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ON SEMI-ANALYTICAL PROCEDURE FOR DETECTING LIMIT CYCLE BIFURCATIONS

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This paper reports some computation of periodic solutions arising from Hopf bifurcations in order to build up a more accurate procedure for semi-analytical approximations to detect limit cycle bifurcations. The approximation formulas are derived using nonlinear feedback systems theory and the harmonic balance method. The monodromy matrix is computed for several simple nonlinear flows to detect the first bifurcation of the cycles in the neighborhood of the original Hopf bifurcation.

Keywords: Monodromy matrix; harmonic balance; limit cycles; Hopf bifurcation; period doubling.

1. Introduction

Approximate analysis methods play an important role in studying periodic solutions in nonlinear systems. Several methods are available today, some of them with a long tradition in nonlinear dynamics arena such as those based on perturbation or averaging theory (see e.g. [Buonomo & Di Bello, 1996] and [Phillipson & Schuster, 2000]), and others with a control theory background such as the frequency domain version of the Hopf bifurcation theorem [Mees & Chua, 1979]. In addition, the evolution of a periodic solution originated by means of the Hopf bifurcation mechanism can be traced out by varying the so-called bifurcation parameter. As the parameter is varied, and depending on the nonlinear terms, the cycle can undergo complicated phenomena such as period-doubling bifurcations and even chaos (see [Seydel, 1994]). One possible way in which a periodic solution evolves to a chaotic motion is by means of a cascade of period-doubling bifurcations. This important scenario has been extensively analyzed in the past years in order to provide an accurate detection of the first period-doubling bifurcation [Belhaq & Houssni, 1995; Rand, 1989; Tesi *et al.*, 1996]. Recently, the study has been extended to the complete cascade and to other complex global behaviors of cycles [Belhaq *et al.*, 2000; Bonani & Gilli, 1999].

In this work, approximations of periodic solutions emerging from Hopf bifurcation points are obtained aiming to develop a more accurate semi-analytical procedure to detect limit cycle bifurcations. These approximations are derived using the frequency domain approach and the harmonic balance method. Then the stability of the periodic solution is studied computing an approximation of the monodromy matrix. Examples of three-dimensional systems are used to illustrate the methodology. Three examples leading to nontrivial realizations in this setting are presented first. The considered examples are the one discovered by Chen and Ueta [2000], the one proposed by Rand [1989] and one of the nineteen systems introduced by Sprott [1994]. These examples contain nonlinear terms involving products of variables, and thus the expressions of the approximations are rather complicated to deal with. Therefore, in the second part, simple flows containing nonlinearities in one variable are studied. The considered chaotic flows are the one introduced by Tesi *et al.* [1996], and two modifications of the celebrated Rössler system taken from Thomas [1999a]. For these systems higher-order approximations formulas [Moiola & Chen, 1996] and the computation of the monodromy matrix have been obtained. This procedure provides a simple and accurate test for the local convergency of the higher-order approximations. In addition, bifurcations of cycles close to the Hopf bifurcation point can be detected. In these simple flows, the method is used to detect period-doubling, pitchfork and Neimark–Sacker bifurcations close to the original Hopf bifurcation. This seems to be a promissory result in the analysis of cycles bifurcations with expansions involving power series [Guckenheimer & Meloon, 2000].

The paper is organized as follows. In Sec. 2 a brief recount of the analysis of Hopf bifurcations using second-order harmonic balance, extensions to higher-order approximations and the stability of periodic solutions are given. In Secs. 3 and 4 the method has been applied to three-dimensional systems of different complexity regarding their nonlinearities. In Sec. 5 comparisons of numerical simulations and results using the frequency domain approach are confronted. Finally, in Sec. 6 some conclusions are collected.

2. The Frequency Domain Approach

Consider the general ordinary differential equation

$$\begin{aligned} \dot{x} &= Ax + Bg(y;\, \mu)\,, \\ y &= Cx\,, \end{aligned} \tag{1}$$

where $x \in R^n$, $y \in R^m$, A, B and C are $n \times n$, $n \times l$, and $m \times n$ matrices, $\mu \in R$ is the bifurcation parameter, y is the output and $g : R^m \times R \to R^l$ is at least a C^4-function in x and μ. There exist infinitely many distinct but equivalent feedback representations of Eq. (1). Toward this end, let us consider an arbitrary $l \times m$ matrix D and rewrite Eq. (1) as

$$\begin{aligned} \dot{x} &= Ax + BDy + B[g(y;\, \mu) - Dy]\,, \\ y &= Cx\,. \end{aligned} \tag{2}$$

By following the notation and results in [Mees & Chua, 1979], the system can be separated into two parts: a linear transfer matrix $G(s;\, \mu)$ and a memoryless nonlinear part $f(\cdot\,;\, \mu)$, as follows

$$\begin{aligned} G(s;\, \mu) &= C[sI - (A + BDC)]^{-1}B\,, \\ u = f(e;\, \mu) &:= g(y;\, \mu) - Dy\,, \end{aligned} \tag{3}$$

where $e = -y$. It can be verified that the equilibrium points of Eq. (2) correspond to the solutions $\hat{e}$ of

$$G(0;\, \mu)f(e;\, \mu) + e = 0\,. \tag{4}$$

The linearization of the feedback path $f(e;\, \mu)$ at the equilibrium point $\hat{e}$ is the Jacobian matrix $J = D_e f(\hat{e};\, \mu)$. Thus, the open-loop matrix associated with the feedback realization [Eq. (2)] is $G(s;\, \mu)J(\mu)$, and the corresponding eigenvalues are given by

$$\begin{aligned} h(\lambda,\, s;\, \mu) &= \det[\lambda I - GJ] \\ &= \lambda^p + a_{p-1}(s;\, \mu)\lambda^{p-1} + \cdots + a_0(s;\, \mu) \\ &= 0\,, \end{aligned} \tag{5}$$

where $p = \min[\text{rank } G, \text{ rank } J]$ and $a_i(\cdot)$ are rational functions of s.

Assuming a single root of $h(\cdot)$ at $\lambda = -1$ and replacing $s = i\omega$ in Eq. (5), a necessary condition for computing a bifurcation point $(\omega_0,\, \mu_0)$ is obtained solving

$$\begin{aligned} h(-1,\, i\omega;\, \mu) &= (-1)^p + \sum_{k=0}^{p-1} (-1)^k a_k(i\omega;\, \mu) \\ &= 0\,, \end{aligned} \tag{6}$$

for ω and μ. If $\omega_0 = 0$, then the bifurcation condition is called *static*, and it is related to the multiplicity of the equilibrium solutions. On the other hand, if $\omega_0 \neq 0$, the bifurcation condition is known as *dynamic* or *Hopf*, and providing that some additional conditions are fulfilled, it is related to the appearance of periodic solutions.

2.1. *Hopf bifurcation*

In order to analyze the Hopf bifurcation let us split Eq. (6) into real and imaginary parts,

$$\begin{aligned} F_1(\omega;\, \mu) &= \text{Re}\{h(-1,\, i\omega;\, \mu)\} \\ &= (-1)^p + \sum_{k=0}^{p-1} (-1)^k \text{Re}\{a_k(i\omega;\, \mu)\} \\ &= 0\,, \\ F_2(\omega;\, \mu) &= \text{Im}\{h(-1,\, i\omega;\, \mu)\} \\ &= \sum_{k=0}^{p-1} (-1)^k \text{Im}\{a_k(i\omega;\, \mu)\} = 0\,. \end{aligned}$$

Then, the three basic statements of the Hopf bifurcation theorem in the frequency domain setting are:

(H1) There is only one eigenvalue of $G(s;\,\mu)J(\mu)$, denoted as $\hat{\lambda}$, passing through the critical point $-1+i0$ when ω varies in $[0,\,\infty)$, reflecting the change in the stability of the equilibrium solution. In addition, there is only one frequency $\omega_0 \neq 0$ satisfying Eq. (6) for a given $\mu = \mu_0$ (avoiding a resonance condition) and $(\partial F_1/\partial\omega)(\omega_0;\,\mu_0)$, $(\partial F_2/\partial\omega)(\omega_0;\,\mu_0)$ are not simultaneously zero.

(H2) The following determinant is nonzero,

$$\det \begin{bmatrix} \dfrac{\partial F_1}{\partial \mu} & \dfrac{\partial F_2}{\partial \mu} \\ \dfrac{\partial F_1}{\partial \omega} & \dfrac{\partial F_2}{\partial \omega} \end{bmatrix}_{(\omega_0,\mu_0)} \neq 0\,.$$

(H3) The expression

$$\sigma_1 = -\mathrm{Re}\left\{ \frac{w^\top G(i\omega_0;\,\mu)p_1(\omega_0;\,\mu)}{w^\top G'(i\omega_0;\,\mu)Jv} \right\}\,, \tag{7}$$

called the *curvature coefficient*, does not change its sign when μ varies near μ_0.

In Eq. (7), $w^\top$ and v are respectively the left and right eigenvectors of the open-loop transfer matrix $G(i\omega;\,\mu)J(\mu)$ associated to $\hat{\lambda}$, $G'(i\omega_0;\,\mu) = dG/ds|_{s=i\omega_0}$,

$$p_1(\omega;\,\mu) = QV_{02} + \frac{1}{2}\overline{Q}V_{22} + \frac{1}{8}L\overline{v}\,, \tag{8}$$

where

$$V_{02} = -\frac{1}{4}H(0;\,\mu)Q\overline{v}\,, \tag{9}$$

$$V_{22} = -\frac{1}{4}H(i2\omega;\,\mu)Qv\,, \tag{10}$$

with

$$H(s;\,\mu) = [G(s;\,\mu)J(\mu) + I]^{-1}G(s;\,\mu)\,, \tag{11}$$

and "$\bar{\cdot}$" denotes the complex conjugate; Q and L are $n \times l$ and $l \times m$ matrices, respectively, which include the information of the second and third derivatives of $f(e;\,\mu)$ evaluated at $\hat{e}$ [Mees & Chua, 1979]. For Q its jkth element is

$$Q_{jk} = \sum_{p=1}^{m} f_{pk}^{j} v_p \quad j = 1,\,2,\ldots,\,l; \quad k = 1,\,2,\ldots,\,m\,,$$

where $f_{pk}^{j} = \partial^2 f_j(e)/\partial e_p \partial e_k|_{\hat{e}}$, and for the matrix L its jkth element is

$$L_{jk} = \sum_{p=1}^{m}\sum_{q=1}^{m} f_{pqk}^{j} v_p v_q \quad j = 1,\,2,\ldots,\,l; \quad k = 1,\,2,\ldots,\,m\,,$$

where $f_{pqk}^{j} = \partial^3 f_j(e)/\partial e_p \partial e_q \partial e_k|_{\hat{e}}$.

