Available online at www.sciencedirect.com
SCIENCE DIRECT®

J. Math. Anal. Appl. 309 (2005) 709–723

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Nullspaces and frames ☆

J. Antezana [a,b], G. Corach [b,c,*], M. Ruiz [a,b], D. Stojanoff [a,b]

[a] *Departamento. de Matemática, FCE-UNLP, La Plata, Argentina*
[b] *IAM-CONICET, Saavedra 15, Piso 3 (1083), Ciudad Autónoma de Buenos Aires, Argentina*
[c] *Departamento de Matemática, FI-UBA, Buenos Aires, Argentina*

Received 17 September 2004

Available online 20 April 2005

Submitted by P.G. Casazza

Abstract

In this paper we give new characterizations of Riesz and conditional Riesz frames in terms of the properties of the nullspace of their synthesis operators. On the other hand, we also study the oblique dual frames whose coefficients in the reconstruction formula minimize different weighted norms.
© 2004 Elsevier Inc. All rights reserved.

Keywords: Frames; Generalized inverses; Riesz frames; Angles

1. Introduction

Frames were introduced by Duffin and Schaeffer [16] in the context of nonharmonic Fourier series, and they have been intensively applied in wavelet and frequency analysis theories since the work of Daubechies et al. [14]. Today, frame-like expansions are fundamental in a wide range of disciplines (see, for example, [16,17] or [25]), including the analysis and design of oversampled filter banks and error corrections codes.

☆ Partially supported by CONICET (PIP 2083/00), UBACYT I030, UNLP (11 X350) and ANPCYT (PICT03-9521).
* Corresponding author.
E-mail addresses: antezana@mate.unlp.edu.ar (J. Antezana), gcorach@fi.uba.ar (G. Corach), mruiz@mate.unlp.edu.ar (M. Ruiz), demetrio@mate.unlp.edu.ar (D. Stojanoff).

0022-247X/$ – see front matter © 2004 Elsevier Inc. All rights reserved.
doi:10.1016/j.jmaa.2004.11.041

A frame is a redundant set of vectors in a Hilbert space that leads to expansions of vectors (signals) in terms of the frame elements. More precisely, a sequence of vectors $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ in a (separable) Hilbert space $\mathcal{H}$ is a *frame* (for $\mathcal{H}$) if there exist numbers $A, B > 0$ such that, for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{N}} \left|\langle f, f_n\rangle\right|^2 \leqslant B\|f\|^2. \tag{1}$$

Associated with each frame there exists an operator $T : \ell^2 \to \mathcal{H}$ defined by $T(e_n) = f_n$, where $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ denotes the canonical basis of ℓ^2; T is called the synthesis operator of $\mathcal{F}$.

The results of this paper can be divided in two parts. The main results of the first part are devoted to the study of Riesz frames and conditional Riesz frames through the structure and geometric properties of the nullspace of their synthesis operators. Riesz and conditional Riesz frames were introduced by Christensen in [9] (see definitions in Section 3). These frames are important because they behave well with respect to the projection method. In general, frame theory describes how to choose the corresponding coefficients to expand a given vector in terms of the frame vectors. However, in applications, to obtain these coefficient requires the inversion of an operator on $\mathcal{H}$. The projection method was introduced by Christensen in [7] to avoid this problem. We refer the interested reader to [6], [7], [8], [9] or [10] for more information about the projection method. In [1] we found a characterization of Riesz frames by studying the nullspace of the synthesis operator. Namely, if the nullspace $N(T)$ has a certain geometric property of compatibility with the closed subspaces spanned by subsets of $\mathcal{B}$, then $\mathcal{F}$ is a Riesz frame, and conversely. In Section 3, we extend these results for conditional Riesz frames and give some new characterizations in terms of angles.

Throughout the second part of this work we study the so-called oblique dual frames. Let $\{f_n\}_{n\in\mathbb{N}}$ be a frame for the closed subspace $\mathcal{W} \subseteq \mathcal{H}$, and let $\mathcal{M} \subseteq \mathcal{H}$ be another closed subspace such that $\mathcal{H} = \mathcal{W} \dotplus \mathcal{M}^\perp$ ($\dotplus$ means a nonnecessarily orthogonal direct sum). The sequence $\{g_n\}_{n\in\mathbb{N}}$ in $\mathcal{M}$ is an oblique dual frame of $\{f_n\}_{n\in\mathbb{N}}$ (see Li [21] or Li and Ogawa [22,23]) if

$$f = \sum_{n=1}^{\infty} \langle f, g_n\rangle f_n \quad \forall f \in \mathcal{W}.$$

Among the oblique dual frames, there exists a particular class with the minimal norm property. Recall that a dual frame $\{g_n\}_{n\in\mathbb{N}}$ has the minimal norm property if the coefficients $\{\langle f, g_n\rangle\}_{n\in\mathbb{N}}$ that appear in the reconstruction formula have minimal ℓ^2 norm.

If $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ denotes the canonical orthonormal basis for ℓ^2 and T is the synthesis operator of $\{f_n\}_{n\in\mathbb{N}}$, then Christensen and Eldar [11] proved that the minimal norm oblique dual frames have the form

$$\{g_n\}_{n\in\mathbb{N}} = \left\{B(T^*B)^\dagger e_n\right\}_{n\in\mathbb{N}},$$

where B is any bounded operator with $R(B) = \mathcal{M}$. From the point of view of sampling theory, the operator B can be interpreted as the synthesis operator associated to the frame used to sample the signals.

In this work, we are interested in duals frames which lead to reconstruction coefficients that have minimal norm, but with respect to some weighted norms. Recall that weighted norms in ℓ^2 arise from inner products obtained by perturbing the original one with invertible positive operators which are diagonal in the canonical basis. In Section 4 we give explicit formulae for dual frames which minimize a given weighted norm, and we prove that in the case of Riesz frames, if the sampling frame is fixed, then the norms of the synthesis operators corresponding to the dual frames which minimize the different weighted norms are uniformly bounded from above.

We thank Ole Christensen for his useful comments.

