International Journal of Bifurcation and Chaos, Vol. 14, No. 10 (2004) 3647–3654

DETECTION OF LIMIT CYCLE BIFURCATIONS USING HARMONIC BALANCE METHODS

FEDERICO I. ROBBIO,* DIEGO M. ALONSO and JORGE L. MOIOLA
CONICET and Departamento de Ingeniería Eléctrica y de Computadoras,
Universidad Nacional del Sur, Av. Alem 1253, B8000CPB Bahía Blanca, Argentina
**frobbio@criba.edu.ar*

Received September 30, 2003; Revised October 28, 2003

In this paper, bifurcations of limit cycles close to certain singularities of the vector fields are explored using an algorithm based on the harmonic balance method, the theory of nonlinear feedback systems and the monodromy matrix. Period-doubling, pitchfork and Neimark–Sacker bifurcations of cycles are detected close to a Gavrilov–Guckenheimer singularity in two modified Rössler systems. This special singularity has a zero eigenvalue and a pair of pure imaginary eigenvalues in the linearization of the flow around its equilibrium. The presented results suggest that the proposed technique can be promising in analyzing limit cycle bifurcations arising in the unfoldings of other complex singularities.

Keywords: Harmonic balance; limit cycle bifurcations; monodromy matrix; Hopf bifurcation; period-doubling.

1. Introduction

Bifurcations of limit cycles announce qualitative changes in the periodic behavior of dynamical systems when varying a certain distinguished parameter. Sometimes, their presence predicts the birth of chaos and thus, the sudden disappearance of the periodic behavior. Since our main interest is in periodic regimes, we focus the attention on studying bifurcations of limit cycles. Toward this end, the stability[1] of the periodic orbit is analyzed by computing the characteristic or Floquet multipliers. In this regards, some sophisticated techniques have been recently developed to reduce dramatically the errors in the computations of the Floquet multipliers aiming to locate more precisely the birth of the cycle bifurcations. For example, in [Choe & Guckenheimer, 1999] and in [Guckenheimer & Meloon, 2000] accurate computations of periodic orbits and their bifurcations have been performed. In the same spirit, Lust [2001] develops efficient numerical methods for linear stability analysis of periodic branches for low-dimensional systems of ordinary differential equations. On the other hand, some methods for computing Floquet multipliers do not have such a great performance in accuracy, but indeed are very useful in engineering for its practicality. In this regard, we are referring to computations of limit cycle bifurcations using harmonic balance methods with first order approximations [Torrini *et al.*, 1998] or with higher-order approximations [Berns *et al.*, 2001].

In this paper higher-order harmonic balance methods and the theory of nonlinear feedback systems are used to approximate limit cycles. The approximated periodic solution is employed to compute the monodromy matrix and the characteristic multipliers to detect the first bifurcations of cycles [Robbio *et al.*, 2004]. We have computed the first bifurcations of the cycles arising from a Hopf bifurcation in two simple three-dimensional dynamical

[1]It is worth mentioning that in some cases a limit cycle can develop a bifurcation without changing the stability.

systems having quadratic and cubic nonlinearities. The analysis is done in the vicinity of a Gavrilov–Guckenheimer singularity (also known as fold-Hopf bifurcation as noted by [Kuznetsov, 1995]) presenting one zero eigenvalue and a pair of pure imaginary eigenvalues in the linearization matrix. Some of the typical bifurcation curves in the unfoldings of this singularity, such as static and Hopf bifurcations of equilibria, and period-doubling, pitchfork and Neimark–Sacker bifurcations of cycles are detected.

This methodology seems to be powerful in detecting the first bifurcation of the cycles close to a Hopf bifurcation curve, since other singularities of vector fields such as double or triple zero eigenvalues, double Hopf bifurcation, etc. involve the presence of the Hopf curve in a two or three parameter space. In addition, the increased higher-order harmonic balance expansion of the cycles extends the local prediction of the standard second-order approximation (and hence, of the locality of the Hopf bifurcation) via the computation of the Floquet multipliers.