It is worth mentioning, that Eq. (7) tells us about the stability of the emerging periodic solution at *criticality*: if σ_1 is negative (positive) the limit cycle is stable (unstable).

2.2. *Approximations of periodic solutions*

Approximations of the amplitude represented by $\hat{\theta}$, and frequency $\hat{\omega}$ of the periodic solution in the neighborhood of the criticality are obtained solving

$$\hat{\lambda}(i\hat{\omega}) = -1 + \xi_1(\omega_R)\hat{\theta}^2\,, \tag{12}$$

where

$$\xi_1(\omega_R) = -\frac{w^\top G(i\omega_R;\,\mu)p_1(\omega_R;\,\mu)}{w^\top v}\,, \tag{13}$$

and ω_R is the frequency at which the eigenlocus $\hat{\lambda}(i\omega)$ crosses the real axis closest to -1. Equation (12) is solved by an iterative process

$$\begin{aligned}
&(\text{STEP 1}) \quad \hat{\lambda}(i\hat{\omega}_a) = -1 + \xi_1(\omega_R)\hat{\theta}_a^2\,,\\
&(\text{STEP 2}) \quad \hat{\lambda}(i\hat{\omega}_b) = -1 + \xi_1(\hat{\omega}_a)\hat{\theta}_b^2\,,\\
&(\text{STEP 3}) \quad \hat{\lambda}(i\hat{\omega}_c) = -1 + \xi_1(\hat{\omega}_b)\hat{\theta}_c^2\,,\\
&\qquad\vdots \qquad\qquad\qquad \vdots\\
&(\text{STEP n}) \quad \hat{\lambda}(i\hat{\omega}_1) = -1 + \xi_1(\hat{\omega}_{n-1})\hat{\theta}_1^2\,.
\end{aligned}$$

Since we want to develop a simple method to obtain the approximation, we stop in the second iteration (so $\hat{\omega}_1 = \hat{\omega}_b$ and $\hat{\theta}_1 = \hat{\theta}_b$). Then, a second-order harmonic balance approximation (HBA) is given by

$$e_{2\mathrm{HB}}(t) = \hat{e} + \mathrm{Re}\left\{ \sum_{k=0}^{2} E^k \exp(ik\hat{\omega}_1 t) \right\}\,,$$

where

$$\begin{aligned}
E^0 &= V_{02}(\hat{\omega}_1)\hat{\theta}_1^2\,,\\
E^1 &= V_{11}(\hat{\omega}_1)\hat{\theta}_1 + V_{13}(\hat{\omega}_1)\hat{\theta}_1^3\,,\\
E^2 &= V_{22}(\hat{\omega}_1)\hat{\theta}_1^2\,,
\end{aligned}$$

and $V_{11} = v$ (the expression of V_{13} can be found in [Mees, 1981]).

In order to obtain a more accurate representation of the periodic solution, higher-order HBAs are pursued. For example, to obtain a fourth-order HBA Eq. (12) is updated to a more accurate form

$$\hat{\lambda}(i\hat{\omega}_2) = -1 + \xi_1(\hat{\omega}_1)\hat{\theta}_2^2 + \xi_2(\hat{\omega}_1)\hat{\theta}_2^4\,, \tag{14}$$

where

$$\xi_2(\hat{\omega}_1) = -\frac{w^\top G(i\hat{\omega}_1)p_2(\hat{\omega}_1;\,\mu)}{w^\top v} - \xi_1(\hat{\omega}_1)w^\top V_{13}(\hat{\omega}_1)\,.$$

The pair $(\hat{\omega}_2,\, \hat{\theta}_2)$ satisfying Eq. (14) correspond to the fourth-order HBA of the periodic solution

$$e_{4\mathrm{HB}}(t) = \hat{e} + \mathrm{Re}\left\{\sum_{k=0}^{4} E^k \exp(ik\hat{\omega}_2 t)\right\}\,,$$

where the expressions for E^k and $p_2(\hat{\omega}_1;\,\mu)$ are given in [Mees, 1981]. Analogously, an algorithm including the sixth- and eighth-order HBAs can be implemented, to find the solutions $(\hat{\omega}_3,\, \hat{\theta}_3)$ and $(\hat{\omega}_4,\, \hat{\theta}_4)$, respectively.

In the general case, the procedure consists in expanding Eq. (12) to

$$\hat{\lambda}(i\hat{\omega}_q) = -1 + \sum_{k=1}^{q} \xi_k(\hat{\omega}_{q-1})\hat{\theta}_q^{2k}\,,$$

where the pair $(\hat{\omega}_q,\, \hat{\theta}_q)$ are obtained by means of an iterative procedure as before. Then the approximation of the periodic solution is updated to

$$e_{2q\mathrm{HB}}(t) = \hat{e} + \mathrm{Re}\left\{\sum_{k=0}^{2q} E^k \exp(ik\hat{\omega}_q t)\right\}\,.$$

For simplicity, let us refer to the approximation L_1 given by second-order HBA, L_2 given by fourth-order HBA, and so on. All of these approximations are calculated for a specific value of μ and the corresponding value of $\hat{\omega}_q$.

2.3. *Stability analysis of periodic solutions*

The stability of a periodic solution $\gamma(t)$ of Eq. (1) is analyzed studying the behavior of the trajectories in the neighborhood of $\gamma(t)$. This behavior can be characterized via the state transition matrix $\Phi(t,\, 0)$ which is the unique solution of

$$\begin{aligned} \dot{\Phi}(t,\, 0) &= J_{\mathrm{var}}(t)\Phi(t,\, 0)\,, \\ \Phi(0) &= I\,, \end{aligned} \tag{15}$$

where $J_{\mathrm{var}}(t)$ is the Jacobian matrix of the system evaluated around the periodic solution $\gamma(t)$, i.e.

$$J_{\mathrm{var}}(t) = A + B\left.\frac{\partial g}{\partial x}\right|_{x=\gamma(t)}\,.$$

Notice that $J_{\mathrm{var}}(t)$ is periodic with period $T = 2\pi/\hat{\omega}$.

The stability of $\gamma(t)$ is determined by the eigenvalues of the *monodromy matrix* M, defined as

$$M(\mu) := \Phi(T,\, 0)\,.$$

The matrix $M(\mu)$ in the general case has n eigenvalues, $\lambda_1(\mu)$, $\lambda_2(\mu) \ldots \lambda_n(\mu)$, which are known as characteristic (or Floquet) multipliers. One of them is always equal to $+1$, say $\lambda_1(\mu)$. The remaining $n-1$ determine the local stability by the following rule:

- The periodic solution is stable if $|\lambda_j| < 1$, for all $j \neq 1$.
- The periodic solution is unstable if $|\lambda_j| > 1$, for some $j \neq 1$.

The multiplier that crosses the unit circle is known as *critical multiplier*. Depending on where the critical multiplier or the pair of complex conjugate multipliers cross the unit circle, different types of branching occur. There are three distinguished ways of crossing the unit circle, with three associated types of branching. Figure 1 shows the path of the critical multiplier only, that is the eigenvalue with $|\lambda_j(\mu_0)| = 1$, where $j \neq 1$. In Fig. 1(a) the eigenvalue crosses the unit circle at the negative real axis, leading to a period-doubling bifurcation. When the eigenvalue crosses the unit circle at the point $+1 + 0i$ [Fig. 1(b)] it may indicate a saddle-node, transcritical or pitchfork bifurcation of cycles. The third type of bifurcation, Neimark–Sacker or *torus* bifurcation, is characterized by a pair of complex-conjugate multipliers crossing the unit circle as shown in Fig. 1(c).

To compute the monodromy matrix M, an integration of two dynamical systems is required: the original nonlinear system [Eq. (1)] and the variational equation [Eq. (15)]. To avoid this computation, it is possible to approximate the limit cycle by using a higher-order HBA and thus to obtain an approximate monodromy matrix M_q after performing a $2q$th harmonic solution. So we only need to deal with the integration of the approximate variational equation

$$\dot{Y}(t) = J_{D_q}(t)Y(t)\,,$$

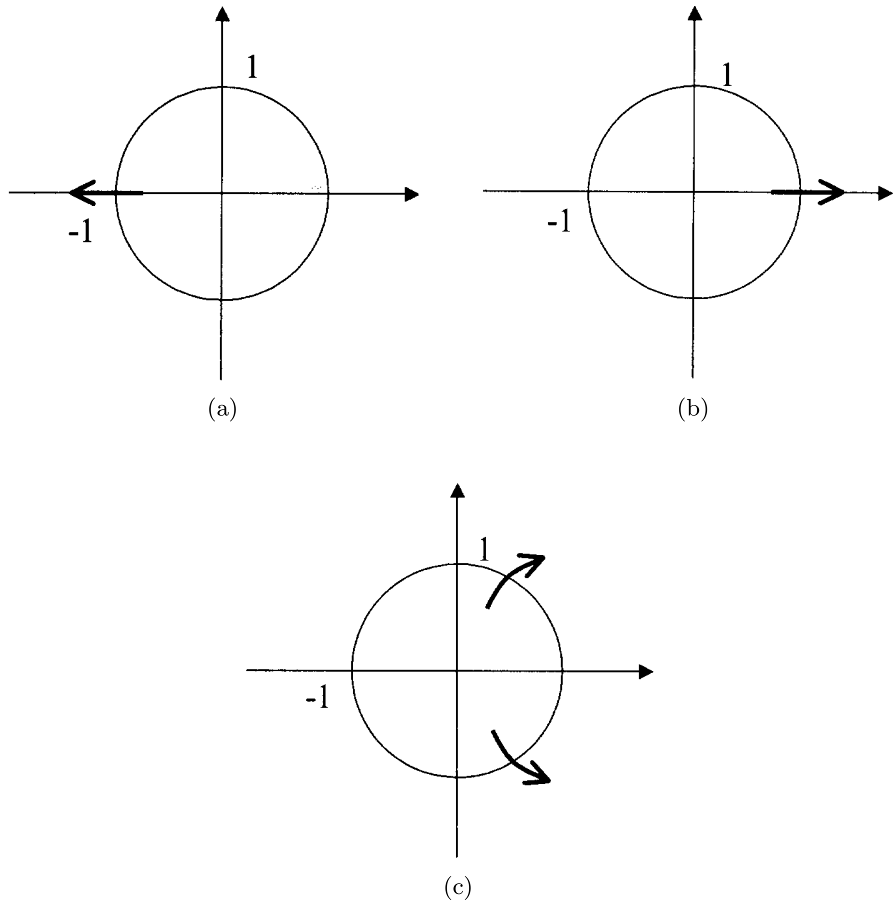

Fig. 1. Three ways in which the multipliers leave the unit circle.

$$Y(0) = I\,,$$
$$M_q = Y\left(\frac{2\pi}{\hat{\omega}_q}\right)\,,$$

where $J_{D_q} = J_{\text{var}}(t)|_{L_q}$ is a periodic matrix obtained by using the information of the approximation of the limit cycle $e_{2q\text{HB}}(t)$.

