

Weyl geometry, anti-de Sitter space, and Φ^4 -theory

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Received: 21 April 2016

Published online: 26 May 2016 – © Società Italiana di Fisica / Springer-Verlag 2016

Abstract. We study the Gaussian approximation to the quantum fluctuations of the metric of the four-dimensional anti-de Sitter spacetime. The associated massless scalar field has a quartic self-interaction, for which we construct the generating functional of the n -point correlation functions. The concomitant infrared divergence is cured by a mass renormalization provided by the cosmological constant, which is also responsible for the renormalization of the coupling constant of the field theory.

1 Introduction

The equivalence of a Weylian geometry with vanishing 1-form or, equivalently, with zero gauge curvature, with a Riemannian geometry, allows us to interpret a conformally flat metric as a massless scalar field in Minkowski space. In particular, we consider the four-dimensional anti-de Sitter spacetime, and study the Gaussian approximation to the quantum fluctuations of the metric. This is done through the computation of the propagator between two conformal metrics along a finite interval in one of the two patches of the AdS space. A change of the field variable, however, shows that the system is equivalent to a massless Φ^4 -theory in one spacetime dimension. We then construct the generating functional of the n -point correlation functions, which give the full quantum fluctuations of the AdS metric around its classical value. The infrared divergence of the scalar theory, due to its massless character, is cured by a finite mass renormalization provided by the cosmological constant, which plays the role of an IR cutoff; this is already exhibited by the two-point function calculated to first order in the coupling constant, which is also finitely renormalized by the four-point function with the same cutoff, at zero momenta, and to second order in the perturbation expansion.

2 Weyl and Riemann geometries

As is well known [1] any Riemannian or pseudo-Riemannian (in particular Lorentz) manifold $(M_{\mathcal{R}}^n, g_{\mu\nu})$ can be considered as a Weylian manifold with zero Weyl field ϕ locally given by the 1-form $\phi = \phi_\mu dx^\mu$, $(M_{\mathcal{W}}^n, g_{\mu\nu}, 0)$, with $M_{\mathcal{R}}^n = M_{\mathcal{W}}^n \equiv M^n$ as differentiable manifolds. Of particular interest is the case when M^n is *Lorentzian* and *conformally flat*, *i.e.*

$$g_{\mu\nu} = e^\varphi \eta_{\mu\nu}, \quad (1)$$

where φ is a real valued scalar function on M^n and $\eta_{\mu\nu}$ is the Lorentz metric; so

$$(M_{\mathcal{W}}^n, g_{\mu\nu}, 0) = (M^n, e^\varphi \eta_{\mu\nu}, 0). \quad (2)$$

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However, any Weyl manifold belongs to the equivalence class of manifolds $[(M^n, g'_{\mu\nu}, \phi')]$ defined by

$$(M^n, g''_{\mu\nu}, \phi'') \sim (M^n, g'_{\mu\nu}, \phi'), \tag{3}$$

if and only if

$$g''_{\mu\nu} = \lambda^{-1} g'_{\mu\nu} \tag{4a}$$

and

$$\phi'' = \phi' - \lambda^{-1} d\lambda, \tag{4b}$$

where λ is a smooth positive real valued function defined on M^n . (4a) is a conformal transformation of the metric, while (4b) is a gauge transformation of the Weyl field. Both (4a) and (4b) constitute what is called a *Weyl transformation*, denoted by w . A choice of a representative of the class is called a *choice of gauge* or *gauge fixing*. Locally, the Weyl connection $\Gamma^\alpha_{\nu\mu}$ is given by the sum of the Levi-Civita connection,

$$\begin{pmatrix} \alpha \\ \nu\mu \end{pmatrix} = \frac{1}{2} g^{\alpha\beta} (\partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} - \partial_\beta g_{\nu\mu}), \tag{5}$$

and the Weyl tensor,

$$W_{\nu\mu}^\alpha = \frac{1}{2} (\delta_\nu^\alpha \phi_\mu + \delta_\mu^\alpha \phi_\nu - g_{\nu\mu} g^{\alpha\rho} \phi_\rho). \tag{6}$$