2. Background Material

Consider the general nonlinear system

$$\begin{aligned} \dot{x} &= Ax + BDy + B[g(y;\,\mu) - Dy]\,, \\ y &= Cx\,, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, A, B, C and D are $n \times n$, $n \times l$, $m \times n$ and $l \times m$ matrices, respectively, $\mu \in \mathbb{R}$ is the bifurcation parameter, y is the output and g: $\mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^l$ is at least a C^4-function in x and μ. It is worth mentioning that the matrix D is arbitrarily added for convenience. The system can be represented in feedback form as a linear transfer matrix $G(s;\,\mu)$ in the direct path and a memoryless nonlinear part u in the feedback path, i.e.

$$\begin{aligned} G(s;\,\mu) &= C[sI - (A + BDC)]^{-1}B\,, \\ u = f(e;\,\mu) &:= g(y;\,\mu) - Dy\,, \end{aligned}$$

where $e = -y$. Notice that the equilibrium points of Eq. (1) correspond to the solutions $\hat{e}$ of

$$G(0;\,\mu)f(e;\,\mu) + e = 0\,.$$

The open-loop linearization matrix associated with the feedback realization is $G(s;\,\mu)J(\mu)$ where $J = (\partial f(e;\,\mu)/\partial e)|_{e=\hat{e}} = D_e f(\hat{e};\,\mu)$, and the corresponding eigenvalues are given by

$$\begin{aligned} h(\lambda,\,s;\,\mu) &= \det[\lambda I - GJ] \\ &= \lambda^p + a_{p-1}(s;\,\mu)\lambda^{p-1} + \cdots + a_0(s;\,\mu) \\ &= 0\,, \end{aligned} \tag{2}$$

where $p = \min[\text{rank } G,\, \text{rank } J]$ and $a_i(\cdot)$ are rational functions of s. Assuming a single root of $h(\cdot)$ at $\lambda = -1$ and replacing $s = i\omega$ in Eq. (2), a necessary condition for computing a bifurcation point $(\omega_0,\, \mu_0)$ is obtained solving

$$h(-1,\, i\omega;\, \mu) = (-1)^p + \sum_{k=0}^{p-1} (-1)^k a_k(i\omega;\,\mu) = 0\,,$$

for ω and μ. If $\omega_0 = 0$ the bifurcation condition is called *static*, and it is related to the multiplicity of the equilibrium solutions. On the other hand, if $\omega_0 \neq 0$, the bifurcation condition is known as *dynamic* or *Hopf*, and provided that some additional conditions are fulfilled, it is related to the appearance of periodic solutions.

Once the birth of a limit cycle due to a Hopf bifurcation is detected, a second-order [Mees & Chua, 1979], fourth-order [Mees, 1981], sixth- and eighth-order [Moiola & Chen, 1996], and, in general, a $2q$-order harmonic balance approximation (HBA) of the periodic solution in the neighborhood of the criticality can be obtained applying the statements of the Hopf bifurcation theorem in the frequency domain [Mees & Chua, 1979]. Toward this end, we must solve the general equation

$$\hat{\lambda}(i\hat{\omega}_q) = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q-1})\hat{\theta}_q^{2k}\,, \tag{3}$$

where $\hat{\lambda}$ is the eigenvalue associated to GJ whose eigenlocus crosses the real axis closest to the critical point $-1 + i0$, $\hat{\theta}$ and $\hat{\omega}$ are approximations of the amplitude and frequency respectively, and ξ_k are complex numbers (see the corresponding expressions in [Robbio *et al.*, 2004]). The pair $(\hat{\omega}_q,\, \hat{\theta}_q)$ are obtained by means of an iterative procedure starting with $\hat{\omega}_R$, the frequency at which the locus of $\hat{\lambda}$ crosses the real axis near to the point $-1 + i0$. The procedure consisting of N iterations over the $2q$-order approximation is

$$\begin{aligned} &\text{STEP 1} \quad \hat{\lambda}(i\hat{\omega}_1) = -1 + \xi_1(\hat{\omega}_R)\hat{\theta}_1^2\,, \\ &\text{STEP 2} \quad \hat{\lambda}(i\hat{\omega}_2) = -1 + \xi_1(\hat{\omega}_1)\hat{\theta}_2^2 + \xi_2(\hat{\omega}_1)\hat{\theta}_2^4\,, \\ &\qquad\vdots \qquad\qquad\qquad \vdots \\ &\text{STEP } q \quad \hat{\lambda}(i\hat{\omega}_{q,0}) = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q-1})\hat{\theta}_{q,0}^{2k}\,, \end{aligned}$$

Table 1. Limit cycle bifurcations and critical multipliers crossing point.

Crossing Point	Cycle Bifurcation
$-1+i0$	Period doubling
$1+i0$	Saddle-node, transcritical or pitchfork
$\alpha \pm i\beta\ (\alpha^2+\beta^2=1)$	Neimark–Sacker or *torus*

$$\text{STEP } q{+}1 \quad \hat{\lambda}(i\hat{\omega}_{q,1}) = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q,0})\hat{\theta}_{q,1}^{2k},$$

$$\vdots \qquad\qquad \vdots$$

$$\text{STEP } q{+}N \quad \hat{\lambda}(i\hat{\omega}_{q,N}) = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q,N-1})\hat{\theta}_{q,N}^{2k}.$$

Finally $\hat{\omega}_q = \hat{\omega}_{q,N}$ and $\hat{\theta}_q = \hat{\theta}_{q,N}$. In the following, let us refer as L_q the approximation given by $2q$-order HBA calculated for a specific value of μ and the corresponding value of $\hat{\omega}_q$.