3. Systems without SISO Realizations

In order to apply the methodology described in Sec. 2 it is necessary to represent the nonlinear system in feedback form [Eq. (2)]. Systems with nonlinearities involving mixed terms, i.e. depending on two of the variables, lead to realizations with multiple outputs, and in some cases with multiple inputs. As a result, the computation of higher-order approximations is cumbersome. In the following, three examples of such kind of systems are treated. The main objective of this section is to illustrate the procedure of finding minimal realizations, i.e. those leading to the analysis of an unique eigenlocus in the frequency domain. In addition expressions for computing a second-order HBA are obtained.

3.1. *Chen system*

Chen system is a chaotic attractor discovered by Chen and Ueta [2000], which looks like the classical Lorenz system though not topologically equivalent to it. The system is described by

$$\begin{aligned}\dot{x}_1 &= -\mu x_1 + \mu x_2\,,\\ \dot{x}_2 &= (c-\mu)x_1 + cx_2 - x_1x_3\,,\\ \dot{x}_3 &= -bx_3 + x_1x_2\,,\end{aligned} \tag{16}$$

where b and c are real constants not equal to zero and μ is the main bifurcation parameter.

A simple choice in the realization of Eq. (2) is

$$A = \begin{bmatrix} -\mu & \mu & 0 \\ 0 & c & 0 \\ 0 & 0 & -b \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = I_3 \,,$$

$$D = 0\,, \quad g(x;\, \mu) = \begin{bmatrix} (c-\mu)x_1 - x_1x_3 \\ x_1x_2 \end{bmatrix}.$$

The corresponding linear and nonlinear paths [Eq. (3)] are

$$G(s;\, \mu) = \frac{1}{\Delta(s)} \begin{bmatrix} \mu(s+b) & 0 \\ (s+\mu)(s+b) & 0 \\ 0 & (s+\mu)(s-c) \end{bmatrix},$$

where $\Delta(s) = (s+\mu)(s-c)(s+b)$, and

$$f(e;\, \mu) = \begin{bmatrix} -(c-\mu)e_1 - e_1e_3 \\ e_1e_2 \end{bmatrix}.$$

After computing Eq. (4) it is easy to obtain the equilibrium points $P = (\hat{e}_1,\, \hat{e}_2,\, \hat{e}_3)$

$$P_0 = (0,\, 0,\, 0)\,, \quad P_{1,2} = (\pm\sqrt{bd},\, \pm\sqrt{bd},\, -d)\,,$$

where $d = 2c - \mu$. In order to study Hopf bifurcations, the equilibrium points $P_{1,2}$ will be considered in the following. At these equilibria the Jacobian matrix of $f(e;\, \mu)$ results

$$J = \begin{bmatrix} c & 0 & -\hat{e}_1 \\ \hat{e}_1 & \hat{e}_1 & 0 \end{bmatrix}.$$

Then the eigenvalues in the frequency domain are obtained from [Eq. (5)]

$$\lambda \left[\lambda^2 - \frac{\mu c}{(s+\mu)(s-c)}\lambda + \frac{\hat{e}_1^2(s+2\mu)}{(s+\mu)(s+b)(s-c)} \right] = 0\,. \tag{17}$$

Notice that the analysis is cumbersome since there are two nontrivial eigenvalues, i.e. an algebraic function instead of a characteristic polynomial like the one in the time domain. However, observe that replacing $\lambda = -1$ in Eq. (17) and $\hat{e}_1$ in terms of the system parameters, it is easy to obtain the equation

$$s^3 + (\mu + b - c)s^2 + bcs + 2\mu b(2c - \mu) = 0\,. \tag{18}$$

This is the characteristic equation in the time domain formulation, and so it is the common contact point with the frequency domain formulation.

Returning to our main concern, in the frequency domain method it is important to deal with only one eigenvalue, so we propose to modify the Jacobian to be of rank 1. Toward this end, let us consider A, B, C and $g(x;\, \mu)$ as before and

$$D = \begin{bmatrix} 0 & 0 & \alpha_{13} \\ 0 & \alpha_{22} & 0 \end{bmatrix}.$$

Then, the nonlinear path results

$$f(e;\, \mu) = [g(y;\, \mu) - Dy]|_{y=-e} = \begin{bmatrix} -(c-\mu)e_1 + \alpha_{13}e_3 - e_1e_3 \\ \alpha_{22}e_2 + e_1e_2 \end{bmatrix},$$

and the corresponding Jacobian matrix at $P_{1,2}$ is

$$J = \begin{bmatrix} c & 0 & \alpha_{13} - \hat{e}_1 \\ \hat{e}_1 & \alpha_{22} + \hat{e}_1 & 0 \end{bmatrix},$$

where we have used the fact $\hat{e}_2 = \hat{e}_1$ and $\hat{e}_3 = \mu - 2c$. Therefore, J has rank 1 if $\alpha_{13} = \hat{e}_1$ and $\alpha_{22} = -\hat{e}_1$, providing that $c \neq 0$ and $b \neq 0$ as it has been assumed before, and $\mu \neq 2c$. Thus,

$$J = \begin{bmatrix} c & 0 & 0 \\ \hat{e}_1 & 0 & 0 \end{bmatrix}.$$

The transfer matrix results

$$G(s;\, \mu) = \frac{1}{\Delta(s)} \begin{bmatrix} \mu(s+b) & \hat{e}_1\mu \\ (s+\mu)(s+b) & \hat{e}_1(s+\mu) \\ -\hat{e}_1(s+\mu) & (s+\mu)(s-c) \end{bmatrix},$$

where $\Delta(s) = (s+\mu)[s^2 + (b-c)s - bc + \hat{e}_1^2]$. Then, the eigenvalues of $G(s)J$ are given by

$$\lambda^2 \left\{ \lambda - \frac{\mu c(s+b) + \mu\hat{e}_1^2}{(s+\mu)[s^2 + (b-c)s - bc + \hat{e}_1^2]} \right\} = 0\,. \tag{19}$$

For $\lambda = -1$ Eq. (19) yields Eq. (18), which is the characteristic equation obtained for the linearization of the system [Eq. (16)] around P_1 or P_2. Moreover, the relevant eigenvalue after replacing the value of P_1 gives

$$\hat{\lambda}(s) = \frac{\mu c(s+3b) - \mu^2 b}{(s+\mu)[s^2 + (b-c)s + bc - \mu b]}\,. \tag{20}$$

The Hopf bifurcation condition is obtained replacing $\hat{\lambda} = -1$ and $s = i\omega_0$ ($\omega_0 \neq 0$) in Eq. (20). After separating in real and imaginary parts this condition results in

$$\omega_0^2(\mu + b - c) - \mu bc + \mu^2 b = 3\mu cb - \mu^2 b\,,$$
$$\omega_0^3 - c(b-\mu)\omega_0 = \mu c\omega_0\,.$$

Then, it is easy to obtain

$$\omega_0 = \sqrt{bc}\,,$$

and

$$\mu_0 = \frac{1}{4}[3c \pm \sqrt{c(-8b + 17c)}]\,.$$

Let us compute the frequency ω_R at which $\hat{\lambda}(i\omega)$ crosses the real axis

$$\begin{aligned}\hat{\lambda}(i\omega_R) &= P_R + i0 \\ &= \frac{\mu c(i\omega_R + 3b) - \mu^2 b}{-i\omega_R^3 - (\mu + b - c)\omega_R^2 + (bc - \mu c)i\omega_R + \mu bc - \mu^2 b},\end{aligned}$$

where P_R is the intersection point (real). Separating the previous equation in real and imaginary parts and considering $\omega_R \neq 0$, it can be shown that

$$P_R = \frac{\mu c}{bc - \mu c - \omega_R^2},$$

and

$$\omega_R = \sqrt{bc\frac{3bc - \mu b - 4\mu c + 2\mu^2}{c^2 - \mu c + 2bc - \mu b}}.$$

The calculation of the vectors for the approximations of the periodic solutions starts with the right v and left $w^\top$ eigenvectors associated to $\hat{\lambda}(s)$

$$v(s) = [\,v_1(s) \quad v_2(s) \quad v_3(s)\,]^\top, \quad w^\top = [1 \quad 0 \quad 0],$$

where

$$v_1(s) = 1,$$

$$v_2(s) = \frac{s + \mu}{\mu},$$

$$v_3(s) = \frac{(s + \mu)(s - 2c)\hat{e}_1}{\mu[c(s + b) + \mu\hat{e}_1^2]}.$$

Then, the closed-loop transfer matrix [Eq. (11)] is computed as

$$H(s) = \frac{1}{\Delta_1(s)}\begin{bmatrix} \mu(s + b) & \hat{e}_1\mu \\ (s + \mu)(s + b) & \hat{e}_1(s + \mu) \\ -\hat{e}_1(s + 2\mu) & s(s + \mu - c) \end{bmatrix},$$

where $\Delta_1(s) = s^3 + (\mu + b - c)s^2 + cbs + 2\mu b(2c - \mu)$ coincides with the characteristic polynomial in the time-domain.

The remaining vectors [Eqs. (8)–(10)] are

$$V_{02} = -\frac{1}{4}H(0)\begin{bmatrix} -2\mathrm{Re}\{v_3(i\omega_R)\} \\ 2\mathrm{Re}\{v_2(i\omega_R)\} \end{bmatrix},$$

$$V_{22} = -\frac{1}{4}H(i2\omega_R)\begin{bmatrix} -2v_3(i\omega_R) \\ 2v_2(i\omega_R) \end{bmatrix},$$

$$p_1(\omega_R;\, \mu) = QV_{02} + \frac{1}{2}\overline{Q}V_{22} + \frac{1}{8}L\overline{v},$$

where

$$Q = \begin{bmatrix} -v_3(i\omega_R) & 0 & -1 \\ v_2(i\omega_R) & 1 & 0 \end{bmatrix},$$

and

$$L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Finally, Eqs. (12) and (13) give the approximations of the amplitude and frequency of the oscillations.

