

AN ULTRAPOWER CONSTRUCTION OF THE MULTIPLIER ALGEBRA OF A C^* -ALGEBRA AND AN APPLICATION TO BOUNDARY AMENABILITY OF GROUPS

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ABSTRACT. Using ultrapowers of C^* -algebras we provide a new construction of the multiplier algebra of a C^* -algebra. This extends the work of Avsec and Goldbring [Houston J. Math., to appear, arXiv:1610.09276.] to the setting of noncommutative and nonseparable C^* -algebras. We also extend their work to give a new proof of the fact that groups that act transitively on locally finite trees with boundary amenable stabilizers are boundary amenable.

1. INTRODUCTION

The multiplier algebra $\mathcal{M}(\mathcal{A})$ of a C^* -algebra \mathcal{A} is a C^* -algebra that contains \mathcal{A} as an essential ideal and satisfies the following universal property: for every C^* -algebra \mathcal{B} containing \mathcal{A} as an ideal, there exists a unique $*$ -homomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{M}(\mathcal{A})$ such that φ is the identity on \mathcal{A} . If \mathcal{A} is abelian, thus of the form $C_0(X)$ for some locally compact Hausdorff space X , then $\mathcal{M}(\mathcal{A})$ is isomorphic to $C_b(X)$ and this in turn can be identified with $C(\beta X)$, where βX is the Stone-Ćech compactification of X (for more about multiplier algebras, see, for instance [4, 6]).

In the article [1], Avsec and Goldbring provided a new construction of the multiplier algebra of the abelian C^* -algebra $C_0(X)$ using ultraproducts of C^* -algebras, in the case when X is a second countable locally compact Hausdorff space. From there, they inferred a new construction of the Stone-Ćech compactification of X , and they used it to give a new proof of the fact that groups that act properly and transitively on trees are boundary amenable. In section 2 of this note, we extend their work providing a construction of the multiplier algebra of any C^* -algebra \mathcal{A} by means of ultraproducts of C^* -algebras. In section 3, we focus on the case of commutative and separable C^* -algebras, and compare our main technical tool with the main technical tool used in [1] to explain why the work done here is indeed a generalization of [1]. Finally, in section 4, we extend the techniques of [1] to show that groups that act transitively on locally finite trees having boundary amenable stabilizers are boundary amenable.

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2. ULTRAPRODUCTS AND MULTIPLIERS

Let \mathcal{I} be a set. An *ultrafilter* over \mathcal{I} is a nonempty collection \mathcal{U} of subsets of \mathcal{I} with the following properties:

- (1) finite intersection property: for every $\mathcal{I}_0, \mathcal{I}_1 \in \mathcal{U}$, then $\mathcal{I}_0 \cap \mathcal{I}_1 \in \mathcal{U}$;
- (2) directness: for every $\mathcal{I}_0 \subset \mathcal{I}_1$, where \mathcal{I}_0 belongs to \mathcal{U} , then $\mathcal{I}_1 \in \mathcal{U}$;
- (3) maximality: for every $\mathcal{I}_0 \subset \mathcal{I}$, either $\mathcal{I}_0 \in \mathcal{U}$ or $\mathcal{I} \setminus \mathcal{I}_0 \in \mathcal{U}$.

An ultrafilter is *principal* if there exists $i \in \mathcal{I}$ such that the subsets of \mathcal{I} that contains i are in the ultrafilter. Ultrafilters not of this form are called *nonprincipal* or *free*. It is easy to show that an ultrafilter is nonprincipal exactly when it contains no finite sets. An ultrafilter is *cofinal* when the index set is a directed set, and the sets $\{i \in \mathcal{I} : i \geq i_0\}$ are in \mathcal{U} for every $i_0 \in \mathcal{I}$.

When dealing with directed sets with the property that there is no maximal element, every cofinal ultrafilter is nonprincipal. Moreover, when the ultrafilter is over \mathbb{N} , being cofinal is the same as being nonprincipal. If a directed set has a maximal element, then every cofinal ultrafilter is principal.

Definition 2.1. Let \mathcal{U} be an ultrafilter over \mathcal{I} . Let (X, d) be a metric space and let $(a_i)_{i \in \mathcal{I}} \subset X$. We say that $(a_i)_{i \in \mathcal{I}}$ is convergent along \mathcal{U} , if there exists an element $a \in X$ such that, for every $\varepsilon > 0$, the set $\{i \in \mathcal{I} : d(a_i, a) < \varepsilon\}$ belongs to \mathcal{U} . The element a is called the \mathcal{U} -limit of $(a_i)_{i \in \mathcal{I}}$ and it is denoted by $\lim_{\mathcal{U}} a_i$.

2.1. Ultraproducts of C^* -algebras. Let \mathcal{U} be an ultrafilter over a set \mathcal{I} and let \mathcal{A} be a C^* -algebra. Denote by $\prod_{\mathcal{I}} \mathcal{A}$ the set $\{(a_i)_{i \in \mathcal{I}} : \sup_{i \in \mathcal{I}} \|a_i\| < \infty\}$ and let $\mathcal{N}_{\mathcal{U}}$ be the subspace generated by those $(a_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ such that $\lim_{\mathcal{U}} \|a_i\| = 0$.

Denote by $\mathcal{A}^{\mathcal{U}}$ the quotient $\prod_{\mathcal{I}} \mathcal{A} / \mathcal{N}_{\mathcal{U}}$. This is a vector space, and with the norm defined by $\|(a_i)_{i \in \mathcal{I}}\|_{\mathcal{U}} := \lim_{\mathcal{U}} \|a_i\|$, and the involution defined by $(a_i)_{i \in \mathcal{I}}^* := (a_i^*)_{i \in \mathcal{I}}$, so $\mathcal{A}^{\mathcal{U}}$ becomes a C^* -algebra.

Remark 2.2. Let $((a_i^n)_{i \in \mathcal{I}})_{n \in \mathbb{N}} \subset \mathcal{A}^{\mathcal{U}}$ be a sequence that converges to $(a_i)_{i \in \mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$. Then, for all $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $n \geq n_0$, then $\|(a_i^n)_{i \in \mathcal{I}} - (a_i)_{i \in \mathcal{I}}\|_{\mathcal{U}} < \varepsilon$. We claim that if $n \geq n_0$, then the set $\Omega_n(\varepsilon) := \{i \in \mathcal{I} : \|a_i^n - a_i\| < \varepsilon\}$ belongs to \mathcal{U} . To show this, set $\alpha_n := \|(a_i^n)_{i \in \mathcal{I}} - (a_i)_{i \in \mathcal{I}}\|_{\mathcal{U}}$. For every $\delta > 0$, we have $\{i \in \mathcal{I} : |\|a_i^n - a_i\| - \alpha_n| < \delta\} \in \mathcal{U}$. Taking $\delta = \varepsilon - \alpha_n > 0$ we get $\varepsilon - \alpha_n > |\|a_i^n - a_i\| - \alpha_n| \geq \|a_i^n - a_i\| - \alpha_n$. It follows that $\{i \in \mathcal{I} : |\|a_i^n - a_i\| - \alpha_n| < \delta\} \subset \Omega_n(\varepsilon)$. By directness, this implies that $\Omega_n(\varepsilon) \in \mathcal{U}$.

From now on we fix a faithful and non-degenerate representation of \mathcal{A} on $B(\mathcal{H})$.

