Critical behavior of a nonlocal ϕ^4 field theory and asymptotic freedom

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The critical behavior of a nonlocal scalar field theory is studied. This theory has a nonlocal kinetic term which involves a real power $1-2\alpha$ of the Laplacian. The interaction term is the usual local ϕ^4 interaction. The lowest order Feynman diagrams corresponding to coupling constant renormalization, mass renormalization, and field renormalization are computed. Particular features appearing in the renormalization of this nonlocal theory that differ from the case of local theories are studied. The previous calculations lead to the perturbative computation of the coupling constant beta function and critical exponents ν and η. In four dimensions for $\alpha < 0$ this beta function presents asymptotic freedom in the UV. This is remarkable since no non-Abelian vector fields are included. However, this comes at the expense of losing reflection positivity.

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I. INTRODUCTION

The computation of critical exponents for the 3-dimensional Ising model using the ϵ-expansion provides a concrete example of the relevance of the renormalization group ideas [1,2]. This is done by considering a self-interacting ϕ^4 theory in $d = 4 - \epsilon$ dimensions, where ϵ is allowed to take real values. This procedure led to a qualitative understanding of the 3-dimensional Ising model physics and to predictions for critical exponents in reasonable agreement with the exact values. The results for the theory in d-dimensions are obtained by computing the theory in an integer number n of dimensions and then replacing n by d.

The renormalization group consists in the study of the evolution of a system under scale transformations. This system involves all possible interactions of any range for all kinds of dynamical variables. Different physical systems correspond to the study of particular fixed points in this huge space of couplings. This paper studies a particular example of system described near the corresponding fix point by a nonlocal field theory. Nonlocal field theories appear in various aspects of physics. These include proposals for dealing with quantum gravity [3], field theories based on noncommutative geometry [4] and in critical phenomena. The use of nonlocal field theories in the description of critical phenomena is not new [5–8]. Such models appear in statistical systems with long range interactions. In this paper the critical behavior of a nonlocal field theory is studied. This nonlocal theory is motivated by an alternative approach to noninteger dimensional spaces (NIDS) [9]. Free scalar theories on these spaces have been studied in this last reference. This theory has been employed to compute loop corrections and compare the results with dimensional regularization [10,11], showing that the structure of singularities is the same as in dimensional regularization. In addition, the fulfillment or not of the requirement of reflection positivity for the corresponding Euclidean field theory has been considered [12]. There, it is shown that for negative values of the noninteger power mentioned above, the theory fulfills reflection positivity. This means that the corresponding theory in Minkowski space is unitary for those values of the noninteger power. The aim in this work is to add a ϕ^4 interaction and study the renormalization and critical properties of the resulting nonlocal theory.[1] This study shows the relevance of this model in describing nontrivial fixed points. The features and results of this work are summarized as follows:

(i) The theory to be considered is the free scalar theory studied in [9] with the addition of a ϕ^4 interaction term.

(ii) The contribution of the lowest order Feynman diagrams corresponding to coupling constant renormalization, mass renormalization, and field renormalization are computed. This computation exemplifies general issues about the renormalization of nonlocal field theories. The procedure employed involves features that do not appear in the local case.

[1] This theory can also be obtained as the analytic regularized [13] version of the usual ϕ^4 local field theory.

(iii) The previous calculation allows to compute the fixed point value for the coupling constant and the critical exponents ν and η, respectively. The corresponding results describe a theory which shows asymptotic freedom in the UV and a nontrivial infrared fixed point at finite coupling. The corresponding theory does not fulfill the condition of reflection positivity.

(iv) In addition, assuming the usual n-dimensional conformal algebra to be a symmetry of the theory, the unitarity bounds are studied. They agree with the ones obtained by requiring the condition of reflection positivity.