3.2. *Rand system*

A system proposed by Rand [1989] and analyzed in [Belhaq & Houssni, 1995] and [Belhaq *et al.*, 2000] is

$$\begin{aligned}\dot{x}_1 &= \mu x_1 - x_2 - x_1 x_3, \\ \dot{x}_2 &= x_1 + \mu x_2, \\ \dot{x}_3 &= -x_3 + x_1^2 x_3 + x_2^2,\end{aligned} \tag{21}$$

where μ is the bifurcation parameter. Providing that x_3 is the control variable the system may be interpreted as a damped linear oscillator in the variables x_1 and x_2.

Let us consider the following realization

$$A = \begin{bmatrix} \mu & -1 & 0 \\ 1 & \mu & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = I_3,$$

$$D = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ \alpha_{21} & 0 & 0 \end{bmatrix}, \quad g(x;\, \mu) = \begin{bmatrix} -x_1x_3 \\ x_1^2x_3 + x_2^2 \end{bmatrix}.$$

The linearization of the nonlinear function

$$f(e;\, \mu) = \begin{bmatrix} \alpha_{11}e_1 - e_1e_3 \\ \alpha_{21}e_1 - e_1^2e_3 + e_2^2 \end{bmatrix},$$

around the equilibrium point $P = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$ is

$$J = \begin{bmatrix} \alpha_{11} - \hat{e}_3 & 0 & -\hat{e}_1 \\ \alpha_{21} - 2\hat{e}_1\hat{e}_3 & 2\hat{e}_2 & -\hat{e}_1^2 \end{bmatrix}.$$

Let us assume $\alpha_{11} = \alpha_{21} = 1$, and thus the linear transfer matrix is

$$G(s) = \frac{1}{\Delta(s)}\begin{bmatrix} (s - \mu)(s + 1) & 0 \\ s + 1 & 0 \\ s - \mu & (s - \mu)(s - \mu - 1) + 1 \end{bmatrix}$$

where $\Delta(s) = (s + 1)[(s - \mu)^2 - (s - \mu) + 1]$. Then, the equilibrium points are

$$P_0 = (0, 0, 0),$$

$$P_{1,2} = \left(\pm\sqrt{\frac{d}{1 + d}}, \mp\frac{1}{\mu}\sqrt{\frac{d}{1 + d}}, \frac{d}{\mu^2}\right),$$

where $d = \mu(1+\mu^2)$. In the following, only the equilibrium point P_0 is considered since by increasing the value of μ from the critical value $\mu = 0$ a limit cycle appears by means of the Hopf bifurcation phenomenon. At P_0 the Jacobian matrix results in

$$J = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},$$

which has rank 1 and thus simplifies the analysis.

The eigenvalues of $G(s)J$ are obtained from

$$\lambda^2 \left\{ \lambda - \frac{(s-\mu)(s+1)}{(s+1)[(s-\mu)^2-(s-\mu)+1]} \right\} = 0\,. \tag{22}$$

Therefore, the eigenvalue to be considered is

$$\hat{\lambda}(s) = \frac{s-\mu}{(s-\mu)^2-(s-\mu)+1}\,.$$

Equation (22) for $\lambda = -1$ yields

$$(s+1)(s^2-2\mu s+\mu^2+1) = 0\,, \tag{23}$$

which is the characteristic equation obtained by linearizing the system [Eq. (21)] around P_0. It is also simple to show that the frequency of the oscillation is $\omega_0 = 1$ (for $\mu = 0$).

The calculation of the vectors for the approximations of the periodic solutions begins with the right v and left $w^\top$ eigenvectors associated to $\hat{\lambda}$

$$v(s) = [\,v_1(s) \quad v_2(s) \quad v_3(s)\,]^\top, \quad w^\top = [1 \quad 0 \quad 0],$$

where

$$v_1(s) = 1\,,$$

$$v_2(s) = \frac{1}{(s-\mu)}\,,$$

$$v_3(s) = \frac{(s-\mu)^2+1}{(s-\mu)(s+1)}\,.$$

Then, the closed-loop transfer matrix is computed as

$$H(s) = \frac{1}{\Delta_1(s)} \begin{bmatrix} (s-\mu)(s+1) & 0 \\ (s+1) & 0 \\ 0 & (s-\mu)^2+1 \end{bmatrix},$$

where $\Delta_1(s) = (s+1)(s^2-2\mu s+\mu^2+1)$. Notice that $H(s)$ contains as the transmission poles the real and the complex conjugate poles given in Eq. (23).

Let us compute the frequency ω_R from

$$\hat{\lambda}(i\omega_R) = P_R + i0 = \frac{-\mu + i\omega_R}{(i\omega_R-\mu)^2-(i\omega_R-\mu)+1}\,.$$

After simple calculations the values of P_R and ω_R are given by

$$P_R = -\frac{1}{1+2\mu}\,, \quad \text{and} \quad \omega_R = \sqrt{1-\mu^2}\,.$$

Notice that the right eigenvector v calculated at ω_R is given completely in terms of the main bifurcation parameter μ. Then, the remaining vectors result as

$$V_{02} = -\frac{1}{4}H(0) \begin{bmatrix} -2\mathrm{Re}\{v_3(i\omega_R)\} \\ 2v_2(i\omega_R)\overline{v}_2(i\omega_R) \end{bmatrix},$$

$$V_{22} = -\frac{1}{4}H(i2\omega_R) \begin{bmatrix} -2v_3(i\omega_R) \\ 2v_2^2(i\omega_R) \end{bmatrix},$$

$$p_1(\omega_R;\,\mu) = QV_{02} + \frac{1}{2}\overline{Q}V_{22} + \frac{1}{8}L\overline{v}\,,$$

where

$$Q = \begin{bmatrix} -v_3(i\omega_R) & 0 & -1 \\ 0 & 2v_2(i\omega_R) & 0 \end{bmatrix},$$

and

$$L = \begin{bmatrix} 0 & 0 & 0 \\ -4v_3(i\omega_R) & 0 & -2 \end{bmatrix}.$$

As in the previous example, the approximations of the periodic solution can be obtained from Eqs. (12) and (13).

3.3. *Sprott system*

Let us consider the following system proposed by Sprott [1994]

$$\begin{aligned} \dot{x}_1 &= \mu x_1 + x_3\,, \\ \dot{x}_2 &= -x_2 + x_1 x_3\,, \\ \dot{x}_3 &= -x_1 + x_2\,, \end{aligned} \tag{24}$$

where we have added the bifurcation parameter μ (in Sprott's paper $\mu = 0.4$). Taking the realization on Eq. (24) given by

$$A = \begin{bmatrix} \mu & 0 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ C = I_3\,,\ D = 0\,,$$

$$g(x;\,\mu) = x_1 + x_1 x_3\,,$$

we end up in a simple form of the linear transfer function

$$G(s;\,\mu) = \frac{1}{\Delta(s)} \begin{bmatrix} 1 \\ s^2-\mu s+1 \\ s-\mu \end{bmatrix},$$

where $\Delta(s) = s^3 + (1-\mu)s^2 + (1-\mu)s + 2$. In addition, the nonlinear function is

$$f(e;\,\mu) = -e_1 + e_1 e_3\,.$$

The equilibrium points $P = (\hat{e}_1,\,\hat{e}_2,\,\hat{e}_3)$ are

$$P_0 = (0,\,0,\,0)\,, \quad P_1 = \left(\frac{1}{\mu},\,\frac{1}{\mu},\,-1\right).$$

We are interested in the equilibrium point at the origin, since a limit cycle emerges under the Hopf bifurcation mechanism. The Jacobian matrix at the origin results in $J = [-1\ \ 0\ \ 0]$ and the eigenvalues are obtained from

$$\lambda^2\left(\lambda + \frac{1}{\Delta(s)}\right) = 0\,.$$

Therefore the nontrivial eigenvalue is

$$\hat{\lambda}(s) = -\frac{1}{s^3 + (1-\mu)s^2 + (1-\mu)s + 2}\,.$$

The necessary condition for the Hopf bifurcation, i.e. $\hat{\lambda} = -1 + i0$, gives $\omega_0 = 1$ and $\mu = 0$. In terms of the bifurcation parameter, the characteristic locus crosses the negative real axis close to the critical point $-1 + i0$ with the following expression

$$\hat{\lambda}(i\omega_R) = \frac{-1}{2 - (1-\mu)^2}\,, \quad \omega_R = \pm\sqrt{1-\mu}\,,$$

where ω_R is the frequency to be used for the computations of the periodic solutions. At criticality $(\mu = 0)$, $\omega_{R(\mu=0)} = \omega_0 = 1$. The eigenlocus $\hat{\lambda}(i\omega)$ is depicted in Fig. 2 when $\omega \in [0,\,\infty)$ and $\mu = 0$, revealing the crossing at the critical point. Since $\Delta(s)$ has two complex roots in the right half-plane, the equilibrium point at the origin loses its stability for $\mu > 0$. A 3D-plot is given in Fig. 3 where the z-axis is μ, and x and y-axes are the same as in Fig. 2.

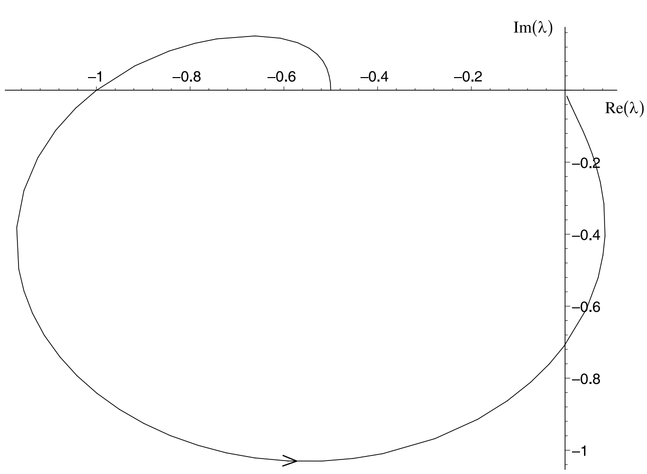

Fig. 2. Relevant eigenlocus at criticality ($\mu = 0$).

