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DIFFUSE AND LOCALIZED FAILURE PREDICTIONS OF PERZYNA VISCOPLASTIC MODELS FOR COHESIVE-FRICTIONAL MATERIALS

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***Abstract*— Viscoplastic constitutive formulations are characterized by instantaneous tangent operators which do not exhibit degradation from the elastic properties. As a consequence viscoplastic material descriptions were often advocated to retrofit the shortcomings of the inviscid elastoplastic formulations such as loss of stability and loss of ellipticity.**

However, when the time integration of viscoplastic material processes is considered within finite time increments, there exists an algorithmic tangent operator which may lead to loss of stability and loss of ellipticity similar to rate–independent elastoplastic materials.

The algorithmic tangent operator follows from the consistent linearization process. Therefore, the numerical method considered for the time integration of the constitutive equations plays a fundamental role in failure analysis of viscoplastic materials.

This paper focuses on the performance of the conditions for diffuse and localized failure of two Perzyna–type viscoplastic models, one of them based on the classical formulation and the other one based on a new proposal by Ponthot (1995) which includes a constrain condition representing a rate dependent generalization of the plasticity's yield condition. Application of Backward Euler method for time integration of both Perzyna formulations leads to quite different form of the consistent tangent material operators. These stiffness tensors are obtained for Perzyna generalizations of the so called Extended Leon Model which is a fracture energy–based elastoplastic constitutive model for concrete.

The results included in the paper illustrate the strong differences between the failure predictions of both Perzyna–type viscoplastic formulations. In this regard, the classical formulation is unable to reproduce the predictions of the inviscid model when the viscosity approaches zero. This case leads to very small values of both failures indicators and their performance are characterized by strong oscillations and even discontinuities. On the other hand the so–called continuous formulation is associated with algorithmic tangent moduli which signals a smooth transition from the elastic operator to the elastoplastic algorithmic one, when the viscosity varies from very large to very small values.

***Keywords*— Viscoplasticity, failure, localization, consistent tangent.**

I. INTRODUCTION

The Perzyna–type viscoplastic models are widely used to characterize rate effects in plastic materials. In the classical form the constitutive equations of Perzyna–type viscoplasticity do not reduce to the rate–independent plasticity formulation as the viscosity parameter approaches zero. This feature had led to some difficulties on the development of efficient time integration algorithm for computational implementation of Perzyna viscoplastic models, both at the local as well as at the global "momentum-balance loop" level. On the local level, the viscoplastic rate equations are numerically integrated for the specified time increment Δt. It is certainly desirable if the integration algorithm employed is unconditionally stable. Therefore, the Backward Euler algorithm is usually considered for the return mapping during Perzyna viscoplastic material processes. On the global or Finite Element level, the use of consistent tangent moduli is crucial in preserving the quadratic rate of convergence of the Newton method in conjunction with nonlinear finite element computations. The lack of a constrain condition in case of the classical formulation of Perzyna viscoplasticity forces the consideration of residual functions for Newton iteration of momentum balance, see Ju (1990) and Etse and Willam (1999). In this way a consistent or algorithmic tangent operator for rate–dependent material formulations is derived which strongly varies from that of the inviscid elastoplastic material when the viscosity tends to zero. In this case the fourth order material operator of the viscoplastic model approaches the fourth order zero tensor.

Recently, an alternative formulation for Perzyna type rate–dependent models was derived by Ponthot (1995) whereby the viscoplastic problem is treated in a similar way to the elastoplastic one. In this proposal the classical viscoplastic constitutive equations are complemented by a new constrain condition which represents a generalization of the inviscid condition $F = 0$ for rate dependent viscoplastic materials.

The generalized constrain condition plays a fundamental role in the alternative on "continuous" viscoplastic formulation. On one hand it allows a generalization of the Kuhn - Tucker conditions. On the other hand, the linearization of its differential form leads to the algorithmic tangent operator. This stiffness tensor takes a similar form to that corresponding to the inviscid material which is fully recover in the extreme case when the viscosity tends to zero.

Aside from measuring canonical degradation of stiffness and strength, the spectral properties of tangent operators of continuum or smeared crack-based models define two types of failure modes: one of them corresponds to a loss of uniqueness of the incremental response and is associated with the singularity of the fourth order material operator, or of its symmetric counterpart, see Runnesson and Mróz (1989). The other one corresponds to a localization in the form of discontinuous bifurcation which signals a change in the character of the governing differential equations from elliptic to hyperbolic and vice-versa in quasi-static and dynamic loading conditions, respectively (Needleman, 1988, Sluys, 1992). Both failure modes may arise due to destabilizing non-symmetric constitutive operators in the case of non–associated flow rules and/or due to softening of the strength properties.

In this paper the fundamental differences between both formulations for Perzyna–type viscoplastic materials are firstly analyzed. After derivation of the consistent stiffness tensors the attention focuses on the performance of the conditions for diffuse and localized failures of both Perzyna viscoplastic formulations. The results obtained for plane strain conditions support the conclusion that the classical Perzyna formulation leads to strong discontinuities of the failure indicators performance when the viscosity approaches zero. In this case both diffuse and localized failure modes may be activated and the prediction of the inviscid material are not reproduced.

On the other hand the continuous Perzyna formulation leads to algorithmic material operators which exhibit a smooth transition from the elastic to the elastoplastic tensor according to the assumed value for the viscosity. Therefore, when this material parameter approaches zero the same failure predictions as the inviscid material are obtained.

II. FAILURE INDICATORS

Diffuse or continuous failure is associated with a singularity of the tangential material stiffness. This condition infers material branching, *i.e.* a singularity of the tangential material operator and leads to limit state or limit point when the critical eigen direction is activated. On the other hand, localized failure corresponds to a state of bifurcation inside the material which is associated with spatial discontinuities of the field variables.

In the following the indicators for diffuse and localized failure are presented.

A. Diffuse Failure Indicator

The condition for diffuse or continuous failure is mathematically expressed as

$$\det\left(\mathbf{E}_{tg}\right) = 0, \tag{1}$$

where $\mathbf{E}_{tg}$ represents the tangential material stiffness.

From the eigenvalue problem for $\mathbf{E}_{tg}$ follows that in case of diffuse failure the critical eigentensor $\dot{\epsilon}_{crit}$ associated with the minimum eigenvalue $\lambda_{\min}(\mathbf{E}_{tg}) = 0$ is the one which renders stationary values of the stress rate

$$\dot{\boldsymbol{\sigma}} = \mathbf{E}_{tg} : \dot{\boldsymbol{\epsilon}} = \mathbf{0} \qquad \text{for} \qquad \dot{\boldsymbol{\epsilon}}_{crit} \neq 0 \tag{2}$$

corresponding to an horizontal asymptote at the maximum strength or limit state of the stress – strain relation, where the (:) symbol defines the doubly contracted tensor product and the dot indicates rate.

