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Use of Piecewise Linear Models for Optimal Control of Nonlinear Systems

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Abstract. The optimal feedback control of nonlinear process is attacked in this paper. The solution of this problem is numerically computed using a Continuous Piecewise Linear (CPWL) approximation of the Ordinary Differential Equations (ODEs) system which describes the dynamics of the plant to control. In order to obtain this solution, the optimal regulation problem of an affine system is obtained. A numerical simulation example of a nonlinear chemical reactor is presented to shown the quality of the obtained response.

Keywords: Optimal Control, Nonlinear Systems, Piecewise Linear Functions.

AMS subject classifications: 49J15, 93C10

1 Introduction

In this paper, let us consider the optimal control problem of a nonlinear system, given as:

$$\dot{x}(t) = f(x(t), u(t)), \tag{1.1}$$

$$y(t) = h(x(t), u(t)), \tag{1.2}$$

where $x \in \Re^n$, $u \in \Re^m$, $y \in \Re^m$ and $f(x,u) : \Re^{n+m} \to \Re^n$ and $h(x,u) : \Re^{n+m} \to \Re^m$ are continuous functions.

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The control objective is to move the system from some initial condition $(u(0), x(0))$ to an stationary point (u_s, x_s) minimizing the following penalty function,

$$\min_{u(t)} \int_{t=0}^{t_f} \left[(x(t)-x_s)^T R_1 (x(t)-x_s) + (u(t)-u_s)^T R_2 (u(t)-u_s)\right] dt + (x(t_f)-x_s)^T P_1 (x(t_f)-x_s). \quad (1.3)$$

where R_1 and P_1 are positive semi-definite matrices, and R_2 is a definite positive matrix.

This optimal control problem could be solved using Pontryagin's maximum principle (Pontryagin *et al.*, 1962; Bertsekas, 2005) or by solving the Hamilton-Jacobi-Bellman equation. In booth cases, the solution involves the integration of a system of ordinary differential equations subject to two-point boundary conditions. The drawback with this ideal solution is its numerical complexity, which in many cases precludes its use on practical situations.

If this problem is constrained to linear systems there is a close solution that could be written as a feedback control (Kwakernaak and Sivan, 1972; Anderson and Moore, 1989). Some results present solutions for this problem for some types of non linear models, for example in the case of bilinear systems (Cebuhar and Costanza, 1984; Costanza and Neuman, 2006).

In this paper, we propose an alternative approach in the continuous time domain. To perform it, a Continuous Piecewise Linear (CPWL) approximation of the nonlinear model of the system is used. Then, the solution of the optimal control problem is obtained.

The paper is organized as follow. In Section 2 the proposed solution for the Optimal Control Problem is presented. In Section 3 a simulation example is developed to show the numerical quality of the proposed method. The paper ends in Section 3 with some conclusions.

2 Optimal control solution

In this section we will consider an approximated solution for the optimal control problem (1.3) subject to the system model (1.1-1.2) . To obtain this solution, the nonlinear system is first approximated by a continuous piecewise linear system, then the optimal control solution of this approximation is computed. We will note that this solution is an extension of the optimal control of linear systems to affine systems.

First, let us consider the CPWL approximation for the state-equation (1.1). Let the sets $X \subset \Re^n$ and $u \subset \Re^m$ be domain of the x and u variables, respectively, and consider the set

$$\aleph = \left\{[x^T, u^T]^T : x \in X, u \in U\right\} \quad (2.1)$$

on which we want to approximate the given system. Consider also the simplicial partition (Julián *et al.*, 1999) of the set $\aleph$ such that

$$\aleph = \bigcup_{i=1}^{r} \aleph^{<i>}, \quad (2.2)$$

where $\aleph^{<i>}$ is the "i-th partition" of the set $\aleph$ and r being the number of simplices considered.

Let us note as $f_{PWL}^{<i>}$ the CPWL approximation of the vectorial function f on each simplex. Then, for $[x^T, u^T]^T \in \aleph^{<i>}$ and $i = 1, \cdots, r$, the CPWL system takes the form (Julián *et al.*, 1999)

$$\dot{x}(t) = f_{PWL}^{<i>}(x) = A^{<i>}x(t) + B^{<i>}u(t) + C^{<i>}, \tag{2.3}$$

where the superscript $< i >$ identifies the sector of the partition where are the variables. Note that the system model depends on the simplices where it is evolving. Then, in order to know $A^{<i>}$, $B^{<i>}$ and $C^{<i>}$, first it is necessary to know the profiles $x(t)$ and $u(t)$. This fact will complicate the solution of the optimal control problem because it requires the cross points from one simplex to the next one.