The normalized ($w^\top v = 1$, $|v| = 1$) right and left eigenvectors v, $w^\top$ of the matrix $G(i\omega_R)J$ are

$$v = \frac{1}{\gamma_1}\begin{bmatrix} 1 \\ \mu(1 - i\sqrt{1-\mu}) \\ -\mu + i\sqrt{1-\mu} \end{bmatrix}, \quad w^\top = [\,\gamma_1 \quad 0 \quad 0\,]\,,$$

where $\gamma_1 = \sqrt{2 - \mu^3 + 3\mu^2 - \mu}$.

The closed-loop transfer function is

$$H(s) = \frac{1}{\Delta(s) - 1}\begin{bmatrix} 1 \\ s^2 - \mu s + 1 \\ s - \mu \end{bmatrix}.$$

Then, we can compute the vectors

$$V_{02} = \frac{\mu}{2\gamma_1^2}\begin{bmatrix} 1 \\ 1 \\ -\mu \end{bmatrix},$$

$$V_{22} = \frac{\mu - i\sqrt{1-\mu}}{2\gamma_2\gamma_1^2}\begin{bmatrix} 1 \\ -3 + 4\mu - i2\mu\sqrt{1-\mu} \\ -\mu + i2\sqrt{1-\mu} \end{bmatrix},$$

where

$$\begin{aligned}\gamma_2 &= \Delta(i2\omega_R) - 1 \\ &= 1 - 4(1-\mu)^2 - i6(1-\mu)\sqrt{1-\mu}\,.\end{aligned}$$

The number $p_1(\omega_R;\,\mu)$ is

$$\begin{aligned}p_1(\omega_R;\,\mu) = &-\frac{\mu}{2\gamma_1^3}(2\mu - i\sqrt{1-\mu}) \\ &- \frac{1}{4\gamma_1^3\gamma_2}(-1 + \mu + 2\mu^2 - i3\mu\sqrt{1-\mu})\,.\end{aligned}$$

In this case

$$Q = \frac{1}{\gamma_1}[\,-\mu + i\sqrt{1-\mu} \quad 0 \quad 1\,]\,,$$

and $L = 0$ since the third partial derivatives of the function $f(e;\,\mu)$ with respect to e are equal to 0. This is an important simplification in order to proceed with higher-order HBA formulas.

The complex number $\xi_1(\omega_R)$ is

$$\begin{aligned}\xi_1(\omega_R) = &\frac{\mu}{2\gamma_1^2\gamma_3}(2\mu - i\sqrt{1-\mu}) \\ &+ \frac{1}{4\gamma_1^2\gamma_3\gamma_2}(-1 + \mu + 2\mu^2 - i3\mu\sqrt{1-\mu})\,,\end{aligned}$$

where $\gamma_3 = 1 + 2\mu - \mu^2$. Then, the amplitude $\hat{\theta}$ of the periodic solution can be computed by solving Eq. (12).

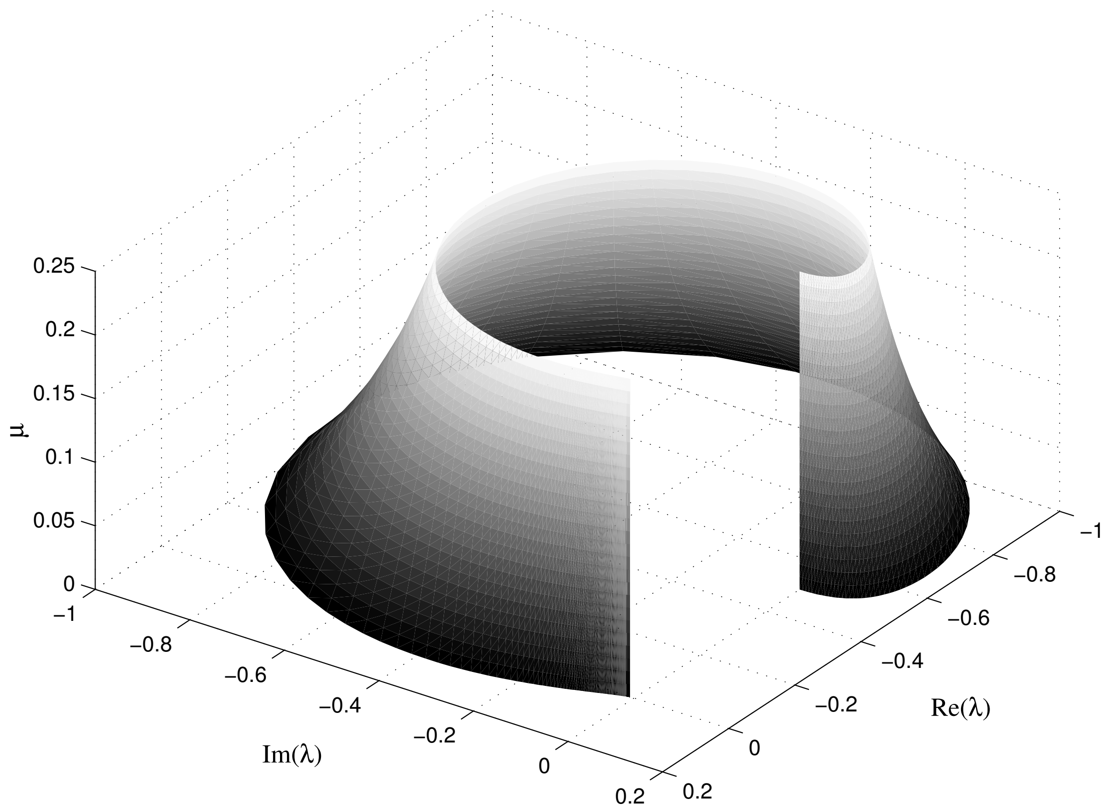

Fig. 3. Relevant eigenlocus by varying μ.

The stability index at criticality [Eq. (7)] yields

$$\sigma_1 = -\frac{1}{160}\,.$$

Therefore, for $\mu > 0$ the emerging limit cycle is stable.

The computations carried out above are very close to the results reported with the celebrated Chua's circuit [Bonani & Gilli, 1999; Phillipson & Schuster, 2000], but in the present paper the example seems to be more easily adapted for computation since it only has one distinguished bifurcation parameter. Furthermore, the expressions for the harmonics are very simple and they are in terms of the bifurcation parameter μ. For example, the vectors $V_{11} = v$, V_{02} and V_{22} for $\mu = 0$ are

$$v = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 0 & i\end{bmatrix}^\top,$$

$$V_{02} = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^\top,$$

$$V_{22} = \frac{2+i}{60}\begin{bmatrix}1 & -3 & 2i\end{bmatrix}^\top,$$

and for $\mu = 1/4$ are

$$v = \frac{8}{\sqrt{123}}\begin{bmatrix}1 \\ \frac{1}{4} - i\frac{1}{8}\sqrt{3} \\ -\frac{1}{4} + i\frac{1}{2}\sqrt{3}\end{bmatrix},$$

$$V_{02} = \frac{8}{123}[1 \quad 1 \quad -1/4]^\top,$$

$$V_{22} = \frac{8}{8241}(49 + i19\sqrt{3})\begin{bmatrix}1 \\ -2 - i\frac{1}{4}\sqrt{3} \\ -\frac{1}{4} + i\sqrt{3}\end{bmatrix}.$$

This shows that the content of the bias is increased at $\mu = 1/4$ when compared to the one at criticality.

The analyzed system has a period-doubling sequence starting at $\mu \cong 0.264427$ that ends in a chaotic attractor for $\mu > 0.3272$. Thus, this system is appropriate to pursue an analytical treatment of the period-doubling bifurcations using the approximations for the periodic solutions given before. However, no conclusion can be drawn from this approximation regarding the first period-doubling bifurcation, since the approximations used here take care of the linearization around the equilibrium point, and for the analysis of the bifurcation of the cycles we need the linearization of the flux around the periodic solution. This step will be considered in the following section for simpler flows such as given by Tesi *et al.* [1996] and Thomas [1999a, 1999b] since they contain the simplest nonlinearity involving one variable as well as the least number of linear terms. These special configurations will help in obtaining the linearized matrix of the flow around the periodic solutions and then the computation of higher-order approximation formulas.

Finally, it is interesting to point out that more sophisticated techniques have been used recently to study semi-analytically the bifurcations of cycles [Sinha & Butcher, 1997; Butcher & Sinha, 1998]. In addition, more accurate numerical techniques have been proposed to solve the continuation problem of the limit cycles [Guckenheimer & Meloon, 2000; Viswanath, 2001] or the computation of the accurate Poincaré maps [Tucker, 2002]. These two current trends are certainly part of the motivation of using higher-order approximation methods in computing periodic solutions and their bifurcations. Furthermore, the potentiality of the power series method has been emphasized in [Guckenheimer & Meloon, 2000] while a computation of the errors has been shown in [Berns *et al.*, 2001] in connection with approximate monodromy matrices.

4. Systems with SISO Realizations

In this section the methodology is applied to simple flows. In all the three cases a Single-Input-Single-Output (SISO) realization is obtained since the nonlinearity involves only one variable. This results in $v = w^\top = 1$. Furthermore, an important simplification of the higher-order corrections in the direction of the eigenvector v is attained.

4.1. *Genesio and Tesi system*

Let us consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \,, \\ \dot{x}_2 &= x_3 \,, \\ \dot{x}_3 &= -x_1 - 1.2x_2 + \mu x_3 + x_1^2 \,, \end{aligned} \tag{25}$$

where μ is the main bifurcation parameter. This system has been studied in [Tesi *et al.*, 1996] concerning the delaying of the period-doubling cascade by using feedback control.

Let us take the realization

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1.2 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = -1\,,$$

$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \quad g(x) = x_1^2 \,.$$

Then, the transfer function $G(s;\, \mu)$ is

$$G(s;\, \mu) = \frac{1}{\Delta(s)}\,,$$

where $\Delta(s) = s^3 - \mu s^2 + 1.2s + 2$, and the nonlinear function reads

$$f(e) = -e_1 + e_1^2 \,.$$

For this realization, the equilibrium points (output) correspond to

$$P_0 = 0\,, \quad P_1 = -1\,.$$

In the following, only the equilibrium point P_0 at the origin is considered. The eigenlocus is given by

$$\hat{\lambda}(s) = G(s)J = -\frac{1}{\Delta(s)}\,,$$

where $J = -1 + 2\hat{e}_1 = -1$.