II. THE ACTION

The free part of the action to be considered is essentially the same as in [9] for[2] $M = 0$. The interaction part is ϕ^4. In terms of the scalar product of form fields mentioned above and described in [9], the action is given by:

$$S = S_0 + S_I, \qquad S_0 = \frac{1}{2}\langle d\phi, d\phi\rangle, \qquad S_I = \frac{\lambda_0}{4!}\langle \phi^2, \phi^2\rangle, \qquad (2.1)$$

evaluating the scalar products appearing in the last equation leads to the following expression in terms of an integral over the integer n-dimensional space:

$$S_0 = \int d^n x \frac{1}{2}\phi(-\Box + m_0^2)(-\Box)^{-2\alpha}\phi, \qquad S_I = \frac{\lambda_0}{4!}\int d^n x \phi^4,$$

where, anticipating renormalization effects, an explicit mass term has been included.[3] In what follows, bare mass and coupling will be indicated by m_0 and λ_0; the corresponding renormalized quantities will be m and λ. The Fourier transform of the free two point function is therefore given by:

$$\begin{aligned}\langle\phi\phi\rangle(p) &= \frac{1}{(p^2 + m_0^2)(p^2)^{-2\alpha}} \\ &= \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\int_0^1 da(1-a)^{-1-2\alpha}\frac{1}{(p^2 + m_0^2 a)^{1-2\alpha}};\end{aligned}$$

the second equality in the last equation is obtained using Feynman parametrization. This last expression will be employed in the computations below.

III. RENORMALIZATION AND THE CRITICAL EXPONENTS

A. Field renormalization

Field renormalization is required at the two loop level. The corresponding correction to the two point function is given by the following sunrise diagram (see Fig. 1).

The integral to be computed is

$$\begin{aligned} a_S(p,\alpha) &= \left(\frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\right)^3 \\ &\quad \times \int_0^1 \left(\prod_{i=1}^3 da_i (1-a_i)^{-1-2\alpha}\right) \\ &\quad \times I_S(p,\alpha,a_1,a_2,a_3) \\ I_S(p,\alpha,a_1,a_2,a_3) &= \lambda_0^2 \int \frac{d^n q_1}{(2\pi)^n}\frac{d^n q_2}{(2\pi)^n} \\ &\quad \times \frac{1}{(q_1^2 + a_1 m_0^2)^{1-2\alpha}(q_2^2 + a_2 m_0^2)^{1-2\alpha}} \\ &\quad \times \frac{1}{[(p+q_1+q_2)^2 + a_3 m_0^2]^{1-2\alpha}}.\end{aligned}$$

The correction to the kinetic term comes from the above integral evaluated at $m_0 = 0$. This integral is,

$$\begin{aligned} I_0(p,\alpha) &= \lambda_0^2 \int \frac{d^n q_1}{(2\pi)^n}\frac{d^n q_2}{(2\pi)^n} \\ &\quad \times \frac{1}{(q_1^2)^{1-2\alpha}(q_2^2)^{1-2\alpha}[(p+q_1+q_2)^2]^{1-2\alpha}}\end{aligned}$$

introducing Feynman parametrizations to rewrite the integrand, performing the momentum integrals and the integrals on the Feynman parameters and taking $n = 4$ leads to,

$$I_0(p,\alpha) = p^{2(6\alpha+1)}\frac{4^{-2\alpha-3}\lambda_0^2 \csc(4\pi\alpha)\Gamma(-6\alpha-1)\Gamma(\frac{3}{2}-2\alpha)B_1(4\alpha+2, 2\alpha+1)}{\pi^{7/2}\Gamma(3-8\alpha)\Gamma(1-2\alpha)^2\Gamma(4\alpha)}$$

The coupling λ_0 has dimension $4 - n - 8\alpha$ in momentum units, therefore, for $n = 4$, it can be written as follows in terms of an adimensional coupling g_0 as follows,

[2] No infrared regulator is required for the following computations.
[3] This way of introducing a mass term is motivated by the calculation of perturbative corrections appearing below.

