $|||\cdot|||$ (i.e., by using the same symmetric norm g). Then, the following properties hold:

1. Let $A \in L_0(\mathcal{H})$, and define $A_1 \in L_0(\mathcal{H}\oplus\mathcal{H})$ as any of the following matrices
$$A_1 = \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}\ .$$
Then $s(A_1) = s(A)$, $|||A_1||| = |||A|||$, and $A_1 \in \mathcal{I}_2$ if and only if $A \in \mathcal{I}$.
2. Under the mentioned identification, $\mathcal{I}_2 = \mathcal{I}^{\,2\times 2}$. △

Given $A = [a_{ij}], B = [b_{ij}] \in \mathcal{M}_n(\mathbb{C})$ denote by $A \circ B$ the Hadamard product $[a_{ij}b_{ij}]$. Every $A \in \mathcal{M}_n(\mathbb{C})$ defines a linear map $\Phi_A : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ given by

$$\Phi_A(B) = A \circ B\ , \quad B \in \mathcal{M}_n(\mathbb{C}). \tag{7}$$

Given a norm $|||\cdot|||$ on $\mathcal{M}_n(\mathbb{C})$, it induces a norm of the linear map Φ_A by means of

$$|||\Phi_A||| = \max\{\,|||A\circ X||| :\ X \in \mathcal{M}_n(\mathbb{C})\ ,\ \ |||X||| \le 1\ \}\,. \tag{8}$$

The following result collects two classical results of Schur about Hadamard (or Schur) products of positive matrices (see [16]), and a generalization of the second one for unitarily invariant norms, proved by Ando in [1, Proposition 7.7] .

Proposition 2.4 (Schur)**.** Let $A \in \mathcal{M}_n(\mathbb{C})^+$ and $B \in \mathcal{M}_n(\mathbb{C})$. Then

1. If $B \in \mathcal{M}_n(\mathbb{C})^+$ then also $A \circ B \in \mathcal{M}_n(\mathbb{C})^+$.

2. Denote by $d_A = \max \{A_{ii} \ : \ 1 \le i \le n\}$. Then
$$|||A \circ B||| \le d_A \ ||| B ||| \quad \text{and} \quad ||| \Phi_A||| = d_A \ , \tag{9}$$
for every unitarily invariant norm $|||\cdot|||$ on $\mathcal{M}_n(\mathbb{C})$. □

3. EQUIVALENT INEQUALITIES

Theorem 3.1. *Let $k \in \mathbb{R}$ and $|||\cdot|||$ be an unitarily invariant norm on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$. Then the following inequalities are equivalent:*

1. $|||PTQ^{-1}+P^{-1}TQ+kT||| \ge (k+2)|||T|||$, for every $T \in \mathcal{I}$ and $P, Q \in Gl(\mathcal{H})^+$.
2. $|||STS^{-1} + S^{-1}TS + kT||| \ge (k+2) \, ||| T|||$, for every $T \in \mathcal{I}$ and $S \in Gl(\mathcal{H})^+$.
3. $|||STS + S^{-1}TS^{-1} + kT||| \ge (k+2) \, ||| T|||$, for every $T \in \mathcal{I}$ and $S \in Gl(\mathcal{H})^+$.

Proof. It is clear that 1 implies 2 and 3. Suppose that 2 holds. Consider the space $L(\mathcal{H} \oplus \mathcal{H}) \cong L(\mathcal{H})^{2\times 2}$. For $P, Q \in Gl(\mathcal{H})^+$ and $T \in \mathcal{I}$, define the operators
$$S_1 = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \quad \text{and} \quad T_1 = \begin{bmatrix} 0 & T \\ 0 & 0 \end{bmatrix} \in \mathcal{I}_2 \ .$$
Then
$$S_1T_1S_1^{-1} + S_1^{-1}T_1S_1 + kT_1 = \begin{bmatrix} 0 & PTQ^{-1} + P^{-1}TQ + kT \\ 0 & 0 \end{bmatrix}.$$
Therefore, as $|||S_1AS_1^{-1} + S_1^{-1}AS_1 + kA||| \ge (k+2) \, ||| A|||$ for every $A \in \mathcal{I}_2$, then
$$|||PTQ^{-1} + P^{-1}TQ + kT||| \ge (2+k) \, ||| T||| \ , \quad T \in \mathcal{I}.$$
This shows $2 \to 1$. The same arguments using $S_1 = \begin{bmatrix} P & 0 \\ 0 & Q^{-1} \end{bmatrix}$ show $3 \to 1$. □

Remark 3.2. As said in the Introduction, the inequality 2 of Theorem 3.1 was proved, for the usual norm, by Corach-Porta-Recht in [6] with $k = 0$ (and S not necessarily positive), and by Ameur Seddik in [15] with $k = 1, 2$. The inequality 1 of Theorem 3.1 was proved, in the finite dimensional case, by X. Zhan in [18], for $k \in (-2, 2]$. In the rest of this section, we give a proof of inequality 2 of Theorem 3.1 for $k \in (-2, 2]$ in the general setting.

Lemma 3.3. *Let $n \in \mathbb{N}$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}_+^n$ and $k \in \ (-2, 2]$. Let $C_n(k, \lambda) \in \mathcal{M}_n(\mathbb{C})$ be given by*
$$C_n(k,\lambda)_{ij} = \frac{\lambda_i\lambda_j}{k\lambda_i\lambda_j + \lambda_i^2 + \lambda_j^2} \ , \quad 1 \le i, j \le n \ .$$
Then $C_n(k, \lambda) \in \mathcal{M}_n(\mathbb{C})^+$ for every $n \in \mathbb{N}$.