The static bifurcation condition is given by $G(0)J = -1$, which is not attained for any value of μ. On the other hand, for Hopf bifurcations ($\omega \neq 0$), the necessary condition $\hat{\lambda} = -1 + i0$ gives

$$\omega_0 = \omega_R = \sqrt{\frac{6}{5}} \quad \text{and} \quad \mu_0 = -\frac{5}{6}\,.$$

The normalized right and left eigenvectors v, $w^\top$ of the transfer function $G(i\omega_R)J$ corresponding to $\hat{\lambda}$ are $v = w^\top = 1$, and the closed-loop transfer function is

$$H(s) = \frac{1}{s^3 - \mu s^2 + 1.2s + 1}\,. \tag{26}$$

Equation (26) calculated at $s = 0$ and $s = i2\omega_R$ yields

$$H(0) = 1\,, \quad H(i2\omega_R) = \frac{1}{1 + \frac{24}{5}\mu - i\frac{36}{5}\sqrt{\frac{6}{5}}}\,.$$

Then, the following numbers are obtained

$$V_{02} = -\frac{1}{2}\,,$$

$$V_{22} = -\frac{1}{2}H(i2\omega_R)\,,$$

and

$$p_1(\omega_R;\, \mu) = \left(-\frac{3}{2} - \frac{24}{5}\mu + i\frac{36}{5}\sqrt{\frac{6}{5}}\right) H(i2\omega_R)\,.$$

It is worth mentioning that $Q = \overline{Q} = 2$ and $L = 0$ again since the function $f(e)$ is quadratic.

Therefore, the vector $\xi_1(\omega_R)$ is shown as

$$\xi_1(\omega_R) = \frac{5}{10+6\mu}\left(\frac{3}{2} + \frac{24}{5}\mu - i\frac{36}{5}\sqrt{\frac{6}{5}}\right) H(i2\omega_R)\,.$$

4.2. *Modified Rössler system with a quadratic nonlinearity*

Let us consider a system with the Rössler type structure [Thomas, 1999a, 1999b] and also identified as one simple system with complex dynamics

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3\,, \\ \dot{x}_2 &= x_1 + \mu x_2\,, \\ \dot{x}_3 &= x_1^2 - cx_3\,, \end{aligned} \tag{27}$$

where μ is the main bifurcation parameter and c is a real constant.

The proposed realization on Eq. (27) is

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \mu & 0 \\ -1 & 0 & -c \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = 0\,,$$

$$C = [1 \quad 0 \quad 0]\,, \quad g(x) = x_1 + x_1^2\,.$$

The transfer function results is

$$G(s;\,\mu) = -\frac{s-\mu}{\Delta(s)}\,, \tag{28}$$

where $\Delta(s) = s^3 + (c-\mu)s^2 - \mu c s + \mu + c$, and the nonlinear function is

$$f(e) = -e_1 + e_1^2\,.$$

Then, the equilibrium points (output) are

$$P_0 = 0\,, \quad P_1 = -\frac{c}{\mu}\,.$$

Again, the equilibrium point P_0 at the origin is considered. The eigenlocus is given by

$$\hat{\lambda}(s) = G(s)J = \frac{s-\mu}{\Delta(s)}\,,$$

where $J = -1 + 2\hat{e}_1 = -1$.

Let us first analyze in the time-domain the types of bifurcations that this system can perform. The linearization around the origin of Eq. (27) gives

$$A_t = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \mu & 0 \\ 0 & 0 & -c \end{bmatrix},$$

which has the eigenvalues given by

$$(\lambda_t + c)(\lambda_t^2 - \mu\lambda_t + 1) = 0\,.$$

Then, it is easy to see that the condition for a Hopf bifurcation is fulfilled when $\mu = 0$, and a type of static bifurcation appears when $c = 0$. Moreover, when both conditions occur simultaneously a bifurcation involving two pure imaginary eigenvalues plus a zero eigenvalue is detected.

These observations can be made analogously in the frequency domain counterpart, by taking $\hat{\lambda}(s) = -1$, which is the bifurcation condition, resulting in

$$\begin{aligned} \hat{\lambda}(i\omega) &= \frac{-\mu + i\omega}{c + \mu - \omega^2(c-\mu) - i\omega(\mu c + \omega^2)} \\ &= -1\,. \end{aligned} \tag{29}$$

It is easily checked that when $\omega = 0$ (static bifurcation condition) Eq. (29) gives

$$\frac{-\mu}{c+\mu} = -1 \Rightarrow c = 0\,, \quad \mu \neq 0\,.$$

On the other hand, for $\omega \neq 0$ (dynamic or Hopf bifurcation), the following equations are obtained by solving the real and the imaginary parts of Eq. (29)

$$\begin{aligned} \omega^2 &= 1 - \mu c\,, \\ \omega^2 &= -\mu^2 + 1 + \frac{\mu}{c}\,. \end{aligned} \tag{30}$$

After simple calculations, we arrive at

$$-\mu^2 + \frac{\mu}{c} + \mu c = 0\,. \tag{31}$$

Then, clearly for the Hopf bifurcation condition, Eq. (31) gives $\mu = 0$, and Eq. (30) $\omega_0 = 1$. In addition, the eigenlocus $\hat{\lambda}(i\omega)$ crosses the real axis closest to the critical point $-1 + i0$ at

$$\omega_R = \sqrt{-\mu^2 + 1 + \frac{\mu}{c}}\,.$$

It is also easily verified that the condition of a simultaneous occurrence of a pure imaginary pair of eigenvalues and a zero eigenvalue when $\mu = c = 0$ in Eq. (29),

$$-1 = \frac{i\omega}{-i\omega^3}\,, \Rightarrow \omega(\omega^2 - 1) = 0\,,$$

giving the frequencies $\omega_S = 0$ and $\omega_H = 1$ as solutions. However, the reader can notice that in the frequency domain the eigenvalue conditions for detecting bifurcations seem more cumbersome than in the classical time-domain formulation.

The normalized right and left eigenvectors v and $w^\top$ of the transfer function $G(i\omega_R)J$ corresponding to $\hat{\lambda}$ result in $v = w^\top = 1$, and the closed-loop transfer function is

$$H(s) = -\frac{s-\mu}{s^3 + (c-\mu)s^2 + (1-\mu c)s + c}\,. \tag{32}$$

Expression (32) evaluated at $s = 0$ and $s = i2\omega_R$ gives

$$H(0) = \frac{\mu}{c}\,,$$

$$H(i2\omega_R) = -\frac{-\mu + i2\omega_R}{c - 4\omega_R^2(c-\mu) + i\omega_R(2 - 2\mu c - 8\omega_R^2)}\,.$$

Then, we can compute the numbers

$$V_{02} = -\frac{\mu}{2c}\,,$$

$$V_{22} = -\frac{1}{2}H(i2\omega_R)\,,$$

$$p_1(\omega_R;\,\mu) = -\frac{\mu}{c} - \frac{1}{2}H(i2\omega_R)\,,$$

and

$$\xi_1(\omega_R) = -\frac{(-\mu + i\omega_R)\left(\dfrac{\mu}{c} + \dfrac{1}{2}H(i2\omega_R)\right)}{c - \mu - \omega_R^2(c-\mu) - i\omega_R(\mu c + \omega_R^2)}\,.$$

As in the previous example, $Q = \overline{Q} = 2$ and $L = 0$.

4.3. *Modified Rössler system with a cubic nonlinearity*

A system with a Rössler type structure and a cubic nonlinearity [Thomas, 1999a, 1999b] is considered. This system is

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3\,, \\ \dot{x}_2 &= x_1 + \mu x_2\,, \\ \dot{x}_3 &= x_1^3 - c x_3\,, \end{aligned} \tag{33}$$

where μ is the main bifurcation parameter and c is a real constant.

The adopted realization is

$$A = \begin{bmatrix} 0 & -1 & -1 \\ 1 & \mu & 0 \\ -1 & 0 & -c \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad D = 0\,,$$

$$C = [1 \quad 0 \quad 0]\,, \quad g(x) = x_1 + x_1^3\,.$$

The transfer function is given by Eq. (28) and the nonlinear function is

$$f(e) = -e_1 - e_1^3\,.$$

The equilibrium points (output) are

$$P_0 = 0\,, \quad P_{1,2} = \pm\sqrt{\frac{c}{\mu}}\,.$$

Let us consider the equilibrium point at the origin. The eigenlocus is given by

$$\hat{\lambda}(s;\,\mu) = G(s)J = \frac{s-\mu}{\Delta(s)}\,,$$

where $J = -1 - 2\hat{e}_1^2 = -1$.

Since the linearized system is the same as in the modified Rössler system with a quadratic nonlinearity, the bifurcation conditions are the same as in the previous example: $\mu_0 = 0$ and $\omega_0 = 1$. In the same way as before $\omega_R = \sqrt{-\mu^2 + 1 + (\mu/c)}$.

The normalized right and left eigenvectors result in $v = w^\top = 1$ and the closed-loop transfer function is Eq. (32). Since $Q = \overline{Q} = 0$ then

$$V_{02} = V_{22} = 0\,,$$

and

$$p_1(\omega_R;\,v) = \frac{1}{8}L\overline{v} = -\frac{3}{4}\,,$$

where $L = -6$. The expression of the number $\xi_1(\omega_R)$ is given by

$$\xi_1(\omega_R) = \frac{3}{4(\omega_R^2 + \mu c)}\,.$$

5. Numerical Results

In this section the proposed technique is applied to the systems described in Sec. 4 to detect different types of bifurcations of limit cycles. To validate the results, standard packages for the continuation of periodic solutions are used. For example LBLC from the LOCBIF library [Khibnik *et al.*, 1993] or AUTO [Doedel *et al.*, 1997; Ermentrout, 2001]. The value of one of the output variables (in oscillatory regime) is plotted as a function of the main bifurcation parameter μ; therefore the values of the parameter where the bifurcations occur can be read very precisely.

In order to obtain the expressions of the higher-order approximations it is important to mention that the examples of Sec. 3 have the right and left eigenvectors v and $w^\top$ of dimension 3×1 and 1×3, respectively. Therefore, the vectors V_{13}, V_{15}, and V_{17} have complicated expressions. For this reason, only the examples of Sec. 4 are considered for which $V_{13} = V_{15} = V_{17} = 0$ (see the higher-order approximation formulas in the Appendix).