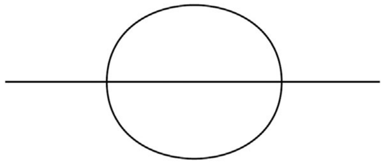


FIG. 1. Lowest order diagram contributing to field renormalization.

$$\lambda_0 = g_0\mu^{-8\alpha} \Rightarrow g = \lambda\mu^{8\alpha} \tag{3.1}$$

noting that,

$$\left(\frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\right)^3 \int_0^1 \left(\prod_{i=1}^3 da_i(1-a_i)^{-1-2\alpha}\right) = 1$$

leads to,

$$a_S(p,\alpha) = \frac{g_0^2}{(4\pi)^4}\mu^{-16\alpha}(p^2)^{6\alpha}\left[\frac{p^2}{12\alpha} + \mathcal{O}(\alpha^0, m)\right]$$
$$= \frac{g_0^2}{(4\pi)^4}\mu^{-16\alpha}\frac{(p^2)^{6\alpha}}{12\alpha}[p^2 + \alpha\mathcal{O}(\alpha^0, m)]$$

where $\mathcal{O}(\alpha^0, m)$ denote terms that vanish when α and m go to zero. At this stage a recurrent situation in the renormalization of these non-local theories shows up. Similar to what happens in dimensional regularization the correction provided by a given diagram, in this case the sunrise diagram, is proportional to a power of the momentum which is not in general the same as the one that originally appears in the Lagrangian. The integral $I_0(p,\alpha)$ gives a contribution[4] proportional to $p^{2(1+6\alpha)}$, while the original Lagrangian has the power $p^{2(1-2\alpha)}$. The choice of the power of p^2 that appears in the kinetic term of the renormalized Lagrangian fixes the finite contribution of this diagram. In other words, if a different power of p^2 is chosen then the finite contribution of the diagram will also be different. The following way of rewriting $a_S(p,\alpha)$ illustrates this point,

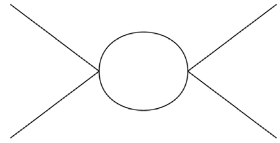


FIG. 2. Lowest order diagram contributing to coupling constant renormalization.

$$a_S(p,\alpha) = \frac{g_0^2}{(4\pi)^4}\mu^{-16\alpha}\frac{(p^2)^{-2\alpha}}{12\alpha}(p^2)^{8\alpha}[p^2 + \alpha\mathcal{O}(\alpha^0, m)]$$
$$= \frac{g_0^2}{(4\pi)^4}\frac{(p^2)^{-2\alpha}}{12\alpha}[p^2 + \alpha\mathcal{O}(\alpha^0, m)]$$
$$\times \left(1 + 8\alpha\log\left(\frac{p^2}{\mu^2}\right) + \mathcal{O}(\alpha^2)\right)$$
$$= \frac{g_0^2}{(4\pi)^4}\frac{(p^2)^{-2\alpha}}{12\alpha}[p^2 + \alpha\mathcal{O}(\alpha^0, m)]. \tag{3.2}$$

For $\alpha \to 0$ the pole term of the last expression multiplied by the symmetry [14] factor $\frac{1}{6}$ is the one to be subtracted. It is given by:

$$\left(\frac{1}{6}a_S(p,\alpha)\right)_{\text{pole}} = \frac{g_0^2}{(4\pi)^4}\frac{p^{2(1-2\alpha)}}{72\alpha},$$

which leads to the renormalization constant,

$$Z_\phi = 1 + \frac{g_0^2}{(4\pi)^4}\frac{1}{72\alpha},$$

the function γ is defined and given by:

$$\gamma(g) = \mu\frac{\partial}{\partial\mu}\log Z_\phi^{\frac{1}{2}}\Big|_{\lambda\text{fixed}} = \frac{1}{2}\frac{\partial}{\partial\log\mu}\log\left(1 + \frac{\lambda_0^2\mu^{16\alpha}}{(4\pi)^4}\frac{1}{72\alpha}\right)$$
$$= \frac{1}{2Z_\phi}\left(\frac{\lambda_0^2 16\alpha}{(4\pi)^4 72\alpha}\mu^{16\alpha}\right) = \frac{g^2}{9(4\pi)^4} + \mathcal{O}(g^4). \tag{3.3}$$

B. The fixed point and coupling constant renormalization

The diagram to be considered is the one corresponding to the one loop correction to the quartic coupling, i.e., Fig. 2.