Proof. Note that $C_n(k, \lambda)_{ij} = (\lambda_i\lambda_j^{-1} + \lambda_i^{-1}\lambda_j + k)^{-1}$. These numbers are well defined because $x+x^{-1} \ge 2$ for every $x \in \mathbb{R}_+$ and $k > -2$. Bhatia and Parthasarathy [4] proved that, for $-2 < k \in \mathbb{R}$ and $\lambda \in \mathbb{R}_+^n$, the matrix $Z_n(k, \lambda) \in \mathcal{M}_n(\mathbb{C})$ with entries
$$Z_n(k,\lambda)_{ij} = \frac{1}{k\lambda_i\lambda_j + \lambda_i^2 + \lambda_j^2} \ , \quad 1 \le i, j \le n, \tag{10}$$
satisfies $Z_n(k, \lambda) \in \mathcal{M}_n(\mathbb{C})^+$ for every $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}_+^n$ if and only if $k \in \ (-2, 2]$.

On the other hand, if $\lambda \in \mathbb{R}^n_+$, then the matrix $L = \lambda\lambda^* = (\lambda_i\lambda_j)_{1\le i,j\le n} \in \mathcal{M}_n(\mathbb{C})^+$. By Propposition 2.4, $C_n(k,\lambda) = Z_n(k,\lambda)\circ L \in \mathcal{M}_n(\mathbb{C})^+$ for every $k \in (-2,2]$. □

Theorem 3.4. *Let $S \in Gl(\mathcal{H})^+$ and $k \in (-2,2]$. Then, for every unitarily invariant norm $|||\cdot|||$ on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$, and for every $T \in \mathcal{I}$,*

$$|||kT + STS^{-1} + S^{-1}TS||| \ge (k+2)\,|||\,T\,|||\ .$$

Proof. We follow the same steps as in [6]. By the spectral theorem, we can suppose that $\sigma(S)$ is finite, since S can be approximated in norm by operators S_n such that each $\sigma(S_n)$ is finite. We can suppose also that $\dim\mathcal{H} < \infty$, by choosing an adequate net of finite rank projections $\{P_F\}_{F\in\mathcal{F}}$ which converges strongly to the identity and replacing S, T by P_FSP_F, P_FTP_F. Indeed, the net may be chosen in such a way that $SP_F = P_FS$ and $\sigma(P_FSP_F) = \sigma(S)$ for every $F \in \mathcal{F}$. Note that, by Proposition 2.2, $|||P_FAP_F|||$ converges to $|||A|||$ for every $A \in \mathcal{I}$.

We can suppose that S is diagonal by a unitary change of basis in $\mathbb{C}^n$. Take $S = \operatorname{diag}(\lambda_1,\lambda_2,...,\lambda_n)$. Then $k\,T + STS^{-1} + S^{-1}TS = B_n(k,\lambda)\circ T$, where

$$B_n(k,\lambda)_{ij} = \left(k + \frac{\lambda_i}{\lambda_j} + \frac{\lambda_j}{\lambda_i}\right) = \left(\frac{k\ \lambda_i\lambda_j + \lambda_i^2 + \lambda_j^2}{\lambda_i\lambda_j}\right)\ ,\quad 1\le i,j\le n. \tag{11}$$

Since $x + x^{-1} \ge 2$ for every $x \in \mathbb{R}$, $x > 0$, it follows that, if $k \in (-2,+\infty)$, then $B_n(k,\lambda)_{ij} > 0$ for every $1 \le i,j \le n$. Consider the matrix $C_n(k,\lambda)$ given by $C_n(k,\lambda)_{ij} = B_n(k,\lambda)^{-1}_{ij}$. Hence, in order to prove inequality (6) for every $T \in \mathcal{M}_n(\mathbb{C})$, it suffices to show that $|||C_n(k,\lambda)\circ A||| \le (k+2)^{-1}|||A|||$ for $A \in \mathcal{M}_n(\mathbb{C})$ and $k \in (-2,2]$. By Lemma 3.3, $C_n(k,\lambda) \in \mathcal{M}_n(\mathbb{C})^+$ for every $k \in (-2,2]$. Finally, note that $C_n(k,\lambda)_{ii} = (k+2)^{-1}$, $1 \le i \le n$. Therefore, inequality (6) holds by Eq. (9) in Proposition 2.4. □

As a consequence of this result and Theorem 3.1, we get an infinite dimensional version, for every unitarily invariant norm, of Zhang inequality:

Corollary 3.5. *Let $P, Q \in Gl(\mathcal{H})^+$ and $k \in (-2,2]$. Then, for every unitarily invariant norm $|||\cdot|||$ on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$, and for every $T \in \mathcal{I}$,*

$$|||PTQ^{-1} + P^{-1}TQ + kT||| \ge (2+k)\,|||\,T\,|||\,.$$

Corollary 3.6. *Let $S \in Gl(\mathcal{H})^+$ and $k \in (-2,2]$. Then, for every unitarily invariant norm $|||\cdot|||$ on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$, and for every $T \in \mathcal{I}$,*

$$|||STS + S^{-1}TS^{-1} + kT||| \ge (2+k)\,|||\,T\,|||\,.$$

4. THE ASSOCIATED OPERATORS

In this section we study the operators associated with the inequalities proved in the previous section. Given $P, Q \in Gl(\mathcal{H})^+$ and $k \in \mathbb{R}$, we consider the bounded operator $\Upsilon_{P,Q,k} : L(\mathcal{H}) \to L(\mathcal{H})$ associated with inequality (4):

$$\Upsilon_{P,Q,k}(T) = PTQ^{-1} + P^{-1}TQ + kT\ ,\quad T \in L(\mathcal{H})\ . \tag{12}$$

Hence, for every unitarily invariant norm $|||\cdot|||$ defined on an ideal $\mathcal{I} \subseteq L(\mathcal{H})$, inequality (4) means that $|||\Upsilon_{P,Q,k}(T)||| \geq (2+k)\,|||T|||$ for $T \in \mathcal{I}$, $-2 < k \leq 2$. Given $S \in L(\mathcal{H})^+$ and $k \in \mathbb{R}$, define the operators $\Phi_{S,k} : L(\mathcal{H}) \to L(\mathcal{H})$ and $\Psi_{S,k} : L(\mathcal{H}) \to L(\mathcal{H})$ associated with inequalities (6) and (5): $\Phi_{S,k} = \Upsilon_{S,S,k}$ and $\Psi_{S,k} = \Upsilon_{S,S^{-1},k}$. In this section we characterize, for fixed $P, Q \in Gl(\mathcal{H})^+$, those $k \in \mathbb{R}$ such that $\Upsilon_{P,Q,k}$ is invertible. In some cases we estimate, for a given norm on some ideal of $L(\mathcal{H})$, the induced norms of their inverses.