From the comparison of the CPWL model (2.3) with a classical linear approximation, it is clear that the difference between them is the term $C^{<i>}$. This term is why it is not possible to use the classical linear solution and it is necessary to develop an optimal control solution for this application.

Now, if we consider that the evolution of the system is constrained in the i-th simplex, the optimal solution of this problem is (Bertsekas, 2005)

$$u^o(t) = -R_2^{-1}(B^{<i>})^T p(t) + u_s \tag{2.4}$$

and

$$\dot{p}(t) = -(A^{<i>})^T p(t) - R_1(x^o(t) - x_s). \tag{2.5}$$

Then, the optimal behavior of the system ($x^o(t)$) is given by the following system,

$$\begin{bmatrix} \dot{x}^o(t) \\ \dot{p}(t) \end{bmatrix} = \tilde{A} \begin{bmatrix} x^o(t) \\ p(t) \end{bmatrix} + \tilde{B} \tag{2.6}$$

with

$$p(t_f) = P_1 \left(x^o(t_f) - x_s \right), \tag{2.7}$$

where

$$\tilde{A} = \begin{bmatrix} A^{<i>} & -B^{<i>}R_2^{-1}(B^{<i>})^T \\ -R_1 & -(A^{<i>})^T \end{bmatrix} \tag{2.8}$$

and

$$\tilde{B} = \begin{bmatrix} C^{<i>} + B^{<i>}u_s \\ R_1 x_s \end{bmatrix}. \tag{2.9}$$

Now, if $\Theta(t_f, t)$ is the transition matrix of $\tilde{A}$, the integral of (2.6) is

$$\begin{bmatrix} x^o(t) \\ p(t) \end{bmatrix} = \Theta(t_f, t) \begin{bmatrix} x^o(t_f) \\ p(t_f) \end{bmatrix} + \int_{t_f}^{t} \Theta(\tau, t_f)\tilde{B} d\tau \tag{2.10}$$

To evaluate this expression let us consider the form of $\Theta(t_f, t)$. Let us assume that the system is moving through the simplices 1, 2, $\cdots$, f at times t_1, t_2, $\cdots$, t_f. Then,

$$\Theta(t_f, t) = \Theta_f(t_{f-1}, t_f) \cdots \Theta_2(t_1, t_2)\Theta_1(t, t_1) = \Theta(t_f, t_1)e^{A^{<1>}(t-t_1)} \tag{2.11}$$

where

$$\Theta(t_f, t_1) = e^{A^{<f>}(t_{f-1}-t_f)} \cdots e^{A^{<2>}(t_1-t_2)}. \tag{2.12}$$

Under this condition, the integral term in Eq. (2.10) is

$$\begin{array}{c} \Theta_I(t_f, t) = \int_{t_f}^{t} \Theta(\tau, t_f)\tilde{B}d\tau = \\ \Theta(t_f, t_1)(A^{<1>})^{-1}\left(e^{A^{<1>}(t-t_1)} - I\right)B^{<1>} + \\ (A^{<2>})^{-1}\left(e^{A^{<2>}(t_1-t_2)} - I\right)B^{<2>} + \cdots \\ +(A^{<f>})^{-1}\left(e^{A^{<f>}(t_{f-1}-t_f)} - I\right)B^{<f>}. \end{array} \tag{2.13}$$

Now, considering the following partition in these matrices,

$$\Theta(t_f, t) = \begin{bmatrix} \Theta_{11}(t_f, t) & \Theta_{12}(t_f, t) \\ \Theta_{21}(t_f, t) & \Theta_{22}(t_f, t) \end{bmatrix} \tag{2.14}$$

and

$$\Theta_I(t_f, t) = \begin{bmatrix} \Theta_{I1}(t_f, t) \\ \Theta_{I2}(t_f, t) \end{bmatrix}. \tag{2.15}$$