The integral to be computed is:

$$a_F(p,\alpha) = \left(\frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\right)^2 \int_0^1 \left(\prod_{i=1}^2 da_i(1-a_i)^{-1-2\alpha}\right) I_F(p,\alpha,a,b)$$
$$I_F(p,\alpha,a,b) = \frac{3}{2}\lambda_0^2 \int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2+m_0^2a)^{1-2\alpha}((p+q)^2+m_0^2b)^{1-2\alpha}};$$

[4]In dimensional regularization of the usual local ϕ^4 theory, the correction provided by the sunrise diagram is proportional to $p^{2(d-3)} = p^{2(1-\epsilon)}$.

the factor $\frac{3}{2}$ coming from the $\frac{1}{2!}$ of the second order term of the exponential and the contributions of 3 diagrams which give the same contribution. Introducing the Feynman parametrization and integrating over the n-moment q, leads to:

$$\begin{aligned} I_F(p,\alpha,a,b) &= \frac{3}{2}\frac{\lambda_0^2}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(2-\frac{n}{2}-4\alpha)}{\Gamma(1-2\alpha)^2}\int_0^1 dx\frac{(m^2[ax+b(1-x)]+p^2(1-x)x)^{\frac{n}{2}-2+4\alpha}}{((1-x)x)^{2\alpha}} \\ &\overset{n=4}{=}\frac{3}{2}\frac{\lambda^2\mu^{8\alpha}}{(4\pi)^2}\frac{\Gamma(-4\alpha)}{\Gamma(1-2\alpha)^2}\int_0^1 dx\left[\frac{(m_0^2+p^2(1-x)x)^2}{\mu^4(1-x)x}\right]^{2\alpha} \\ &\overset{\alpha\ll 1}{=}\frac{3(\lambda_0\mu^{4\alpha})^2}{4(4\pi)^2}\left[-\frac{1}{2\alpha}+\int_0^1 dx\log\left(\frac{\mu^4(1-x)x}{(m_0^2-p^2(x-1)x)^2}\right)\right], \end{aligned}$$

where in the second equality a parameter μ with dimensions of mass has been introduced in order to make adimensional the argument of the logarithm. In addition, in the last equality only terms up to $\mathcal{O}(\alpha^0)$ have been kept. In the minimal subtraction scheme only the first term in the square bracket of the last expression will be relevant in defining the renormalized coupling λ_R. This term is independent of a and b; therefore,

$$a_F(p,\alpha)=\left(\frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\right)^2\int_0^1 dadb(1-a)^{-1-2\alpha}(1-b)^{-1-2\alpha}\frac{3(\lambda_0\mu^{4\alpha})^2}{4(4\pi)^2}\left(-\frac{1}{2\alpha}\right)=\frac{3(\lambda_0\mu^{4\alpha})^2}{4(4\pi)^2}\left(-\frac{1}{2\alpha}\right).$$

Taking into account the computation in the last subsection, this leads to the following renormalized coupling:

$$\lambda=\frac{\lambda_0 Z_\phi^2}{Z_g},\qquad Z_g=1+\frac{3}{4}\frac{\lambda_0}{(4\pi)^2}\frac{(\mu^2)^{4\alpha}}{(-2\alpha)}+\mathcal{O}(\lambda_0^2),\qquad Z_\phi=1+\frac{g_0^2}{(4\pi)^4}\frac{1}{72\alpha};$$

the beta function corresponding to the renormalized adimensional coupling g fulfills

$$\begin{aligned} \beta(g_R)=\mu\frac{d}{d\mu}g_R=\mu\frac{d}{d\mu}\left(\frac{\lambda Z_\phi^2}{Z_g}\mu^{8\alpha}\right) \\ =8\alpha g_R+2g_R Z_\phi^{-1}\mu\frac{d}{d\mu}Z_\phi+g_R Z_g^{-1}\mu\frac{d}{d\mu}Z_g \\ =8\alpha g_R+4g_R\gamma+g_R\frac{3}{8\alpha}\frac{1}{(4\pi)^2}\beta(g_R), \end{aligned}$$

which implies

$$\beta(g_R)=8\alpha g_R+\frac{6g_R^2}{2(4\pi)^2}+4\frac{g_R^3}{9(4\pi)^4}.$$

Neglecting negative values of g, which make the theory unstable, the figure below shows a plot of this function for $\alpha=\pm 0.01$ (see Fig. 3).