It is easily verified that

$$\Gamma_{\nu\mu}^\alpha = \begin{pmatrix} \alpha \\ \nu\mu \end{pmatrix} + W_{\nu\mu}^\alpha \tag{7}$$

is invariant under w , and so are the curvature tensor

$$R_{\sigma\mu\nu}^\rho = \partial_\mu \Gamma_{\nu\sigma}^\rho - \partial_\nu \Gamma_{\mu\sigma}^\rho + \Gamma_{\mu\beta}^\rho \Gamma_{\nu\sigma}^\beta - \Gamma_{\nu\beta}^\rho \Gamma_{\mu\sigma}^\beta \tag{8}$$

and the Ricci tensor

$$R_{\nu\sigma} = R_{\nu\mu\sigma}^\mu. \tag{9}$$

However, since

$$g^{\mu\nu} \xrightarrow{w} g'^{\mu\nu} = \lambda g^{\mu\nu}, \tag{10}$$

the curvature scalar

$$R = g^{\mu\nu} R_{\mu\nu} \tag{11}$$

is not an invariant but transforms to λR , *i.e.*

$$R \xrightarrow{w} R' = \lambda R, \tag{12}$$

leaving the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R \tag{13}$$

invariant.

Applying (3), (4a) and (4b) to (2) with $\phi' = 0$ and $\lambda = e^\varphi$ one obtains

$$(M^n, e^\varphi \eta_{\mu\nu}, 0) \sim (M^n, \eta_{\mu\nu}, -d\varphi). \tag{14}$$

That is, a conformally flat Lorentzian manifold is equivalent to a flat Weylian manifold $(M^n, \eta_{\mu\nu}, \phi'')$ with Weyl field

$$\phi''_\mu = -\partial_\mu \varphi \equiv -\varphi_{,\mu}. \tag{15}$$

Since $\phi'' = -d\varphi$, its curvature vanishes:

$$F = d\phi'' = 0. \tag{16}$$

So, *classically*, the whole gravitational information in $(M^n, e^\varphi \eta_{\mu\nu})$ is encoded in the scalar φ living in Minkowski space $Mink^n$.

For any constant scalar curvature Riemannian or pseudo-Riemannian manifold, with curvature tensor

$$R_{\mu\nu\rho\sigma} = \frac{R}{n(n-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \tag{17}$$

it is easy to verify that the Weyl curvature tensor [2] $C_{\mu\nu\rho\sigma}$, defined by

$$R_{\beta\sigma\mu\nu} = C_{\beta\sigma\mu\nu} - \frac{R}{(n-1)(n-2)} (g_{\mu\beta} g_{\nu\sigma} - g_{\nu\beta} g_{\mu\sigma}) + \frac{1}{(n-2)} (g_{\mu\beta} R_{\nu\sigma} - g_{\nu\beta} R_{\mu\sigma} - g_{\mu\sigma} R_{\nu\beta} + g_{\nu\sigma} R_{\mu\beta}), \tag{18}$$

vanishes everywhere, and, since a necessary and sufficient condition for a metric to be conformally flat is that its Weyl curvature tensor vanishes everywhere [3], then all maximal symmetric spaces are conformally flat.

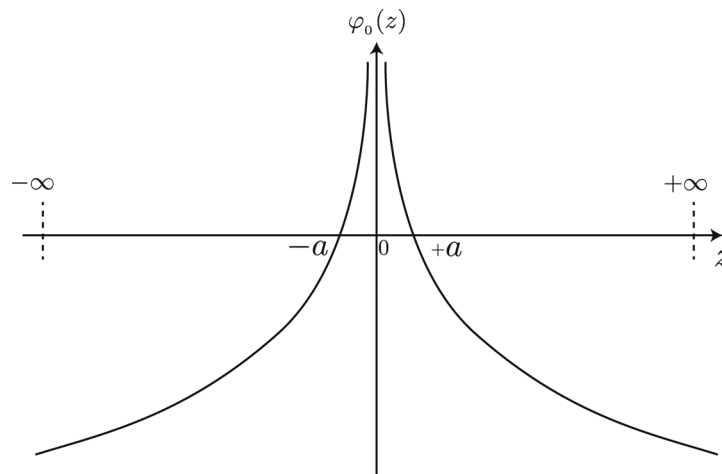


Fig. 1. Classical massless conformal scalar field.