In order to study the stability of the periodic solution, the $2q$-order HBA approximation of the limit cycle is used to obtain an approximate *monodromy matrix* M_q. In the general case M_q has n eigenvalues, $\lambda_1(\mu)$, $\lambda_2(\mu), \ldots, \lambda_n(\mu)$, which are known as characteristic (or Floquet) multipliers. One of them is always equal to $+1$, say $\lambda_1(\mu)$ and the remaining $n-1$ determine the local stability of the limit cycle. The multiplier that crosses the unit circle is known as *critical multiplier* and the three distinguished ways of crossing the unit circle determine three associated types of branching as shown in Table 1.

To compute the monodromy matrix M_q we need to solve the variational equation

$$\dot{Y}(t) = J_{D_q}(t)Y(t),$$
$$Y(0) = I,$$
$$M_q = Y\left(\frac{2\pi}{\hat{\omega}_q}\right),$$

where J_{D_q} is the Jacobian matrix of the system [Eq. (1)] evaluated at the periodic solution L_q.

3. Application Examples

In order to illustrate the methodology a system with the Rössler type structure is considered. The system was proposed by Thomas [1999] and notwithstanding its simplicity it can develop complex dynamical behaviors. This system is

$$\begin{aligned}\dot{x}_1 &= -x_2 - x_3,\\ \dot{x}_2 &= x_1 + \mu x_2,\\ \dot{x}_3 &= \varphi(x_1) - cx_3,\end{aligned} \tag{4}$$

where μ is the main bifurcation parameter, c is a real constant and $\varphi(x_1)$ is a nonlinear function that, in this paper, can be quadratic $\varphi(x_1) = x_1^2$ or cubic $\varphi(x_1) = x_1^3$. In both cases a Single-Input–Single-Output (SISO) feedback realization can be obtained since the nonlinearity involves only one state variable. Therefore, the formulas for obtaining the higher-order approximations of the periodic solutions are considerably simplified.

The proposed feedback realization on Eq. (4) is

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \mu & 0 \\ -1 & 0 & -c \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \qquad D = 0,$$
$$g(x_1) = x_1 + \varphi(x_1).$$

The corresponding transfer function results in

$$G(s;\,\mu) = -\frac{s-\mu}{\Delta(s)},$$

where $\Delta(s) = s^3 + (c-\mu)s^2 - \mu cs + \mu + c$, and the nonlinear function in terms of the output $e_1 = -x_1$ is

$$f(e_1) = -e_1 + \varphi(e_1).$$

The equilibrium solutions for the output are

$$\hat{e}_1^0 = 0,$$

and

$$\hat{e}_1^1 = -\frac{c}{\mu}, \quad \text{when} \quad \varphi(x_1) = x_1^2,$$

or

$$\hat{e}_1^{1,2} = \pm\sqrt{\frac{c}{\mu}}, \quad \text{when} \quad \varphi(x_1) = x_1^3.$$

We will focus the attention on the equilibrium point $\hat{e}_1^0$ since it undergoes a Hopf bifurcation when $\mu = 0$ and thus a limit cycle arises near $\mu = 0$. Then, the eigenlocus to be considered in the frequency domain method is

$$\hat{\lambda}(s) = G(s)J = \frac{s-\mu}{\Delta(s)},$$

where $J = -1 + (\partial\varphi(e_1)/\partial e_1)|_{e_1=\hat{e}_1^0} = -1$.

In addition to the Hopf bifurcation, the equilibrium point $\hat{e}_1^0$ develops a type of static bifurcation when $c = 0$: a saddle-node when $\varphi(x_1) = x_1^2$ or a pitchfork when $\varphi(x_1) = x_1^3$. Moreover, when both conditions occur simultaneously ($\mu = c = 0$) a Gavrilov–Guckenheimer bifurcation takes place, i.e. the linearization matrix of Eq. (4) evaluated at $\hat{e}_1^0$ has two pure imaginary eigenvalues plus a zero eigenvalue.

In the following, the proposed technique is applied to the system considering both nonlinearities to obtain the first bifurcation of the cycle emerged at the Hopf bifurcation of $\hat{e}_1^0$ when $\mu = 0$. The variational system is easily obtained from Eq. (4) as

$$\dot{Y}(t) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \mu & 0 \\ \varphi'(x_1) & 0 & -c \end{bmatrix} Y(t)\,,$$

where $\varphi'(x_1) = 2x_1(t)$ for the quadratic system or $\varphi'(x_1) = 3x_1^2(t)$ for the cubic system, and $x_1(t)$ is the periodic solution obtained using the eighth-order HBA.