In case of non–associated elastoplasticity the minimum eigenvalue of $\mathbf{E}_{tg}$ is defined by the bilinear form below, which is normalized with regard to the corresponding elastic value

$$\lambda_{\min}^{norm} = \frac{\mathbf{n} : \mathbf{E}_{tg} : \mathbf{m}}{\mathbf{n} : \mathbf{E} : \mathbf{m}} = 1 - \frac{1}{1 - \frac{E_p}{\mathbf{n}:\mathbf{E}:\mathbf{m}}}, \tag{3}$$

where $\mathbf{E}$ is the elastic tensor, $\mathbf{m}$ and $\mathbf{n}$ the gradients of the plastic potential and the yield function respectively and E_p represents the so called hardening modulus.

We observe that the diffuse failure condition $\det(\mathbf{E}_{tg}) = 0$ is fulfilled only if the hardening modulus diminishes to zero, as long as the bilinear form $(\mathbf{n} : \mathbf{E}_{tg} : \mathbf{m}) > 0$ remains positive.

B. Localized Failure Indicator

Localized failure involves the formation of weak discontinuities in the strain rates or rather velocity gradient fields. Assuming that the deformed solid can be divided in two subdomains named as "+" and "-" by an internal boundary with normal $\vec{\mathbf{N}}$, and according to Maxwell's compatibility theorem (Truesdell & Toupin, 1960) this jump of the velocity gradient field is a rank one second order tensor which must satisfy

$$[[\nabla \dot{\mathbf{u}}]] = \dot{\gamma} \vec{\mathbf{M}} \otimes \vec{\mathbf{N}}, \tag{4}$$

where $\vec{\mathbf{M}}$ denotes the polarization vector and $\dot{\gamma}$ an scalar factor defining the magnitude of the jump. The situation described by Eq. (4) has its analogous counterpart in fracture mechanics, corresponding to a Mode I fracture for $\vec{\mathbf{M}} \perp \vec{\mathbf{N}}$ and Mode II fracture for $\vec{\mathbf{M}} \parallel \vec{\mathbf{N}}$. The strain rate discontinuity is then defined by the symmetrized dyadic of the unit vectors $\vec{\mathbf{M}}$ and $\vec{\mathbf{N}}$, *i.e.*,

$$[[\,\dot{\boldsymbol{\epsilon}}]] = [[\nabla^s \dot{\mathbf{u}}]] = \tfrac{1}{2}\dot{\gamma}\left(\vec{\mathbf{M}} \otimes \vec{\mathbf{N}} + \vec{\mathbf{N}} \otimes \vec{\mathbf{M}}\right) = \dot{\gamma}\left(\vec{\mathbf{M}} \otimes \vec{\mathbf{N}}\right)^s \quad (5)$$

Balance of linear momentum across the discontinuity surface leads to the localization condition

$$\dot{\vec{\mathbf{t}}} = \vec{\mathbf{N}} \cdot [[\,\dot{\boldsymbol{\sigma}}]] = \vec{\mathbf{N}} \cdot \mathbf{E}_{tg} : [[\,\dot{\boldsymbol{\epsilon}}]] = \dot{\gamma}\vec{\mathbf{N}} \cdot \mathbf{E}_{tg} : \left(\vec{\mathbf{M}} \otimes \vec{\mathbf{N}}\right)^s = \dot{\gamma}\mathbf{Q} \cdot \vec{\mathbf{M}} \doteq \mathbf{0} \quad (6)$$

where the localization (acoustic) tensor is defined as $\mathbf{Q} = \vec{\mathbf{N}} \cdot \mathbf{E}_{tg} \cdot \vec{\mathbf{N}}$. According to (6), discontinuous bifurcation initiates when the localization tensor exhibits a singularity, *i.e.*, $\det(\mathbf{Q}) = 0$ for some directions $\vec{\mathbf{M}}$ and $\vec{\mathbf{N}}$.

III. CONSTITUTIVE EQUATIONS FOR PERZYNA VISCOPLASTICITY

In this section the constitutive equations of Perzyna type viscoplastic models are presented. We distinguish between the *classical formulation* and the *continuous formulation* of Perzyna viscoplastic constitutive equations. The second one leads to a constrain condition which plays a fundamental role in the algebraic problem when finite time increments are considered, as we will see in Section 4.

A. The Classical Formulation

Similar to the flow theory of plasticity, the constitutive relations of Perzyna (1963, 1966) type elasto–viscoplastic material formulations may be written

$$\dot{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}}_e - \dot{\boldsymbol{\sigma}}_{vp} = \mathbf{E} : (\,\dot{\boldsymbol{\epsilon}} - \dot{\boldsymbol{\epsilon}}_{vp}) \quad (7)$$

$$\dot{\boldsymbol{\epsilon}}_{vp} = \mathbf{g}(\psi, F,\, \boldsymbol{\sigma}) = \frac{1}{\eta}\langle\psi(F)\rangle\mathbf{m} \quad (8)$$

$$\mathbf{m} = \mathbf{A}^{-1} : \mathbf{n} = \mathbf{A}^{-1} : \frac{\partial F}{\partial\, \boldsymbol{\sigma}} \quad (9)$$

$$\psi(F) = \left[\frac{F(\,\boldsymbol{\sigma}, \mathbf{q},)}{F_o}\right]^N \quad (10)$$

$$\dot{\mathbf{q}} = \frac{1}{\eta}\,\langle\psi(F)\rangle\mathbf{H} : \mathbf{m}, \quad (11)$$

where $\boldsymbol{\epsilon}_{vp}$ represents the viscoplastic portion of the total strain tensor $\boldsymbol{\epsilon}$, η the viscosity and $\mathbf{q}$ the set of hardening/softening variables defined as a tensor of arbitrary order. The relations (7) follow the additive decomposition of the total strain rate into an elastic and a viscoplastic part $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_e + \dot{\boldsymbol{\epsilon}}_{vp}$, quite similar to the Prandtl-Reuss equations in case of inviscid elasto-plastic constitutive relations. Equations (8) and (9) describe a general non-associated flow rule, whereby the direction of the viscoplastic strains $\mathbf{m}$, is obtained by a modification of the gradient tensor $\mathbf{n}$ of the yield surface F by means of the fourth order transformation tensor $\mathbf{A}$. Moreover, $\psi(F)$ is a dimensionless monotonically increasing over–stress function whereby F_o represents a normalizing factor. The Mc Cauley brackets in Eq. (8) defines the features of the over–stress function as

$$\langle\psi(F)\rangle = \begin{cases} F & \text{if} \quad F > 0 \\ 0 & \text{if} \quad F \leq 0 \end{cases} \quad (12)$$

being $F = F(\,\boldsymbol{\sigma}, \mathbf{q})$ a convex yield function which defines the limit of the elastic domain.