3 Anti-de Sitter space

As a concrete example, we consider the anti-de Sitter space in four space-time dimensions (AdS₄), whose metric is given by

$$ds^2 = \left(\frac{a}{z}\right)^2 (dt^2 - dy_1^2 - dy_2^2 - dz^2) = \left(\frac{a}{z}\right)^2 \eta_{\mu\nu} dx^\mu dx^\nu, \tag{19}$$

with $x^0 = t$, $x^i = y^i \in (-\infty, +\infty)$, $i = 1, 2$, $x^3 = z \in (-\infty, 0) \cup (0, +\infty)$ ($(-\infty, 0)$ and $(0, +\infty)$ defining the left and right patches of AdS₄, respectively), $[t] = [y^i] = [z] = [a] = [\text{length}]$, and a related to the attractive cosmological constant Λ through

$$\Lambda = -\frac{3}{a^2}. \tag{20}$$

It is clear that from (1) and (19), $e^\varphi = \left(\frac{a}{z}\right)^2 = \frac{a^2}{|z|^2}$ and therefore

$$\varphi \equiv \varphi_0(z) = 2 \ln \left(\frac{a}{|z|}\right), \tag{21}$$

which is plotted in fig. 1. $z = 0$ corresponds to the boundary with radial coordinate $r = +\infty$ and $z = \pm\infty$ are the coordinate values of the “right” and “left” horizons, $\varphi_0(z)$ obeys the non-linear partial differential equation

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \varphi + \frac{1}{2} \eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) + \frac{4}{a^2} e^\varphi = 0, \tag{22}$$

with Lagrangian density

$$\mathcal{L} = \frac{1}{a^2} \left(e^\varphi \eta^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) - \frac{4}{a^2} e^{2\varphi} \right) \tag{23}$$

and action

$$S = \int d^4x \mathcal{L}(z). \tag{24}$$

Equation (22) is obtained from the vacuum Einstein equation for a conformally flat metric $e^\varphi \eta_{\mu\nu}$ in the presence of a cosmological constant Λ :

$$\partial_\mu \partial_\nu \varphi - \eta_{\mu\nu} \eta^{\alpha\beta} \partial_\alpha \partial_\beta \varphi - \frac{1}{2} (\partial_\mu \varphi)(\partial_\nu \varphi) - \frac{1}{4} \eta_{\mu\nu} (\partial_\alpha \varphi)(\partial^\alpha \varphi) + \Lambda \eta_{\mu\nu} e^\varphi = 0, \tag{25}$$

through its contraction with the inverse Minkowski metric $\eta^{\mu\nu}$.

From the statement immediately after eq. (16), the quantum fluctuations of the metric (19) are given by the quantum theory of the field φ with action (24) in *Min*k⁴. An important element which must then be studied in this context, is the propagator from the conformal metric $g_{\mu\nu}^{(1)}(z) = e^{\varphi_1(z)} \eta_{\mu\nu}$ to the conformal metric $g_{\mu\nu}^{(2)}(z) = e^{\varphi_2(z)} \eta_{\mu\nu}$,

with $z_1 \leq z \leq z_2$, $0 \notin [z_1, z_2]$, $\varphi_i(z) = \varphi_0(z) + \psi_i(z)$, $\psi_i(z_k) = 0$, $i = 1, 2$, $k = 1, 2$, which is represented by the Feynman path integral over the field φ , *i.e.*

$$K(g_{\mu\nu}^{(2)}, g_{\mu\nu}^{(1)}) \equiv K(2, 1) = \int_{g_{\mu\nu}^{(1)}}^{g_{\mu\nu}^{(2)}} \mathcal{D}\varphi e^{iS[\varphi]}. \tag{26}$$

From the decomposition

$$\varphi(z) = \varphi_0(z) + \psi(z), \tag{27}$$

(26) becomes

$$\begin{aligned} K(2, 1) &= \int_{\psi_1}^{\psi_2} \mathcal{D}\psi e^{i \int_{z_1}^{z_2} d^4x \mathcal{L}(\varphi_0 + \psi)} = \int_{\psi_1}^{\psi_2} \mathcal{D}\psi e^{-4i \int_{z_1}^{z_2} d^4x \frac{e^\psi}{z^4} (1 - z\psi' + \frac{z^2}{4}\psi'^2 + e^\psi)} \\ &= \int_{\psi_1}^{\psi_2} \mathcal{D}\psi e^{-4i \mathcal{V}_\infty \int_{z_1}^{z_2} dz \frac{e^\psi}{z^4} (1 - z\psi' + \frac{z^2}{4}\psi'^2 + e^\psi)}, \end{aligned} \tag{28}$$

where we assumed that the quantum fluctuation ψ only depends on z , with $\psi' = \frac{d\psi}{dz}$, and $\mathcal{V}_\infty = \int_{-\infty}^{+\infty} dt dx^1 dx^2$ is an infinite volume transverse to z which will be considered below.