Lemma 2.3. *Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. For each $(a_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$, there exists a unique element $a_{\mathcal{U}\text{-WOT}} \in B(\mathcal{H})$ such that for every $\xi, \eta \in \mathcal{H}$, it holds that $\langle a_{\mathcal{U}\text{-WOT}} \xi, \eta \rangle = \lim_{\mathcal{U}} \langle a_i \xi, \eta \rangle$. The operator $a_{\mathcal{U}\text{-WOT}}$ is called the \mathcal{U} -WOT-limit of $(a_i)_{i \in \mathcal{I}}$.*

Proof. Let $(a_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ and let $\xi, \eta \in \mathcal{H}$. Then $(\langle a_i \xi, \eta \rangle)_{i \in \mathcal{I}} \subset \mathbb{C}$ is bounded, hence it has a \mathcal{U} -limit, which is denoted by $b_{\xi, \eta}$. It is easy to see that the map

$(\xi, \eta) \mapsto b_{\xi, \eta}$ is a bounded sesquilinear form on $\mathcal{H} \times \mathcal{H}$. Take $a_{\mathcal{U}\text{-WOT}} \in B(\mathcal{H})$ the unique operator associated to it. \square

Definition 2.4. Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. An element $(a_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ is \mathcal{U} -strict convergent if there exists an operator $a_{\mathcal{U}} \in B(\mathcal{H})$ such that, for every $x \in \mathcal{A}$, and every $\varepsilon > 0$ we have $\{i \in \mathcal{I} : \|a_i x - a_{\mathcal{U}} x\| < \varepsilon, \|x a_i - x a_{\mathcal{U}}\| < \varepsilon\} \in \mathcal{U}$. The operator $a_{\mathcal{U}}$ is called the \mathcal{U} -strict limit of $(a_i)_{i \in \mathcal{I}}$. Observe that $a_{\mathcal{U}} x$ and $x a_{\mathcal{U}}$ are elements of \mathcal{A} for every $x \in \mathcal{A}$.

In what follows, it will be convenient to have the following notation at hand.

Notation 2.5. Let $(a_i)_{i \in \mathcal{I}}, (b_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ that are \mathcal{U} -strict convergent to $a_{\mathcal{U}}$ and $b_{\mathcal{U}}$, respectively. For every $x \in \mathcal{A}$ and every $\varepsilon > 0$, put

$$\begin{aligned} A_x(\varepsilon) &:= \{i \in \mathcal{I} : \|x(a_i - a_{\mathcal{U}})\|, \|(a_i - a_{\mathcal{U}})x\| < \varepsilon\} \in \mathcal{U}, \\ B_x(\varepsilon) &:= \{i \in \mathcal{I} : \|x(b_i - b_{\mathcal{U}})\|, \|(b_i - b_{\mathcal{U}})x\| < \varepsilon\} \in \mathcal{U}. \end{aligned}$$

Proposition 2.6. *Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. If $(a_i)_{i \in \mathcal{I}}, (b_i)_{i \in \mathcal{I}} \in \prod_{\mathcal{I}} \mathcal{A}$ define the same element in $\mathcal{A}^{\mathcal{U}}$, then*

- (1) *if $(a_i)_{i \in \mathcal{I}}$ is \mathcal{U} -WOT convergent to $a_{\mathcal{U}\text{-WOT}}$, then $(b_i)_{i \in \mathcal{I}}$ is \mathcal{U} -WOT convergent to $a_{\mathcal{U}\text{-WOT}}$;*
- (2) *if $(a_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $a_{\mathcal{U}}$, then $(b_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $a_{\mathcal{U}}$.*

Proof. To prove (1), take $\xi, \eta \in \mathcal{H}$ of norm 1, let $\varepsilon > 0$, and take i in the set $\{i \in \mathcal{I} : \|a_i - b_i\| < \frac{\varepsilon}{2}\} \cap \{i \in \mathcal{I} : |\langle a_{\mathcal{U}\text{-WOT}} \xi, \eta \rangle - \langle a_i \xi, \eta \rangle| < \frac{\varepsilon}{2}\} \in \mathcal{U}$. Then

$$|\langle a_{\mathcal{U}\text{-WOT}} \xi, \eta \rangle - \langle b_i \xi, \eta \rangle| \leq |\langle (a_{\mathcal{U}\text{-WOT}} - a_i) \xi, \eta \rangle| + |\langle (a_i - b_i) \xi, \eta \rangle| < \varepsilon.$$

It follows that the set $\{i \in \mathcal{I} : |\langle a_{\mathcal{U}\text{-WOT}} \xi, \eta \rangle - \langle b_i \xi, \eta \rangle| < \varepsilon\}$ belongs to \mathcal{U} .

To prove (2), take $\varepsilon > 0$, $x \in \mathcal{A}$ and $i \in \{i \in \mathcal{I} : \|a_i - b_i\| < \frac{\varepsilon}{2\|x\|}\} \cap A_x(\frac{\varepsilon}{2}) \in \mathcal{U}$.

Then

$$\begin{aligned} \|x(b_i - a_{\mathcal{U}})\| &\leq \|x(b_i - a_i)\| + \|x(a_i - a_{\mathcal{U}})\| < \varepsilon, \\ \|(b_i - a_{\mathcal{U}})x\| &\leq \|(b_i - a_i)x\| + \|(a_i - a_{\mathcal{U}})x\| < \varepsilon. \end{aligned}$$

It follows that the set $\{i \in \mathcal{I} : \|x(b_i - a_{\mathcal{U}})\|, \|(b_i - a_{\mathcal{U}})x\| < \varepsilon\}$ belongs to \mathcal{U} . \square

Proposition 2.7. *Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. The set*

$\mathcal{A}^{s\mathcal{U}} := \{(a_i)_{i \in \mathcal{I}} \in \mathcal{A}^{\mathcal{U}} : \text{there exists } a_{\mathcal{U}} \in B(\mathcal{H}) : (a_i)_{i \in \mathcal{I}} \text{ is } \mathcal{U}\text{-strict convergent to } a_{\mathcal{U}}\}$
is a C^* -algebra.

Proof. Let $(a_i)_{i \in \mathcal{I}}, (b_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$ that are \mathcal{U} -strict convergent to $a_{\mathcal{U}}$ and $b_{\mathcal{U}}$, respectively, let $\lambda \neq 0$ be a complex number, and let $x \in \mathcal{A}$. By following Notation 2.5, if $i \in A_x\left(\frac{\varepsilon}{2}\right) \cap B_x\left(\frac{\varepsilon}{2|\lambda|}\right) \in \mathcal{U}$, then

$$\|(a_i + \lambda b_i - a_{\mathcal{U}} - \lambda b_{\mathcal{U}})x\| \leq \|(a_i - a_{\mathcal{U}})x\| + |\lambda| \|(b_i - b_{\mathcal{U}})x\| < \varepsilon,$$

and

$$\|x(a_i + \lambda b_i - a_{\mathcal{U}} - \lambda b_{\mathcal{U}})\| \leq \|x(a_i - a_{\mathcal{U}})\| + |\lambda| \|x(b_i - b_{\mathcal{U}})\| < \varepsilon.$$

It follows that

$$A_x\left(\frac{\varepsilon}{2}\right) \cap B_x\left(\frac{\varepsilon}{2|\lambda|}\right) \subset \{i \in \mathcal{I} : \|(a_i + \lambda b_i - a_{\mathcal{U}} - \lambda b_{\mathcal{U}})x\| < \varepsilon, \|x(a_i + \lambda b_i - a_{\mathcal{U}} - \lambda b_{\mathcal{U}})\| < \varepsilon\},$$

so $(a_i + \lambda b_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $a_{\mathcal{U}} + \lambda b_{\mathcal{U}}$.