Remark 4.1. Let $u, v \in \mathcal{H}$. Denote by $u \otimes v \in L(\mathcal{H})$ the operator given by $u \otimes v(z) = \langle z, v\rangle u$, $z \in \mathcal{H}$. It is clear that $R(u \otimes v) = \operatorname{span}\{u\}$. Hence $u \otimes v \in \mathcal{I}$ for every norm ideal induced by a unitarily invariant norm $|||\cdot|||$. The following properties are easy to see:

1. $|||u \otimes v||| = \|u \otimes v\| = \|u\|\,\|v\|$.
2. The map $(u, v) \mapsto u \otimes v$ is sesqui linear.
3. If $T \in L(\mathcal{H})$, then $T(u \otimes v) = (Tu) \otimes v$ and $(u \otimes v)T = u \otimes (T^*v)$. △

Proposition 4.2. *Let $P, Q \in Gl(\mathcal{H})^+$. Denote by $\Upsilon_{P,Q} = \Upsilon_{P,Q,0}$. Then*

$$\sigma(\Upsilon_{P,Q}) = \{\lambda\mu^{-1} + \lambda^{-1}\mu \ : \ \lambda \in \sigma(P), \mu \in \sigma(Q)\} .$$

Moreover, $\Upsilon_{P,Q}$ has the same spectrum, if it is considered as acting on any norm ideal $\mathcal{I}$ associated with a unitarily invariant norm.

Proof. Fix the norm ideal $\mathcal{I}$ and consider the restriction $\Upsilon_{P,Q} : \mathcal{I} \to \mathcal{I}$. Let $A_{P,Q} : \mathcal{I} \to \mathcal{I}$ be given by $A_{P,Q}(T) = PTQ^{-1}$, $T \in \mathcal{I}$. Note that $\Upsilon_{P,Q} = A_{P,Q} + A_{P,Q}^{-1}$. Therefore, by the known properties of the Riesz functional calculus for operators on Banach spaces (in this case, the Banach space is $\mathcal{I}$ and the map is $f(z) = z + z^{-1}$), it suffices to show that $\sigma(A_{P,Q}) = \{\lambda\mu^{-1} : \ \lambda \in \sigma(P), \ \mu \in \sigma(Q)\}$.
Given $C \in L(\mathcal{H})$, denote by $L_C : \mathcal{I} \to \mathcal{I}$ (resp. R_C) the operator given by $L_C(T) = CT$ (resp. $R_C(T) = TC$), $T \in \mathcal{I}$. By Remark 2.1, these operators are bounded. If $C \in \mathcal{G}l(\mathcal{H})$, then $L_{C^{-1}} = (L_C)^{-1}$, and similarly for R_C. Hence $\sigma(L_C) \subseteq \sigma(C)$ and $\sigma(R_C) \subseteq \sigma(C)$. Note that $A_{P,Q} = L_P R_{Q^{-1}} = R_{Q^{-1}} L_P$. Therefore

$$\sigma(A_{P,Q}) \subseteq \sigma(L_P)\sigma(R_{Q^{-1}}) \subseteq \{\lambda\mu^{-1} : \ \lambda \in \sigma(P), \ \mu \in \sigma(Q)\}.$$

Given $\lambda \in \sigma(P)$, $\mu \in \sigma(Q)$ and $\varepsilon > 0$, let $x, y \in \mathcal{H}$ be unit vectors such that $\|Px - \lambda x\| < \varepsilon$ and $\|Q^{-1}y - \mu y\| < \varepsilon$. Such vectors exist because $P - \lambda I$ and $Q^{-1} - \mu^{-1}I$ are selfadjoint operators. Consider the rank one operator $x \otimes y \in \mathcal{I}$. Then, by Remark 4.1, $A_{P,Q}(x \otimes y) = Px \otimes (Q^{-1})^*y = Px \otimes Q^{-1}y$. Hence

$$\begin{aligned} |||(A_{P,Q} - \lambda\mu^{-1}\,\mathrm{Id}_{\mathcal{I}})\, x \otimes y||| &= |||Px \otimes Q^{-1}y - \lambda x \otimes \mu^{-1}y||| \\ &\leq |||(Px - \lambda x) \otimes Q^{-1}y\,||| + |||\,\lambda x \otimes (Q^{-1}y - \mu^{-1}y)||| \\ &= \|Px - \lambda x\|\,\|Q^{-1}y\| + \|\lambda x\|\,\|Q^{-1}y - \mu^{-1}y\| \\ &< (\|Q^{-1}y\| + \|\lambda x\|)\,\varepsilon \leq (\|Q^{-1}\| + \|P\|)\,\varepsilon\ . \end{aligned}$$

Therefore $\lambda\mu^{-1} \in \sigma(A_{P,Q})$, because $|||x \otimes y||| = \|x\|\,\|y\| = 1$. This shows that $\sigma(A_{P,Q}) = \{\lambda\mu^{-1} : \lambda \in \sigma(P), \ \mu \in \sigma(Q)\}$, and the proof is complete. □

Corollary 4.3. *Let $S \in Gl(\mathcal{H})^+$ and $k \in \mathbb{C}$. Then $\Upsilon_{P,Q,k}$ is invertible if and only if $-k \notin \{\lambda\mu^{-1} + \lambda^{-1}\mu \ : \ \lambda \in \sigma(P), \mu \in \sigma(Q)\}$.*

Proof. Just note that $\Upsilon_{P,Q,k} = \Upsilon_{P,Q} + k \ \mathrm{Id}_{\mathcal{I}}$. Then apply Proposition 4.2. □