Finally Eq. (11) represents the evolution law of the hardening/softening variables $\mathbf{q}$ by means of a suitable tensor function $\mathbf{H}$ of the state variables.

A consistency condition similar to the flow theory of plasticity can not be obtained in the classical formulation of viscoplastic materials. However, if the viscoplastic problem is treated in a similar way to the elastoplastic one, as we will see in Section 3.2, a constrain condition can be obtained which represents a generalization of the inviscid yield condition for viscoplastic materials.

Remark: instantaneous viscoplastic material response do not exhibit deterioration of the elastic properties. Therefore a viscoplastic continuum tangent stiffness tensor $\mathbf{E}_{vp}$ similar to inviscid elastoplastic materials can not be obtained in case of viscoplastic formulations.

B. The Continuous Formulation

In this formulation the Eqs. (7) to (11) are complemented by a consistency parameter $\dot{\lambda}$, see Ponthot (1995), defined as an increasing function of the over–stress

$$\dot{\lambda} = \frac{1}{\eta}\langle\psi(F)\rangle. \quad (13)$$

So that the evolutions Eqs. (8) and (11) take now the classical forms

$$\dot{\boldsymbol{\epsilon}}_{vp} = \dot{\lambda}\,\mathbf{m} \quad (14)$$

$$\dot{\mathbf{q}} = \dot{\lambda}\,\mathbf{H} : \mathbf{m} = \dot{\lambda}\,\mathbf{h}. \quad (15)$$

being $\mathbf{h} = \mathbf{H} : \mathbf{m}$. Thus, from Eqs. (8) and (14) follows

$$F = \psi^{-1}\left(\frac{\|\,\dot{\boldsymbol{\epsilon}}_{vp}\,\|}{\|\,\mathbf{m}\,\|}\eta\right) = \psi^{-1}\left(\dot{\lambda} \cdot \eta\right) \quad (16)$$

We may now define for the viscoplastic range, the new constrain condition where $\vec{M}$ denotes the polarization vector and $\dot{\gamma}$ an scalar factor defining the magnitude of the jump. The situation described by Eq. (4) has its analogous counterpart in fracture mechanics, corresponding to a Mode I fracture for $\vec{M} \perp \vec{N}$ and Mode II fracture for $\vec{M} \parallel \vec{N}$. The strain rate discontinuity is then defined by the symmetrized dyadic of the unit vectors $\vec{M}$ and $\vec{N}$, *i.e.*,

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\end{aligned}$$

where $\boldsymbol{\epsilon}_{vp}$ represents the viscoplastic portion of the total strain tensor $\boldsymbol{\epsilon}$, η the viscosity and $\mathbf{q}$ the set of hardening/softening variables defined as a tensor of arbitrary order. The relations (7) follow the additive decomposition of the total strain rate into an elastic and a viscoplastic part $\dot{\boldsymbol{\epsilon}} = \dot{\boldsymbol{\epsilon}}_e + \dot{\boldsymbol{\epsilon}}_{vp}$, quite similar to the Prandtl-Reuss equations in case of inviscid elasto-plastic constitutive relations. Equations (8) and (9) describe a general non-associated flow rule, whereby the direction of the viscoplastic strains $\mathbf{m}$, is obtained by a modification of the gradient tensor $\mathbf{n}$ of the yield surface F by means of the fourth order transformation tensor $\mathbf{A}$. Moreover, $\psi(F)$ is a dimensionless monotonically increasing over–stress function whereby F_o represents a normalizing factor. The Mc Cauley brackets in Eq. (8) defines the features of the over–stress function as

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$$F = \psi^{-1}\left(\frac{\|\dot{\boldsymbol{\epsilon}}_{vp}\|}{\|\mathbf{m}\|}\eta\right) = \psi^{-1}\left(\dot{\lambda}\cdot\eta\right) \tag{16}$$

We may now define for the viscoplastic range, the new constrain condition

$$\bar{F} = F - \psi^{-1}\left(\dot{\lambda}\cdot\eta\right) = 0 \tag{17}$$

which represents a generalization of the inviscid yield condition $F = 0$ for rate-dependent Perzyna viscoplastic materials. The name *continuous formulation* is due to the fact that the condition $\eta = 0$ (no viscosity effect) leads to the elastoplastic yield condition $F = 0$. Moreover, from (13) follows that when $\eta \to 0$ the consistency parameter remains finite and positive since also the over-stress goes to zero. The other extreme case, $\eta \to \infty$, leads to the inequality $\bar{F} < 0$ for every possible stress state, indicating that only elastic response may be activated.

The constrain defined by Eq.(17) allows a generalization of the Kuhn-Tucker conditions which may be now written as

$$\dot{\lambda}\bar{F} = 0\,, \quad \dot{\lambda} \geq 0, \quad \bar{F} \leq 0. \tag{18}$$

Finally, the viscoplastic consistency condition expands into

$$\dot{\bar{F}} = \mathbf{n} : \dot{\boldsymbol{\sigma}} + \bar{\mathbf{r}} : \dot{\mathbf{q}} - \frac{\partial \psi^{-1}(\dot{\lambda}\,\eta)}{\partial \dot{\lambda}}\ddot{\lambda} = 0 \tag{19}$$

where

$$\bar{\mathbf{r}} = \frac{\partial \bar{F}}{\partial \mathbf{q}} = \frac{\partial F}{\partial \mathbf{q}} - \frac{\partial \psi^{-1}(\dot{\lambda}\,\eta)}{\partial \mathbf{q}}. \tag{20}$$

Other recent and interesting approach to this problem is due to Wang (see Wang et al., 1997), which includes the strain rate as state variable into the flow and viscoplastic potential function, *i.e.*

$$F^{vp} = F^{vp}\,(\,\boldsymbol{\sigma}, \mathbf{q}, \,\dot{\boldsymbol{\epsilon}}) \tag{21}$$

this also leads to a rate dependent Kuhn-Tucker conditions as in case of the continuous Perzyna formulation.

IV. CONSISTENT TANGENT STIFFNESS TENSOR

In this section the consistent tangent tensors of both Perzyna viscoplastic material formulations described in section 4 are derived. These operators will be used in the next section for the analysis of diffuse and localized failure predictions.

The lack of a constrain condition in case of the classical formulation of Perzyna viscoplasticity forces the consideration of the stress residual to derive the consistent tangent operator. On the other hand, in the continuous formulation this stiffness tensor follows from the linearization process of the differential form of the generalized consistency condition, similar to the case of the inviscid elastoplastic formulation.