4 Semiclassical quantization

The path integral (28) can not be done exactly. So, we consider here its Gaussian or semiclassical (quadratic) approximation, which we call $K_2(2, 1)$. From

$$e^\psi \left(1 - z\psi' + \frac{z^2}{4}\psi'^2 \right) + e^{2\psi} = 2 + 3\psi - z\psi' + \frac{5}{2}\psi^2 - z\psi\psi' + \frac{z^2}{4}\psi'^2 + \text{cubic terms}, \tag{29}$$

we have

$$K_2(2, 1) = \int_{\psi_1}^{\psi_2} \mathcal{D}\psi e^{-4i \mathcal{V}_\infty \int_{z_1}^{z_2} dz \frac{e^\psi}{z^4} (2 + 3\psi - z\psi' + \frac{5}{2}\psi^2 - z\psi\psi' + \frac{z^2}{4}\psi'^2)} = \int_{\psi_1}^{\psi_2} \mathcal{D}\psi e^{iS}, \tag{30}$$

where

$$S = -4 \left(\frac{\mathcal{V}_\infty}{a^3} \right) \int_{z_1}^{z_2} dz \mathcal{L}_2(z), \tag{31}$$

with

$$\mathcal{L}_2(z) = \frac{a^3}{z^4} \left(2 + 3\psi - z\psi' + \frac{5}{2}\psi^2 - z\psi\psi' + \frac{z^2}{4}\psi'^2 \right). \tag{32}$$

The Lagrange equation for \mathcal{L}_2 ,

$$\frac{d}{dz} \left(\frac{\partial \mathcal{L}_2}{\partial \psi'} \right) - \frac{\partial \mathcal{L}_2}{\partial \psi} = 0, \tag{33}$$

is

$$\psi'' - \frac{2}{z}\psi' - \frac{4}{z^2}\psi = 0 \tag{34}$$

($\psi'' = \frac{d^2\psi}{dz^2}$) which, in terms of the dimensionless coordinate

$$\hat{z} = \frac{z}{a}, \tag{35}$$

becomes

$$\ddot{\psi} - \frac{2}{\hat{z}}\dot{\psi} - \frac{4}{\hat{z}^2}\psi = 0, \tag{36}$$

with $\dot{\psi} = \frac{d\psi}{d\hat{z}}$ and $\ddot{\psi} = \frac{d^2\psi}{d\hat{z}^2}$. Its general solution is

$$\bar{\psi}(\hat{z}) = C_1 \hat{z}^4 + \frac{C_2}{\hat{z}}, \quad C_1, C_2 \in \mathbb{R}. \tag{37}$$

In terms of z ,

$$\bar{\psi}(z) = \frac{C_1}{a^4} z^4 + \frac{C_2 a}{z}, \tag{38}$$

leading to

$$\int_{z_1}^{z_2} \frac{dz}{z^4} \left(2 + 3\bar{\psi} - z\bar{\psi}' + \frac{5}{2}\bar{\psi}^2 - z\bar{\psi}\bar{\psi}' + \frac{z^2}{4}(\bar{\psi}')^2 \right) = \left[-\frac{2}{3z^3} - \frac{C_1}{a^4} z - \frac{C_2 a}{z^4} + \frac{C_1^2}{2a^8} z^5 - \frac{3}{4} \frac{C_2^2 a^2}{z^5} \right]_{z_1}^{z_2}. \tag{39}$$

Since $\bar{\psi}$ has to obey the same boundary conditions as ψ , *i.e.* $\bar{\psi}(z_1) = \bar{\psi}(z_2) = 0$, the unique solution (38) satisfying these conditions with $z_1 \neq z_2$ is $\bar{\psi}(z) = 0$, *i.e.* $C_1 = C_2 = 0$. So

$$\int_{z_1}^{z_2} \frac{dz}{z^4} \left(2 + 3\bar{\psi} - z\bar{\psi}' + \frac{5}{2}\bar{\psi}^2 - z\bar{\psi}\bar{\psi}' + \frac{z^2}{4}(\bar{\psi}')^2 \right) = \frac{2}{3} \left(\frac{1}{z_1^3} - \frac{1}{z_2^3} \right), \tag{40}$$

which gives the action

$$\bar{S}(\bar{\psi} = 0) = -\frac{8}{3} \left(\frac{\mathcal{V}_\infty}{a^3} \right) a^3 \left(\frac{1}{z_1^3} - \frac{1}{z_2^3} \right). \tag{41}$$