2. Preliminaries

Let $\mathcal{H}$ be a separable Hilbert space and $L(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$. $Gl(\mathcal{H})$ denotes the group of invertible operators in $L(\mathcal{H})$, and $Gl(\mathcal{H})^+$ the set of positive definite invertible operators on $\mathcal{H}$. For an operator $A \in L(\mathcal{H})$, $R(A)$ denotes the range of A, $N(A)$ the nullspace of A, $\sigma(A)$ the spectrum of A, A^* the adjoint of A, $\rho(A)$ the spectral radius of A and $\|A\|$ the operator norm of A; if $R(A)$ is closed, $A^\dagger$ is the Moore–Penrose pseudoinverse of A. We use the fact that A is an isometry (respectively coisometry) if $A^*A = I$ (respectively $AA^* = I$). Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_\mathcal{M}$ denotes the orthogonal (i.e., selfadjoint) projection onto $\mathcal{M}$. If $B \in L(\mathcal{H})$ satisfies $P_\mathcal{M} B P_\mathcal{M} = B$, we consider the compression of B to $\mathcal{M}$ (i.e., the restriction of B to $\mathcal{M}$, which is an operator on $\mathcal{M}$), and we say that we consider B as *acting* on $\mathcal{M}$. Given a subspace $\mathcal{M}$ of $\mathcal{H}$, its unit ball is denoted by $\mathcal{M}_1$, and its closure by $\bar{\mathcal{M}}$ or $\mathrm{cl}(\mathcal{M})$. If $\mathcal{N}$ is another subspace of $\mathcal{H}$, we denote $\mathcal{M} \ominus \mathcal{N} := \mathcal{M} \cap \mathcal{N}^\perp$. If $\mathcal{M} \cap \mathcal{N} = \{0\}$, we denote by $\mathcal{M} \dot{+} \mathcal{N}$ the (direct) sum of the two subspaces. If the sum is orthogonal, we write $\mathcal{M} \oplus \mathcal{N}$. The distance between two subsets $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ is $\mathrm{d}(\mathcal{M}, \mathcal{N}) = \inf\{\|x - y\|:\ x \in \mathcal{M},\ y \in \mathcal{N}\}$.

2.1. *Angle between closed subspaces*

We shall recall the definition of angle between closed subspaces of $\mathcal{H}$. We refer the reader to the nice survey of Deutsch [15] and the books by Kato [19] and Havin and Jöricke [18] for details and proofs.

Definition 2.1. Given two closed subspaces $\mathcal{M}$ and $\mathcal{N}$, let $\tilde{\mathcal{N}} = \mathcal{N} \ominus (\mathcal{M} \cap \mathcal{N})$ and $\tilde{\mathcal{M}} = \mathcal{M} \ominus (\mathcal{M} \cap \mathcal{N})$. The *angle* between $\mathcal{M}$ and $\mathcal{N}$ is the angle in $[0, \pi/2]$ whose cosine is

$$c[\mathcal{M}, \mathcal{N}] = \sup\bigl\{\bigl|\langle x, y\rangle\bigr|:\ x \in \tilde{\mathcal{M}},\ y \in \tilde{\mathcal{N}} \text{ and } \|x\| = \|y\| = 1\bigr\}.$$

The *sine* of this angle is denoted by $s[\mathcal{M}, \mathcal{N}]$.

Now, we state some known results concerning angles and closed range operators (see [15]).

Proposition 2.2. *Let $\mathcal{M}$ and $\mathcal{N}$ be two closed subspaces of $\mathcal{H}$. Then*

(1) $c[\mathcal{M},\mathcal{N}] = c[\mathcal{N},\mathcal{M}] = c[\tilde{\mathcal{M}},\mathcal{N}] = c[\mathcal{M},\tilde{\mathcal{N}}]$.
(2) $c[\mathcal{M},\mathcal{N}] < 1$ *if and only if* $\mathcal{M}+\mathcal{N}$ *is closed.*
(3) $c[\mathcal{M},\mathcal{N}] = c[\mathcal{M}^\perp,\mathcal{N}^\perp]$.
(4) $c[\mathcal{M},\mathcal{N}] = \|P_\mathcal{M} P_{\tilde{\mathcal{N}}}\| = \|P_{\tilde{\mathcal{M}}} P_\mathcal{N}\| = \|P_\mathcal{M} P_\mathcal{N} P_{(\mathcal{M}\cap\mathcal{N})^\perp}\| = \|P_\mathcal{M} P_\mathcal{N} - P_{\mathcal{M}\cap\mathcal{N}}\|$.

Proposition 2.3 (Bouldin [2]; see also [15]). *Let $A, B \in L(\mathcal{H})$ such that $R(A)$ and $R(B)$ are closed. Then, AB has closed range if and only if $c[R(B), N(A)] < 1$.*

Proposition 2.4 (Kayalar and Weinert [20]; see also [15]). *Let P and Q be two orthogonal projections defined on $\mathcal{H}$. Then,*

$$\left\|(PQ)^k - P \wedge Q\right\| = c\big[R(P), R(Q)\big]^{2k-1},$$

where $P \wedge Q$ is the orthogonal projection onto $R(P) \cap R(Q)$.

Finally, we give a characterization of $s[\mathcal{M},\mathcal{N}]$ in terms of distances.

Proposition 2.5. *Let $\mathcal{M}$ and $\mathcal{N}$ be to closed subspaces of $\mathcal{H}$. Denote $\tilde{\mathcal{N}} = \mathcal{N} \ominus (\mathcal{M}\cap\mathcal{N})$ and $\tilde{\mathcal{M}} = \mathcal{M} \ominus (\mathcal{M}\cap\mathcal{N})$. Then,*

$$s[\mathcal{M},\mathcal{N}] = \mathrm{d}(\tilde{\mathcal{M}}_1, \mathcal{N}) = \mathrm{d}(\tilde{\mathcal{N}}_1, \mathcal{M}).$$

Proof. By Proposition 2.2, we can suppose that $\mathcal{M}\cap\mathcal{N} = \{0\}$, i.e., $\mathcal{M} = \tilde{\mathcal{M}}$. By the definition of the sine and Proposition 2.2, $s[\mathcal{M},\mathcal{N}]^2 = 1 - \|P_\mathcal{M} P_\mathcal{N}\|^2$. On the other hand, as $\mathrm{d}(x,\mathcal{N}) = \|P_{\mathcal{N}^\perp} x\|$ for every $x \in \mathcal{H}$, we have that

$$\begin{aligned}\mathrm{d}(\mathcal{M}_1,\mathcal{N})^2 &= \inf\big\{\|P_{\mathcal{N}^\perp} x\|^2\colon\ x \in \mathcal{M}_1\big\} = \inf\big\{1 - \|P_\mathcal{N} x\|^2\colon\ x \in \mathcal{M}_1\big\}\\ &= 1 - \sup\big\{\|P_\mathcal{N} x\|^2\colon\ x \in \mathcal{M}_1\big\} = 1 - \|P_\mathcal{N} P_\mathcal{M}\|^2\\ &= 1 - \|P_\mathcal{M} P_\mathcal{N}\|^2. \qquad \square\end{aligned}$$

2.2. The reduced minimum modulus

Definition 2.6. The *reduced minimum modulus* $\gamma(T)$ of an operator $T \in L(\mathcal{H})$ is defined by

$$\gamma(T) = \inf\big\{\|Tx\|\colon\ \|x\| = 1,\ x \in N(T)^\perp\big\}. \tag{2}$$

It is well known that $\gamma(T) = \gamma(T^*) = \gamma(T^*T)^{1/2}$. Also, it can be shown that an operator T has closed range if and only if $\gamma(T) > 0$. In this case, $\gamma(T) = \|T^\dagger\|^{-1}$.

The following result is an easy consequence of Eq. (2).

Lemma 2.7. *Let $B \in L(\mathcal{H})$ with B invertible. Then,*

$$\|B^{-1}\|^{-1}\gamma(T) \leqslant \gamma(BT) \leqslant \|B\|\gamma(T).$$

Moreover, the same formula follows, replacing $\|B^{-1}\|^{-1}$ by $\gamma(B)$, if $R(B)$ is closed and $R(T) \subseteq N(B)^\perp$.