The formulas for calculating the higher order approximations of the limit cycles are given in [Robbio *et al.*, 2004]. To validate the semi-analytical results, standard packages for the continuation of periodic solutions such as LOCBIF library [Khibnik *et al.*, 1993] and AUTO [Doedel *et al.*, 1997; Ermentrout, 2001] are used. For the proposed procedure, an eighth-order HBA is used, and the number of iterations is $N = 100$ in all the cases.

3.1. *Quadratic nonlinearity*

Let us consider the modified Rössler system represented by Eq. (4) with $\varphi(x_1) = x_1^2$. The performed numerical study reveals that the limit cycle arising from the Hopf bifurcation of the origin when $\mu = 0$ exhibits a Neimark–Sacker bifurcation (NS for brevity) when $\mu = c$ in the interval (0, 0.74361). That is, a pair of complex conjugate multipliers of the monodromy matrix cross the unit circle when $\mu = c$ in that interval. This fact can also be determined via the Liouville formula relating the product of the multipliers and the exponential of the integral of the divergence of the vector field [Torrini *et al.*, 1998], but without defining the end point 0.74361. It is worth mentioning that the bifurcating multipliers sweep the unit circle starting with both multipliers at +1 when $\mu = c = 0$ and ending with both multipliers at −1 when $\mu = c = 0.74361$. The locus corresponding to the NS bifurcation in the parameter plane $\mu - c$ is shown in Fig. 1. In addition to the NS curve, a period-doubling (PD) bifurcation locus is depicted in the same figure. At the PD curve the stable limit cycle arising from HB ($\mu = 0$) bifurcates into a period-2 cycle. Notice that in Fig. 1 only the first bifurcation curves of the limit cycle are shown. Thus, a region of the parameter plane $\mu - c$ where only one stable limit cycle exists is delimited by HB, NS and PD curves (dashed region).

In order to illustrate the complex behavior evidenced beyond the first bifurcation curves, a continuation of the limit cycle (using AUTO program) varying parameter μ and fixing c at $c = 2$ is presented in Fig. 2. The cycle ends-up in a homoclinic loop (HOM) after developing a series of period-doubling (PD) and saddle-node (SN) bifurcations.

The procedure based on the approximate monodromy matrix is capable of detecting completely the NS curve and the first PD points. Nevertheless, NS points are obtained with a certain error due to the limit cycle approximation. A measure of the error is given by the separation of the multiplier that must be at +1 (see [Guckenheimer & Meloon, 2000] for more details) and in this case is of order 1.5×10^{-2}. The ending point of the NS bifurcation curve, i.e. when it meets the PD curve (two multipliers at −1), is $\mu = 0.7144355$, $c = 0.717257$ which differs from $\mu = c = 0.74361$ computed with LOCBIF. The computed bifurcation points are marked by asterisks in Fig. 1. The NS curve is obtained fixing c and finding the corresponding value of μ. Notice that parameter c is arbitrarily increased on steps of 0.1 until the point where the NS curve meets the PD curve. At this point the step is reduced in order to obtain more accurately the resonance point (see the enlargement in Fig. 1). A similar procedure is employed for the PD curve. In this case the last detected point is $\mu = 0.26074$, $c = 2.5963$. Beyond this value the method fails since not all the computed amplitude loci

$$L_q = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q-1})\hat{\theta}_q^{2k}\,,$$

intersect the locus of $\hat{\lambda}(i\omega)$ for a specific value of μ, i.e. it is not possible to solve Eq. (3). For example, for $c = 2.6182$, by increasing the value of μ the eigenvalues of the monodromy matrix are given in Table 2.