This figure shows that for $\alpha>0$ the theory has asymptotic freedom in the infrared. However for $\alpha<0$ this 4-dimensional theory presents asymptotic freedom (AF) in the ultraviolet (UV). This is a remarkable result since it is usually believed that non-Abelian gauge bosons are required in order to get AS in the UV. However, as the analysis in [12] shows, the theory for $\alpha<0$ does not satisfy the requirement of reflection positivity (RP). This means that the Wick rotated theory in Minkowski space does not provide a unitary representation of the Poincaré group, which implies that no unitary evolution can be defined in this space. Alternatively, as will be shown in the next

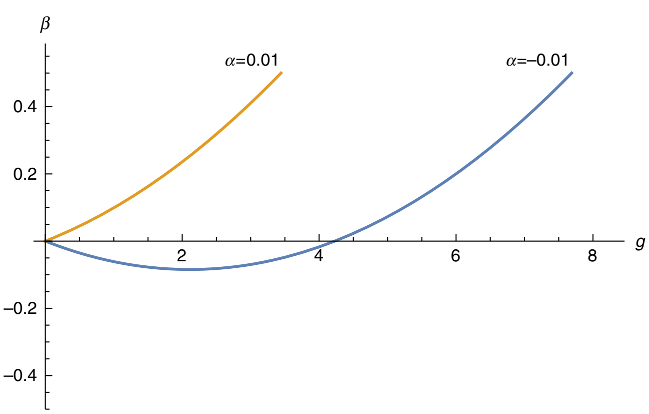


FIG. 3. Quartic coupling beta function for $\alpha=\pm 0.01$.

section, the unitarity bounds are violated for $\alpha < 0$. This does not mean that the Euclidean theory is useless; indeed many useful statistical mechanical models fail to satisfy RP.

The fixed point $g^\star$ is defined by $\beta(g^\star) = 0$. Writing the solution of this last equation as a power series in α,

$$g^\star = g_0 + g_1\alpha + g_2\alpha^2 + \cdots$$

leads to two solutions, the Gaussian fixed point $g^\star = 0$ and

$$g^\star = -\frac{8}{3}(4\pi)^2\alpha - \frac{256}{243}(4\pi)^2\alpha^2.$$

C. Mass renormalization

The one loop correction to the two point function is given by the diagram in Fig. 4.

The integral to be computed is

$$a_T(\alpha) = \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\int_0^1 da(1-a)^{-1-2\alpha} I_T(\alpha, a)$$
$$I_T(\alpha, a) = -\lambda_0 \int \frac{d^n q}{(2\pi)^n}\frac{1}{(q^2 + m_0^2 a)^{1-2\alpha}}, \quad (3.4)$$

leading to

$$I_T(\alpha, a) = -\frac{\lambda_0}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(1-\frac{n}{2}-2\alpha)}{\Gamma(1-2\alpha)}(m_0^2 a)^{\frac{n}{2}-1+2\alpha},$$

replacing in (3.4) leads to

$$\begin{aligned} a_T(\alpha) &= \frac{\Gamma(1-2\alpha)}{\Gamma(-2\alpha)}\int_0^1 da(1-a)^{-1-2\alpha} \\ &\quad \times\left(-\frac{\lambda_0}{(4\pi)^{\frac{n}{2}}}\frac{\Gamma(1-\frac{n}{2}-2\alpha)}{\Gamma(1-2\alpha)}(m_0^2 a)^{\frac{n}{2}-1+2\alpha}\right) \\ &\overset{n=4}{=} \frac{-g_0\mu^{-8\alpha}}{16\pi\sin(2\pi\alpha)}(m_0^2)^{1+2\alpha} \\ &= \frac{-g_0}{16\pi\sin(2\pi\alpha)}(m_0^2)^{1-2\alpha}\left(\frac{m_0^2}{\mu^2}\right)^{4\alpha} \\ &= (m_0^2)^{1-2\alpha}\left(\frac{-g_0}{(4\pi)^2 2\alpha} + \mathcal{O}(\alpha^0)\right), \end{aligned}$$

where in the second equality the dimensional coupling λ has been expressed in terms of the adimensional coupling g by means of (3.1). As was mentioned for the case of the sunrise diagram, in this case also the power of p^2 appearing in the correction is different from the one appearing in the Lagrangian. In a similar way as for the sunrise, the choice of the power to appear in the renormalized Lagrangian fixes the finite contribution of this diagram. This last point is illustrated by the following computation of the correction to the proper two point function[5]:

$$\begin{aligned} \Gamma_2(p) &= (p^2 + m_0^2)(p^2)^{-2\alpha} - \frac{1}{2}a_T(\alpha) \\ &= (p^2 + m_0^2)(p^2)^{-2\alpha} + (m_0^2)^{1-2\alpha}\frac{g_0}{(4\pi)^2 4\alpha} + C \\ &= (p^2 + m_0^2)(p^2)^{-2\alpha} + m_0^2(p^2)^{-2\alpha}\left(\frac{m_0^2}{p^2}\right)^{-2\alpha} \\ &\quad \times\frac{g_0}{(4\pi)^2 4\alpha} + C \\ &= \left(p^2 + m_0^2\left(1 + \frac{g_0}{(4\pi)^2 4\alpha}\right)\right)(p^2)^{-2\alpha} + C', \end{aligned}$$

where C and C' denote terms that converge for $\alpha \to 0$. Therefore, taking into account the computation in subsection III A, the renormalized mass m in the minimal subtraction scheme is given by

$$m^2 = m_0^2\frac{Z_\phi}{Z_{m^2}},$$

where, up to $\mathcal{O}(g^2)$,

$$Z_{m^2} = 1 - \frac{g_0}{(4\pi)^{\frac{n}{2}}}\frac{1}{4\alpha};$$

the beta function γ_m for the mass is given by

$$\begin{aligned} \gamma_m(g) = \frac{\mu}{m}\frac{\partial m}{\partial\mu} &= \frac{1}{2}\left(\frac{\mu}{Z_\phi}\frac{\partial \log Z_\phi}{\partial\mu} - \mu\frac{\partial Z_{m^2}}{\partial\mu}\right) \\ &= \gamma + \frac{1}{2}\frac{\beta(g)}{(4\pi)^{\frac{n}{2}}4\alpha} \\ &= \frac{1}{2}\beta(g)\left(\frac{1}{(4\pi)^{\frac{n}{2}}4\alpha} + \frac{2g_0}{(4\pi)^4 72\alpha}\right) \\ &= \frac{g}{(4\pi)^2} + \frac{g^2}{9(4\pi)^4} + \mathcal{O}(g^3). \end{aligned}$$

D. The critical exponents ν and η

These critical exponents are related to the fixed point values $\gamma^\star$ and $\gamma_m^\star$ of the functions γ and γ_m. They are given by

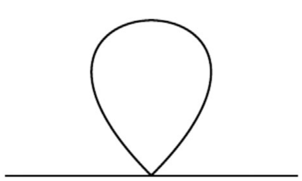
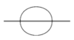


FIG. 4. Lowest order diagram contributing to mass renormalization.

[5]It is worth noting that if the mass was included with a kinetic term of the form, $\mathcal{L}_0 = \phi(-\Box + m^2)^{1-2\alpha}\phi$, then the singular contribution of this diagram when $\alpha \to 0$ could not be absorbed by mass renormalization; in other words the counterterm required to cancel the divergence when $\alpha \to 0$ would not be of the form $\mathcal{L}_0$.

TABLE I. Relation between degree of divergence and renormalization constants for the ϵ and α theories.

Diagram	ω_ϵ	ω_α	$Z(\epsilon)$	$Z(\alpha)$	$\omega_\epsilon \to \omega_\epsilon$ / $Z(\epsilon) \to Z(\alpha)$
	$2-\epsilon$	$2+4\alpha$	$Z_{m^2}(\epsilon) = 1 + \frac{g}{(4\pi)^2\epsilon}$	$Z_{m^2}(\alpha) = 1 - \frac{g}{(4\pi)^2 4\alpha}$	$\epsilon \to -4\alpha$
	$-\epsilon$	8α	$Z_g(\epsilon) = 1 + \frac{3g}{(4\pi)^2\epsilon}$	$Z_g(\alpha) = 1 - \frac{3g}{(4\pi)^2 8\alpha}$	$\epsilon \to -8\alpha$
	$2-2\epsilon$	$2+12\alpha$	$Z_\phi(\epsilon) = 1 - \frac{g^2}{(4\pi)^4 12\epsilon}$	$Z_\phi(\alpha) = 1 + \frac{g^2}{(4\pi)^4 72\alpha}$	$\epsilon \to -6\alpha$