It is clear that $\mathcal{A}^{s\mathcal{U}}$ is $*$ -closed. To show that $\mathcal{A}^{s\mathcal{U}}$ is closed under taking products, set $M = \sup_{i \in \mathcal{I}} \{\|a_i\|, \|b_i\|\}$ and take $i \in A_x\left(\frac{\varepsilon}{2M}\right) \cap A_{b_{\mathcal{U}}x}\left(\frac{\varepsilon}{2}\right) \cap B_x\left(\frac{\varepsilon}{2M}\right) \cap B_{xa_{\mathcal{U}}}\left(\frac{\varepsilon}{2}\right) \in \mathcal{U}$. Then

$$\|(a_i b_i - a_{\mathcal{U}} b_{\mathcal{U}})x\| \leq \|(a_i b_i - a_i b_{\mathcal{U}})x\| + \|(a_i b_{\mathcal{U}} - a_{\mathcal{U}} b_{\mathcal{U}})x\| \leq \varepsilon$$

and

$$\|x(a_i b_i - a_{\mathcal{U}} b_{\mathcal{U}})\| \leq \|x(a_i b_i - a_{\mathcal{U}} b_i)\| + \|x(a_{\mathcal{U}} b_i - a_{\mathcal{U}} b_{\mathcal{U}})\| \leq \varepsilon,$$

which means that $(a_i b_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $a_{\mathcal{U}} b_{\mathcal{U}}$.

It is left to show that $\mathcal{A}^{s\mathcal{U}}$ is norm closed. Let $((a_i^n)_{i \in \mathcal{I}})_{n \in \mathbb{N}}$ be a sequence in $\mathcal{A}^{s\mathcal{U}}$ that converges to $(\alpha_i)_{i \in \mathcal{I}}$ in $\mathcal{A}^{\mathcal{U}}$. We need to see that $(\alpha_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent.

As a first step, we will show that, for a fixed element $x \in \mathcal{A}$, $(\alpha_i x)_{i \in \mathcal{I}}$ and $(x \alpha_i)_{i \in \mathcal{I}}$ have \mathcal{U} -limit in \mathcal{A} (in the sense of Definition 2.1).

Let $x \in \mathcal{A}$ be fixed and $x \neq 0$. For each $n \in \mathbb{N}$ let $a_{\mathcal{U}}^n$ be the \mathcal{U} -strict limit of $(a_i^n)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$. We proceed to show that $(a_{\mathcal{U}}^n x)_{n \in \mathbb{N}}$ is a Cauchy sequence in \mathcal{A} . By Remark 2.2, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, the sets $\Omega_n\left(\frac{\varepsilon}{4\|x\|}\right)$ are elements of \mathcal{U} . It follows that the set

$$\{i \in \mathcal{I} : \|a_i^n x - a_{\mathcal{U}}^n x\| < \frac{\varepsilon}{4}\} \cap \{i \in \mathcal{I} : \|a_i^m x - a_{\mathcal{U}}^m x\| < \frac{\varepsilon}{4}\} \cap \Omega_n\left(\frac{\varepsilon}{4\|x\|}\right) \cap \Omega_m\left(\frac{\varepsilon}{4\|x\|}\right)$$

is an element of \mathcal{U} for all $n, m \geq n_0$. Take i in this set. Then

$$\begin{aligned} \|a_{\mathcal{U}}^n x - a_{\mathcal{U}}^m x\| &\leq \|a_{\mathcal{U}}^n x - a_i^n x\| + \|a_i^n x - a_i^m x\| + \|a_i^m x - a_{\mathcal{U}}^m x\| \\ &\leq \frac{\varepsilon}{4} + \|a_i^n - \alpha_i\| \|x\| + \|\alpha_i - a_i^m\| \|x\| + \frac{\varepsilon}{4} < \varepsilon. \end{aligned}$$

Let $\rho(x) := \lim_{n \in \mathbb{N}} a_{\mathcal{U}}^n x \in \mathcal{A}$. We will show that $\lim_{\mathcal{U}} \alpha_i x = \rho(x)$, that is, $\{i \in \mathcal{I} : \|\rho(x) - \alpha_i x\| < \varepsilon\} \in \mathcal{U}$ for each $\varepsilon > 0$. Let $n \in \mathbb{N}$ large enough such that

$$\|\rho(x) - a_{\mathcal{U}}^n x\| < \frac{\varepsilon}{3} \quad \text{and} \quad \Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}.$$

For such $n \in \mathbb{N}$, take $i \in \{i \in \mathcal{I} : \|a_{\mathcal{U}}^n x - a_i^n x\| < \frac{\varepsilon}{3}\} \cap \Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}$. Then

$$\|\rho(x) - \alpha_i x\| \leq \|\rho(x) - a_{\mathcal{U}}^n x\| + \|a_{\mathcal{U}}^n x - a_i^n x\| + \|a_i^n x - \alpha_i x\| \leq \varepsilon.$$

Repeating this with $(x \alpha_i)_{i \in \mathcal{I}}$ concludes the first step.

Let $\alpha_{\mathcal{U}\text{-WOT}} \in B(\mathcal{H})$ be the \mathcal{U} -WOT-limit of $(\alpha_i)_{i \in \mathcal{I}}$. We will show that $\alpha_{\mathcal{U}\text{-WOT}} x = \rho(x)$. Take $\eta, \xi \in \mathcal{H}$ of norm 1, $\varepsilon > 0$, and

$$i \in \{i \in \mathcal{I} : |\langle (\alpha_i - \alpha_{\mathcal{U}\text{-WOT}}) x \xi, \eta \rangle| < \frac{\varepsilon}{2}\} \cap \{i \in \mathcal{I} : \|\rho(x) - \alpha_i x\| < \frac{\varepsilon}{2}\} \in \mathcal{U}.$$

We then have that

$$|\langle (\rho(x) - \alpha_{\mathcal{U}\text{-WOT}} x) \xi, \eta \rangle| \leq |\langle (\rho(x) - \alpha_i x) \xi, \eta \rangle| + |\langle (\alpha_i x - \alpha_{\mathcal{U}\text{-WOT}} x) \xi, \eta \rangle| < \varepsilon,$$

which implies that $\alpha_{\mathcal{U}\text{-WOT}}x = \rho(x) = \lim_{\mathcal{U}} \alpha_i x$. Therefore, for all $x \in \mathcal{A}$ and $\varepsilon > 0$, we have $\{i \in \mathcal{I} : \|\alpha_{\mathcal{U}\text{-WOT}}x - \alpha_i x\| < \varepsilon\} \in \mathcal{U}$. In a similar manner, one shows that $\{i \in \mathcal{I} : \|x\alpha_{\mathcal{U}\text{-WOT}} - x\alpha_i\| < \varepsilon\} \in \mathcal{U}$. It follows that $(\alpha_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $\alpha_{\mathcal{U}\text{-WOT}}$. \square

Proposition 2.8. *Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. The set*

$$J := \{(a_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}} : a_i \text{ is } \mathcal{U}\text{-strict convergent to } 0\}$$

is an ideal of $\mathcal{A}^{s\mathcal{U}}$.