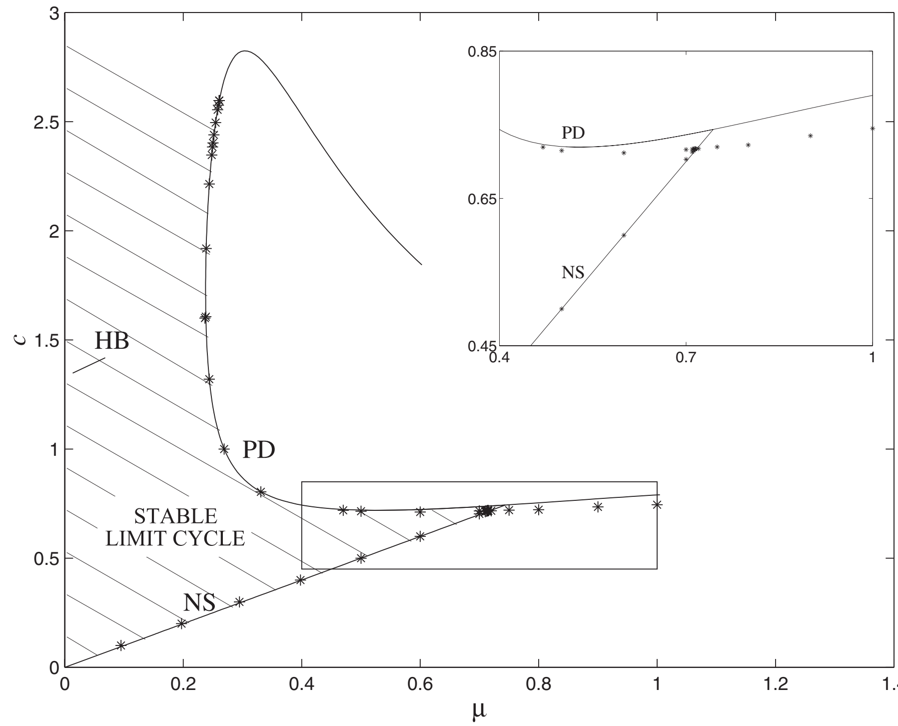

Fig. 1. First limit cycle bifurcation curves for the system with a quadratic nonlinearity. The origin in the parameter space corresponds to the condition of the Gavrilov–Guckenheimer bifurcation. The asterisks correspond to eighth-order HBA which are contrasted to the solutions obtained with LOCBIF (solid lines).

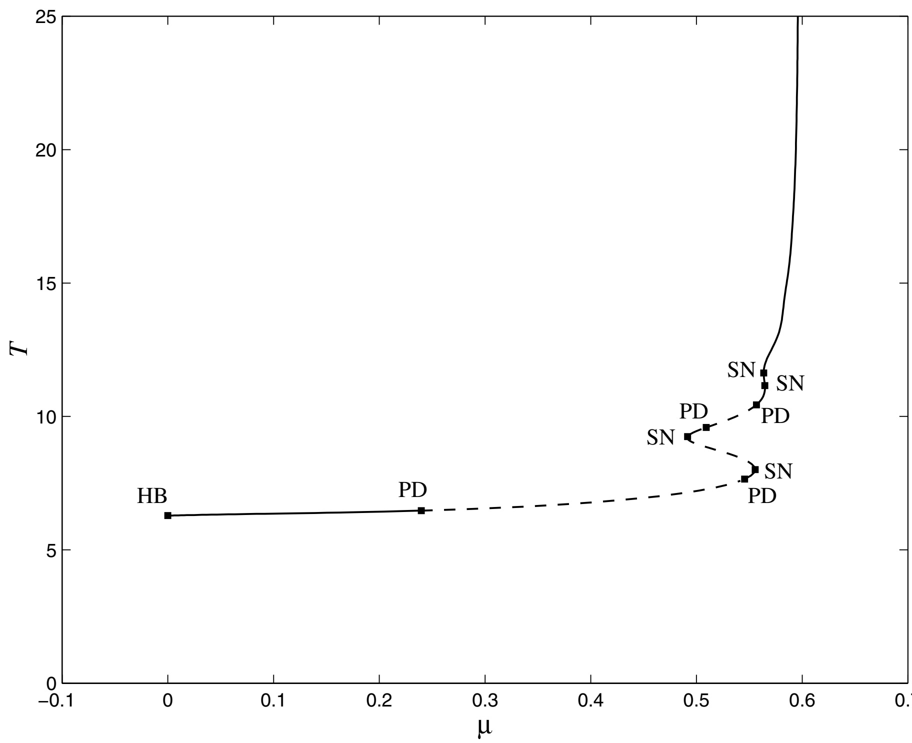

Fig. 2. Continuation of the limit cycle arising at the Hopf bifurcation for the system with a quadratic nonlinearity using AUTO program with $c = 2$. T is the period of the oscillation. — Stable limit cycle; - - unstable limit cycle.