\mathcal{V}_∞/a^3 is a dimensionless infinite transverse volume. Since the problem is essentially 1-dimensional —only the z coordinate is relevant— we define the action per unit dimensionless transverse volume

$$\sigma = \frac{\bar{S}}{\mathcal{V}_\infty/a^3} = -\frac{8}{3} a^3 \left(\frac{1}{z_1^3} - \frac{1}{z_2^3} \right). \tag{42}$$

The van-Vleck determinant [4, 5] associated with σ is

$$\Delta = \det \begin{pmatrix} \frac{\partial^2 \sigma}{\partial z_1^2} & \frac{\partial^2 \sigma}{\partial z_1 \partial z_2} \\ \frac{\partial^2 \sigma}{\partial z_2 \partial z_1} & \frac{\partial^2 \sigma}{\partial z_2^2} \end{pmatrix} = \frac{(32)^2 a^{10}}{(z_1 z_2)^5}, \tag{43}$$

which, together with the exponentiation of σ , gives the desired propagator (k_2) for the 1-dimensional problem, *i.e.* that ignoring the irrelevant transverse coordinates t , x^1 and x^2 :

$$k_2(2, 1) = \sqrt{\frac{i\Delta}{2\pi}} e^{i\sigma} = \sqrt{\frac{i}{2\pi}} \frac{32a^5}{(z_1 z_2)^{5/2}} e^{-\frac{8}{3} i a^3 \left(\frac{1}{z_1^3} - \frac{1}{z_2^3} \right)}. \tag{44}$$

5 Φ^4 -theory

A simple change of field variable, $\varphi \rightarrow \Omega$, shows that the Lagrangian (23) corresponds to a Φ^4 -theory for the real massless scalar field Ω in one spacetime dimension, that corresponding to the z -coordinate. (Quantum field theory in (0+1) dimensions is discussed in ref. [6], where the space coordinate z is replaced by a time coordinate t .) In fact, with

$$e^\varphi = \Omega^2, \tag{45}$$

i.e. the field Ω gives the standard form for the conformal factor in (1), the Lagrangian density (23) becomes

$$\mathcal{L} = \frac{4}{a^2} \left(\eta^{\mu\nu} (\partial_\mu \Omega) (\partial_\nu \Omega) - \frac{1}{a^2} \Omega^4 \right). \tag{46}$$

As for φ , $[\Omega] = [L]^0$ and $[\mathcal{L}] = [L]^{-4}$. Assuming again that $\Omega = \Omega(z)$, \mathcal{L} becomes

$$\mathcal{L} = -\frac{4}{a^2} \left(\left(\frac{d\Omega}{dz} \right)^2 + \frac{1}{a^2} \Omega^4 \right), \tag{47}$$

with action

$$S = \int d^4 x \mathcal{L} = -4a \left(\frac{\mathcal{V}_\infty}{a^3} \right) \int dz \left(\left(\frac{d\Omega}{dz} \right)^2 + \frac{1}{a^2} \Omega^4 \right). \tag{48}$$

In terms of the dimensionless coordinate

$$\zeta = \frac{z}{a}, \tag{49}$$

$\Omega = \Omega(\zeta)$ and the classical action becomes

$$S = -4 \left(\frac{\mathcal{V}_\infty}{a^3} \right) \int d\zeta \left(\left(\frac{d\Omega}{d\zeta} \right)^2 + \Omega^4 \right). \tag{50}$$

Again, defining an action per unit dimensionless transverse volume, one obtains

$$\sigma = \int d\zeta l = -4 \int d\zeta \left(\left(\frac{d\Omega}{d\zeta} \right)^2 + \Omega^4 \right), \tag{51}$$

with Lagrangian density

$$l = -4 \left(\left(\frac{d\Omega}{d\zeta} \right)^2 + \Omega^4 \right) \tag{52}$$

and classical equation of motion

$$\Delta_\zeta \Omega + 2\Omega^3 = 0, \quad \Delta_\zeta = -\frac{d^2}{d\zeta^2}, \tag{53}$$

with solution

$$\Omega_0(\zeta) = \zeta^{-1} \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow \pm\infty \tag{54}$$

(cf. (21)).