Proof. We only have to show that J is norm closed. Consider $((a_i^n)_{i \in \mathcal{I}})_{n \in \mathbb{N}} \subset J$ a sequence that converges in norm to $(\alpha_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$. Let $\alpha_{\mathcal{U}}$ be the \mathcal{U} -strict limit of $(\alpha_i)_{i \in \mathcal{I}}$. Let $\varepsilon > 0$. Take $n \in \mathbb{N}$ large enough such that $\Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}$, and $i \in \{i \in \mathcal{I} : \|(\alpha_i - \alpha_{\mathcal{U}})x\| < \frac{\varepsilon}{3}\} \cap \{i \in \mathcal{I} : \|a_i^n x\| < \frac{\varepsilon}{3}\} \cap \Omega_n\left(\frac{\varepsilon}{3\|x\|}\right) \in \mathcal{U}$. We then have that $\|\alpha_{\mathcal{U}}x\| \leq \|(\alpha_{\mathcal{U}} - \alpha_i)x\| + \|\alpha_i x\| \leq \frac{\varepsilon}{3} + \|(\alpha_i - a_i^n)x\| + \|a_i^n x\| \leq \varepsilon$. Since the action of \mathcal{A} on \mathcal{H} is nondegenerate, $\alpha_{\mathcal{U}} = 0$. \square

There exists a natural embedding of \mathcal{A} in $\mathcal{A}^{s\mathcal{U}}$, via the constant sequences $a \mapsto (a)_{i \in \mathcal{I}}$. It is clear that this element is \mathcal{U} -strict convergent to a . Moreover, since the representation of \mathcal{A} in $B(\mathcal{H})$ is faithful and nondegenerate, it follows that there exists a natural embedding of \mathcal{A} in $\mathcal{A}^{s\mathcal{U}}/J$.

Recall that an ideal I in a C^* -algebra \mathcal{A} is *essential* if $I \cap K$ is nontrivial for every ideal $K \neq \{0\}$, or equivalently, $aI = 0$ implies $a = 0$.

Lemma 2.9. *Let \mathcal{U} be an ultrafilter defined over \mathcal{I} and let \mathcal{A} be a C^* -algebra. Consider the C^* -algebra $\mathcal{A}^{s\mathcal{U}}/J$. The image of \mathcal{A} in $\mathcal{A}^{s\mathcal{U}}/J$ is an essential ideal.*

Proof. Take $(b_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$, let $b_{\mathcal{U}}$ be its \mathcal{U} -strict limit, and take $a \in \mathcal{A}$. Then $(b_i a)_{i \in \mathcal{I}}$ and $(b_{\mathcal{U}} a)_{i \in \mathcal{I}}$ are both \mathcal{U} -strict convergent to $b_{\mathcal{U}} a$. It follows that $(b_i a)_{i \in \mathcal{I}}$ and $(b_{\mathcal{U}} a)_{i \in \mathcal{I}}$ are equal in $\mathcal{A}^{s\mathcal{U}}/J$. Analogously, $(ab_i)_{i \in \mathcal{I}}$ is equal to $(ab_{\mathcal{U}})_{i \in \mathcal{I}}$ in $\mathcal{A}^{s\mathcal{U}}/J$.

Suppose that $J' \subset \mathcal{A}^{s\mathcal{U}}/J$ is an ideal such that $J' \cap \mathcal{A} = \{0\}$. If $(b_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$ projects to J' , then $(b_i x)_{i \in \mathcal{I}} \in J'$, for each $x \in \mathcal{A}$. Let $b_{\mathcal{U}}$ be the \mathcal{U} -strict limit of $(b_i)_{i \in \mathcal{I}}$. Hence $(b_i x)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $b_{\mathcal{U}} x$. Then $b_{\mathcal{U}} x = 0$ for all $x \in \mathcal{A}$. It follows that $b_{\mathcal{U}} = 0$ and then $J' = \{0\}$. \square

2.2. Ultrafilters and approximate units. Every C^* -algebra \mathcal{A} has an approximate unit, namely, there exist a directed set \mathcal{I} and a net $(e_i)_{i \in \mathcal{I}} \subset \mathcal{A}$ such that for every $x \in \mathcal{A}$, the nets $(xe_i)_{i \in \mathcal{I}}$ and $(e_i x)_{i \in \mathcal{I}}$ converge to x (see [6, Chapter I.4]). Approximate units can be taken to be positive and uniformly bounded, in which case they are elements of $\prod_{\mathcal{I}} \mathcal{A}$. In what follows, we will focus in the case where the ultrafilters are defined over this directed set \mathcal{I} . Observe that for a cofinal ultrafilter \mathcal{U} , the sets $\{i \in \mathcal{I} : \|xe_i - x\| < \varepsilon\}$ and $\{i \in \mathcal{I} : \|e_i x - x\| < \varepsilon\}$ belong to \mathcal{U} , for every $x \in \mathcal{A}$ and for every $\varepsilon > 0$. Moreover, when \mathcal{A} is unital, \mathcal{I} can be taken to be the set with one element $\{1_{\mathcal{A}}\}$ and the approximate unit to be equal to $\{1_{\mathcal{A}}\}$. In this case, the only ultrafilter is the set $\{1_{\mathcal{A}}\}$, which is cofinal.

Theorem 2.10. *Let \mathcal{A} be a C^* -algebra and let $(e_i)_{i \in \mathcal{I}} \in \prod_{i \in \mathcal{I}} \mathcal{A}$ be an approximate unit for \mathcal{A} . Let \mathcal{U} be a cofinal ultrafilter over \mathcal{I} . Then the C^* -algebra $\mathcal{A}^{s\mathcal{U}}/J$ is the multiplier algebra of \mathcal{A} .*

Proof. We saw in Lemma 2.9 that \mathcal{A} is an essential ideal in $\mathcal{A}^{s\mathcal{U}}/J$. We are left to show that for any C^* -algebra \mathcal{B} containing \mathcal{A} as an ideal, there is a unique C^* -homomorphism $\varphi : \mathcal{B} \rightarrow \mathcal{A}^{s\mathcal{U}}/J$ such that $\varphi(a) = a$.

To this end, let $b \in \mathcal{B}$ and consider $\psi : \mathcal{B} \rightarrow \mathcal{A}^{s\mathcal{U}}$ defined by $\psi(b) = (be_i)_{i \in \mathcal{I}}$. To see that ψ is well defined, let $\varepsilon > 0$ and let $x \in \mathcal{A}$. Let $i_0 \in \mathcal{I}$ such that if $i \geq i_0$, then $\|xbe_i - xb\| < \varepsilon$. Let $i_1 \in \mathcal{I}$ such that if $i \geq i_1$, then $\|e_ix - x\| < \frac{\varepsilon}{\|b\|}$. By the

cofinality of \mathcal{U} one obtains that $\{i \in \mathcal{I} : \|be_ix - bx\| < \varepsilon, \|xbe_i - xb\| < \varepsilon\} \in \mathcal{U}$. Observe that since $b \notin B(\mathcal{H})$, the last line does not imply that $(be_i)_{i \in \mathcal{I}} \in \mathcal{A}^{s\mathcal{U}}$. We must “represent” b in $B(\mathcal{H})$. To this end, let $b_{\mathcal{U}\text{-WOT}} \in B(\mathcal{H})$ be the \mathcal{U} -WOT-limit of $(be_i)_{i \in \mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$, an argument similar to one given in the proof of Proposition 2.7, shows that $b_{\mathcal{U}\text{-WOT}}x = bx$ and $xb_{\mathcal{U}\text{-WOT}} = xb$. Thus $(be_i)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $b_{\mathcal{U}\text{-WOT}}$. The same procedure shows that $(e_ib)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to $b_{\mathcal{U}\text{-WOT}}$.