Then,

$$\begin{bmatrix} x^o(t) \\ p(t) \end{bmatrix} = \begin{bmatrix} \Theta_{11}(t_f, t) & \Theta_{12}(t_f, t) \\ \Theta_{21}(t_f, t) & \Theta_{22}(t_f, t) \end{bmatrix} \begin{bmatrix} x^o(t_f) \\ p(t_f) \end{bmatrix} + \begin{bmatrix} \Theta_{I1}(t_f, t) \\ \Theta_{I2}(t_f, t) \end{bmatrix} \tag{2.16}$$

and since $p(t_f) = P_1\left(x^o(t_f) - x_s\right)$, it is possible to write

$$x^o(t) = \left[\Theta_{11}(t_f, t) + \Theta_{12}(t_f, t)P_1\right]x^o(t_f) - \Theta_{12}(t_f, t)P_1x_s + \Theta_{I1}(t_f, t) \tag{2.17}$$

$$p(t) = \left[\Theta_{21}(t_f, t) + \Theta_{22}(t_f, t)P_1\right]x^o(t_f) - \Theta_{22}(t_f, t)P_1x_s + \Theta_{I2}(t_f, t). \tag{2.18}$$

Then,

$$\begin{array}{l} p(t) = \left[\Theta_{21}(t_f, t) + \Theta_{22}(t_f, t)P_1\right] \times \left[\Theta_{11}(t_f, t) + \Theta_{12}(t_f, t)P_1\right]^{-1} \times \\ \left[x^o(t) + \Theta_{12}(t_f, t)P_1x_s - \Theta_{I1}(t_f, t)\right] - \Theta_{22}(t_f, t)P_1x_s + \Theta_{I2}(t_f, t). \end{array} \tag{2.19}$$

From this expression, it is possible to obtain $p(t)$ as function of the present state $x^o(t)$, then using Eq. (2.4) the control law is obtained as a function of the present state.

Note that this expressions implies the generalization of the linear optimal control problem to CPWL models (i.e., including the term $C^{<i>}$ in the model).

A problem of this control scheme is that it depends on the sectors visited by the process trajectory and at which times the process changes of sector. In other words, this method replaces the numerical complexity of solving a problem with two-points boundary conditions with an iterative procedure that determines the points of sector change as the system evolves with time. This is performed by evaluating the pair ($x^o(t)$,$u^o(t)$) using Eqs. (2.17) and (2.4), and verifying in which sector $\aleph^{<i>}$ this point is. To verify if the pair belongs to a given set is equivalent to satisfying a linear inequality (Figueroa, 2000; Figueroa, 2001), that is performed in a simple way.

2.1 Initialization

The data for the proposed control problem are R_1, P_1, R_2 and t_f.

For the solution of the problem, we should initialize the simplex that process switches to during its evolution and the time at which switches occurs (i.e. $\aleph^{<i>}$ and t_i for $i = 1, \cdots, f$). A good selection for this initialization is to consider that the system moves from $u(0)$ to u_s as

$$u(t) = (1 - t/t_f)u(0) + t/t_f u_s \quad t \in [0, t_f]. \tag{2.20}$$

Then, from the pairs $(x(t), u(t))$ it is possible to evaluate the simplex $\aleph^{<i>}$ where the process is, the time t_i at which the simplex switch and the model of the process ($A^{<i>}$, $B^{<i>}$ and $C^{<i>}$) at each simplex (Figueroa, 2000).

In conclusion, the optimal control solution of PWL systems is solved by applying the algorithm summarized in Table 1. Note that this algorithm involves an iteration because to compute the optimal control we need to know within which simplex that the process lies during its evolution, which, in turn, also depends on the applied control.

It is not possible to prove the convergence of this iteration, however, our experience shows that in the almost all cases converge is achieved in few iterations. Then, a sub-optimal solution is found with lower computational complexity than the required for the nonlinear optimal control problem. If this is not the case, a possible solution is to initializes the manipulated variable in a different way that the expressed in (2.20), for example as $u(t) = u_s$.

3 Example: simulation of a CSTR

Consider an adiabatic CSTR in which the exothermic first-order irreversible Van de Vusse reaction is taking place (Sistu and Bequette, 1995; Costanza and Neuman, 2006). The dimensionless equations for the mass and heat balances are

$$\dot{x_1} = -\theta x_1 exp\left(\frac{x_2}{1 + x_2\gamma}\right) + q\left(x_{1f} - x_1\right) \tag{3.1}$$

Table 1: Learning algorithm for the PWL-DFANN realization.