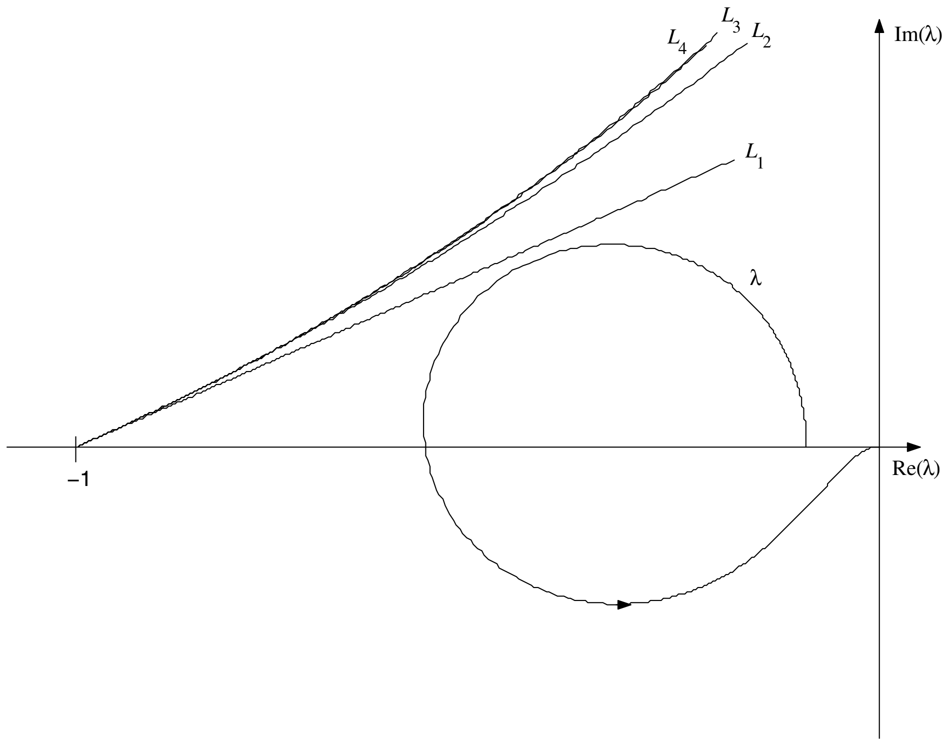

Fig. 3. Characteristic gain locus $\hat{\lambda}$ and the amplitude loci $L_1, \ldots, L_4$ when there are no tranversal intersection, $\mu = 0.265$, $c = 2.6182$ for the system with a quadratic nonlinearity.

Notice that $\lambda_1 \approx 1$ and λ_2 approaches -1 (PD condition) but for $\mu = 0.2593$, it is not possible to solve Eq. (3) for the sixth- and eighth-order HBA (L_3 and L_4, respectively) and therefore it is not possible to compute the monodromy matrix. The situation is worst if the value of c is increased, until there are no intersections with any of the curves L_i, as shown in Fig. 3. This is considered as the limit of the approximation of the method since the amplitude of the limit cycle is far away from criticality and then the results are not local. Finally, it is worth mentioning that a quite similar behavior has been analyzed by other researchers with the connection of degenerate Hopf and pitchfork bifurcations (a type of Gavrilov–Guckenheimer singularity) as reported in [Algaba *et al.*, 1999, 2000]. In particular, in [Algaba *et al.*, 2000] the appearance of period-doubling and torus bifurcations of the cycles takes place near the singularity too.

Table 2. Eigenvalues of the monodromy matrix for $c = 2.6182$ for the system with a quadratic nonlinearity.

μ	λ_1	λ_2
0.259	1.0004	−0.979
0.2592	1.0004	−0.980257
0.2593	—	—

3.2. *Cubic nonlinearity*

Let us now consider the modified Rössler system with a cubic nonlinearity, i.e. $\varphi(x_1) = x_1^3$. The numerical analysis performed with AUTO exhibits that the limit cycle emerging at the Hopf bifurcation at the origin undergoes a pitchfork bifurcation (PB). This curve is depicted in Fig. 4 as a solid line.

The proposed technique detects the PB from the origin until $c = 1.9065$ with $\mu = 0.7665$. Beyond this point, this technique fails but in a different way compared to the failure of the first example. Here, all the L_i are obtained but the critical multiplier, computed with the semi-analytical technique, does not cross the unit circle at $+1$ as is required for the PB bifurcation. In addition, the method is not capable of detecting other secondary bifurcations of the cycle. However, we do not emphasize this fact since the purpose of the paper is to describe the first bifurcation of the cycles close to the singularity. As in the first example, we fix a value of c and look for a value of μ in order to detect the PB condition by using HBA (points marked by asterisks in Fig. 4).