Call π the quotient projection to $\mathcal{A}^{s\mathcal{U}}/J$, and let $\varphi = \pi \circ \psi$. It is clear that φ is linear and bounded. Since $\psi(b^*) = (b^*e_i)_{i \in \mathcal{I}}$ and $\psi(b)^* = (e_ib^*)_{i \in \mathcal{I}}$ and they are both \mathcal{U} -strict convergent to $(b^*)_{\mathcal{U}\text{-WOT}}$, hence $\psi(b^*) - \psi(b)^*$ is an element of J , so φ is a $*$ -preserving homomorphism.

To see that φ is multiplicative, fix $b, b' \in \mathcal{B}$ of norm 1 and take $x \in \mathcal{A}$, $\varepsilon > 0$, $M = \sup_{i \in \mathcal{I}} \{\|e_i\|\}$ and i in the set

$$\{i \in \mathcal{I} : \|e_ix - x\| < \frac{\varepsilon}{3M}\} \cap \{i \in \mathcal{I} : \|x - e_ix\| < \frac{\varepsilon}{3M}\} \cap \{i \in \mathcal{I} : \|e_ib'x - b'x\| < \frac{\varepsilon}{3M}\},$$

which is an element of \mathcal{U} . Then

$$\begin{aligned} \|(be_ib'e_i - bb'e_i)x\| &\leq \|e_ib'e_ix - e_ib'x\| + \|e_ib'x - b'x\| + \|b'x - b'e_ix\| \\ &\leq \|e_ib'\| \|e_ix - x\| + \|e_ib'x - b'x\| + \|b'\| \|x - e_ix\| < \varepsilon. \end{aligned}$$

Take $i \in \{i \in \mathcal{I} : \|xbe_i - xb\| < \frac{\varepsilon}{M}\} \in \mathcal{U}$. Then $\|x(be_ib'e_i - bb'e_i)\| \leq \|xbe_i - xb\| \|b'e_i\| < \varepsilon$. It follows that $\psi(b)\psi(b') - \psi(bb') = (be_ib'e_i - bb'e_i)_{i \in \mathcal{I}}$ is an element of J .

Since $(ae_i - a)_{i \in \mathcal{I}}$ is \mathcal{U} -strict convergent to 0, for all $a \in \mathcal{A}$, so $\varphi(a) = a$ in $\mathcal{A}^{s\mathcal{U}}/J$.

Suppose that there exists another C^* -homomorphism $\varphi' : \mathcal{B} \rightarrow \mathcal{A}^{s\mathcal{U}}/J$ such that $\varphi'(a) = a$ for all $a \in \mathcal{A}$. Then

$$\varphi'(b)a = \varphi'(b)\varphi'(a) = ba = \varphi(b)\varphi(a) = \varphi(b)a.$$

By Lemma 2.9, $\varphi(b) = \varphi'(b)$. □

Ultraproducts provide a new point of view for dealing with multiplier algebras. For instance, the identification of $\mathcal{M}(\mathcal{A})$ with $\mathcal{A}^{s\mathcal{U}}/J$ yields an easy proof of the next characterization of multipliers, without using double centralizers.

Corollary 2.11. *$\mathcal{M}(\mathcal{A})$ is isomorphic to $\mathcal{M} := \{m \in B(\mathcal{H}) : \text{for all } a \in \mathcal{A}, am \in \mathcal{A}, ma \in \mathcal{A}\}$. In particular, $\mathcal{M}(\mathcal{A})$ is unital.*

Proof. Consider $\varphi : \mathcal{A}^{s\mathcal{U}} \rightarrow \mathcal{M}$ defined by $\varphi((a_i)_{i \in \mathcal{I}}) = \lim_{\mathcal{U}\text{-strict}} a_i$. This map is well defined, it is a C^* -homomorphism (Proposition 2.7), and $\ker(\varphi) = J$. To show that φ is surjective, let $m \in \mathcal{M}$. Then $am \in \mathcal{A}$ and $ma \in \mathcal{A}$ for every $a \in \mathcal{A}$. Hence, for all $\varepsilon > 0$, the sets $\{i \in \mathcal{I} : \|a(me_i - m)\| < \varepsilon\}$ and $\{i \in \mathcal{I} : \|(me_i - m)a\| < \varepsilon\}$ are elements of \mathcal{U} . Then $(me_i)_{i \in \mathcal{I}} \in \mathcal{A}^{\mathcal{U}}$ is \mathcal{U} -strict convergent to m .

Taking $m = 1$, it follows that the image of $(e_i)_{i \in \mathcal{I}}$ in $\mathcal{A}^{s\mathcal{U}}/J$ is the unit of $\mathcal{A}^{s\mathcal{U}}/J$. \square

For a second application, observe that every C^* -homomorphism $\phi : \mathcal{A} \rightarrow \mathcal{B}$ defines a natural C^* -homomorphism $\phi' : \mathcal{A}^{\mathcal{U}} \rightarrow \mathcal{B}^{\mathcal{U}}$. When ϕ is surjective, a proof similar to one given in Proposition 2.7 shows that $\phi'(\mathcal{A}^{s\mathcal{U}}) \subset \mathcal{B}^{s\mathcal{U}}$. This together with Theorem 2.10 immediately gives the following known result.

Proposition 2.12. *Let \mathcal{A}, \mathcal{B} be C^* -algebras and let $\phi : \mathcal{A} \rightarrow \mathcal{B}$ be a surjective homomorphism. The natural extension $\phi' : \mathcal{A}^{\mathcal{U}} \rightarrow \mathcal{B}^{\mathcal{U}}$ induces the following commutative diagram:*

$$\begin{array}{ccccccc} \mathcal{A} & \longrightarrow & \mathcal{A}^{s\mathcal{U}} & \longrightarrow & \mathcal{M}(\mathcal{A}) & \longrightarrow & \mathcal{M}(\mathcal{A})/\mathcal{A} \\ \downarrow \phi & & \downarrow \phi' & & \downarrow \phi'' & & \downarrow \phi''' \\ \mathcal{B} & \longrightarrow & \mathcal{B}^{s\mathcal{U}} & \longrightarrow & \mathcal{M}(\mathcal{B}) & \longrightarrow & \mathcal{M}(\mathcal{B})/\mathcal{B}. \end{array}$$

3. THE CASE OF COMMUTATIVE AND SEPARABLE C^* -ALGEBRAS

Recall that when a C^* -algebra \mathcal{A} is separable, it is σ -unital, namely, there exists a countable approximate unit (see [6, Chapter I.4]). That entails that the index set \mathcal{I} of the previous section can be taken to be equal to \mathbb{N} , in which case nonprincipal ultrafilters are cofinal and $\prod_{\mathcal{I}} \mathcal{A}$ is $\ell^\infty(\mathcal{A})$.

In [1], the authors built the multiplier algebra for commutative and separable C^* -algebras using ultraproducts of C^* -algebras. More precisely they considered $\mathcal{A} = C_0(X)$ where X is a second countable, locally compact topological space and took \mathcal{U} a nonprincipal ultrafilter defined over \mathbb{N} to construct the multiplier algebra of \mathcal{A} , $C_b(X)$, identifying it with a quotient of a sub- C^* -algebra of $\prod_{\mathcal{I}} \mathcal{A}$. For that, the authors use the key fact that the hypothesis on X entails the existence of a proper metric compatible [8]. In what follows, we will then identify the second countable, locally compact topological space X with the metric space (X, d) , where d is a proper metric on X . The closed ball of radius $r > 0$ centered at a fixed base-point $o \in X$, will be denoted by $B_o(r)$. The main technical tool of [1] is the following definition.