$$\nu = \frac{1}{2-2\gamma_m^\star}, \qquad \eta = 2\gamma^\star.$$

The nontrivial fixed point is given by

$$g^\star = -\frac{8}{3}(4\pi)^2\alpha - \frac{256}{243}(4\pi)^2\alpha^2;$$

the fixed point values $\gamma_m^\star$ and $\gamma^\star$ are therefore given by

$$\gamma_m^\star = \gamma_m(g^\star) = -\frac{8\alpha}{3} + \frac{256}{243}(12\pi^2-1)\alpha^2 + \frac{65536\pi^2\alpha^3}{6561} + \mathcal{O}(\alpha^3)$$

$$\gamma^\star = \gamma(g^\star) = \frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561},$$

which implies

$$\nu = \frac{1}{2} - \frac{4\alpha}{3} + \frac{32}{243}(23+48\pi^2)\alpha^2 - \frac{256(171+736\pi^2)\alpha^3}{6561}$$
$$\eta = 2\left(\frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561}\right). \tag{3.5}$$

It is worth noting that the value of α is related to the dimension of space. A free propagator at the Gaussian fixed point in 4-dimensions should behave as $\frac{1}{|x|^2}$; this corresponds to small values of α, as the ones employed in the last figure. The critical exponents for the non-Gaussian fixed point for $\alpha = -0.01$ are

$$\nu \overset{\alpha=-0.01}{=} 0.52, \qquad \eta \overset{\alpha=-0.01}{=} 0.0001.$$

Following the same reasoning in 3 dimensions, the $\frac{1}{|x|}$ behavior of the free propagator gives $\alpha = -\frac{1}{4}$. This is the value of α which corresponds to the $\epsilon = 1$ in the ϵ-expansion. In the same spirit as in the case of the ϵ-expansion, the critical exponents for the non-Gaussian fixed point can be computed for this last value of α. Replacing $\alpha = -\frac{1}{4}$ in (3.5) leads to the following values for the critical exponents:

$$\nu \overset{\alpha=-\frac{1}{4}}{=} 4.92, \qquad \eta \overset{\alpha=-\frac{1}{4}}{=} 0.079,$$

which, in comparison with the values obtained with the ϵ-expansion, significantly differs from the $3d$-Ising model critical exponents. This shows that this fixed point does not describe the $3d$-Ising model critical point.

IV. RELATION WITH ϵ EXPANSION

For each diagram there is a way to obtain its divergent contribution (when $\alpha \to 0$) from the corresponding one in the ϵ expansion. In order to show this let us consider the superficial degree of divergence (SDD) for both theories: the one considered in this paper described by the action (2.1), from now on the α-theory; and the usual ϕ^4 theory dimensionally regularized to a dimension $d = 4-\epsilon$, from now on the ϵ-theory. The SDD for a proper graph G in the ϵ-theory is given by

$$\omega_\epsilon(G) = 4 - \epsilon(1+V) + \left(\frac{\epsilon}{2}-1\right)E,$$

where V denotes the number of vertices and E the number of external legs. For the case of the α-theory the SDD can be computed to give

$$\omega_\alpha(G) = 4 + 8\alpha V - (1+2\alpha)E,$$

which of course coincides for $\alpha = \epsilon = 0$. Note that there is no replacement of ϵ as a function of α such that for any V and E the following equality holds[6]:

$$\omega_{\epsilon(\alpha)}(G) = \omega_\alpha(G).$$

However for each given V and E there is a replacement. This is shown in the Table I, which compares the SDD and the renormalization constants for the diagrams considered in the previous section:

$$Z(\epsilon) \to Z(\alpha).$$

This table shows that knowing the SDD of a given diagram in both theories allows to obtain the renormalization constant in one theory knowing the renormalization

[6]If such a replacement were possible then an expansion in powers of α would be the same as the ϵ expansion.

constant in the other. In other words for a given diagram G the same replacement that sends $\omega_\epsilon(G)$ to $\omega_\alpha(G)$, sends $Z_G(\epsilon)$ to $Z_G(\alpha)$. This fact shows that an expansion in powers of α and the ϵ expansion are not same and describe different critical theories; this is so because of the nontrivial dependence of this replacement on the diagram considered.