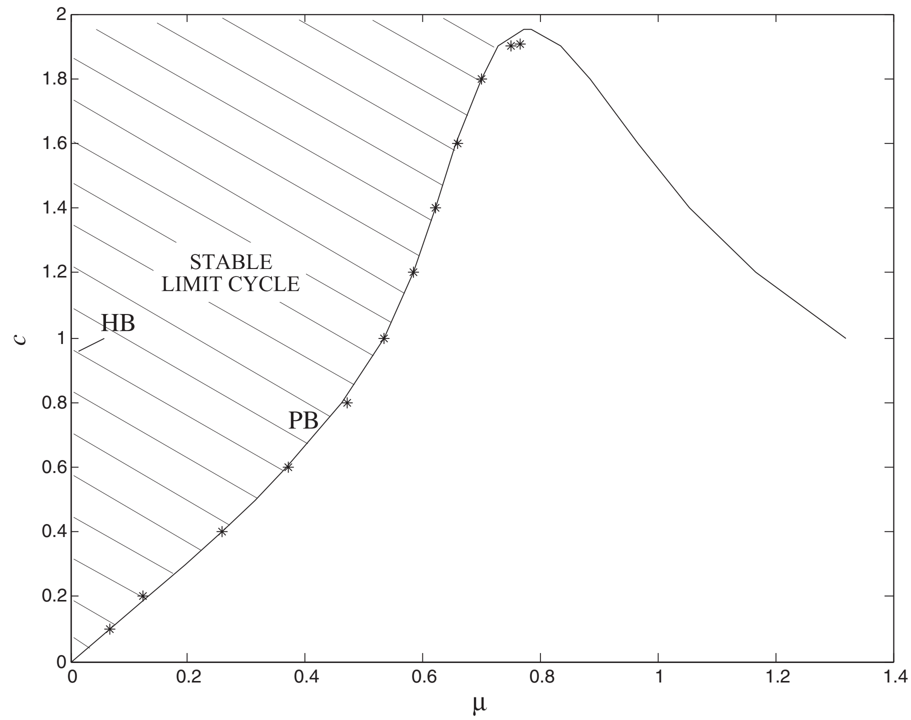

Fig. 4. First limit cycle bifurcation curves for the system with a cubic nonlinearity. The origin in the parameter space corresponds to the condition of the Gavrilov–Guckenheimer bifurcation. The asterisks correspond to eighth-order HBA which are contrasted to the solutions obtained with AUTO (solid line).

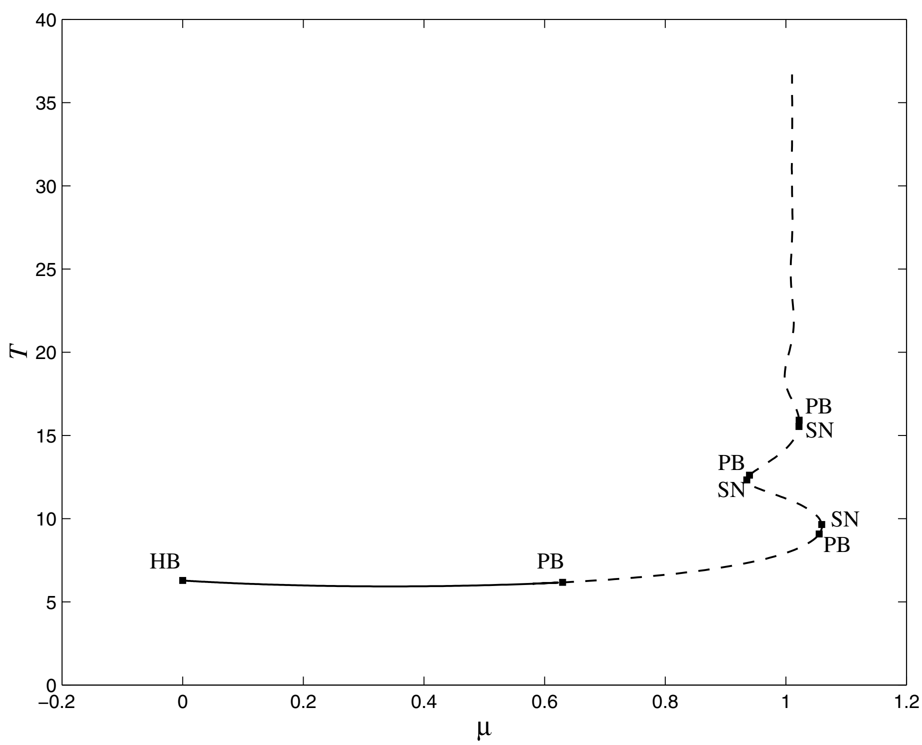

Fig. 5. Continuation of the limit cycle arising in the Hopf bifurcation at the origin for the system with a cubic nonlinearity using AUTO program with parameter $c = 1.4$. T is the period of the oscillation. — Stable limit cycle; -- unstable limit cycle.

Notice that in this example between the Hopf and the PB bifurcation curves there is one stable limit cycle encircling the origin whose amplitude increases as parameter μ moves from Hopf to PB. Two stable limit cycles emerge to the right of PB and the existing one changes its stability. The current approach detects the stability change of the main cycle but the stability of the bifurcated cycles (or, equivalently, the direction of the pitchfork bifurcation) can be determined via other techniques such as the Poincaré map, numerical simulation or continuation of the periodic branches via specific software like AUTO, LOCBIF, etc.