Data:
  R_1, P_1, R_2, t_f, u_s, x_s.
Initial Conditions: $x(0)$, $u(0)$
Initialization:
  Perform a simulation for $u(t) = (1 - t/t_f)u(0) + t/t_f u_s$ for $t \in [0, t_f]$.
Step 0: From $(x(t), u(t))$ compute $\aleph^{<i>}$ and t_i for $i = 1, \cdots, f$.
Step 1: Compute the model $A^{<i>}$, $B^{<i>}$ and $C^{<i>}$ for $i = 1, \cdots, f$.
Step 2: Compute $p(t)$ from Eq. (2.19).
Step 3: Compute $u^o(t)$ from Eq. (2.4).
Step 4: From de model and $u^o(t)$, compute the pair $x(t)$, $u^o(t)$.
Step 5: From $(x(t), u^o(t))$ compute $\aleph_n^{<i>}$ and t_i^n for $i = 1, \cdots, f$.
Step 6: If the sectors $\aleph^{<i>}$ and times t_i are close to $\aleph_n^{<i>}$ and t_i^n,
  then stop the algorithm, the solution is found.
  Otherwise, go to Step 1.

$$\dot{x_2} = \beta\theta x_1 exp\left(\frac{x_2}{1 + x_2\gamma}\right) + q\left(x_{2f} - x_2\right)\delta x_2 \tag{3.2}$$

Typical values for the parameters are $\theta = 1.135$, $\gamma = 20$, $x_{1f} = 1$, $\beta = 11$, $x_{2f} = 0$ and $\delta = 1.5$; the variable x_1 is the dimensionless extent of reaction and x_2 is the dimensionless reaction temperature. The dimensionless feed flow rate q is the only variable that could be manipulated.

The objective function is to control the reactor from an initial state[‡] $x(t_0) = [0.2569 \quad 1.4870]^T$ to a final state $x(t_f) = [0. \quad 0.]^T$ using the control strategy described in the previous section.

The optimal control parameters for problem 1.3 are $t_0 = 0$, $t_f = 10$; $R_1 = I$, $R_2 = 20$ and $P_1 = 1000$. To perform the PWL approximation the domain ($0 \leq x_1 \leq 1$, $0 \leq x_2 \leq 2$ and $-1 \leq q \leq 1$) is divided in 125 sectors. The parameters of the optimization problem were determined to obtain a good performance of the complete nonlinear problem and the number of sectors in the PWL approximation was selected to ensure close responses between both models.

To perform the computation of the optimal control, a first set of sectors and cross times is computed considering the free evolution of the process from the initial to final state points. Using these data, in two iterations, the control law converges to a curve that is shown in Figure 1. In this plot, the results are compared with the solution of the complete nonlinear problem. Figure 2 shows the manipulated variable for the PWL controller. The objective function for the proposed methodology is 2506 that is

[‡]Steady state corresponding to the control input $q = 0.6$.

comparable with the value of 2486 obtained for the optimal case.