Remark 4.4. Let $|||\cdot|||$ be a unitarily invariant norm defined on a norm ideal $\mathcal{I} \subseteq L(\mathcal{H})$. By Remark 2.1, $|||AB||| \le \|A\| \ |||B|||$ for $A \in L(\mathcal{H})$ and $B \in \mathcal{I}$. Given a linear operator $\Upsilon : \mathcal{I} \to \mathcal{I}$, we denote by $|||\Upsilon|||$ the induced norm:

$$|||\Upsilon||| = \sup\{|||\Upsilon(T)||| : T \in \mathcal{I}, |||T||| = 1\} .$$

By a standard $\varepsilon/2$ argument and using the continuity of "taking inverse", one can show that, for $k \in (-2, 2]$ fixed, the map $Gl(\mathcal{H})^+ \times Gl(\mathcal{H})^+ \ni (P,Q) \mapsto |||\Upsilon^{-1}_{P,Q,k}|||$ is continuous. △

Proposition 4.5. *Let $P, Q \in Gl(\mathcal{H})^+$ and $k \in (-2, 2]$. Let $|||\cdot|||$ be a unitarily invariant norm defined on a norm ideal $\mathcal{I} \subseteq L(\mathcal{H})$. Then $|||\Upsilon^{-1}_{P,Q,k}||| \le (k+2)^{-1}$. Moreover, if $\sigma(P) \cap \sigma(Q) \ne \emptyset$, then $|||\Upsilon^{-1}_{P,Q,k}||| = (k+2)^{-1}$.*

Proof. The inequality follows from Corollary 3.5. Suppose that $\lambda \in \sigma(P) \cap \sigma(Q)$ is an eigenvalue for both P and Q. Let $x, y \in \mathcal{H}$ be unit vectors such that $Px = \lambda x$ and $Qy = \lambda y$. Consider $x \otimes y \in \mathcal{I}$. Then, since $Q = Q^*$ and $\lambda \in R$, it is easy to see that $\Upsilon_{P,Q,k}(x \otimes y) = (2 + k)\ (x \otimes y)$. Hence, $|||\Upsilon^{-1}_{S,k}||| = (k+2)^{-1}$ in this case. An easy consequence of spectral theory is that every $S \in Gl(\mathcal{H})^+$ such that $\lambda \in \sigma(S)$ can be arbitrarily approximated by positive invertible operators such that λ is a isolated point of their spectra, hence an eigenvalue. Applying this, jointly for P and Q, for some $\lambda \in \sigma(P) \cap \sigma(Q)$, and using the fact that the map $(P,Q) \mapsto |||\Upsilon^{-1}_{P,Q,k}|||$ is continuous, the proof is completed. □

Corollary 4.6. *Let $S \in Gl(\mathcal{H})^+$ and $k \in (-2, 2]$. Let $|||\cdot|||$ be a unitarily invariant norm defined on a norm ideal $\mathcal{I} \subseteq L(\mathcal{H})$. Then $|||\Phi^{-1}_{S,k}||| = (k+2)^{-1}$. If there exists $\lambda \in \sigma(S)$ such that also $\lambda^{-1} \in \sigma(S)$, then also $|||\Psi^{-1}_{S,k}||| = (k+2)^{-1}$.*

Proof. The first case follows applying Proposition 4.5 with $P = Q = S$. Note that the hypothesis $\sigma(P) \cap \sigma(Q) \ne \emptyset$ becomes obvious. For the second, take $P = S$ and $Q = S^{-1}$. Note that $\sigma\left(S^{-1}\right) = \{\lambda^{-1} : \lambda \in \sigma(S)\}$. □

Remark 4.7. The Forbenius norm $|||A|||_2^2 = \operatorname{tr} A^*A$ works on the ideal of Hilbert Schmit operators, which is a Hilbert space with this norm. In this case, the operator $\Upsilon_{P,Q,k}$ defined in Eq. (12) is positive, so that $|||\Upsilon^{-1}_{P,Q,k}|||_2 = \rho(\Upsilon^{-1}_{P,Q,k})$. Therefore, Proposition 4.2 gives the sharp constant for inequality (4) for this norm. Observe that $|||\Upsilon^{-1}_{P,Q,k}|||_2 = (k+2)^{-1}$ if and only if $\sigma(P) \cap \sigma(Q) \ne \emptyset$. △

5. SHARP CONSTANTS AND HADAMARD INDEX

Preliminary results. In this subsection, we shall give a brief exposition of the definitions and results of the theory of Hadamard index, which we shall use in the rest of the section. All the results are taken from [7].

Denote by $p = (1, \ldots, 1) \in \mathbb{R}^n$ and $P = pp^* \in \mathcal{M}_n(\mathbb{C})^+$, the matrix with all its entries equal to 1. Given $A \in \mathcal{M}_n(\mathbb{C})^+$ and N a norm on $\mathcal{M}_n(\mathbb{C})$, we define the N-index of A by

$$I(N, A) = \max\{\lambda \ge 0 : N(A \circ B) \ge \lambda N(B) \ \forall \ B \in P(n)\}$$

and the minimal index of A by

$$\begin{aligned} I(A) &= \max\ \{\ \lambda \geq 0\ :\ A \circ B \geq \lambda B \quad \forall\ B \in \mathcal{M}_n(\mathbb{C})^+\ \} \\ &= \max\ \{\ \lambda \geq 0\ :\ A - \lambda P \geq 0\ \}\ . \end{aligned}$$

The index relative to the spectral norm on $\mathcal{M}_n(\mathbb{C})$ will be denoted by $I(sp, A)$, and the index relative to the 2-norm $\|B\|_2^2 = \text{tr}\ B^*B$, $B \in \mathcal{M}_n(\mathbb{C})$, is denoted by $I_2(A)$.