Lemma 2.8. *Let $T \in L(\mathcal{H})$ be a partial isometry (i.e., TT^* is a projection), $\mathcal{M}$ a closed subspace of $\mathcal{H}$ and $P_{\mathcal{M}}$ the orthogonal projection onto $\mathcal{M}$. Then*

$$\gamma(TP_{\mathcal{M}}) = s\big[N(T), \mathcal{M}\big].$$

Proof. Denote $\mathcal{N} = N(T)$ and $\mathcal{R} = \mathcal{N}^{\perp}$. Since T acts isometrically on $\mathcal{R}$, it is clear by Eq. (2) that

$$\gamma(TP_{\mathcal{M}}) = \gamma(TP_{\mathcal{R}}P_{\mathcal{M}}) = \gamma(P_{\mathcal{R}}P_{\mathcal{M}}).$$

Since $N(P_{\mathcal{R}}P_{\mathcal{M}}) = \mathcal{M}^{\perp} \oplus (\mathcal{M} \cap \mathcal{N})$, it follows that $N(P_{\mathcal{R}}P_{\mathcal{M}})^{\perp} = \mathcal{M} \cap (\mathcal{M} \cap \mathcal{N})^{\perp} = \tilde{\mathcal{M}}$. Then, by Proposition 2.5,

$$\gamma(P_{\mathcal{R}}P_{\mathcal{M}}) = \inf_{x \in \tilde{\mathcal{M}}_1} \|P_{\mathcal{R}}x\| = \inf_{x \in \tilde{\mathcal{M}}_1} \mathrm{d}(x, \mathcal{N}) = \mathrm{d}(\tilde{\mathcal{M}}_1, \mathcal{N}) = s[\mathcal{N}, \mathcal{M}]. \qquad \square$$

The next result was proved in [1]. We include a short proof for the sake of completeness.

Proposition 2.9. *If $T \in L(\mathcal{H})$ has closed range and $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ such that $c[N(T), \mathcal{M}] < 1$ (so that $TP_{\mathcal{M}}$ has closed range), then*

$$\gamma(T)s\big[N(T), \mathcal{M}\big] \leqslant \gamma(TP_{\mathcal{M}}) \leqslant \|T\|s\big[N(T), \mathcal{M}\big]. \tag{3}$$

Proof. Take $B = |T^*| = (TT^*)^{1/2}$. It is well known that $R(B) = R(T)$ which is closed by hypothesis. It is easy to see that $\gamma(T) = \gamma(B)$ and $\|B\| = \|T\|$. Also, $B^{\dagger}T$ is a coisometry, with the same nullspace as T. So, by Lemma 2.8, $\gamma(B^{\dagger}TP_{\mathcal{M}}) = s[N(T), \mathcal{M}]$. Now, using Lemma 2.7 for B and $B^{\dagger}TP_{\mathcal{M}}$ and the fact that $BB^{\dagger}TP_{\mathcal{M}} = P_{R(T)}TP_{\mathcal{M}} = TP_{\mathcal{M}}$, we get

$$\gamma(T)s\big[N(T), \mathcal{M}\big] \leqslant \gamma(TP_{\mathcal{M}}) \leqslant \|T\|s\big[N(T), \mathcal{M}\big],$$

because $R(B) = R(B^{\dagger})$, so that $R(B^{\dagger}TP_{\mathcal{M}}) \subseteq R(B) = N(B)^{\perp}$. $\square$

Remark 2.10. With the same ideas, the following formulae generalizing Lemma 2.8 and Proposition 2.9, can be proved.

(1) Let $U, V \in L(\mathcal{H})$ be partial isometries. Then, $\gamma(UV) = s[N(U), R(V)]$.
(2) If $A, B \in L(\mathcal{H})$ have closed ranges, then

$$\gamma(A)\gamma(B)s\big[N(A), R(B)\big] \leqslant \gamma(AB) \leqslant \|A\|\|B\|s\big[N(A), R(B)\big].$$

Note that the first inequality implies Proposition 2.3.

In particular, this gives the following formula for the sine of an angle: given $\mathcal{M}$ and $\mathcal{N}$ two closed subspaces of $\mathcal{H}$, it holds

$$s[\mathcal{N}, \mathcal{M}] = \gamma(P_{\mathcal{N}^{\perp}}P_{\mathcal{M}}).$$

2.3. Frames

We introduce some basic facts about frames in Hilbert spaces. For complete descriptions of frame theory and applications, the reader is referred to the survey by Heil and Walnut [17] or the books by Young [25] and Christensen [10].

Definition 2.11. Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ a sequence in $\mathcal{H}$.

(1) $\mathcal{F}$ is called a *frame* if there exist numbers $A, B > 0$ such that, for every $f \in \mathcal{H}$,

$$A\|f\|^2 \leqslant \sum_{n\in\mathbb{N}} \big|\langle f, f_n\rangle\big|^2 \leqslant B\|f\|^2. \tag{4}$$

(2) The optimal constants A, B for Eq. (4) are called the *frame bounds* for $\mathcal{F}$.
(3) The frame $\mathcal{F}$ is called *tight* if $A = B$, and *Parseval* if $A = B = 1$.
(4) Associated with $\mathcal{F}$ there exist an operator $T : \ell^2 \to \mathcal{H}$ such that $T(e_n) = f_n$, where $\{e_n\}_{n\in\mathbb{N}}$ denotes the canonical basis of ℓ^2. This operator is called the *synthesis* operator of $\mathcal{F}$. For finite frames we assume that the domain of the synthesis operator is $\mathbb{C}^m$, where m is the number of vectors of the frame.

Remark 2.12. Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame in $\mathcal{H}$ and T its synthesis operator.

(1) The frame bounds of $\mathcal{F}$ can be computed in terms of the synthesis operator

$$A = \gamma(T)^2 \quad \text{and} \quad B = \|T\|^2. \tag{5}$$

(2) The adjoint $T^* \in L(\mathcal{H}, \ell^2)$ of T, is given by $T^*(x) = \sum_{n\in\mathbb{N}} \langle x, f_n\rangle e_n$, $x \in \mathcal{H}$. It is called the *analysis operator* for $\mathcal{F}$.
(3) The operator $S = TT^*$ is usually called frame operator and it is easy to see that

$$Sf = \sum_{n\in\mathbb{N}} \langle f, f_n\rangle f_n, \quad f \in \mathcal{H}. \tag{6}$$

It follows from (4) that $A.I \leqslant S \leqslant B.I$, so that $S \in Gl(\mathcal{H})^+$. Moreover, the optimal constants A, B for Eq. (4) are

$$B = \|S\| = \rho(S) \quad \text{and} \quad A = \gamma(S) = \|S^{-1}\|^{-1} = \min\big\{\lambda\colon \lambda \in \sigma(S)\big\}.$$

Finally, from (6) we get

$$f = \sum_{n\in\mathbb{N}} \langle f, S^{-1} f_n\rangle f_n \quad \forall f \in \mathcal{H}.$$

(4) The numbers $\{\langle f, S^{-1} f_n\rangle\}$ are called the *frame coefficients* of f. They have the following optimal property: if $f = \sum_{n\in\mathbb{N}} c_n f_n$, for a sequence $(c_n)_{n\in\mathbb{N}}$, then

$$\sum_{n\in\mathbb{N}} \big|\langle f, S^{-1} f_n\rangle\big|^2 \leqslant \sum_{n\in\mathbb{N}} |c_n|^2.$$

The frame $\{S^{-1} f_n\}_{n\in\mathbb{N}}$ is called canonical dual frame. We shall return to dual frames in Section 4.