Finally, Fig. 5 shows the dynamical scenario obtained with AUTO by varying μ with a fixed value of $c = 1.4$. The cycle exhibits other secondary bifurcations in addition to PB, like period doubling (PD) or saddle-node (SN) bifurcations. As we have discussed before, these bifurcations are far away from the original Hopf bifurcation as well as from the equilibrium point, and they are impossible to detect with the proposed method due to the large errors in the calculation of the multipliers. This observation is useful to determine the limit of the HBA. Nevertheless, the method contributes to extend the local results of the classical Hopf bifurcation approach.

4. Conclusions

In this paper, the potentiality of a semi-analytical method for detecting the first bifurcation of limit cycles is illustrated with two examples. The proposed method detects Neimark–Saker, period doubling and pitchfork bifurcations of cycles accurately even when the cycle is not so close to the Hopf bifurcation point. This technique seems to be very powerful in analyzing the unfoldings of certain singularities involving limit cycle bifurcations, such as Gavrilov–Guckenheimer bifurcation, a triple-zero bifurcation, double Hopf bifurcation and so on.

Acknowledgments

The authors acknowledge the support of the CONICET, the Universidad Nacional del Sur and ANPCyT through grant PICT-2002-11-12524.

References

Algaba, A., Freire, E., Gamero, E. & Rodríguez-Luis, A. J. [1999] "On a codimension-three unfolding of the interaction of degenerate Hopf and pitchfork bifurcations," *Int. J. Bifurcation and Chaos* **9**, 1333–1362.

Algaba, A., Merino, M., Freire, E., Gamero, E. & Rodríguez-Luis, A. J. [2000] "On the Hopf-pitchfork bifurcation in the Chua's equation," *Int. J. Bifurcation and Chaos* **10**, 291–305.

Berns, D. W., Moiola, J. L. & Chen, G. [2001] "Detecting period-doubling bifurcation: An approximate monodromy matrix approach," *Automatica* **37**, 1787–1795.

Choe, W. G. & Guckenheimer, J. [1999] *Computing Periodic Orbits with High Accuracy*, Computer Methods in Applied Mechanics and Engineering, Vol. 170, pp. 331–341.

Doedel, E. J., Champneys, A. R., Fairgrieve, T. F., Kuznetsov, Yu, A., Sandstede, B. & Wang, X. J. [1997] "AUTO 97: Continuation and bifurcation software for ordinary differential equations," (available via FTP at: ftp.cs.concordia.ca/pub/doedel/auto), Concordia University, Montreal, Canada.

Ermentrout, B. [2001] "XPPAUT5.0 — The Differential Equation Tool" (available via http://www.math.pitt.edu/∼bard/xpp/xpp.html), University of Pittsburg, Pittsburg, USA.

Guckenheimer, J. & Meloon, B. [2000] "Computing periodic orbits and their bifurcations with automatic differentiation," *SIAM J. Sci. Comput.* **22**, 951–985.

Khibnik, A. I., Kuznetsov, Yu. A., Levitin, V. V. & Nikolaev, E. V. [1993] "Continuation techniques and interactive software for bifurcation analysis of ODE's and iterated maps," *Physica* **D62**, 360–371.

Kuznetsov, Y. A. [1995] *Elements of Applied Bifurcation Theory*, Applied Mathematical Sciences, Vol. 112 (Springer-Verlag).

Lust, K. [2001] "Improved numerical Floquet multipliers," *Int. J. Bifurcation and Chaos* **11**, 2389–2410.

Mees, A. I. & Chua, L. O. [1979] "The Hopf bifurcation theorem and its applications to nonlinear oscillations in circuits and systems," *IEEE Trans. Circuits Syst.* **26**, 235–254.

Mees, A. I. [1981] *Dynamics of Feedback Systems* (John Wiley & Sons, Chichester, UK).

Moiola, J. L. & Chen, G. [1996] *Hopf Bifurcation Analysis – A Frequency Domain Approach*, World Scientific Series on Nolinear Science, Series A, Vol. 21 (World Scientific, Singapore).

Robbio, F., Moiola, J. L. & Alonso, D. [2004] "On semi-analytical procedure for detecting limit cycle bifurcations," *Int. J. Bifurcation and Chaos* **14**, 951–970.

Thomas, R. [1999] "Deterministic chaos seen in terms of feedback circuits: Analysis, synthesis, 'labyrinth chaos'," *Int. J. Bifurcation and Chaos* **9**, 1889–1905.

Torrini, G., Genesio, R. & Tesi, A. [1998] "On the computation of characteristic multipliers for predicting limit cycle bifurcations," *Chaos Solit. Fract.* **9**, 121–133.