The start point for the derivation of the consistent tangent moduli in both formulations of Perzyna viscoplasticity is the Backward Euler stress equation.

A. Classical Perzyna Formulation

Integrating Eqs. (8) and (11) during the finite time step Δt with the unconditionally stable Backward-Euler (BE) or Closest Point Projection (CPP) algorithm and considering Perzyna formulations of first order, *i.e.* $N = 1$ in Eq. (10), we obtain the algebraic format:

$$\begin{aligned} \Delta\,\boldsymbol{\sigma}_{n+1} &= \mathbf{E} : \Delta\,\boldsymbol{\epsilon}_{n+1} - \Delta t\mathbf{E} : \mathbf{g}_{n+1} \\ \Delta\mathbf{q}_{n+1} &= \Delta t\mathbf{H}_{n+1} : \mathbf{g}_{n+1} \end{aligned} \tag{22}$$

where the Perzyna viscoplastic evolution law is evaluated at $t = t_{n+1}$

$$\mathbf{g}_{n+1} = \frac{1}{\eta}\langle\psi(F)_{n+1}\rangle\mathbf{m}_{n+1}. \tag{23}$$

To derive consistent tangent moduli of the Perzyna description we define the stress residual at $t = t_{n+1}$ as

$$\mathbf{R}_{n+1} = \mathbf{E} : \Delta\,\boldsymbol{\epsilon}_{n+1} - \Delta t\,\mathbf{E} : \mathbf{g}_{n+1} - \Delta\,\boldsymbol{\sigma}_{n+1}. \tag{24}$$

The root $\mathbf{R}_{n+1} = 0$ of Eq. (24) is determined via Newton-Raphson iteration, in the form

$$\mathbf{R}^{k+1}_{n+1} = \mathbf{R}^{k}_{n+1} + \Delta\mathbf{R}^{k+1}_{n+1} = \mathbf{0}, \tag{25}$$

where the superscript on the right indicates the current iteration cycle. Linearization of the residual in Eq. (25) yields

$$\Delta\mathbf{R}^{k+1}_{n+1} = \frac{\partial \mathbf{R}^{k}_{n+1}}{\partial\,\boldsymbol{\sigma}} : \Delta\,\boldsymbol{\sigma} + \frac{\partial \mathbf{R}^{k}_{n+1}}{\partial\,\boldsymbol{\epsilon}} : \Delta\,\boldsymbol{\epsilon}, \tag{26}$$

where the individual terms of the Jacobian involve

$$\begin{aligned} \frac{\partial \mathbf{R}^{k}_{n+1}}{\partial\,\boldsymbol{\sigma}} &= -\mathbf{I} - \Delta t\mathbf{E} : \frac{\partial \mathbf{g}^{k}_{n+1}}{\partial\,\boldsymbol{\sigma}} \\ &= -\mathbf{I} - \frac{\Delta t}{\eta}\mathbf{E} : \left[\boldsymbol{\Psi}\otimes\mathbf{m} + \psi(F)\frac{\partial \mathbf{m}}{\partial\,\boldsymbol{\sigma}}\right]_{n+1} \end{aligned} \tag{27}$$

$$\frac{\partial \mathbf{R}^{k}_{n+1}}{\partial\,\boldsymbol{\epsilon}} = \mathbf{E} \tag{28}$$

with

$$\boldsymbol{\Psi} = \frac{\partial \psi(F)}{\partial\,\boldsymbol{\sigma}} \tag{29}$$

and $\mathbf{I}$ is the fourth order identity tensor.

Substituting Eqs.(27) and (28) into Eq. (26) and subsequently into Eq. (25) we obtain

$$\frac{d\,\boldsymbol{\epsilon}}{d\,\boldsymbol{\sigma}} = \mathbf{C} : \left[\mathbf{I} + \frac{\Delta t}{\eta}\mathbf{E} : (\boldsymbol{\Psi}\otimes\mathbf{m} + \psi(F)\mathbf{M})\right] \tag{30}$$

where

$$\mathbf{C} = \mathbf{E}^{-1}, \quad (31)$$

$$\mathbf{M} = \frac{\partial \mathbf{m}}{\partial \boldsymbol{\sigma}} = \mathbf{A}^{-1} : \frac{\partial^2 F}{\partial \boldsymbol{\sigma} \otimes \partial \boldsymbol{\sigma}}. \quad (32)$$

Equation (30) can be alternatively expressed as

$$\frac{d\,\boldsymbol{\epsilon}}{d\,\boldsymbol{\sigma}} = \mathbf{D}^{-1} + \frac{\Delta t}{\eta} \boldsymbol{\Psi} \otimes \mathbf{m} \quad (33)$$

whereby

$$\mathbf{D}^{-1} = \mathbf{C} + \frac{\Delta t}{\eta} \psi(F) \mathbf{M}. \quad (34)$$

The final expression of the consistent tangent moduli of classical Perzyna viscoplasticity takes then the form

$$\left[\mathbf{E}^{alg}_{Per}\right]^{class} = \frac{d\,\boldsymbol{\sigma}}{d\,\boldsymbol{\epsilon}} = \mathbf{D} - \frac{\mathbf{D} : \boldsymbol{\Psi} \otimes \mathbf{m} : \mathbf{D}}{\frac{\eta}{\Delta t} + \boldsymbol{\Psi} : \mathbf{D} : \mathbf{m}}. \quad (35)$$

Note: the algorithmic tangent operator obtained with the classical Perzyna formulation does not require the determination of the elastoplastic moduli tensor, as in case of Duvaut and Lions (1972) viscoplasticity, see Etse and Willam (1999). From Eqs. (30) and (33) follow that the limiting case $\eta \to \infty$ results in instantaneous elasticity $\Delta\,\boldsymbol{\sigma} = \mathbf{E} : \Delta\,\boldsymbol{\epsilon}$ like the Duvaut-Lions viscoplastic material. On the other hand, when $\eta \to 0$ it also implies that the overstress measure $\psi(F)$ tends to zero for a finite viscoplastic strain value, as an analysis of Eq.(8) can readily show. In consequence, as the coefficient affecting $\mathbf{M}$ in the linear combination Eq.(34) becomes undeterminated, the material operator has no defined limit for this case. In spite of the "apparent" regularization capability of the rate-dependent classical Perzyna formulation we will see in Section 5 that in this extreme case numerical instability may arise due to the particular form which takes the consistent material moduli.