3. Riesz frames and conditional Riesz frames

It was remarked by Christensen [10, p. 65], that given a frame $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$, in practice it can be difficult to use the frame decomposition $f = \sum\langle f, S^{-1} f_n\rangle f_n$ because it requires the calculation of S^{-1} or, at least, the frame coefficients $\langle f, S^{-1} f_n\rangle$. In order to get some of the advantages of Riesz bases, Christensen introduced in [7] the *projection method*, approximating S and S^{-1} by finite rank operators, acting on certain finite dimensional spaces $\mathcal{H}_n$ approaching $\mathcal{H}$. Later on, Christensen [9] introduced two special classes of frames, namely *Riesz frames* and *conditional Riesz frames*, which are well adapted to some of these problems (see also [3–5]).

We need to fix some notations: Let $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be the canonical orthonormal basis of ℓ^2 and $I \subseteq \mathbb{N}$.

(1) We denote $\mathcal{M}_I = \overline{\operatorname{span}}\{e_n\colon n \in I\}$ and $P_I = P_{\mathcal{M}_I}$, the orthogonal projection onto $\mathcal{M}_I$.
(2) If $I = \mathbb{I}_n := \{1, 2, \dots, n\}$, we put $\mathcal{M}_n$ for $\mathcal{M}_I$.
(3) Given $\mathcal{N}$ a closed subspace of ℓ^2, we denote $\mathcal{N}_n = \mathcal{N} \cap \mathcal{M}_n$, $n \in \mathbb{N}$.
(4) If $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is a frame for $\mathcal{H}$, we denote by $\mathcal{F}_I = \{f_n\}_{n\in I}$.
(5) We say that $\mathcal{F}_I$ is a *frame sequence* if it is a frame for $\overline{\operatorname{span}}\{\mathcal{F}_I\}$.
(6) $\mathcal{F}_I$ is called a *subframe* of $\mathcal{F}$ if it is itself a frame for $\mathcal{H}$.

Recall the definitions of Riesz frames and conditional Riesz frames.

Definition 3.1. A frame $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ is called a *Riesz frame* if there exists $A, B > 0$ such that, for every $I \subset \mathbb{N}$, the subfamily $\mathcal{F}_I$ is a frame sequence with bounds A, B (not necessarily optimal).

The sequence $\mathcal{F}$ is called a *conditional Riesz frame* if there are common bounds for the frame sequences $\mathcal{F}_{I_n}$, where $\{I_n\}_{n=1}^{\infty}$ is a sequence of finite subsets of $\mathbb{N}$ such that $I_n \subseteq I_{n+1}$ for every $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$.

Remark 3.2. Let $\mathcal{F}$ be a frame, and T its synthesis operator. Given $I \subseteq \mathbb{N}$, then $\mathcal{F}_I$ is a frame sequence if and only if $R(TP_I)$ is closed, and $\mathcal{F}_I$ is a subframe if and only if $R(TP_I) = \mathcal{H}$. In both cases the frame bounds for $\mathcal{F}_I$ are $A = \gamma(TP_I)^2$ and $B = \|TP_I\|^2$. Using these facts we get an equivalent definition of Riesz frames: $\mathcal{F}$ is a Riesz frame if there exists $\varepsilon > 0$ such that $\gamma(TP_I) \geqslant \varepsilon$ for every $I \subseteq \mathbb{N}$.

Proposition 2.9 can be used to characterize Riesz frames in terms of the angles between the nullspace of the synthesis operator T and the closed subspaces of ℓ^2 which are spanned by subsets of $\mathcal{B}$.

Proposition 3.3. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame, and T be its synthesis operator. Let $\mathcal{N} = N(T)$. Then $\mathcal{F}$ is a Riesz frame if and only if*

$$c = \sup_{I\subseteq\mathbb{N}} c[\mathcal{N}, \mathcal{M}_I] < 1. \tag{7}$$

Proof. By Proposition 2.3, TP_I has closed range iff $c[\mathcal{N}, \mathcal{M}_I] < 1$. By Proposition 2.9, $\gamma(TP_I)$ has an uniform lower bound if and only ifthere exists a constant $c < 1$ such that, for every $I \subseteq \mathbb{N}$, $c[\mathcal{N}, \mathcal{M}_I] \leqslant c$. $\square$

Remark 3.4. Let $\mathcal{N}$ be a closed subspace of ℓ^2 and $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ be the canonical orthonormal basis of ℓ^2. If Eq. (7) holds, following the terminology of [1], we say that $\mathcal{N}$ is *$\mathcal{B}$-compatible*.

In the following proposition, we state a characterization of $\mathcal{B}$-compatible subspaces of $\mathcal{H}$, proved in [1].

Proposition 3.5. *Let $\mathcal{N}$ be a closed subspace of ℓ^2 and let $\mathcal{B} = \{e_k\}_{k\in\mathbb{N}}$ be the canonical orthonormal basis of ℓ^2. For $n \in \mathbb{N}$, denote by $c_n = \sup_{J\subseteq\mathbb{I}_n} c[\mathcal{N}_n, \mathcal{M}_J]$. Then the following conditions are equivalent*:

(1) *$\mathcal{N}$ is $\mathcal{B}$-compatible.*
(2) $c = \sup_{n\in\mathbb{N}} c[\mathcal{N}, \mathcal{M}_n] < 1$, *and* $\sup_{n\in\mathbb{N}} c_n < 1$.
(3) $\mathrm{cl}(\bigcup_{n\in\mathbb{N}} \mathcal{N}_n) = \mathcal{N}$ *and* $\sup_{n\in\mathbb{N}} c_n < 1$.
(4) *There exists a constant $c < 1$ such that $c[\mathcal{N}, \mathcal{H}_I] \leqslant c$ for every finite subset I of $\mathbb{N}$ with $\mathcal{N} \cap \mathcal{M}_I = \{0\}$.*

Proposition 3.5 can be "translated" to frame language to get a characterization of Riesz frames, similar to the one obtained by Christensen and Lindner in [13]:

Theorem 3.6. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame and T its synthesis operator. Denote $\mathcal{N} = N(T)$. Then the following conditions are equivalent*:

(1) *$\mathcal{F}$ is a Riesz frame.*
(2) *$\mathcal{N}$ is $\mathcal{B}$-compatible.*
(3) *There exists an uniform lower frame bound for every finite linearly independent frame sequence $\mathcal{F}_J$, $J \subset \mathbb{N}$.*
(4) *There exists $d > 0$ such that $\gamma(TP_J) \geqslant d$, for every $J \in \mathbb{N}$ finite such that $\mathcal{N} \cap \mathcal{M}_J = \{0\}$.*

Proof. If I is a finite subset of $\mathbb{N}$ then $\mathcal{M}_I \cap \mathcal{N} = \{0\}$ if and only if $\mathcal{F}_I$ is linearly independent. Then, conditions (3) and (4) are equivalent. By Propositions 2.9 and 3.5, they are also equivalent to the $\mathcal{B}$-compatibility of $\mathcal{N}$.