Table 1. Characteristic multipliers of M_3 for Genesio and Tesi system for detecting a period-doubling bifurcation.

	μ	λ_1	λ_2	λ_3
	−0.4880	1.00272	−1.00548	−0.05884
M_3	−0.4885	1.00270	−1.00057	−0.05896
	−0.4890	1.00269	−0.99567	−0.05909

5.1. *Genesio and Tesi system*

For the system of Eq. (25), a branch of periodic solutions emerges from the origin when a *supercritical* Hopf bifurcation occurs at $\mu_{\text{HB}} = -5/6$. Using AUTO, the prediction for the first period-doubling bifurcation is $\mu_{\text{PD}} \approx -0.48096$. At this point the maximum value of x_1 is $x_{1\,\text{max}} \approx 0.77944$ and the period is $T \approx 5.7830$.

Evaluating the approximation of the monodromy matrix M_3, by using the periodic prediction L_3 (six harmonics), we obtain the characteristic multipliers shown in Table 1. From this table, the approximate monodromy matrix M_3 gives one multiplier (λ_2) close to -1 at $\mu_{\text{PD}M_3} \approx -0.4885$, also at this point, $x_{1\,\max M_3} \approx 0.7994$ and $T_{M_3} \approx 5.7882$. All of these results are close to those obtained by AUTO.

It is important to mention that one of the multipliers must be $+1.0000$, since it is a requirement for the technical construction of the Poincaré section. In other words, it indicates a measure of the error of the approximation [Guckenheimer & Meloon, 2000]. For this case, the error of the approximation of M_3 is of order 2.7×10^{-3}.

Figure 4 shows the continuation of the limit cycle from the Hopf bifurcation to the first period-doubling bifurcation, where the L_3-approximation is plotted and compared with that obtained with AUTO.

5.2. *Modified Rössler system with a quadratic nonlinearity*

The behavior of the modified Rössler system with a quadratic nonlinearity [Eq. (27)] is evaluated for $c = 2$, $c = 0.74$ and $c = 0.6$. Two kinds of bifurcations are obtained: a period-doubling bifurcation ($c = 2$ and $c = 0.74$), and a Neimark–Sacker bifurcation ($c = 0.6$).

For $c = 2$, a branch of periodic solutions develops when the parameter μ is varied from 0, after the occurrence of a Hopf bifurcation. The prediction of the first period-doubling is $\mu_{\text{PD}} \approx 0.23971$, with $x_{1\,\max} \approx 1.901518$ and $T \approx 6.4718$. The multipliers of the approximate monodromy matrices M_3 (sixth-order) and M_4 (eighth-order) are shown in Table 2. From this table, M_3 determines that the period-doubling bifurcation takes place at $\mu_{\text{PD}M_3} \approx 0.2358$, with $x_{1\,\max M_3} \approx 1.910537$ and $T_{M_3} \approx 6.4843$. For M_4, $\mu_{\text{PD}M_4} \approx 0.2415$, with $x_{1\,\max M_4} \approx 1.89912$ and $T_{M_4} \approx 6.4665$. For these cases, the errors of the approximation are $e_{M_3} \approx 6.83 \times 10^{-3}$ and $e_{M_4} \approx 5.51 \times 10^{-3}$. Although the difference in the order is small, the error in the estimation of $x_{1\,\max}$ is appreciable as shown in Fig. 5.

The system with $c = 0.74$, has a Hopf bifurcation at $\mu = 0$, and a period-doubling bifurcation at $\mu_{\text{PD}1} \approx 0.407716$, with $x_{1\,\max 1} \approx 1.28816$ and $T_1 \approx 5.452903$. A second period-doubling bifurcation occurs at $\mu_{\text{PD}2} \approx 0.7223$, with $x_{1\,\max 2} \approx 1.0711$ and $T_2 \approx 5.2858$. A sixth-order approximation (L_3) is used to estimate the period-doubling, and the results are shown in Table 3. For this case, the period-doubling bifurcation is approximated at $\mu_{\text{PD}1M_3} \approx 0.399$, $x_{1\,\max 1M_3} \approx 1.295102$ and $T_{1M_3} \approx 5.47364$. The error in the approximation is $e_{M_3} \approx 3.52\times10^{-2}$. The second period-doubling bifurcation cannot be detected with this technique. The curves of the

Table 2. Characteristic multipliers of M_3 and M_4 for detecting a period-doubling bifurcation in the modified quadratic Rössler system with $c = 2$.

	μ	λ_1	λ_2	λ_3
	0.2357	0.99317	−0.99960	-0.10817×10^{-4}
M_3	0.2358	0.99317	−1.00057	-0.10835×10^{-4}
	0.2360	0.99315	−1.00251	-0.10829×10^{-4}
	0.24100	1.00546	−0.99596	-0.1148×10^{-4}
M_4	0.24150	1.00551	−1.00055	-0.1145×10^{-4}
	0.24225	1.00557	−1.00743	-0.1141×10^{-4}

Table 3. Characteristic multipliers of M_3 for detecting a period-doubling bifurcation in the modified quadratic Rössler system with $c = 0.74$.

	μ	λ_1	λ_2	λ_3
	0.398	0.9651	−0.99590	−0.15998
M_3	0.399	0.9648	−0.99974	−0.16034
	0.400	0.9646	−1.00358	−0.16071

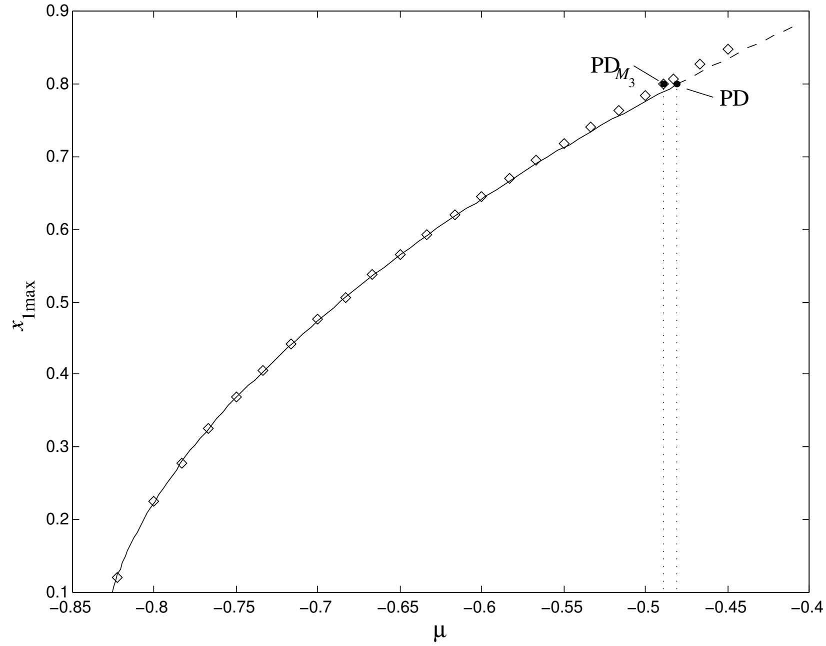

Fig. 4. Periodic branch starting from Hopf bifurcation for Genesio and Tesi system. — Stable, --- unstable branch: AUTO; ◊ : L_3-approximation. PD is the first period-doubling bifurcation.

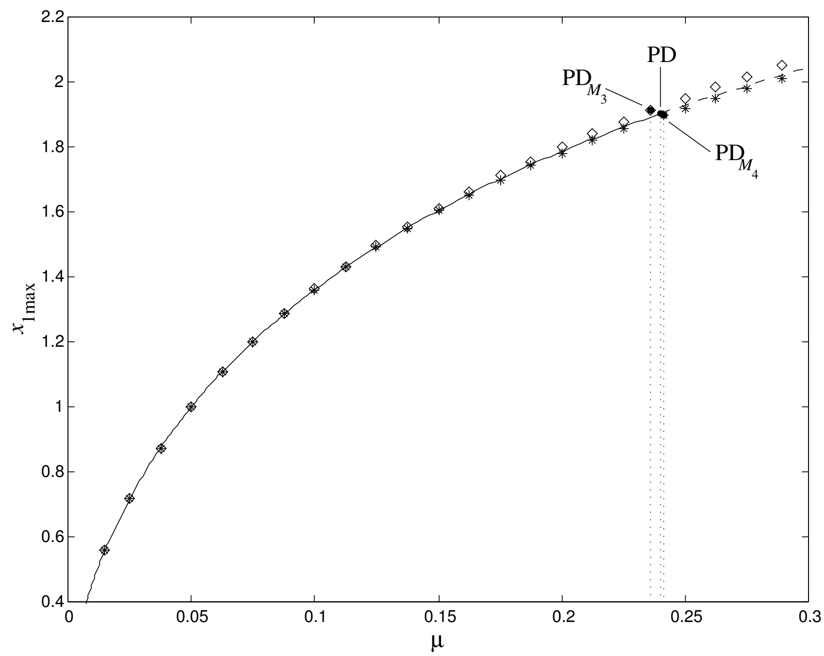

Fig. 5. Branch of periodic solutions in the modified Rössler system with a quadratic nonlinearity and $c = 2$. — Stable, --- unstable branch: AUTO; ◊ : L_3-approximation; $*$: L_4-approximation. PD is the first period-doubling bifurcation.

approximation and the one obtained with AUTO are shown in Fig. 6.