Definition 3.1. [1, Section 3] Let (X, d) be as in the preceding discussion. Let $\mathcal{A} = C_0(X)$ and let \mathcal{U} be a nonprincipal ultrafilter over \mathbb{N} . For $(f_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{A})$, we say that $(f_n)_{n \in \mathbb{N}}$ is \mathcal{U} -equicontinuous on bounded sets if, for every $r, \varepsilon > 0$, there is $\delta > 0$ such that the set $\{n \in \mathbb{N} : \text{for all } s, t \in B_o(r) \text{ with } d(s, t) < \delta \implies |f_n(s) - f_n(t)| < \varepsilon\}$ belongs to \mathcal{U} .

Given $(f_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{A})$, and a fixed $x \in X$, the \mathcal{U} -limit of the sequence $(f_n(x))_{n \in \mathbb{N}}$ is well defined. We denote $f_{\mathcal{U}} : X \rightarrow \mathbb{C}$, $f_{\mathcal{U}}(x) := \lim_{\mathcal{U}}(f_n(x))$. The following fact was observed in [1].

Lemma 3.2. *If $(f_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{A})$ is \mathcal{U} -equicontinuous on bounded sets, then $f_{\mathcal{U}}$ is uniformly continuous on bounded sets.*

The next proposition shows that our notion of \mathcal{U} -strict convergence coincides with the notion of \mathcal{U} -equicontinuity on bounded sets in the case of $\mathcal{A} = C_0(X)$, where X is a locally compact, second countable, topological space. This entails that the work done here in section 2 is indeed a generalization of the work done in [1, Section 3].

Proposition 3.3. *Take $(f_n) \in \ell^\infty(\mathcal{A})$ and let $f_{\mathcal{U}}(x) = \lim_{\mathcal{U}}(f_n(x))$. The following two conditions are equivalent:*

- (1) *The sequence $(f_n)_{n \in \mathbb{N}}$ is \mathcal{U} -equicontinuous on bounded sets.*
- (2) *The sequence $(f_n)_{n \in \mathbb{N}}$ is \mathcal{U} -strict convergent to $f_{\mathcal{U}}$.*

Proof. To show that (1) implies (2), let $(f_n)_{n \in \mathbb{N}} \in \ell^\infty(\mathcal{A})$ be \mathcal{U} -equicontinuous on bounded sets. Let $\varepsilon > 0$ and let $g \in C_0(X)$. Set $M = \sup_{n \in \mathbb{N}}\{\|f_n\|, \|g\|\}$, and take $K \subset X$ a compact set such that $|g(x)| < \frac{\varepsilon}{2M}$ if $x \notin K$. There exists δ_1 such that for $x, y \in K$ with $d(x, y) < \delta_1$, $\{n \in \mathbb{N} : |f_n(x) - f_n(y)| \leq \frac{\varepsilon}{3M}\} \in \mathcal{U}$. By Lemma 3.2, there exists $\delta_2 > 0$ such that $|f_{\mathcal{U}}(x) - f_{\mathcal{U}}(y)| < \frac{\varepsilon}{3}$, for $x, y \in K$ with $d(x, y) < \delta_2$. Take $\delta = \min\{\delta_1, \delta_2\}$ and cover K with a finite number of balls $B_{x_j}(\delta)$ of radius δ centered at x_j , $j = 1, \dots, m$. Since $f_{\mathcal{U}}(x_j) = \lim_{\mathcal{U}} f_n(x_j)$, it follows that the sets $A_j = \{n \in \mathbb{N} : |f_n(x_j) - f_{\mathcal{U}}(x_j)| < \frac{\varepsilon}{3M}\}$ belong to \mathcal{U} . Therefore if $n \in \bigcap_{j=1}^m A_j \in \mathcal{U}$, and $x \in K$, taking x_j with $d(x, x_j) < \delta$ we get

$$\begin{aligned} |(f_n(x) - f_{\mathcal{U}}(x))g(x)| &\leq |f_n(x) - f_n(x_j)|\|g\| + |f_n(x_j) - f_{\mathcal{U}}(x_j)|\|g\| + \\ &\quad |f_{\mathcal{U}}(x_j) - f_{\mathcal{U}}(x)|\|g\| < \varepsilon. \end{aligned}$$

On the other hand, if $x \notin K$, then $|(f_n - f_{\mathcal{U}})g(x)| < \varepsilon$. This shows that $\{n \in \mathbb{N} : \|f_n g - f_{\mathcal{U}} g\| < \varepsilon\} \in \mathcal{U}$.

To show the converse, take $g \in C_0(X)$ such that $g = 1$ in $B_o(r)$. By hypothesis, for all $\varepsilon > 0$ the sets $\{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \varepsilon\}$ are in \mathcal{U} and are infinite. So we can build a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ such that $(f_{n_k})_{k \in \mathbb{N}}$ is uniformly convergent to $f_{\mathcal{U}}$ in $B_o(r)$. By hypothesis, closed balls are compact, so $f_{\mathcal{U}}$ is uniformly continuous on $B_o(r)$. Let $\delta > 0$ such that $d(x, y) < \delta$ implies $|f_{\mathcal{U}}(y) - f_{\mathcal{U}}(x)| < \varepsilon$ in $B_o(r)$. Take $n \in \{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \frac{\varepsilon}{3}\} \in \mathcal{U}$. Then for $x, y \in B_o(r)$ such that $d(x, y) < \delta$ we have

$$|f_n(x) - f_n(y)| \leq |f_n(x) - f_{\mathcal{U}}(x)| + |f_{\mathcal{U}}(x) - f_{\mathcal{U}}(y)| + |f_n(y) - f_{\mathcal{U}}(y)| \leq \varepsilon,$$

therefore $\{n \in \mathbb{N} : \|(f_n - f_{\mathcal{U}})g\| < \frac{\varepsilon}{3}\}$ is a subset of $\{n \in \mathbb{N} : \text{for all } x, y \in B_o(r) \text{ with } d(x, y) < \delta \implies |f_n(x) - f_n(y)| < \varepsilon\}$. This shows that the last set belongs to \mathcal{U} . \square

4. AN APPLICATION TO BOUNDARY AMENABILITY OF GROUPS

Let Γ be a countable group. Let \mathcal{A} be a unital C^* -algebra endowed with a Γ -action by $*$ -automorphisms. Let $C_c(\Gamma, \mathcal{A})$ be the space of finitely supported functions from Γ to \mathcal{A} . This is a $*$ -algebra with the product given by

$$T * S(\gamma) = \sum_{\gamma_1 \gamma_2 = \gamma} T(\gamma_1)(\gamma_1 \cdot S(\gamma_2))$$

and the involution given by

$$T^*(\gamma) = \gamma \cdot T(\gamma^{-1})^*.$$

The space $C_c(\Gamma, \mathcal{A})$ has a pre-Hilbert \mathcal{A} -module structure via the inner product $\langle T, S \rangle_{\mathcal{A}} := \sum_{\gamma \in \Gamma} T(\gamma)^* S(\gamma)$, and the corresponding norm $\|T\|_{\mathcal{A}} := \langle T, T \rangle_{\mathcal{A}}^{1/2}$.