Suppose that there exists a constant d such that $0 < d \leqslant \gamma(TP_{\mathcal{M}_I})$ for every finite subset $I \subseteq \mathbb{N}$ such that $\mathcal{M}_I \cap \mathcal{N} = \{0\}$. This is equivalent to saying that there is a constant $c < 1$ such that $c[\mathcal{N}, \mathcal{M}_I] \leqslant c$ for such kind of sets I. Using Propositions 3.3 and 3.5, we conclude that $\mathcal{F}$ is a Riesz frame. The converse is clear. $\square$

Now, we consider conditional Riesz frames. First of all, we state a result for this class of frames which is similar to Proposition 3.3, and whose proof follows essentially the same lines.

Proposition 3.7. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ and $\mathcal{N}$ the nullspace of its synthesis operator. Then $\mathcal{F}$ is a conditional Riesz frame if and only ifthere exists a sequence $\{I_n\}$ of finite subsets of $\mathbb{N}$ such that $I_n \subseteq I_{n+1}$,*

$$\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N} \quad \textit{and} \quad c = \sup_{n\in\mathbb{N}} c[\mathcal{N}, \mathcal{M}_{I_n}] < 1, \quad n \in \mathbb{N}. \tag{8}$$

As a corollary of this proposition we get the following result.

Proposition 3.8. *Let $\mathcal{F}$ be a conditional Riesz frame, and T its synthesis operator for $\mathcal{F}$. Denote $\mathcal{N} = N(T)$. Then*

$$\mathrm{cl}\left(\bigcup_{n=1}^{\infty} \mathcal{N}_n\right) = \mathcal{N}. \tag{9}$$

In order to prove this proposition, we need the following technical lemma.

Lemma 3.9. *Let $\mathcal{N}$ be a closed subspace of ℓ^2, a constant $c < 1$ and a sequence $\{I_n\}$ of finite subsets of $\mathbb{N}$ such that $I_n \subseteq I_{n+1}$, $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$ and $c[\mathcal{N}, \mathcal{M}_{I_n}] \leqslant c$, for every $n \in \mathbb{N}$. Then*

$$\mathrm{cl}\left(\bigcup_{n\in\mathbb{N}} \mathcal{N} \cap \mathcal{M}_{I_n}\right) = \mathcal{N}.$$

Proof. Denote $Q_n = P_{I_n}$, $n \in \mathbb{N}$. The assertion of the lemma is equivalent to

$$P_{\mathcal{N}} \wedge Q_n \underset{n\to\infty}{\overset{\text{SOT}}{\nearrow}} P_{\mathcal{N}}.$$

Let $x \in \ell^2$ be a unit vector and let $\varepsilon > 0$. Let $k \in \mathbb{N}$ such that $c^{2k-1} \leqslant \varepsilon/2$. By Proposition 2.4, for every $n \geqslant 1$ it holds that

$$\left\|(P_{\mathcal{N}} Q_n)^k - P_{\mathcal{N}} \wedge Q_n\right\| \leqslant \frac{\varepsilon}{2}.$$

On the other hand, since $Q_n P_{\mathcal{N}} \xrightarrow[n\to\infty]{\text{SOT}} P_{\mathcal{N}}$ and the function $f(x) = x^k$ is SOT-continuous on bounded sets (see, for example, 2.3.2 of [24]), there exists $n_0 \geqslant 1$ such that, for every $n \geqslant n_0$,

$$\left\|\left[(Q_n P_{\mathcal{N}})^k - P_{\mathcal{N}}\right]x\right\| < \frac{\varepsilon}{2}.$$

Then, for every $n \geqslant n_0$,

$$\begin{aligned} &\left\|(P_{\mathcal{N}} - P_{\mathcal{N}} \wedge Q_n)x\right\| \\ &\quad \leqslant \left\|\left[P_{\mathcal{N}} - (P_{\mathcal{N}} Q_n)^k\right]x\right\| + \left\|\left((P_{\mathcal{N}} Q_n)^k - P_{\mathcal{N}} \wedge Q_n\right)x\right\| < \varepsilon. \qquad \square \end{aligned}$$

Proof of Proposition 3.8. Since $\mathcal{F}$ is a conditional Riesz frame, there exist $c < 1$ and a sequence $\{I_n\}$ of finite subsets of $\mathbb{N}$ such that $I_n \subseteq I_{n+1}$, $\bigcup_{n\in\mathbb{N}} I_n = \mathbb{N}$ and $c[\mathcal{N}, \mathcal{M}_{I_n}] \leqslant c$, for every $n \in \mathbb{N}$. By Lemma 3.9, $\bigcup_{n\in\mathbb{N}} \mathcal{N} \cap \mathcal{M}_{I_n}$ is dense in $\mathcal{N}$. Finally, for every

$n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $I_n \subseteq \mathbb{I}_m = \{1, 2, \dots, m\}$. Thus, $\bigcup_{n\in\mathbb{N}} \mathcal{N} \cap \mathcal{M}_{I_n} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{N}_m$. $\square$

As a consequence of Proposition 3.8 we obtain the following corollaries.

Corollary 3.10. *Let $\mathcal{F}$ be a conditional Riesz frame with synthesis operator T and suppose that* $\dim N(T) < \infty$. *Then $\mathcal{F}$ is a Riesz frame. Moreover, there exists $m \in \mathbb{N}$ such that $N(T) \subseteq \mathcal{M}_m$.*

Proof. Denote by $\mathcal{N} = N(T)$. By Proposition 3.8, $\mathcal{N}$ satisfies Eq. (9). Since $\dim \mathcal{N} < \infty$, then there exists $m \in \mathbb{N}$ such that $\mathcal{N} = N(T) \subseteq \mathcal{M}_m$. Thus, in the terminology of Proposition 3.5, if $c_n = \sup_{J\subseteq \mathbb{I}_n} c[\mathcal{N}_n, \mathcal{M}_J]$, then $c_n = c_m$ for every $n \geqslant m$. Therefore, by Proposition 3.5, $\mathcal{F}$ is a Riesz frame. $\square$

Corollary 3.11. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a conditional Riesz frame. Given $n \in \mathbb{N}$, denote by S_n the frame operator of $\{f_k\}_{k=1}^n$ and let A_n be the minimum of the lower frame bounds of all frame subsequences of $\{S_n^{-1/2} f_k\}_{k=1}^n$. If* $\inf_n A_n > 0$, *then $\mathcal{F}$ is a Riesz frame.*