V. UNITARITY BOUNDS

A. The conformal algebra in n-dimensions

The action (2.1) is invariant under conformal transformations.[7] It is assumed that there exists conserved charges implementing these transformations at the level of the field. The conformal algebra for dimensions $n \geq 3$ is given by

$$\begin{aligned} [D, P_\mu] &= iP_\mu \\ [P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\ [K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\ [L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}), \end{aligned}$$

where P_μ are the generators of translations, $L_{\mu\nu}$ the generators of rotations in the $\mu - \nu$ plane, D the generator of dilatations, and K_μ the generators of special conformal transformations. In cylindrical coordinates the Hermiticity properties of operators are such that [15]:

$$P_\mu^\dagger = K_\mu.$$

B. Positive definite inner products and bounds for α

For a spinless primary state $|\Delta>$ the commutation relation between P_μ and K_ν can be used to show that

$$\begin{aligned} &||P_\mu|\Delta>|^2 > 0 \Rightarrow \Delta > 0 \\ &||P_\mu P_\nu|\Delta>|^2 > 0 \Rightarrow \Delta > \frac{n-2}{2}, \end{aligned} \tag{5.1}$$

for a space of dimension n. For the free theory $\lambda = 0$, the dimension of the field ϕ is

$$[\phi] = \frac{n-2+4\alpha}{2};$$

thus the unitarity bound (5.1) implies

$$\alpha > 0.$$

For the interacting theory,

$$[\phi] = \frac{n-2+4\alpha}{2} + \frac{\eta}{2};$$

thus the unitarity bound implies

$$\alpha > -\frac{\eta}{4};$$

therefore, using (3.5), this implies that

$$\alpha + \frac{1}{2}\left(\frac{64\alpha^2}{81} + \frac{4096\alpha^3}{6561}\right) > 0;$$

the polynomial on the l.h.s. of the last inequality has only one real root at $\alpha = 0$, and the last inequality is equivalent to $\alpha > 0$. This shows that the free theory unitarity bound is stable under the corrections computed in this work.

VI. CONCLUDING REMARKS

Conclusions and further research motivated by this work are summarized in the series of remarks given below:

(i) It was shown that introducing a nonlocal kinetic term which involves a power 1–2α of the Laplacian for a scalar field with interaction ϕ^4, it is possible for $\alpha < 0$ to get asymptotic freedom in the UV without including non-Abelian vector fields. However for these values of α, the resulting theory can not be Wick rotated to obtain a field theory over Minkowski space realizing a unitary representation of the Poincaré group. This is so because the condition of reflection positivity is not fulfilled for $\alpha < 0$.

(ii) From the Wilson renormalization group point of view, there is no restriction on the range of interactions. Therefore it makes sense to study the renormalization of a nonlocal field theory. In this work, this has been done using the field theoretic version of this procedure. This was done for the first corrections to the two and four point functions. The concrete renormalization procedure shows features different from the case of local theories, which all the same make sense. In particular the choice of the power of the Laplacian for the renormalized kinetic term affects the finite contribution of the corresponding diagram[8] and the form of the counterterms required for mass renormalization dictate the nonlocal version of the mass term to be employed.[9]

[7]Given that this theory can be thought of as a theory depending on derivatives of the field of any order, then there should be an infinite number of conserved charges. This assertion is not analyzed in this paper.

[8]See the paragraph before Eq. (3.2).

[9]See the footnote in subsection III C.

(iii) The fact that reflection positivity does not hold for $\alpha < 0$ is confirmed by the violation for $\alpha < 0$ of the unitarity bounds obtained, assuming that the theory provides a representation of the conformal algebra. This raises a question about which are the symmetries for the nonlocal action (2.1), which is an interesting subject to be considered.

Summarizing, it is believed that the study of nonlocal field theories can enlarge our knowledge about the fixed points and renormalization group flows in the space of all possible couplings mentioned in the introduction.

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