Proposition 5.1. Let $A \in \mathcal{M}_n(\mathbb{C})^+$. Then $I(A) \neq 0$ if and only if $p \in R(A)$. □

If $A = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \in P_2$, then

$$0 \neq I(sp, A) = I(A) \quad \Leftrightarrow \quad b \in \mathbb{R} \ \text{ and } \ 0 \leq b \leq \min\{a, c\} \neq 0. \tag{13}$$

Theorem 5.2. Let $A \in \mathcal{M}_n(\mathbb{C})^+$ with nonnegative entries such that all $A_{ii} \neq 0$. Then the following conditions are equivalent:

1. There exist $u \in \mathbb{R}^n$ with **nonnegative entries** such that $Au = p$.
2. $I(sp, A) = I(A)$. □

Theorem 5.3. Let $B \in \mathcal{M}_n(\mathbb{C})^+$ such that $b_{ij} \geq 0$ for all i, j. Then there exists a subset J_0 of $\mathbb{I}_n = \{1, 2, \ldots, n\}$ such that $I(sp, B) = I(sp, B_{J_0}) = I(B_{J_0})$. Therefore $I(sp, B) = \min\{\ I(sp, B_J)\ :\ J \subseteq \mathbb{I}_n \text{ and } I(sp, B_J) = I(B_J)\ \}$. □

Let $x = (\lambda_1, \ldots, \lambda_n)^* \in \mathbb{R}_+^n$, $L = \{\lambda_1, \ldots, \lambda_n\}$ and $k \in \mathbb{R}$. We consider

$$\Lambda_x = \Big(\lambda_i\lambda_j + \frac{1}{\lambda_i\lambda_j}\ \Big)_{ij} \in \mathcal{M}_n(\mathbb{C})^+ \quad \text{ and }$$

$$E_n(x, k) = \Big(\lambda_i\lambda_j + \frac{1}{\lambda_i\lambda_j} + k\ \Big)_{ij} = \Lambda_x + kP \in \mathcal{M}_n(\mathbb{C})^+. \tag{14}$$

Proposition 5.4. Let $x = (\lambda_1, \ldots, \lambda_n)^* \in \mathbb{R}_+^n$, $L = \{\lambda_1, \ldots, \lambda_n\}$. Consider the matrix Λ_x defined before. Then $I(\Lambda_x) \neq 0$ if and only $\#L \leq 2$. More precisely,

1. If $L = \{\lambda_1\}$ (i.e. #L = 1), then $\Lambda_x = (\lambda_1^2 + \lambda_1^{-2})\ P$ and $I(\Lambda_x) = \lambda_1^2 + \lambda_1^{-2}$.
2. If $\#L > 1$, then the image of Λ_x is the subspace generated by the vectors $x = (\lambda_1, \ldots, \lambda_n)$ and $y = (\lambda_1^{-1}, \ldots, \lambda_n^{-1})$.
3. If $\#L = 2$, say $L = \{\lambda, \mu\}$, denote $\Lambda_0 = \Lambda_{\{\lambda,\mu\}} \in \mathcal{M}_2(\mathbb{C})^+$. Then
$$I(\Lambda_x) = \frac{(\lambda + \mu)^2}{1 + \lambda^2\mu^2} = I(\Lambda_0), \tag{15}$$
4. If $\#L > 2$ then $I(\Lambda_x) = 0$, because $p \notin \text{span}\,\{x, y\}$. □

Sharp constants for positive operators. In the rest of this section, we shall use the results above in order to get the sharp constants for inequality (5), if we consider only *positive* operators T.

Remark 5.5. Proposition 5.4 says that, in most cases (i.e. $\#L > 2$), $I(\Lambda_x) = 0$. On the other hand, in order to apply index theory for the matrix $E_n(x, k) = \Lambda_x + kP$, we need that $E_n(x, k) \in \mathcal{M}_n(\mathbb{C})^+$. Since $\Lambda_x + kP \geq 0$ if and only if

$k \geq -I(\Lambda_x)$ (by the definition of the minimal index), we can only consider the case $k \geq 0$, in general. In this case, it is easy to check that

$$I(E_n(x,k)) = I(\Lambda_x) + k \ . \tag{16}$$

Hence, applying Proposition 5.4, we get formulas for $I(E_n(x,k))$ in any case. △

We shall compute $I(sp, E_n(x,k))$ using Theorem 5.3. Hence, we shall use the principal minors of $E_n(x,k)$, which are matrices of the same type. Let $J \subseteq \{1, 2, \dots, n\}$, $L_J = \{\lambda_j : j \in J\}$ and $p_J, x_J \in R^J$ the induced vectors. Then

$$E_n(x,k)_J = E_{|J|}(x_J, k) = (\Lambda_x)_J + kP_J = \Lambda_{x_J} + kP_J \ .$$

Claim: If $\#L_J > 2$ then, $I(sp, E_n(x,k)_J) \neq I(E_n(x,k)_J)$.

Indeed, by Proposition 5.4, $I(\Lambda_{x_J}) = 0$, because $p_J \notin R(\Lambda_{x_J})$. Note that every $u \in E_n(x,k)_J^{-1}\{p_J\}$ must satisfy $\Lambda_{x_J} u = 0$, because $R(P_J) = \text{span } (p_J)$. By Proposition 5.4, $\ker(\Lambda_{x_J}) = R(\Lambda_{x_J})^\perp = \{x_J, y_J\}^\perp$, where $y = (\lambda_1^{-1}, \dots, \lambda_n^{-1})$. As all entries of x_J are strictly positive, if $u \geq 0$ and $\langle u, x_J \rangle = 0$, then $u = 0$. Therefore, the Claim follows from Theorem 5.2.