Definition 4.1. A group Γ is boundary amenable (or exact) if there exist a compact space Y and a sequence $(S_i)_{i \in \mathbb{N}} \subset C_c(\Gamma, C(Y))$ such that

- (1) S_i is positive, that is, $S_i(\gamma) \geq 0$ for all $\gamma \in \Gamma$;
- (2) $\sum_{\gamma \in \Gamma} S_i^2(\gamma) = 1$;
- (3) for every $\gamma_1 \in \Gamma$, it follows that $\|S_i - \gamma_1 * S_i\|$ goes to 0 when i goes to infinity.

In [1, Section 4], the authors applied their ultraproduct construction of the multiplier algebra of a separable C^* algebra to show that groups acting properly and transitively on a locally finite tree are boundary amenable. The assumption that the action is proper and transitive implies that the vertex stabilizers are all isomorphic finite groups of cardinal m . In the proof given in [1], this constant m contains all the information that is needed about the stabilizers. Since it is known that groups acting on a locally finite tree with exact stabilizers are boundary amenable (see, for instance, [2, 3, 5, 7]), it is interesting to see if the strategy developed in [1] can be extended to prove this more general result. We partially achieve this in the proof of the next theorem.

Theorem 4.2. *If a countable group Γ acts transitively on a locally finite tree \mathcal{T} with boundary amenable stabilizers, then Γ is boundary amenable.*

Proof. Fix a vertex $o \in \mathcal{T}$ as a base-point. We will denote with $B(i)$ the closed ball of radius r centered in o . The geodesic that connects o with t will be denoted by $[o, t]$. In what follows $Stab\{o\}$ will be denoted with Λ .

Since the action of Γ on \mathcal{T} is transitive, we choose a cross section for it, namely, for each $v \in \mathcal{T}$ we choose $g_v \in \Gamma$ such that $g_v o = v$.

Set $(\Gamma_i)_{i \in \mathbb{N}}$ an increasing sequence of finite subsets of Γ that covers Γ and let

$$\Lambda_i := \bigcup_{v \in B(i)} \bigcup_{w \in B(i)} (g_v^{-1} \Gamma_i g_w \cap \Lambda).$$

Since \mathcal{T} is locally finite, Λ_i is finite.

Since Λ is boundary amenable, there exist a compact set Y and a sequence $(S_i)_{i \in \mathbb{N}} \subset C_c((\Lambda, C(Y)))$ satisfying the three conditions of Definition 4.1. The compact space Y will be taken to be a Γ -space (see [3, pg. 178]). Moreover, we

can assume that S_i is defined over all Γ by extending it by 0 in $\Gamma \setminus \Lambda$. For every $i \in \mathbb{N}$, let

$$\kappa(i) := \min\{k \in \mathbb{N} : \|S_l - \lambda * S_l\| < \frac{1}{i} \text{ for all } l \geq k \text{ and for all } \lambda \in \Lambda_i\}.$$

Consider the C^* -algebra $\mathcal{A} := C_0(\mathcal{T} \times Y)$. Since Y is compact and \mathcal{T} is countable, \mathcal{A} is a σ -unital C^* -algebra, hence we are in the case discussed at the beginning of Section 3. Let \mathcal{U} be a non principal ultrafilter over \mathbb{N} and consider the corona algebra $\mathcal{M}(\mathcal{A})/\mathcal{A}$, which we identify with $\mathcal{A}^{s\mathcal{U}}/J/\mathcal{A}$. This is a unital C^* -algebra.

For each $i \in \mathbb{N}$, set $T_i : \Gamma \rightarrow \mathcal{A}^{s\mathcal{U}}$, where $T_i(\gamma) = (T_i^n(\gamma))_{n \in \mathbb{N}}$ is defined by

$$T_i^n(\gamma)(t, y) = \frac{1}{\sqrt{|[o, t] \cap B(i)|}} \chi_{B(n)}(t) \sum_{v \in B(i)} \chi_{[o, t]}(v) S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y),$$

where $|[o, t] \cap B(i)|$ denotes the number of points in the geodesic $[o, t]$ that lie inside the ball $B(i)$. Note that $(T_i^n(\gamma))_{n \in \mathbb{N}}$ is \mathcal{U} -strict convergent to

$$T_i = T_i(\gamma)(t, y) = \frac{1}{\sqrt{|[o, t] \cap B(i)|}} \sum_{v \in B(i)} \chi_{[o, t]}(v) S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y)$$

We will denote by T_i its projection to $\mathcal{A}^{s\mathcal{U}}/J/\mathcal{A}$. We remark that the definition of T_i is inspired by the definition of $\mu_{x, y}$ given by Ozawa in [5].

Claim: The sequence $(T_i)_{i \in \mathbb{N}} \subseteq C(\Gamma; \mathcal{A}^{s\mathcal{U}}/J/\mathcal{A})$ satisfies the conditions of Definition 4.1.

To show this, first observe that if $\Omega_{\kappa(i)} \subseteq \Lambda$ denotes the support of $S_{\kappa(i)}$, then the support of T_i is contained in $\bigcup_{v \in B(i)} g_v \Omega_{\kappa(i)}$. Since \mathcal{T} is locally finite, this is a finite set.

To show that condition (1) holds true, it is enough to notice that T_i is sum and product of positive functions.

To show that condition (2) holds true, note that for each fixed $\gamma \in \Gamma$, there exists only one g_v such that $g_v^{-1}\gamma \in \Lambda$. Then there exists at most one g_v such that $S_{\kappa(i)}(g_v^{-1}\gamma)$ is nonzero. Then

$$\begin{aligned} \sum_{\gamma \in \Gamma} (T_i)^2(t, y) &= \sum_{\gamma \in \Gamma} \frac{1}{|[o, t] \cap B(i)|} \sum_{v \in B(i)} \chi_{[o, t]}(v) S_{\kappa(i)}^2(g_v^{-1}\gamma)(g_v^{-1}y) \\ &= \frac{1}{|[o, t] \cap B(i)|} \sum_{v \in B(i)} \chi_{[o, t]}(v) \sum_{\gamma \in \Gamma} S_{\kappa(i)}^2(g_v^{-1}\gamma)(g_v^{-1}y) \\ &= \frac{1}{|[o, t] \cap B(i)|} \sum_{v \in B(i)} \chi_{[o, t]}(v) \sum_{\lambda \in \Lambda} S_{\kappa(i)}^2(\lambda)(g_v^{-1}y) \\ &= \frac{1}{|[o, t] \cap B(i)|} \sum_{v \in B(i)} \chi_{[o, t]}(v) = 1. \end{aligned}$$

It remains to show that condition (3) holds true. To this end, fix $\gamma_1 \in \Gamma$. Observe that if $(t, y) \in \mathcal{T} \times Y$, then $\gamma_1 * T_i(\gamma)(t, y) = T_i(\gamma_1^{-1}\gamma)(\gamma_1^{-1}t, \gamma_1^{-1}y)$.