Proof. Let T be the synthesis operator of $\mathcal{F}$ and $\mathcal{N} = N(T)$. For each $n \in \mathbb{N}$, denote $\mathcal{F}_n = \{f_k\}_{k=1}^n$, $\mathcal{B}_n = \{e_1, \dots, e_n\}$ and $P_n = P_{\mathcal{M}_n}$. Note that $TP_n : \mathcal{M}_n \to \overline{\operatorname{span}}\{f_k\colon k = 1, \dots, n\}$ can be considered, modulo an unitary operator, as the synthesis operator of $\mathcal{F}_n$. In this way, it holds that $S_n = TP_nT^*$. Also note that $\{S_n^{-1/2} f_k\}_{k=1}^n$ is a Parseval frame, and $N(TP_n) = N(S_n^{-1/2}TP_n) = \mathcal{N} \cap \mathcal{M}_n = \mathcal{N}_n$. So, by Lemma 2.8, if $J \subset \{1, \dots, n\}$, the lower frame bound A_J of $\{S_n^{-1/2} f_k\}_{k\in J}$ satisfies $A_J = 1 - c[\mathcal{N}_n, \mathcal{M}_J]^2$. Using Propositions 3.8 and 3.5, the corollary follows. $\square$

3.1. A counterexample

The nullspace $\mathcal{N}$ of the synthesis operator of a conditional Riesz frame has the property of "density": $\operatorname{cl}(\bigcup_{n=1}^\infty \mathcal{N}_n) = \mathcal{N}$, where $\mathcal{N}_n$ is $\mathcal{N} \cap \mathcal{M}_n$. In the following example we show that the converse is not true, i.e., we construct a frame which is not a conditional Riesz frame such that its synthesis nullspace $\mathcal{N}$ satisfies $\operatorname{cl}(\bigcup_{n=1}^\infty \mathcal{N}_n) = \mathcal{N}$.

We shall prove the assertion in an indirect way, by using Proposition 3.7 and the following fact: if $\mathcal{N}$ is a closed subspace of ℓ^2 such that $\dim \mathcal{N}^\perp = \infty$, then there exists a frame $\mathcal{F}$ with synthesis operator T such that $\mathcal{N} = N(T)$.

Example 3.12. Given $r > 1$, if $\mathcal{B} = \{e_n\}_{n\in\mathbb{N}}$ denotes the canonical basis of ℓ^2, let us define the following orthogonal system:

$$\begin{aligned}
x_1 &= e_1 - re_2 + \frac{1}{r}e_3 + \frac{1}{r^2}e_4 + \frac{1}{r^3}e_5 + \frac{1}{r^4}e_6, \\
x_2 &= e_5 - re_6 + \frac{1}{r^5}e_7 + \frac{1}{r^6}e_8 + \frac{1}{r^7}e_9 + \frac{1}{r^8}e_{10}, \\
&\vdots
\end{aligned}$$

$$x_n = e_{4n-3} - re_{4n-2} + \frac{1}{r^{4n-3}}e_{4n-1} + \frac{1}{r^{4n-2}}e_{4n} + \frac{1}{r^{4n-1}}e_{4n+1} + \frac{1}{r^{4n}}e_{4n+2}.$$

Let $\mathcal{N}$ be the closed subspace generated by $\{x_n\}_{n\in\mathbb{N}}$. By construction, $\mathrm{cl}(\bigcup_{n=1}^{\infty}\mathcal{N}_n) = \mathcal{N}$. Moreover $\{e_{4n-1} - re_{4n}\colon n \in \mathbb{N}\} \subset \mathcal{N}^{\perp}$, so $\dim \mathcal{N}^{\perp} = \infty$. By the remarks above, there exists a frame $\mathcal{F}$ such that the nullspace of its synthesis operator is $\mathcal{N}$. We claim that this frame is not a conditional Riesz frame. By Proposition 3.7, it suffices to verify that for every sequence $J_1 \subseteq J_2 \subseteq J_3 \subseteq \cdots \subseteq J_n \nearrow \mathbb{N}$, it holds that $c[\mathcal{N}, \mathcal{M}_{J_k}] \underset{k\to\infty}{\longrightarrow} 1$. Hence, fix such a sequence $\{J_k\}_{k\in\mathbb{N}}$ and take $0 < \varepsilon < 1$.

Since $\|x_n\|^2 \leqslant 1 + r^2 + 4/r^{8n-6}$ for every $n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - \varepsilon < \frac{1 + r^2}{\|x_n\|^2} \quad \forall n \geqslant n_0.$$

Note that, for $y \in \mathcal{N}$ and $i \in \mathbb{N}$, if $\mathcal{M}_i = \mathrm{span}\{e_{4i-3}, e_{4i-2}\}$, then

$$\langle y, x_i\rangle = 0 \quad \Leftrightarrow \quad P_{\mathcal{M}_i} y = 0, \tag{10}$$

because $P_{\mathcal{M}_i} x_j \neq 0$ if and only if $j = i$. Let $k \in \mathbb{N}$ be such that

$$j = \max\bigl\{i \in \mathbb{N}\colon\ P_{\mathcal{M}_i}(\mathcal{N} \cap \mathcal{M}_{J_k}) \neq 0\bigr\} \geqslant n_0.$$

By Eq. (10), $x_h \in (\mathcal{N} \cap \mathcal{M}_{J_k})^{\perp}$ for every $h > j$. In particular, $x_{j+1} \in \mathcal{N} \ominus (\mathcal{N} \cap \mathcal{M}_{J_k})$ and

$$1 - \varepsilon < \frac{1 + r^2}{\|x_{j+1}\|^2} \leqslant \frac{\|P_{J_k} x_{j+1}\|^2}{\|x_{j+1}\|^2} \leqslant \left\langle \frac{x_{j+1}}{\|x_{j+1}\|}, \frac{P_{J_k} x_{j+1}}{\|P_{J_k} x_{j+1}\|}\right\rangle \leqslant c[\mathcal{N}, \mathcal{M}_{J_k}].$$

A similar argument shows that $1 - \varepsilon \leqslant c[\mathcal{N}, \mathcal{M}_{J_m}]$, for every $m \geqslant k$. This implies that $\liminf_{n\to\infty} c[\mathcal{N}, \mathcal{M}_{J_n}] \geqslant 1 - \varepsilon$. Finally, as ε is arbitrary, we get $c[\mathcal{N}, \mathcal{M}_{J_k}] \underset{k\to\infty}{\longrightarrow} 1$.

4. Weighted dual frames

Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a fixed frame for a closed subspace $\mathcal{W}$ of $\mathcal{H}$ and let $\mathcal{M} \subseteq \mathcal{H}$ be another closed subspace such that $\mathcal{H} = \mathcal{W} \dot{+} \mathcal{M}^{\perp}$. As we have mentioned in the introduction, an oblique dual frame of $\mathcal{F}$ in $\mathcal{M}$ is a frame $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ for $\mathcal{M}$ such that for every $f \in \mathcal{W}$ it holds that

$$f = \sum_{n=1}^{\infty} \langle f, g_n\rangle f_n \quad \forall f \in \mathcal{W}. \tag{11}$$

Such a dual frame has the minimal norm property if for every $f \in \mathcal{W}$ the coefficients $\{\langle f, g_n\rangle\}_{n\in\mathbb{N}}$ have minimal ℓ^2 norm. Christensen and Eldar proved in [12] that the duals frames with the minimal norm property have the form

$$\{g_n\}_{n\in\mathbb{N}} = \bigl\{B(T^*B)^{\dagger} e_n\bigr\}_{n\in\mathbb{N}}, \tag{12}$$

where $\{e_n\}_{n\in\mathbb{N}}$ denote the canonical orthonormal basis of ℓ^2, and B is a bounded operator with $R(B) = \mathcal{M}$.