The Feynman “propagator” is the Green’s function of the free one-dimensional Klein-Gordon equation with a delta function source:

$$(\Delta_\zeta - i\epsilon)\Delta_F(\zeta) = -\delta(\zeta), \tag{55}$$

namely [7]

$$\Delta_F(\zeta) = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi\zeta}}{\xi^2 + i\epsilon}. \tag{56}$$

As it stands, $\Delta_F(\zeta)$ is infrared (IR) divergent (as $\xi \rightarrow 0$); this is so because the bare mass of the Ω field is zero. A natural IR cutoff is

$$\tilde{\mu}^2 = 3a^{-2}, \tag{57}$$

i.e. minus the cosmological constant Λ . In dimensionless units,

$$\mu^2 = a^2 \tilde{\mu}^2 = 3 \tag{58}$$

and so the IR finite propagator is

$$\Delta_F(\zeta)' = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi\zeta}}{\xi^2 - \mu^2 + i\epsilon} = \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi\zeta}}{(\xi - \xi_+)(\xi - \xi_-)}, \tag{59}$$

with $\xi_\pm = \pm(\mu - \frac{i\epsilon}{2\mu}) = \pm(\sqrt{3} - \frac{i\epsilon}{2\sqrt{3}})$. According to $\zeta < 0$ or $\zeta > 0$, we integrate in the upper or lower complex ξ -plane, respectively, obtaining

$$\Delta_F(\zeta)' = -\frac{i}{2\sqrt{3}} e^{\pm i\sqrt{3}\zeta}. \tag{60}$$

Continuing to $\zeta = 0$,

$$\Delta_F(0)' = -\frac{i}{2\mu} = -\frac{i}{2\sqrt{3}}, \tag{61}$$

which, as will be shortly seen, provides a mass renormalization to the bare theory.

To construct the generating functional Z of the n -point correlation functions, one must to introduce a source factor $J(\zeta)$ to the Lagrangian l , defining

$$l_J = -4 \left(\left(\frac{d\Omega}{d\zeta} \right)^2 + \Omega^4 - J\Omega \right). \tag{62}$$

In order to take the theory to the standard $\frac{\lambda}{4!}\Phi^4$ form, where λ is the bare coupling constant, we redefine again the field function as

$$\omega(\zeta) = 2\sqrt{2}\Omega(\zeta), \tag{63}$$

and the source function,

$$j(\zeta) = \sqrt{2}J(\zeta), \tag{64}$$

obtaining the action

$$\sigma[\omega, j] = \int d\zeta (l_0 + l_{int} + j\omega), \tag{65}$$

with

$$l_0 = -\frac{1}{2}(\partial_\zeta \omega)^2, \quad l_{int} = -\frac{3/2}{4!}\omega^4, \tag{66}$$

i.e. $\lambda = 3/2$.

Then $Z[j]$ is given by (see [7] p. 200)

$$Z[j] = \frac{\exp\left(i \int d\zeta l_{int} \left(-i \frac{\delta}{\delta j(\zeta)}\right)\right) Z_0[j]}{\exp\left(i \int d\zeta' l_{int} \left(-i \frac{\delta}{\delta j(\zeta')}\right)\right) Z_0[j]|_{j=0}}, \tag{67}$$

where

$$Z_0[j] = \exp\left(-\frac{i}{2} \int d\zeta \int d\zeta' j(\zeta) \Delta_F(\zeta - \zeta') j(\zeta')\right) \tag{68}$$

is the generating functional of the free theory, *i.e.* that with $\lambda = 0$.

The n -point correlation functions are given by

$$\pi(\zeta_1, \dots, \zeta_n) = \langle \text{vac} | \Pi(\hat{\omega}(\zeta_1) \dots \hat{\omega}(\zeta_n)) | \text{vac} \rangle = (-i)^n \frac{\delta^n Z[j]}{\delta j(\zeta_1) \dots \delta j(\zeta_n)} \Big|_{j=0}, \tag{69}$$

where Π denotes the path ordered product of the quantum field operators $\hat{\omega}(\zeta_k)$ associated with the classical fields $\omega(\zeta_k)$, for $k = 1, \dots, n$. The normalization

$$Z[0] = 1, \tag{70}$$

i.e. the presence of the denominator in (67), guarantees the cancellation of vacuum diagrams in $\pi(\zeta_1, \dots, \zeta_n)$ in the formal perturbation expansion of the exponential function in (67).