From Eq.(35) follows that, when finite time increments are considered, the eigenvalue of the algorithmic tangent operator normalized with respect to the fourth order tensor $\mathbf{D}$ (see 34) yields

$$[\lambda^{norm}_{\min}]^{Per} = 1 - \frac{1}{1 - \frac{\frac{\eta}{\Delta t}}{\mathbf{n}:\mathbf{D}:\mathbf{m}}}. \quad (36)$$

Thus, the condition for diffuse failure is fulfilled only when $\eta = 0$ (non viscosity effects) as long as $\mathbf{n} : \mathbf{D} : \mathbf{m}$ remains positive. However, due to the particular form of $\mathbf{D}$ in Eq. (34), the extreme case $\eta \to 0$ may lead to quite different values of the bilinear form $\mathbf{n} : \mathbf{D} : \mathbf{m}$

B. Continuous Perzyna Formulation

The algorithmic tangent operator can be formulated from the linearization of the viscoplastic consistency condition, see Eq.(19), for a finite increment d, quite similar to rate independent plasticity,

$$d\bar{F} = \mathbf{n} : d\,\boldsymbol{\sigma} + \bar{\mathbf{r}} : d\mathbf{q} - \frac{\partial \psi^{-1}\left(\dot{\lambda} \cdot \eta\right)}{\partial \dot{\lambda}} d\dot{\lambda} = 0. \quad (37)$$

In order to avoid further complications, it's supposed that $\dot{\lambda}$ is accurately approximated by $\dot{\lambda} = \frac{\Delta\lambda}{\Delta t}$, *i.e.* $\Delta\lambda = \frac{\Delta t}{\eta}\langle\psi(F)\rangle$, this leads to $d\dot{\lambda} = \frac{d\Delta\lambda}{\Delta t}$. The consequences of this assumption are analyzed in other work of the authors (Carosio et al., 2000)

Proceeding in a similar form to the algebraic elastoplastic problem, *i.e.* substituting in Eq. (37) the differential changes of the stress tensor and of the state variables evaluated in a consistent form with the BE

$$d\,\boldsymbol{\sigma} = \mathbf{E}^m : (d\,\boldsymbol{\epsilon} - d\Delta\lambda \mathbf{m}) \quad (38)$$

$$d\mathbf{q} = d\Delta\lambda \mathbf{h} + \Delta\lambda\, \mathbf{p} : \mathbf{E}^m : (d\,\boldsymbol{\epsilon} - d\Delta\lambda \mathbf{m}) \quad (39)$$

where

$$[\mathbf{E}^m]^{-1} = (\mathbf{E}^{-1} + \Delta\lambda \mathbf{M}) \quad (40)$$

$$\mathbf{p} = \frac{\partial \mathbf{h}}{\partial \boldsymbol{\sigma}} \quad (41)$$

we obtain the relations $d\,\boldsymbol{\sigma} = \left[\mathbf{E}^{alg}_{Per}\right]^{cont} : d\,\boldsymbol{\epsilon}$, with the algorithmic operator

$$\left[\mathbf{E}^{alg}_{Per}\right]^{cont} = \mathbf{E}^m - \frac{\bar{\mathbf{m}} \otimes \bar{\mathbf{n}} + \Delta\lambda \bar{\mathbf{r}} : \bar{\mathbf{m}} \otimes \bar{\mathbf{p}}}{\bar{E}^m_m + \Delta\lambda E^m_p + E_i} \quad (42)$$

where $\bar{\mathbf{m}} = \mathbf{E}^m : \mathbf{m}$, $\bar{\mathbf{n}} = \mathbf{n} : \mathbf{E}^m$, $\bar{\mathbf{p}} = \mathbf{p} : \mathbf{E}^m$ and the scalar values $\bar{E}^m_m$, E^m_p and E_i defined as

$$\bar{E}^m_m = \mathbf{n} : \mathbf{E}^m : \mathbf{m} - \bar{\mathbf{r}} : \mathbf{h} \quad (43)$$

$$E^m_p = \bar{\mathbf{r}} : \mathbf{p} : \mathbf{E}^m : \mathbf{m} \quad (44)$$

$$E_i = \frac{1}{\Delta t} \frac{\partial \psi^{-1}}{\partial \dot{\lambda}}. \quad (45)$$

The last three equations are similar to the elastoplastic case.

Note 1: Eq.(39) is valid for every possible order n of the tensor $\mathbf{q}$ of state variables. From Eqs.(15) and (41) follow that the order of the tensor $\mathbf{h}$ is equal to that of $\mathbf{q}$, *i.e.* n, while the order of $\mathbf{p}$ is $n + 2$.

Note 2: the algorithmic tangent operator of the continuous formulation approaches the consistency operator of the rate–independent elastoplastic case when $\eta \to 0$. The other extreme case, when $\eta \to \infty$ leads to the elastic tensor. With other words, the continuous formulation of Perzyna viscoplastic materials leads to algorithmic tangent tensors which signal a smooth transition between the elastic one and that of the elastoplastic case.

V. COHESIVE/FRICTIONAL VISCOPLASTIC MODEL

To analyze the failure conditions of viscoplastic material descriptions attention is focused on the elasto–(visco–)plastic Extended Leon concrete model, cf. Etse & Willam (1994). In this section the relevant constitutive equations are briefly summarized.

The rate-independent triaxial Extended Leon Model (ELM) for concrete features a non-associated flow theory of plasticity with isotropic hardening in the pre-peak regime and isotropic fracture energy-based softening in the post-peak regime. The encompassing loading surface in the hardening and softening regimes is described as follows:

$$F = \left[\frac{(1-k)}{f_c^2}\left(\sigma + \frac{\rho g(\theta)}{\sqrt{6}}\right)^2 + \sqrt{\frac{3}{2}}\frac{\rho g(\theta)}{f_c'}\right]^2 + \frac{k^2 m}{f_c'}\left(\sigma + \frac{\rho g(\theta)}{\sqrt{6}}\right) - k^2 c = 0, \quad (46)$$

where σ, ρ and θ designates the three High Westergard stress coordinates and f_c' the uniaxial compressive strength. $g(\theta)$ describes the variation of the deviatoric shear strength $\rho = \rho(\theta, e)$ as a function of the Lode angle and of the eccentricity $1/2 \le e = \rho_t/\rho_c \le 1$, whereby the elliptic approximation of the *5 Parameter Model* of Willam & Warnke (1974) is defined

$$g(\theta) = \frac{2a(\theta,e)\cos\theta + b(e)^2}{a(\theta,e) + b(e)\sqrt{2a(\theta,e)\cos\theta + c(e)}}. \quad (47)$$

with the functions $a(\theta, e)$, $b(e)$ and $c(e)$ defined as

$$a(\theta,e) = 2(1-e^2)\cos\theta \quad (48)$$

$$b(e) = (2e-1) \quad (49)$$

$$c(e) = 5e^2 - 4e \quad (50)$$

The different loading surfaces in the hardening regime are generated by the strength parameter k which varies between 0.1 and 1.0, while c and m, the cohesion and friction parameters, remain constant. k is defined in terms of the strain–hardening parameter κ_h as

$$k = k(\kappa_h) = k_o + (1-k_o)\sqrt{\kappa_h(2-\kappa_h)} \quad \text{with } \dot{\kappa}_h = \frac{1}{x_p}\dot{\lambda}\,\|\,\mathbf{m}\,\|\,. \quad (51)$$

Thereby x_p represents the hardening ductility measure which account for the level of confining stress in the hardening behavior.