Hence

$$\begin{aligned}
\|T_i - \gamma_1 * T_i\|_{C_c(\Gamma, \mathcal{A}^{s\mathcal{U}}/J/\mathcal{A})}^2 &= \left\| \sum_{\gamma \in \Gamma} (T_i(\gamma) - \gamma_1 * T_i(\gamma))^2 \right\|_{\mathcal{A}^{s\mathcal{U}}/J/\mathcal{A}} \\
&= \inf_{a \in \mathcal{A}} \left\| \sum_{\gamma \in \Gamma} (T_i(\gamma) - \gamma_1 * T_i(\gamma))^2 - a \right\|_{\mathcal{A}^{s\mathcal{U}}/J} \\
&= \inf_{a \in \mathcal{A}} \left\| 2 - 2 \sum_{\gamma \in \Gamma} T_i(\gamma)(\gamma_1 * T_i(\gamma)) - a \right\|_{\mathcal{A}^{s\mathcal{U}}/J} \quad (4.1)
\end{aligned}$$

If we set $\theta(i, t) := \left| [o, t] \cap B(i) \right|^{-1/2} \left| [o, \gamma_1^{-1}t] \cap B(i) \right|^{-1/2}$, then

$$\begin{aligned}
&\sum_{\gamma \in \Gamma} T_i(\gamma)(t, y) T_i(\gamma_1^{-1}\gamma)(\gamma_1^{-1}t, \gamma_1^{-1}y) \\
&= \theta(i, t) \sum_{\substack{v \in B(i) \\ w \in B(i)}} \chi_{[o, t]}(v) \chi_{[o, \gamma_1^{-1}t]}(w) \sum_{\gamma \in \Gamma} S_{\kappa(i)}(g_v^{-1}\gamma)(g_v^{-1}y) S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}\gamma)(g_w^{-1}\gamma_1^{-1}y) \\
&= \theta(i, t) \sum_{\substack{v \in B(i) \\ w \in B(i)}} \chi_{[o, t]}(v) \chi_{[o, \gamma_1^{-1}t]}(w) \sum_{\lambda \in \Lambda} S_{\kappa(i)}(\lambda)(z) S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_vz), \quad (4.2)
\end{aligned}$$

where the last equality follows from the changes of variables $\lambda := g_v^{-1}\gamma$ and $z := g_v^{-1}y$, and by recalling that $S_{\kappa(i)}(\lambda) = 0$ if $\lambda \notin \Lambda$.

In order to get that $S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda) \neq 0$, it is necessary that $g_w^{-1}\gamma_1^{-1}g_v$ belongs to Λ . Note that for a fixed v , there exists only one g_w such that $g_w^{-1}\gamma_1^{-1}g_v \in \Lambda$. For this g_w , we have that $w = \gamma_1^{-1}v$ and $\chi_{[o, \gamma_1^{-1}t]}(w) = \chi_{[o, \gamma_1^{-1}t]}(\gamma_1^{-1}v) = \chi_{[\gamma_1 o, t]}(v)$. Therefore, (4.2) is equal to the following sum, only depending on v .

$$\theta(i, t) \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) \sum_{\lambda \in \Lambda} S_{\kappa(i)}(\lambda)(z) S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_vz).$$

Replacing this in (4.1), we get that $\|T_i - \gamma_1 * T_i\|_{C_c(\Gamma, \mathcal{A}^{s\mathcal{U}}/J/\mathcal{A})}^2$ is equal to

$$\inf_{a \in \mathcal{A}} \sup_{(t, y) \in T \times Y} \left\{ \left| 2 - a(t, y) - \right. \right. \\
\left. \left. 2\theta(i, t) \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) \sum_{\lambda \in \Lambda} S_{\kappa(i)}(\lambda)(z) S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_vz) \right| \right\}. \quad (4.3)$$

By the triangle inequality, to estimate (4.3), it is enough to estimate the following two quantities

$$2\theta(i, t) \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) \left| 1 - \sum_{\lambda \in \Lambda} S_{\kappa(i)}(\lambda)(z) S_{\kappa(i)}(g_w^{-1}\gamma_1^{-1}g_v\lambda)(g_w^{-1}\gamma_1^{-1}g_vz) \right| \quad (4.4)$$

and

$$\left| 2 - 2\theta(i, t) \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) - a(t, y) \right| \quad (4.5)$$

In order to bound (4.4), first note that

$$2 \left| 1 - \sum_{\lambda \in \Lambda} S_{\kappa(i)}(\lambda)(z) S_{\kappa(i)}(g_w^{-1} \gamma_1^{-1} g_v \lambda)(g_w^{-1} \gamma_1^{-1} g_v z) \right| = \|S_{\kappa(i)} - (g_v^{-1} \gamma_1 g_w) * S_{\kappa(i)}\|^2.$$

Then note that there exists $r \in \mathbb{N}$ such that $\gamma_1 \in \Gamma_i$ for all $i \geq r$. It follows that for all $i \geq r$, $g_v^{-1} \gamma_1 g_w \in \Lambda_i$, whenever $v, w \in B_i$. Then, by the definition of $\kappa(i)$, it follows that

$$\|S_{\kappa(i)} - (g_v^{-1} \gamma_1 g_w) * S_{\kappa(i)}\|^2 \leq \frac{1}{i}, \text{ for all } i \geq r, \text{ whenever } v \in B(i) \cap \gamma_1 B(i).$$

Then, for all $i \geq r$, (4.4) is bounded by

$$\begin{aligned} \frac{1}{i} \theta(i, t) & \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) \\ & \leq \frac{1}{i} \theta(i, t) \left(\sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \right)^{1/2} \left(\sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[\gamma_1 o, t]}(v) \right)^{1/2} \\ & \leq \frac{1}{i} \theta(i, t) \left| [o, t] \cap B(i) \right|^{1/2} \left| [\gamma_1 o, t] \cap \gamma_1 B(i) \right|^{1/2} \\ & = \frac{1}{i} \theta(i, t) \left| [o, t] \cap B(i) \right|^{1/2} \left| [o, \gamma_1^{-1} t] \cap B(i) \right|^{1/2} \\ & = \frac{1}{i}. \end{aligned}$$

In order to bound (4.5), for each $i \in \mathbb{N}$, we choose

$$a(t, y) = a(t) := \left(2 - 2\theta(i, t) \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) \right) \chi_{B(i) \cup \gamma_1 B(i)}(t).$$

This choice of $a \in \mathcal{A}$ left us to bound (4.5) only when $t \notin B(i) \cup \gamma_1 B(i)$. In this case $\theta(i, t) = \frac{1}{i}$. Moreover, if $i \geq d(o, \gamma_1 o)$ then

$$\left| [o, t] \cap [\gamma_1 o, t] \cap B(i) \cap \gamma_1 B(i) \right| \geq i - d(o, \gamma_1 o).$$

It follows that in (4.5) we have the bound

$$\begin{aligned} 2 - \frac{2}{i} \sum_{v \in B(i) \cap \gamma_1 B(i)} \chi_{[o, t]}(v) \chi_{[\gamma_1 o, t]}(v) & = 2 - \frac{2}{i} \left| [o, t] \cap [\gamma_1 o, t] \cap B(i) \cap \gamma_1 B(i) \right| \\ & \leq 2 - \frac{2}{i} (i - d(o, \gamma_1 o)) \\ & = \frac{d(o, \gamma_1 o)}{i}. \end{aligned}$$

All this combined entails that if $\varepsilon > 0$, and $i \geq \max\{r; d(o, \gamma_1 o), 2/\varepsilon; 2d(o, \gamma_1 o)/\varepsilon\}$, then $\|T_i - \gamma_1 * T_i\|_{C_c(\Gamma, \mathcal{A}^{su}/J/\mathcal{A})}^2 < \varepsilon$. \square

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