On the other hand, let $\mathcal{D}(\ell^2)$ be the set of all $D \in Gl(\ell^2)^+$ which are diagonal in the canonical basis $\{e_n\}_{n\in\mathbb{N}}$. Each $D \in \mathcal{D}(\ell^2)$ defines an inner product $\langle \cdot, \ \cdot \rangle_D$ by means of

$$\langle x, y\rangle_D = \langle Dx, y\rangle, \quad x, y \in \ell^2.$$

This inner product induces a *weighted* norm $\|\cdot\|_D$ which is equivalent to the original one.

In this section, we are interested in dual frames such that their coefficients in the reconstruction formula (11) minimize different weighted norms. We shall give explicit formulae for this class of dual frames that we call weighted dual frames. We also consider the particular case of weighted dual frames associated to a Riesz frame.

First of all, let us recall some preliminary facts on generalized inverses.

Definition 4.1. Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, let $A \in L(\mathcal{H}, \mathcal{K})$ be an operator with closed range. We say that $B \in L(\mathcal{K}, \mathcal{H})$ is a generalized inverse of A if $ABA = A$ and $BAB = B$.

Remarks 4.2. Let $A \in L(\mathcal{H}, \mathcal{K})$ with closed range, and let $B \in L(\mathcal{K}, \mathcal{H})$ be a generalized inverse of A. Then

(1) Both AB and BA are oblique projections, i.e., idempotent operators.
(2) $R(B)$ is also closed.
(3) The idempotent AB and BA induce decompositions of the Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$: $\mathcal{H} = N(A) \dotplus R(B)$ and $\mathcal{K} = R(A) \dotplus N(B)$.
(4) If $(AB)^* = AB$ and $(BA)^* = BA$, then B is called the Moore–Penrose generalized inverse for A. It is usually denoted by $A^\dagger$. In this case, $AA^\dagger$ is the orthogonal projection onto $R(A)$ and $A^\dagger A$ is the orthogonal projection onto $N(A)^\perp$.

Among the generalized inverses of an operator $A \in L(\ell^2, \mathcal{H})$, the following ones will be particularly important for us. In order to clarify the next statement, given a subspace $\mathcal{T}$ of ℓ^2 and $D \in \mathcal{D}(\ell^2)$, the orthogonal complement of $\mathcal{T}$ with respect to the inner product $\langle \cdot, \cdot\rangle_D$ will be denoted by $\mathcal{T}^{\perp_D}$.

Lemma 4.3. *Let $A \in L(\ell^2, \mathcal{H})$ be an operator with closed range, and $D \in \mathcal{D}(\ell^2)$. Then, the operator $\chi_D(A) = D^{-1/2}(AD^{-1/2})^\dagger$ is a generalized inverse of A such that $\chi_D(A)A$ is the orthogonal projection with respect to the weighted inner product $\langle \cdot, \cdot\rangle_D$ onto $N(A)^{\perp_D}$.*

Proof. Since $R(AD^{1/2}) = R(A)$ it follows that

$$A\chi_D(A)A = P_{R(AD^{1/2})}A = A.$$

On the other hand,

$$\begin{aligned}\chi_D(A)A\chi_D(A) &= D^{-1/2}(AD^{-1/2})^\dagger AD^{-1/2}(AD^{-1/2})^\dagger = D^{-1/2}(AD^{-1/2})^\dagger \\ &= \chi_D(A).\end{aligned}$$

Finally, some easy computation shows that an oblique projection Q is D-orthogonal if and only if DQ is selfadjoint. In our case

$$D\big(\chi_D(A)A\big) = D^{1/2}(AD^{-1/2})^\dagger A = D^{1/2}\big(D^{-1/2}A^*(AD^{-1}A^*)^\dagger\big)A$$
$$= A^*(AD^{-1}A^*)^\dagger A,$$

which is clearly selfadjoint. Therefore, $\chi_D(A)A$ is a D-orthogonal projection and clearly $N(\chi_D(A)A) = N(A)$. □

Now, we are ready to give the explicit form of weighted dual frames.

Proposition 4.4. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a fixed frame for a closed subspace $\mathcal{W}$ of $\mathcal{H}$, T its synthesis operator and let $\mathcal{M}$ be another closed subspace of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \dotplus \mathcal{M}^\perp$. Then, given $D \in \mathcal{D}(\ell^2)$, the oblique dual frames such that for every $f \in \mathcal{W}$ their coefficient in the reconstruction formula minimize the weighted norm $\|\cdot\|_D$ have the form*

$$\mathcal{G} = \{g_n\}_{n\in\mathbb{N}} = \big\{B(D^{-1/2}T^*B)^\dagger D^{-1/2}e_n\big\}_{n\in\mathbb{N}},$$

where $\{e_n\}_{n\in\mathbb{N}}$ denotes the canonical orthonormal basis of ℓ^2 and $B \in L(\ell^2, \mathcal{H})$ is any operator with $R(B) = \mathcal{M}$.

Proof. Fix $B \in L(\ell^2, \mathcal{H})$ with range $\mathcal{M}$ and let $\hat{T} = B(D^{-1/2}T^*B)^\dagger D^{-1/2}$. First of all, note that $N(D^{-1/2}T^*B) = N(B)$. So, $R(\hat{T}) = R(B) = \mathcal{M}$ and therefore $\mathcal{G}$ is a frame.

In order to prove that $\mathcal{G}$ is an oblique dual frame it is enough to prove that $T\hat{T}^*$ is an oblique projection onto $\mathcal{W}$. Actually, $T\hat{T}^*$ is the projection onto $\mathcal{W}$ parallel to $\mathcal{M}^\perp$. Indeed, on one hand

$$(T\hat{T}^*)^2 = \big(TD^{-1/2}(B^*TD^{-1/2})^\dagger B^*\big)^2$$
$$= TD^{-1/2}\big((B^*TD^{-1/2})^\dagger(B^*TD^{-1/2})(B^*TD^{-1/2})^\dagger\big)B^*$$
$$= T\big(D^{-1/2}(B^*TD^{-1/2})^\dagger B^*\big) = (T\hat{T}^*),$$

which shows that $T\hat{T}^*$ is a projection. On the other hand, since $N(D^{-1/2}(B^*TD^{-1/2})^\dagger B^*) = \mathcal{M}^\perp$ and $R(D^{-1/2}(B^*TD^{-1/2})^\dagger B^*) = N(T)^\perp$, it holds that $T\hat{T}^*$ is the projection onto $\mathcal{W}$ with nullspace $\mathcal{M}^\perp$.