The maximum strength surface is reached when $k = 1$. Continuous plastic loading in the failure regime below the transition point of brittle–ductile fracture leads to softening which is defined by a fracture energy–based formulation involving a characteristic length to account for mesh objectivity. The overall softening mechanism is described by decohesion and equivalent degradation of tensile bond stress σ_t, expressed in terms of the strain–softening parameter κ_f which represents the difference between crack spacing in mode I and mode II type fracture

$$c = \frac{\sigma_t(\kappa_f)}{f_t'} = \exp\left(-5\frac{\kappa_f}{u_r}\right), \text{with} \dot{\kappa}_f = \frac{1}{x_f}\dot{\lambda}\,\|\,\langle \mathbf{m}^p \rangle\,\| \quad (52)$$

f_t' designates the uniaxial tensile strength, u_r the rupture displacement for mode I type fracture and x_f the softening ductility measure. The fracture energy concepts are introduced in the definition of x_f which depends on the ratio between the fracture energy release rate for mode I and for mode II type of fracture, and of the characteristic length. Finally, $\mathbf{m}^p$ represents the gradient tensor of the plastic potential in the principal stresses space.

To reduce excessive dilatation in the low confinement region, a non-associated flow rule is included. The plastic potential is formulated on the basis of a volumetric modification of the yield condition

$$Q = \left[(1-k)\left(\frac{\sigma}{f_c'} + \frac{\rho g(\theta)}{\sqrt{6}f_c'}\right)^2 + \sqrt{\frac{3}{2}}\frac{\rho g(\theta)}{f_c'}\right]^2 + \frac{k^2}{f_c'}\left(m_Q + m\frac{\rho g(\theta)}{\sqrt{6}}\right) - k^2 c = 0 \quad (53)$$

with the friction parameter $m \to m_Q = m_Q(\sigma)$ being redefined in terms of the volumetric stress $\sigma = \frac{1}{3}\mathbf{I} : \boldsymbol{\sigma}$, see Etse & Willam (1994).

To preserve optimal convergence rate of the iterative solution in the Newton method the algorithmic or consistent tangent operator of the elastoplastic concrete model was developed by Etse (1992) and is presented by Etse & Willam (1996).

The constitutive and algebraic equations of the elastic-plastic backbone description for concrete were used for the evaluation of the algorithmic tangent moduli of the Extended Leon–Perzyna viscoplastic model based on both the classical and the continuous formulation.

VI. NUMERICAL ANALYSIS OF FAILURE INDICATORS

In this section the performance of the diffuse failure indicator and of the condition for localized failure are analyzed. The failure indicators are evaluated from the consistent operator obtained both with the classical and with the continuous Perzyna generalization of the ELM.

Figure 1 illustrates the maximum strength surface of the ELM, which corresponds to the strength parameter $k = 1$ and the cohesion parameter $c = 1$, in the particular case of plane strain condition whereby the relation $\sigma_3 = \nu(\sigma_1 + \sigma_2)$ between the out of plane stress (σ_3) and the other two normal stresses is valid.

Softening behavior takes place below the so-called Transition Point (TP) of brittle-ductile behavior. Figure 2 shows a comparison between the maximum strength surface and the yield surface in softening corresponding to $c = 0.5$.

Due to the symmetry with respect to the hydrostatic axis, the performance of failure indicators are analyzed for stress states located along one half of the yield surface in softening.

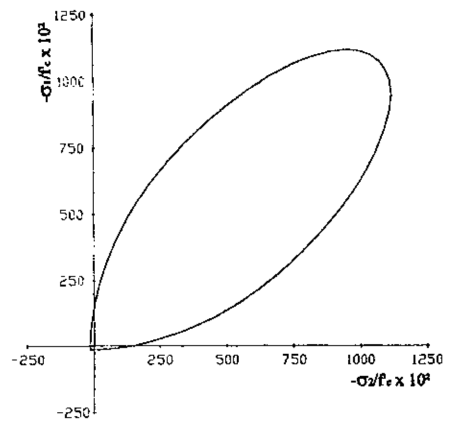

Figure 1: Extended Leon Model. Maximum strength surface in plane strain condition for $\sigma_3 = \nu(\sigma_1 + \sigma_2)$.

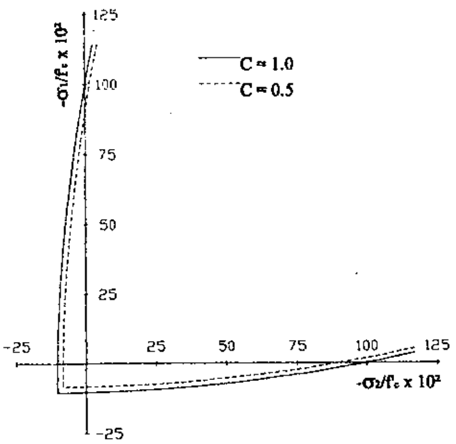

Figure 2: Extended Leon Model. Maximum strength surface and yield surface for $c = 0.5$. Plane strain condition with $\sigma_3 = \nu(\sigma_1 + \sigma_2)$.

A. Performance of the Diffuse Failure Indicator

Figure 3 illustrates the variation of the normalized diffuse failure indicator $\det \mathbf{E}_{ep}^{alg} / \det \mathbf{E}$ for the stress points located on the yield surface characterized by $c = 0.5$ of the inviscid elastoplastic model under plane strain condition. As before, the relation $\sigma_3 = \nu(\sigma_1 + \sigma_2)$ between the out of plane stress (σ_3) and the other two normal stresses is considered. In the evaluation of $\det \mathbf{E}_{ep}^{alg}$ the load is applied in terms of the plastic multiplier $\Delta\lambda$, which defines the amount of plastification in a single time step.

The normalized determinant of the material operator at every stress point is represented by a line segment normal to the load surface in a proper scale. The results in Figure 3 indicate that in the high confinement zone, perfect plasticity and no softening takes place. On the other hand, the maximum softening is obtained in the biaxial tensile region.