6 Mass and coupling constant renormalization

We shall not proceed to the derivation of the diagrams involved in the perturbation series, since the result is completely analogous to the usual spacetime $\frac{\lambda}{4!}\Phi^4$ -theory, where in (69) π is replaced by τ and Π by the time ordered product operator T . Instead, we want to emphasize three points, which suggest a possible further generalization to conformally flat metrics other than the four dimensional AdS case:

- i) The quantum fluctuation of the classical metric hides a Φ^4 -theory in a one-dimensional spacetime.
- ii) The Φ^4 -theory is massless at the classical limit but, at the quantum level, it suffers a finite mass renormalization generated by the cosmological constant in the AdS spacetime, which in the Φ^4 -theory plays the role of a natural IR cutoff. In particular, to first order in λ , from (66), (67), (68) and (69), for the 2-point function one obtains

$$\pi(\zeta_1, \zeta_2) = i \int_{-\infty}^{+\infty} \frac{d\xi}{2\pi} \frac{e^{-i\xi(\zeta_1 - \zeta_2)}}{\xi^2 - m_r^2 + i\epsilon}, \tag{71}$$

where

$$m_r^2 = \delta m^2 = \frac{i\lambda}{2} \Delta_F(0)' = \frac{\lambda}{4\mu} = \frac{\lambda}{4a\sqrt{-\Lambda}} = \frac{\sqrt{3}}{8}. \tag{72}$$

- iii) The coupling constant of the scalar field is also finitely renormalized by the IR cutoff μ^2 . In fact, to second order of the perturbation expansion, the connected 4-point function in momentum space is given by

$$\pi_4(p) = -i\lambda + 3(-i\lambda)^2 I(p), \tag{73}$$

where

$$I(p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi^2 - \mu^2 + i\epsilon)((p - \xi)^2 - \mu^2 + i\epsilon)} = \int_0^1 \frac{dt}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi^2 - \mu^2 + p^2 t(1-t) + i\epsilon)^2}, \tag{74}$$

with $p = p_1 + p_2$ ($p_i, i = 1, 2$: ingoing momenta) (see [7] p. 315). Then

$$I(0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi^2 - \mu^2 + i\epsilon)^2} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi}{(\xi - \xi_+)^2(\xi - \xi_-)^2} := \lim_{\zeta \rightarrow 0} I(0, \zeta), \quad (75)$$

with

$$I(0, \zeta) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{d\xi e^{-i\xi\zeta}}{(\xi - \xi_+)^2(\xi - \xi_-)^2}, \quad (76)$$

and $\xi_{\pm} = \pm(\mu - \frac{i\epsilon}{2\mu}) = \pm(\sqrt{3} - \frac{i\epsilon}{2\sqrt{3}})$. Making the complex integrations as for the case of $\Delta_F(\zeta)'$, one obtains

$$I(0) = \frac{i}{4\mu^3}. \quad (77)$$

So, the renormalized coupling constant results

$$i\pi_4(0) = \lambda_r = \lambda \left(1 + \frac{3\lambda}{4\mu^3}\right) = \lambda \left(1 + \frac{3\lambda}{4a^3(-\Lambda)^{3/2}}\right) = \frac{3}{2} \left(1 + \frac{\sqrt{3}}{8}\right). \quad (78)$$

7 Final comments

It is interesting to see the natural appearance of a massless scalar field derived from a conformally flat metric in four dimensions via a Weyl transformation. This unexpected result suggests several questions which could serve as motivation for further research. Firstly, since this result is valid for all conformally flat metrics in four spacetime dimensions, one could wonder what the consequences of the appearance of a massless scalar field are for all Robertson-Walker geometries, which have all been shown to be conformally flat [8] and which describe the large-scale structure of a homogeneous and isotropic Universe. Could this massless scalar field be interpreted as a natural-appearing inflaton field? If so, what are the implications of this fact for the field of Quantum Cosmology? Secondly, is it possible to obtain a similar result in higher dimensional conformally flat spacetimes within the framework of General Relativity? In particular, can one construct a quantum theory of such a field in five-dimensional Anti-de Sitter spacetime? This would be an interesting question related to the well known AdS/CFT correspondence, and which could help physicists to better understand this duality. And finally, can one obtain a similar higher-dimensional result considering alternative theories of gravity such as Einstein-Gauss-Bonnet gravity or Lovelock theories? We leave all these questions open to further research.

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