Hence, if $I(sp, E_n(x,k)_J) = I(E_n(x,k)_J)$, then $\#L_J \leq 2$. If $\#L_J = 2$, let $i_1, i_2 \in J$ such that $\lambda_{i_1} \neq \lambda_{i_2}$. By Theorem 5.2 there exists a vector $0 \leq u \in \mathbb{R}^J$ such that $E_n(x,k)_J \ u = p_J$. Let $z_1 = \sum\{u_k : k \in J \text{ and } \lambda_k = \lambda_{i_1}\} \geq 0$ and $z_2 = \sum\{u_j : j \in J, \text{ and } \lambda_j = \lambda_{i_2}\} \geq 0$. Easy computations show that, if we denote $E_0 = E_2(\{\lambda_{i_1}, \lambda_{i_2}\}, k) = E_n(x,k)_{\{i_1,i_2\}}$, then $E_0(z_1, z_2) = (1,1)$ and, by Theorem 5.2, $I(E_0) = I(sp, E_0)$. Moreover, by equations (15) and (16),

$$I(sp, E_n(x,k)_J) = I(E_n(x,k)_J) = \frac{(\lambda_{i_1} + \lambda_{i_2})^2}{1 + \lambda_{i_1}^2 \lambda_{i_2}^2} + k = I(E_0) = I(sp, E_0).$$

Therefore, in order to compute $I(sp, E_n(x,k))$ using Theorem 5.3, we have to consider only the diagonal entries of $E_n(x,k)$ and some of its principal minors of size 2×2. If $\lambda_i \neq \lambda_j$ and $E_0 = E_n(x,k)_{\{i,j\}}$ then, by equations (13) and (16),

$$\begin{aligned} I(E_0) = I(sp, E_0) \quad &\iff \quad I(sp, \Lambda_{\{\lambda_i, \lambda_j\}}) = I(\Lambda_{\{\lambda_i, \lambda_j\}}) \\ &\iff \quad \lambda_i \lambda_j + \tfrac{1}{\lambda_i \lambda_j} \leq \min\{\lambda_i^2 + \tfrac{1}{\lambda_i^2} \ , \ \lambda_j^2 + \tfrac{1}{\lambda_j^2}\}. \end{aligned}$$

Easy computations show that, if $\lambda_i < \lambda_j$, this is equivalent to $\lambda_i^2 \leq \frac{1}{\lambda_i \lambda_j} \leq \lambda_j^2$, which implies that $\lambda_i < 1 < \lambda_j$. So, by Theorem 5.3,

$$I(sp, E_n(x,k)) = \min\{M_1, M_2\} \tag{17}$$

where $M_1 = \min_i \lambda_i^2 + \lambda_i^{-2} + k = \min_i E_n(x,k)_{ii}$ and

$$M_2 = \min\left\{ \frac{(\lambda_i + \lambda_j)^2}{1 + \lambda_i^2 \lambda_j^2} + k \ : \ \lambda_i < 1 < \lambda_j \ \text{ and } \ \lambda_i^2 \leq \frac{1}{\lambda_i \lambda_j} \leq \lambda_j^2 \right\}.$$

Now we can characterize the sharp constant for inequality (5), if we consider only *positive* operators T.

Proposition 5.6. *Given $S \in Gl(\mathcal{H})^+$ and $k \geq 0$, denote by $M(S,k)$ the greatest number such that $\|kT + STS + S^{-1}TS^{-1}\| \geq M(S,k)\|T\|$ for every $T \in L(\mathcal{H})^+$. Then $M(S,k) = \min\left\{M_1(S,k), M_2(S,k)\right\}$, where $M_1(S,k) = \min_{\lambda\in\ \sigma(S)} \lambda^2 + \lambda^{-2} + k$ and*

$$M_2(S,k) = \inf\left\{\frac{(\lambda+\mu)^2}{1+\lambda^2\mu^2} + k : \lambda,\mu \in \sigma(S), \lambda < \mu \text{ and } \lambda^2 \leq \frac{1}{\lambda\mu} \leq \mu^2\right\}.$$

In particular, if $\|S\| \leq 1$, then $M(S,k) = \|S\|^2 + \|S\|^{-2} + k$.

Proof. We shall use the same steps as in the proof of Theorem 3.4 (and [6]). By the spectral theorem, we can suppose that $\sigma(S)$ is finite, since S can be approximated in norm by operators $S_n \in Gl(\mathcal{H})^+$ such that $\sigma(S_n)$ is a finite subset of $\sigma(S)$, $\sigma(S_n) \subset \sigma(S_{n+1})$ for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}}\sigma(S_n)$ is dense in $\sigma(S)$. So $M(S_n,k)$ (and $M_i(S_n,k),\ i=1,2$) converge to $M(S,k)$ (resp. $M_i(S,k),\ i=1,2$).
We can also suppose that $\dim\mathcal{H} < \infty$, by choosing an adequate net of finite rank projections $\{P_F\}_{F\in\mathcal{F}}$ which converges strongly to the identity and replacing S, T by P_FSP_F, P_FTP_F. Indeed, the net may be chosen in such a way that $SP_F = P_FS$ and $\sigma(P_FSP_F) = \sigma(S)$ for all $F \in \mathcal{F}$. Note that for every $A \in L(\mathcal{H})$, $\|P_FAP_F\|$ converges to $\|A\|$.
Finally, we can suppose that S is diagonal by a unitary change of basis in $\mathbb{C}^n$. In this case, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of S (with multiplicity) and $x = (\lambda_1, \ldots, \lambda_n)$, then $STS + S^{-1}TS^{-1} + kT = E_n(x,k) \circ T$. Note that all our reductions (unitary equivalences and compressions) preserve the fact that $T \geq 0$. Now the statement follows from formula (17). If $\|S\| \leq 1$ then $M(S) = M_1(S)$, since $M_2(S)$ is the infimum of the empty set. Note that $M_1(S)$ is attained at $\lambda = \|S\|$, because the map $f(x) = x + x^{-1}$ is decreasing on $(0,1]$. ☐

6. NUMERICAL RESULTS

Let $S \in \mathcal{G}l(n)^+$ and $k \in (-2, +\infty)$. Denote by

$$N_n(S,k) \quad = \quad \max\left\{c \geq 0 : \|kT + STS + S^{-1}TS^{-1}\| \geq c\|T\|, \quad T \in \mathcal{M}_n(\mathbb{C})\right\}.$$