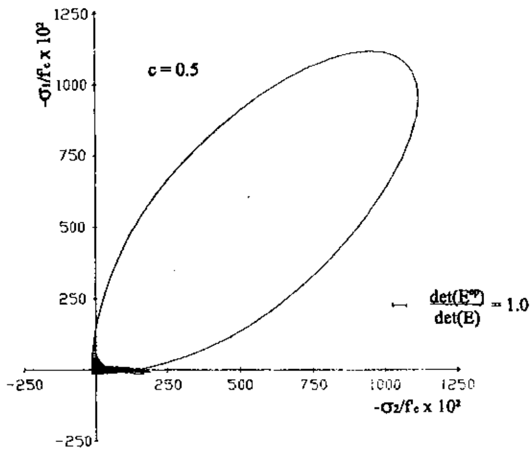

Figure 3: Elastoplastic Extended Leon Model. Normalized diffuse failure indicator on the yield surface $c = 0.5$

Figure 4 shows the performance of the indicator for diffuse failure $\det (\mathbf{E}_{Per}^{alg})^{class} / \det \mathbf{E}$ for the case of the classical Perzyna model. In all studies the time step was held constant, $\Delta t = 1$, while for the viscosity the values 10 and 100 were considered. With increasing η the normalized diffuse failure indicator approaches the value 1 indicating elastic behavior. For the smaller value of η the normalized determinant decreases but remain positive.

Figure 5 illustrates clearly that when η approaches zero also the normalized determinant of the material operator tends to zero. However, strong oscillations and discontinuities appear. Moreover, in some regions the indicator turns negative and the hold performance is quite different from that obtained with the inviscid model. With other words, in case of the classical Perzyna formulation the no viscosity effect does

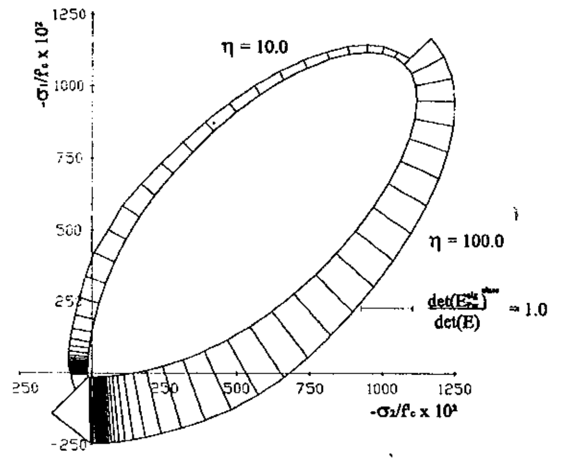

Figure 4: Classical Perzyna-based Extended Leon Model. Normalized diffuse failure indicator on the yield surface $c = 0.5$ for $\eta = 100.0$ and $\eta = 10.0$

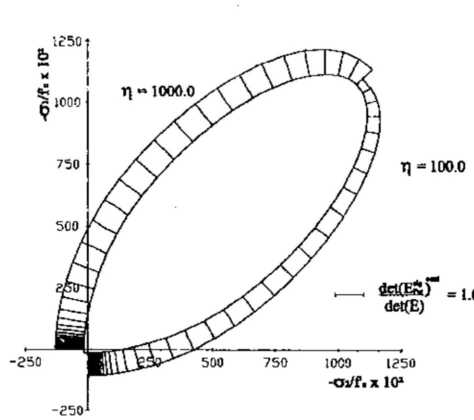

Figure 6: Continuous Perzyna-based Extended Leon Model. Normalized diffuse failure indicators on the yield surface $c = 0.5$ for $\eta = 1000.0$ and $\eta = 100.0$.

not lead to the same failure predictions as the inviscid model.

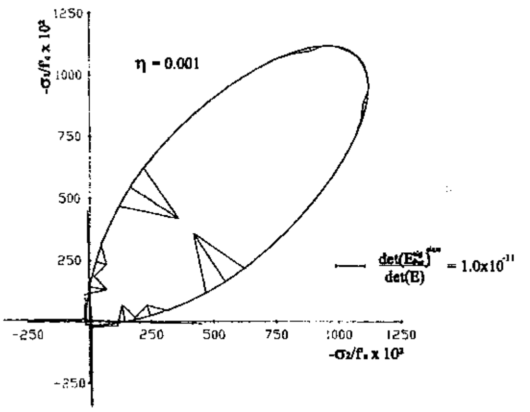

Figure 5: Classical Perzyna-based Extended Leon Model. Normalized diffuse failure indicator on the yield surface $c = 0.5$ for $\eta = 1.0$.

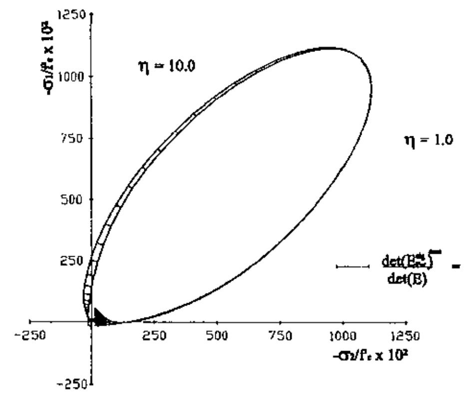

Figure 7: Continuous Perzyna-based Extended Leon Model. Normalized diffuse failure indicators on the yield surface $c = 0.5$ for $\eta = 10.0$ and $\eta = 1.0$.

The performance of the indicator for diffuse failure in case of the continuous Perzyna model is illustrated in Figures 6 and 7. Contrary to the case of the classical Perzyna model, a smooth transition from the elastic to the elastoplastic predictions of the normalized diffuse failure indicator is obtained when η varies from very large to very small values. This performance is quite similar to that obtained with the Duvaut-Lions viscoplastic formulation, see Etse & Willam (1999).

B. Performance of the Localization Indicator

Figure 8 illustrates the variation of the normalized localization indicator along the yield surface $c = 0.5$ of the inviscid ELM. In the high confinement zone no localization takes place and only diffuse failure is obtained. Localized failure occur in the uniaxial compression and low confinement zone as well as in the biaxial tensile zone.

The performance of the localization indicator for classical Perzyna generalization of the ELM is illustrated in Figures 9 and 10 for different values of η. Similar to the performance of the diffuse failure diagnostic corresponding to the classical Perzyna model, oscillations and even discontinuities can be observed for decreasing values of the viscosity. This effect can be clearly recognized in the diagram depicted in Figure 10 corresponding to the performance of normalized localization indicator when a very small value of η was used.