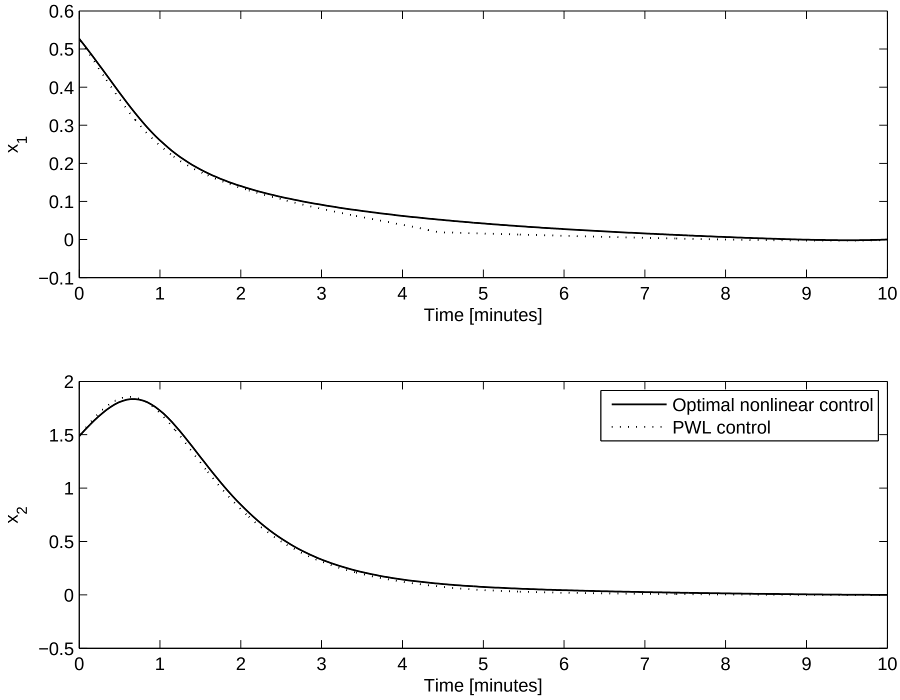

Figure 1: Time Simulation for Reactor States: States.

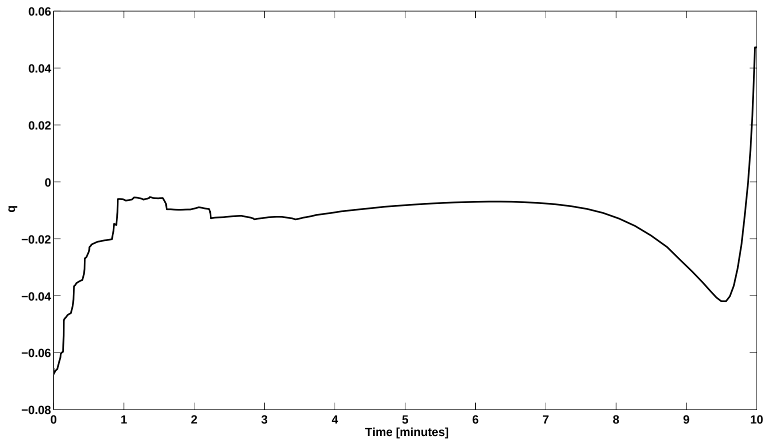

Figure 2: Time Simulation for Reactor States: Manipulated variable.

Figure 2 shows the behavior of the states in the plane $x_1 - x_2$. In this plot are include the bounds of the simplices to show how the algorithm perform in a smooth way ever at the sector switches, despite of small discontinuities in the manipulated variables.

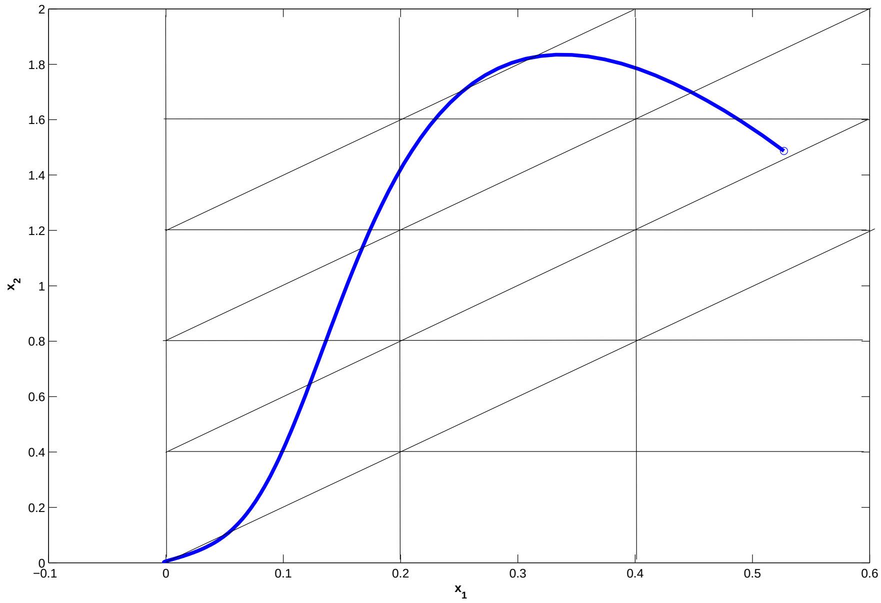

Figure 3: Phase plane with simplicial division.

4 CONCLUSIONS

A methodology for obtaining an optimal control law for a nonlinear system by approximating the functions using Piecewise Linear models is developed. In particular, the expression for the optimal solution of an affine model are developed and generalized to general PWL systems. The proposed method is numerically efficient.

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