Corollary 3.6 says that, if $k \leq 2$, then $N_n(S,k) \geq k+2$ for every $S \in \mathcal{G}l(n)^+$. On the other hand, Corollary 4.6 says that, if there exists $\lambda \in \sigma(S) \cap \sigma\left(S^{-1}\right)$, then $N_n(S,k) = k+2$. In this section we search conditions for S which assure that $N_n(S,k) = \min_{\lambda\in\ \sigma(S)} \lambda^2 + \lambda^{-2} + k$. As in the proof of Theorem 3.4, we can assume that $S = \operatorname{diag}(\lambda)$ for some $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n_{>0}$. In this case, we have that $k\,T + STS + S^{-1}TS^{-1} = E_n(\lambda,k) \circ T$, where $E_n(\lambda,k) \in \mathcal{M}_n(\mathbb{C})^+$ is the matrix defined in Eq. (14). Consider the matrix $G \in \mathcal{M}_n(\mathbb{C})_h$ with entries

$$g_{ij} = (e_n(\lambda,k)_{ij})^{-1} = \frac{1}{k + \lambda_i\lambda_j + \lambda_i^{-1}\lambda_j^{-1}} = \frac{\lambda_i\lambda_j}{1 + k\lambda_i\lambda_j + \lambda_i^2\lambda_j^2}, \qquad (18)$$

$1 \leq i,j \leq n$. Then, for $T \in \mathcal{M}_n(\mathbb{C})$, $T = G \circ (T + STS^{-1} + S^{-1}TS)$. Denote by $\Phi_G : \mathcal{M}_n(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ the map given by $\Phi_G(B) = G \circ B$, for $B \in \mathcal{M}_n(\mathbb{C})$. We conclude that $N_n(S,k)^{-1} = \|\Phi_G\|$.

Remark 6.1. There exists an extensive bibliography concerning methods for computing the norm of a Hadamard multiplier like Φ_G. The oldest result in this direction is Schur Theorem (Proposition 2.4) for the positive case. We have applied this result in the proof of Theorem 3.4, but it is not useful in this case, because $G \notin \mathcal{M}_n(\mathbb{C})^+$. The most general result is 1983's Haagerup theorem [10], which gives a complete characterization, but it is not effective. There exist also several fast algorithms (see, for example, [9]) which allow to make numerical experimentation for this problem. For example, we have observed that the behavior of the map $\mathbb{R} \ni t \longmapsto N_n(tS, k)$, for any fixed S, is chaotic. But, as a great number of examples suggest, it seems that $N_n(tS, k) = (2+k)^{-1}$ if and only if $t\sigma(S) \cap t^{-1}\sigma\left(S^{-1}\right) \neq \emptyset$. Note that these cases are exactly those considered in Corollary 4.6. △

Cowen and others [8] and [9] proved the following result for hermitian matrices:

Theorem 6.2. Let $B = (b_{ij}) \in \mathcal{M}_n(\mathbb{C})_h$ such that $0 < b_{11} = \max\limits_{1 \leq j \leq n} b_{jj}$. Suppose that B has rank two, or that B has one positive eigenvalue and $n-1$ non positive eigenvalues. Then the following conditions are equivalent:

1. $\|\Phi_B\| = b_{11}$.
2. If $1 < j \leq n$ and $J = \{1, j\}$, then $\|\Phi_{B_J}\| = b_{11}$.
3. For $1 \leq k \leq n$, it holds that $b_{11}^2 + b_{11}b_{kk} - 2|b_{1k}|^2 \geq 0$. □

Suppose that $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n) \in \mathbb{R}^n_{>0}$ satisfies $1 \geq \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$ or $1 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Since that map $f(t) = t + t^{-1}$ is increasing on $(1, +\infty)$ and it is decreasing on $(0, 1)$, it follows that the matrix G of Eq. (18) satisfies $0 < g_{11} = \max\limits_{1 \leq j \leq n} g_{jj}$. This suggests that we could apply Theorem 6.2 for our problem. Unfortunately, for $n > 2$, G has rank greater than two, and it can have more that one positive eigenvalue. We prove the following result:

Proposition 6.3. *Let $S \in \mathcal{G}l(2)^+$. Suppose that $S \leq I$, or $I \leq S$. Then*

$$N_n(S, 0) = \min_{\lambda \in \sigma(S)} \lambda^2 + \lambda^{-2} .$$

Proof. Suppose that $\sigma(S) = \{\lambda_1, \lambda_2\}$ with $1 \leq \lambda_1 \leq \lambda_2$ or $1 \geq \lambda_1 \geq \lambda_2 > 0$. Let $G \in \mathcal{M}_2(\mathbb{C})$ as in equation (18), for $k = 0$. Then $\det(G) \neq 0$. Hence, since $g_{11} \geq g_{22}$, can apply Theorem 6.2. Then, in order to prove that $g_{11} = \|\Phi_G\| = N_n(S, 0)^{-1}$, it suffices to verify the inequality $C = g_{11}^2 + g_{11}g_{22} - 2g_{12}^2 \geq 0$. Note that

$$\begin{aligned} C &= \left(\frac{\lambda_1^2}{\lambda_1^4+1}\right)^2 + \frac{\lambda_1^2}{(\lambda_1^4+1)}\frac{\lambda_2^2}{(\lambda_2^4+1)} - 2\left(\frac{\lambda_2\lambda_1}{\lambda_1\lambda_2+1}\right)^2 \\ &= \frac{\lambda_1^2(\lambda_2-\lambda_1)(\lambda_1+\lambda_2)(-1+\lambda_1^2\lambda_2^2+2\lambda_1^6\lambda_2^2-2\lambda_2^4-\lambda_1^4\lambda_2^4+\lambda_1^6\lambda_2^6)}{(\lambda_1^4+1)^2(\lambda_2^4+1)(1+\lambda_1^2\lambda_2^2)^2} . \end{aligned}$$

Straightforward computations with $1 \leq \lambda_1 \leq \lambda_2$ show that $C > 0$, since the polynomial $P(\lambda_1, \lambda_2) = -1 + \lambda_1^2\lambda_2^2 + 2\lambda_1^6\lambda_2^2 - 2\lambda_2^4 - \lambda_1^4\lambda_2^4 + \lambda_1^6\lambda_2^6 > 0$ in this case. A similar analysis shows that still $C > 0$ for $1 \geq \lambda_1 \geq \lambda_2 > 0$. The result follows by applying Theorem 6.2. □

It was proved by Kwong (see [13]) that, if $\lambda \in \mathbb{R}_+^n$, then the matrix $Z_n(k,\lambda) \in \mathcal{M}_n(\mathbb{C})$ defined in Eq. (10), is positive semidefinite in the following cases: $n=2$ and $k \in (-2,\infty)$, $n=3$ and $k \in (-2,8)$, and $n=4$ and $k \in (-2,4)$. Therefore, by the proof of Theorem 3.4, inequality (6) holds in $\mathcal{M}_n(\mathbb{C})$ in these cases, for every unitarily invariant norm. Note that the proof Theorem 3.1 does not give similar estimates for the inequalities (4) and (5), because one needs to duplicate dimensions.