Finally, in order to prove that the reconstruction coefficients minimize the weighted norm $\|\cdot\|_D$ we have to prove that $R(\hat{T}^*) \subseteq N(T)^{\perp_D}$. But, using the notation of Lemma 4.3, we get $\hat{T}^*T = \chi_D(B^*T)B^*T$ and, therefore, using the same lemma, $R(\hat{T}^*) = N(B^*T)^{\perp_D} = N(T)^{\perp_D}$. □

As we have already mentioned in the previous section, $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz frames if and only if $N(T)$ is compatible with the canonical base (see Remark 3.4). If $P_{D,\mathcal{N}}$ denote the (unique) orthogonal projection onto the closed subspace $\mathcal{N}$ of ℓ^2 with respect to the inner product $\langle\cdot,\cdot\rangle_D$, it was proved in [1] that $\mathcal{N}$ is compatible if and only if

$$\sup_{D\in\mathcal{D}(\ell^2)} \|P_{D,\mathcal{N}}\| < \infty.$$

As a consequence of this result we obtain the following

Theorem 4.5. *Let $\mathcal{F} = \{f_n\}_{n\in\mathbb{N}}$ be a frame for a closed subspace $\mathcal{W}$ of $\mathcal{H}$, T its synthesis operator, $\mathcal{M}$ another closed subspace of $\mathcal{H}$ such that $\mathcal{H} = \mathcal{W} \dot{+} \mathcal{M}^\perp$ and $\mathcal{G} = \{g_n\}_{n\in\mathbb{N}}$ a fixed* (*sampling*) *frame for $\mathcal{M}$ with synthesis operator B. Then, the following conditions are equivalent*:

(1) *$\mathcal{F}$ is a Riesz frame on $\mathcal{W}$.*
(2) *The oblique dual frames of T with respect to B that minimize the different weighted norms are bounded from above. In other words*

$$\sup_{D\in\mathcal{D}(\ell^2)} \left\| B(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\| < \infty.$$

Proof. Fix $D \in \mathcal{D}(\ell^2)$. We have already proved in Lemma 4.3 that

$$\begin{aligned}\left(B(D^{-1/2}T^*B)^\dagger D^{-1/2}\right)^* T = T^*B(D^{-1/2}T^*B)^\dagger D^{-1/2} &= T^*B\chi_D(T^*B)\\ &= 1 - P_{D,N(T)}.\end{aligned}$$

Hence

$$\begin{aligned}\left\| B(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\| &\leqslant \|B\| \left\| (D^{-1/2}T^*B)^\dagger D^{-1/2} \right\|\\ &= \|B\| \left\| (T^*B)^\dagger (T^*B)(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\|\\ &\leqslant \|B\| \left\| (T^*B)^\dagger \right\| \left\| (T^*B)(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\|\\ &= \|B\| \left\| (T^*B)^\dagger \right\| \left\| 1 - P^*_{D,N(T)} \right\|,\end{aligned}$$

and

$$\|1 - P_{D,N(T)}\| = \left\| T^*B(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\| \leqslant \|T^*\| \left\| B(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\|.$$

Therefore

$$\sup_{D\in\mathcal{D}(\ell^2)} \left\| B(D^{-1/2}T^*B)^\dagger D^{-1/2} \right\| < \infty \quad \Leftrightarrow \quad \sup_{D\in\mathcal{D}(\ell^2)} \|1 - P_{D,N(T)}\| < \infty,$$

which proves the proposition. □

References

[1] J.A. Antezana, G. Corach, M. Ruiz, D. Stojanoff, Weighted projections and Riesz frames, Linear Algebra Appl., in press.
[2] R. Bouldin, The product of operators with closed range, Tôhoku Math. J. 25 (1973) 359–363.
[3] P.G. Casazza, Characterizing Hilbert space frames with the subframe property, Illinois J. Math. 41 (1997) 648–666.
[4] P.G. Casazza, O. Christensen, Hilbert space frames containing a Riesz basis and Banach spaces which have no subspace isomorphic to c_0, J. Math. Anal. Appl. 202 (1996) 940–950.
[5] P.G. Casazza, O. Christensen, Frames containing a Riesz basis and preservation of this property under perturbations, SIAM J. Math. Anal. 29 (1998) 266–278.
[6] P.G. Casazza, O. Christensen, Riesz frames and approximation of the frame coefficients, Approx. Theory Appl. (N.S.) 14 (1998) 1–11.
[7] O. Christensen, Frames and the projection method, Appl. Comput. Harmon. Anal. 1 (1993) 50–53.

[8] O. Christensen, Frames and pseudo-inverses, J. Math. Anal. Appl. 195 (1995) 401–414.
[9] O. Christensen, Frames containing a Riesz basis and approximation of the frame coefficients using finite-dimensional methods, J. Math. Anal. Appl. 199 (1996) 256–270.
[10] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhäuser, Boston, 2003.
[11] O. Christensen, Y. Eldar, Characterization of oblique dual frames pairs, preprint.
[12] O. Christensen, Y. Eldar, Oblique dual frames and shift-invariant spaces, Appl. Comput. Harmon. Anal. 17 (2004) 48–68.
[13] O. Christensen, A. Lindner, Decomposition of Riesz frames and wavelets into a finite union of linearly independent sets, Linear Algebra Appl. 355 (2002) 147–159.
[14] I. Daubechies, A. Grossmann, Y. Meyer, Painless nonorthogonal expansions, J. Math. Phys. 27 (1986) 1271–1283.
[15] F. Deutsch, The angle between subspaces in Hilbert space, in: S.P. Singh (Ed.), Approximation Theory, Wavelets and Applications, Kluwer Academic, Dordrecht, 1995, pp. 107–130.
[16] R.J. Duffin, A.C. Schaeffer, A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 72 (1952) 314–366.
[17] C.E. Heil, D.F. Walnut, Continuous and discrete wavelet transforms, SIAM Rev. 31 (1989) 628–666.
[18] V. Havin, B. Jöricke, The Uncertainty Principle in Harmonic Analysis, Ergeb. Math. Grenzgeb., vol. 28, Springer-Verlag, Berlin, 1994.
[19] T. Kato, Perturbation Theory of Linear Operators, second ed., Springer, New York, 1984.
[20] S. Kayalar, H. Weinert, Error bounds for the method of alternating projections, Math. Control Signal Systems 1 (1988) 43–59.
[21] S. Li, A theory of generalized multiresolution structure and pseudoframes of translates, J. Fourier Anal. Appl. 7 (2001) 23–40.
[22] S. Li, H. Ogawa, Pseudoframes for subspaces with applications, J. Fourier Anal. Appl. 10 (2004) 409–431.
[23] S. Li, H. Ogawa, Pseudo-duals of frames with applications, Appl. Comput. Harmon. Anal. 11 (2001) 289–304.
[24] G.K. Pedersen, C^*-Algebras and Their Automorphism Groups, Academic Press, London, 1979.
[25] R.M. Young, An Introduction to Nonharmonic Fourier Series, revised first ed., Academic Press, San Diego, 2001.