Figures 11 and 12 illustrates the variation of the normalized acoustic tensor determinant of the continuous

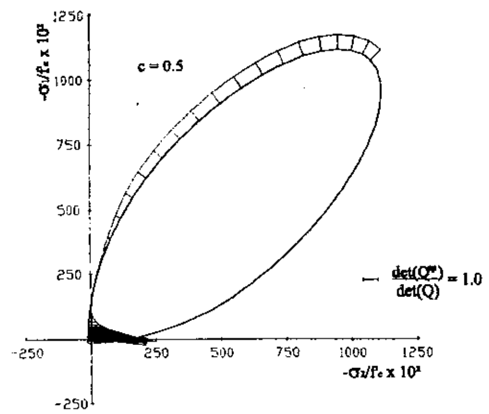

Figure 8: Elastoplastic Extended Leon Model. Normalized localized failure indicator on the yield surface $c = 0.5$

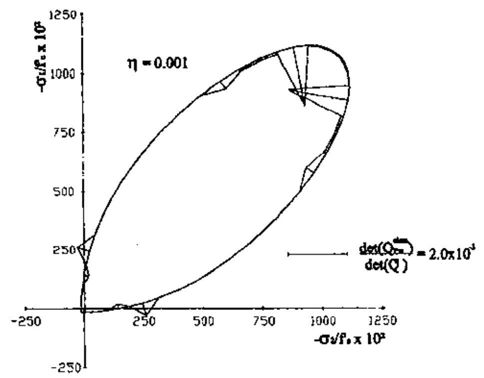

Figure 10: Classical Perzyna-based Extended Leon Model. Normalized localized failure indicator on the yield surface $c = 0.5$ for $\eta = 0.001$

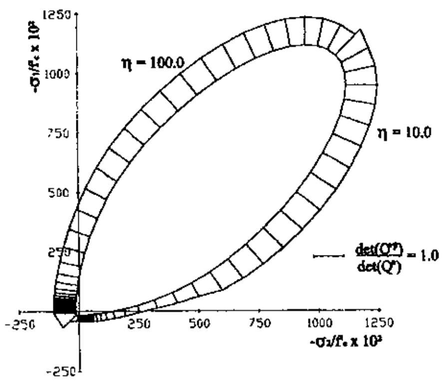

Figure 9: Classical Perzyna-based Extended Leon Model. Normalized localized failure indicators on the yield surface $c = 0.5$ for $\eta = 100.0$ and $\eta = 1.0$.

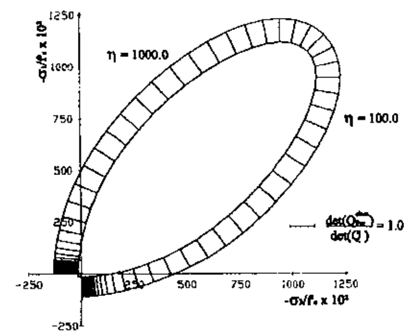

Figure 11: Continuous Perzyna-based Extended Leon Model. Normalized localized failure indicators on the yield surface $c = 0.5$ for $\eta = 1000.0$ and $\eta = 100.0$.

Perzyna model. These results demonstrate again that the continuous Perzyna formulation leads to a smooth transition from elastic to elastoplastic behavior when η varies from large to small values and for every possible stress state.

The evolution of the indicator for localized failure and the predictions of critical bifurcation direction for the limit point stress state of the uniaxial compression test in plane strain condition are illustrated in Figures 13 and 14 for both Perzyna formulations. The plots in Figure 14 corresponding to the continuous Perzyna formulation show that for decreasing values of the viscosity a continuous increase of the elastic properties degradation of the localized failure indicator is obtained. Moreover, for very small values of η the continuous Perzyna model renders same bifurcation directions as the inviscid elastoplastic material. This is not the case of the classical Perzyna model as it can be observed in Figure 13. These results further illustrate the fundamental differences between failure predictions of both Perzyna viscoplastic formulations.

VII. CONCLUSIONS

In this paper two different formulations for Perzyna viscoplasticity were analyzed with regard to the predictions of diffuse and localized failure for stress states under plane strain conditions. Considering the time integration of real viscoplastic material processes within finite time increments, then the algorithmic tangent operator replace the instantaneous one which do not exhibit degradation of the elastic properties. From the consistent linearization process based on the Backward Euler method for time integration of the differential equations, the algorithmic tangent operator of both Perzyna formulations were obtained which exhibit quite different features. The numerical results

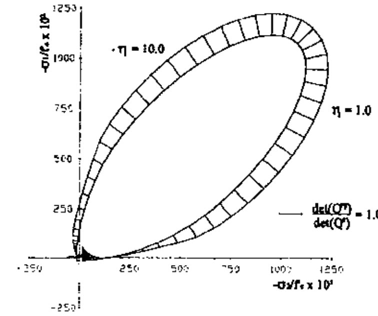

Figure 12: Continuous Perzyna-based Extended Leon Model. Normalized localized failure indicators on the yield surface $c = 0.5$ for $\eta = 10.0$ and $\eta = 1.0$.

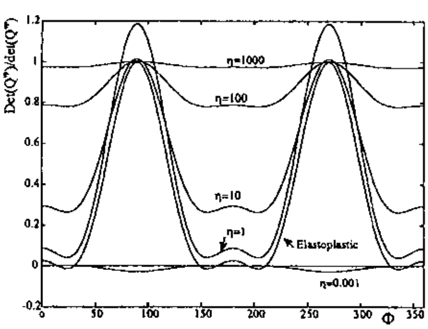

Figure 13: Normalized acoustic tensor determinant plot for Classical Perzyna Viscoplastic and for Elastoplastic Formulations of Extended Leon Model. Limit point stress state of uniaxial compression test in plane strain condition.

in this paper support the conclusion that these differences are responsible for failure predictions which show considerable disagreement when the viscosity approaches zero.

In this extreme case the classical Perzyna model do not reproduce the predictions of the inviscid material and the performance of the diffuse and localized failure indicators exhibit strong oscillations and even discontinuities due to numerical instabilities which arise from the time integration process.

On the other hand the continuous Perzyna formulation leads to algorithmic material operators which exhibit a smooth transition from the elastic to the elastoplastic tensor according to the assumed value for the viscosity. Therefore, when η approaches zero the same localization predictions as the inviscid material are obtained.

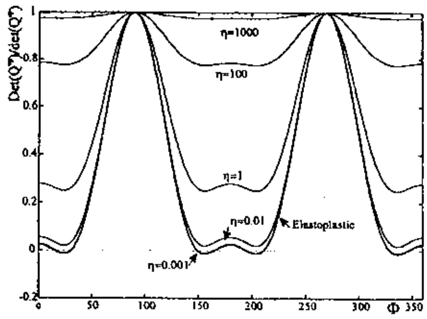

Figure 14: Normalized acoustic tensor determinant plot for Continuous Perzyna Viscoplastic and for Elastoplastic Formulations of Extended Leon Model. Limit point stress state of uniaxial compression test in plane strain condition.

VIII. ACKNOWLEDGMENTS

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