A numerical approach suggests that these intervals are optimal, both for the positivity of the matrix $B_n(k,\lambda)$, defined in Eq. (11), and for inequality (6). In the case of $\mathcal{M}_3(\mathbb{C})$, if $-2 < k \in \mathbb{R}$ and $\lambda = (\lambda_1,\lambda_2,\lambda_3) \in \mathbb{R}^3_{>0}$ then, using symbolic computation with the software Mathematica, one obtains a kind of "proof" of the fact that $\det B_3(k,\lambda) \geq 0$ for every $\lambda \in \mathbb{R}^3_{>0}$ if and only if $-2 < k \leq 8$. The 2×2 principal sub matrices of $B_3(k,\lambda)$ have the form $B_2(k,(\lambda_i,\lambda_j)\,)$, and they live in $\mathcal{M}_2(\mathbb{C})^+$ for every $k > -2$. Therefore,

$$B_3(k,\lambda) \in \mathcal{M}_3(\mathbb{C})^+ \quad \text{for every} \quad \lambda \in \mathbb{R}^3_{>0} \iff -2 < k \leq 8.$$

Likewise, for the 4×4 matrix case, it suffices to study $\det B_4(k,\lambda)$, for $\lambda \in \mathbb{R}^4_{>0}$, and one obtains similar results.

Denote by k_n the maximum number $k \in \mathbb{R}$ such that inequality (6) holds in $\mathcal{M}_n(\mathbb{C})$ for the spectral norm. By the preceding comments, and the proof of Theorem 3.4,

$$k_2 = +\infty, \quad k_3 \geq 8, \quad k_4 \geq 4, \quad \text{and} \quad k_n \geq 2 \quad \text{for} \quad n \geq 5.$$

Computer experimentation using the softwares Mathematica and Matlab suggests that, also in this case, $k_3 = 8$, $k_4 = 4$, and $k_n = 2$, for $n \geq 5$. In other words, inequality (6) holds for $\mathcal{M}_n(\mathbb{C})$ in the same intervals as it holds that $B_n(k,\lambda) \in \mathcal{M}_n(\mathbb{C})^+$ for every $\lambda \in \mathbb{R}^n_{>0}$.

References

[1] T. Ando, Majorization and inequalities in matrix theory, *Linear Algebra Appl.* 199: 17-67, 1994.

[2] E. Andruchow, G. Corach and D. Stojanoff, Geometric operator inequalities, *Linear Algebra Appl.* 258: 295-310, 1997.

[3] R. Bhatia, and CH. Davis, More matrix forms of the Arithmetic-Geometric mean inequality, *SIAM J. Matrix Anal. Appl.* 14: 132-136, 1993.

[4] R. Bhatia and K. R. Parthasarathy, Positive definite functions and operator inequalities, *Bull. London Math. Soc.* 32: 214-228, 2000.

[5] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[6] G. Corach, H. Porta and L. Recht, An operator inequality, *Linear Algebra Appl.* 142: 153-158, 1990.

[7] G. Corach and D. Stojanoff, Index of Hadamard multiplication by positive matrices II, *Linear Algebra Appl.* 332-334 (2001), 503-517.

[8] C. C. Cowen, K. E. Debro and P. D. Sepansky, Geometry and the norms of Hadamard multipliers, *Linear Algebra Appl.* 218: 239-249, 1995.

[9] C. C. Cowen, P. A. Ferguson, D. K. Jackman, E. A. Sexauer, C. Vogt y H. J. Woolf , Finding Norms of hadamard multipliers, *Linear Algebra Appl.* 247, 217-235 (1996).

[10] U. Haagerup, Decompositions of completely bounded maps on operator algebras, unpublished manuscript.

[11] F. Kittaneh, A note on the arithmetic-geometric inequality for matrices, Linear Algebra Appl. 171 (1992), 1-8.

[12] F. Kittaneh, On some operator inequalities, *Linear Algebra Appl.* 208/209:19-28, 1994.

[13] M.K. Kwong, On the definiteness of the solutions of certain matrix equations, *Linear Algebra Appl.* 108: 177-197, 1988.

[14] L. Livshits and S. C. Ong, On the invertibility properties of the map $T \mapsto STS^{-1} + S^{-1}TS$ and operator-norm inequalities, *Linear Algebra Appl.* 183: 117-129, 1993.

[15] A. Seddik, Some results related to the Corach-Porta-Recht inequality, *Proceedings of the American Mathematical Society*, 129: 3009-3015, 2001.

[16] I. Schur, Bemerkungen zur theorie de beschränkten bilineareformen mit unendlich vielen veränderlichen, *J. Reine Angew. Math.* 140: 1-28, 1911.

[17] B. Simon, *Trace ideals and their applications*, London Mathematical Society Lecture Note Series, 35, Cambridge University Press, Cambridge-New York, 1979.

[18] X. Zhan, Inequalities for unitarily invariant norms, *SIAM J. Matrix Anal. Appl.* 20:466-470, 1999.

Cristina Cano
Depto. de Matemática,
FaEA-UNC,
Neuquén, Argentina.
cbcano@uncoma.edu.ar

Irene Mosconi
Depto. de Matemática,
FaEA-UNC,
Neuquén, Argentina.
imosconi@uncoma.edu.ar

Demetrio Stojanoff
Depto. de Matemática,
FCE-UNLP, La Plata, Argentina
and IAM-CONICET.
demetrio@mate.unlp.edu.ar

Recibido: 3 de junio de 2005
Aceptado: 21 de noviembre de 2005