The Rössler system with a quadratic nonlinearity and $c = 0.6$ undergoes a Hopf bifurcation when $\mu = 0$. The cycle experiments a Neimark–Sacker bifurcation at $\mu_{\rm NS} \approx 0.6$, with $x_{1\,\max} \approx 1.218946$ and $T \approx 4.9538$. The approximate characteristic multipliers for this case are shown in Table 4. In both cases the approximation of one of the multipliers (λ_1) must be $+1.0000$, but $\lambda_{1M_3} = 0.8650$

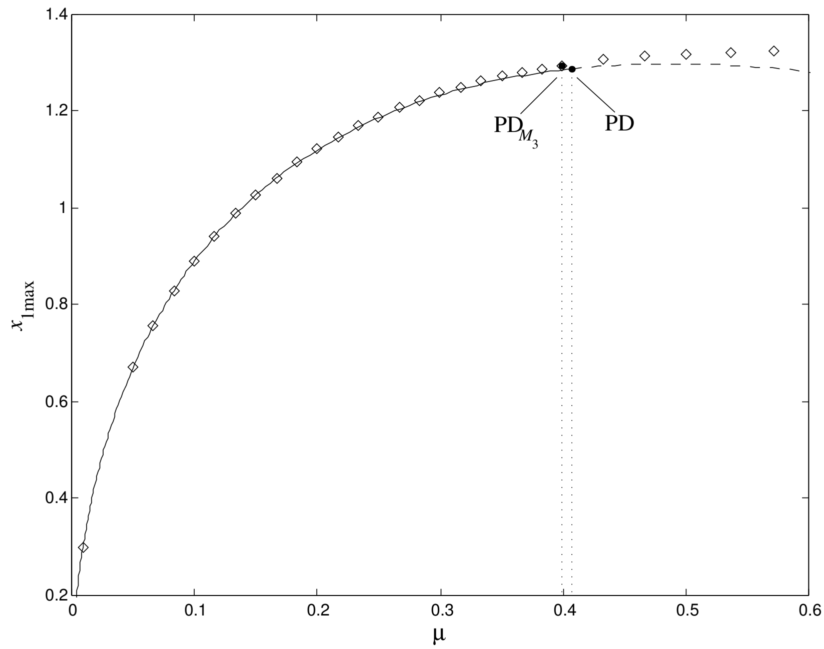

Fig. 6. Branch of periodic solutions in the modified Rössler system with a quadratic nonlinearity and $c = 0.74$. — Stable, - - - unstable branch: AUTO; ◊ : L_3-approximation. PD is the first period-doubling bifurcation.

($e_{M_3} \approx 1.35 \times 10^{-1}$) and $\lambda_{1M_4} = 1.12153$ ($e_{M_4} \approx 1.12 \times 10^{-1}$). This error in the approximation is reflected in the estimation of the Neimark–Sacker bifurcation, for which the eigenvalues of M_3 cross the unit circle at $\mu_{\mathrm{NS}M_3} \approx 0.5715$, with $x_{1\max M_3} \approx 1.30293$ and $T_{M_3} \approx 5.01989$; for the M_4 approximation, $\mu_{\mathrm{NS}M_4} \approx 0.6235$, with $x_{1\max M_4} \approx 1.27279$ and $T_{M_4} \approx 4.90041$. This large error indicates that the

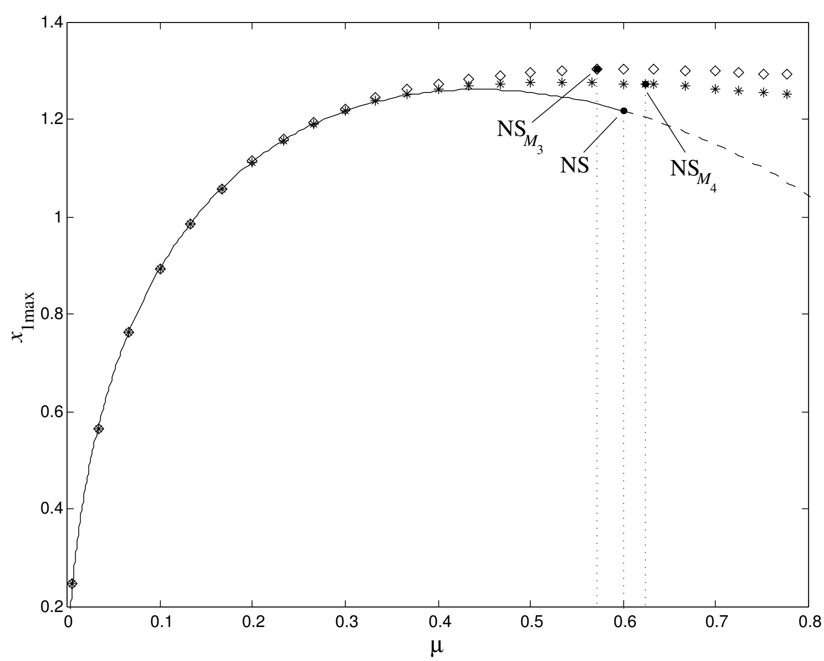

Fig. 7. Branch of periodic solutions in the modified Rössler system with a quadratic nonlinearity and $c = 0.6$. — Stable, - - - unstable branch: AUTO; ◊ : L_3-approximation; ∗ : L_4-approximation. NS is the Neimark–Sacker bifurcation.

Table 4. Characteristic multipliers of M_3 and M_4 for detecting a Neimark–Sacker bifurcation in the modified quadratic Rössler system with $c = 0.6$.

	μ	λ_1	$\lambda_{2,3}$
	0.5700	0.86580	$0.99674\ e^{\pm j95.21}$
M_3	0.5715	0.86507	$1.00093\ e^{\pm j95.12}$
	0.5717	0.86498	$1.00149\ e^{\pm j95.10}$
	0.6226	1.12091	$0.99831\ e^{\pm j93.34}$
M_4	0.6235	1.12153	$1.00023\ e^{\pm j91.11}$
	0.6250	1.12256	$1.00343\ e^{\pm j90.99}$

approximation of the amplitude of one of the variables is not close to the one obtained with AUTO, as shown in Fig. 7.

5.3. *Modified Rössler system with a cubic nonlinearity*

Finally, the proposed technique is applied to the modified Rössler system with a cubic nonlinearity [Eq. (33)]. The Hopf bifurcation occurs at $\mu = 0$, and by increasing the value of μ a pitchfork bifurcation of cycles occurs at $\mu_{\text{PB}} \approx 0.53316$, with $x_{1\,\max} \approx 1.030691$ and $T \approx 5.642649$. The approximate characteristic multipliers M_4 (eighth-order (L_4) approximation) are shown in Table 5.

Table 5. Characteristic multipliers of M_4 for detecting a pitchfork bifurcation of cycles in the modified cubic Rössler system.

	μ	λ_1	λ_2	λ_3
	0.53290	0.994204	0.99993	0.071937
M_4	0.53292	0.994203	1.00008	0.071935
	0.53316	0.994193	1.00184	0.071904

The pitchfork bifurcation of cycles is estimated at $\mu_{\text{PB}M_4} \approx 0.53292$, with $x_{1\,\max M_4} \approx 1.030703$ and $T_{M_4} \approx 5.64726$, very close to the one obtained with AUTO. In this case the approximation error is $e_{M_4} \approx 5.8 \times 10^{-3}$. The curves of the L_4-approximation and the one obtained with AUTO are shown in Fig. 8, where both curves are close to each other.

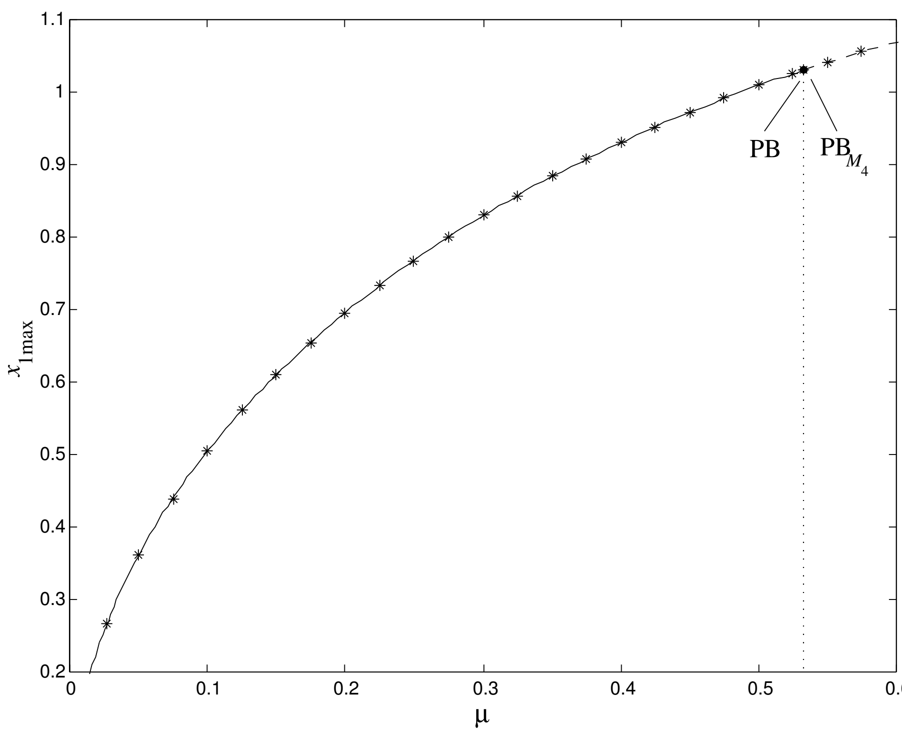

Fig. 8. Branch of periodic solutions in the modified Rössler system with a cubic nonlinearity and $c = 1$. —— Stable, - - - unstable branch: AUTO; $*$: L_4-approximation. PB is the pitchfork bifurcation.

6. Discussions and Conclusions

The main purpose of the present work is to draw the attention on simple formulas to analyze the first limit cycle bifurcations by varying the main bifurcation parameter. It has been observed that in the Sprott and the modified Rössler systems the correction of the bias increases proportionally to the main bifurcation parameter, i.e. $V_{02} \propto \mu$. This is an interesting observation since for period-doubling bifurcation an extra vector, taking care of the frequency $\omega_R/2$, should appear between V_{02} and v. This shift of the energy of the signal to smaller values of frequencies is a typical phenomenon in the period-doubling bifurcation scenario.

The computation of the monodromy matrix has been carried out for simple oscillatory systems containing one nonlinear — quadratic or cubic — term. In order to compute the monodromy matrices, higher-order approximation formulas have been used. The bifurcations of cycles are well predicted (regarding accuracy in engineering terms) even with a fourth-order HBA, except for the Neimark–Sacker bifurcation. This methodology seems to be very powerful to explore the bifurcations of cycles close to Hopf bifurcations.

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Appendix A

A.1. *Sixth-order formulas for the examples of Secs. 4.1 and 4.2*

The corresponding Fourier coefficients for a sixth-order HBA are

$$E^0 = V_{02}(\hat{\omega}_3)\hat{\theta}_3^2 + V_{04}(\hat{\omega}_3)\hat{\theta}_3^4 + V_{06}(\hat{\omega}_3)\hat{\theta}_3^6\,,$$
$$E^1 = v(\hat{\omega}_3)\hat{\theta}_3 + V_{13}(\hat{\omega}_3)\hat{\theta}_3^3 + V_{15}(\hat{\omega}_3)\hat{\theta}_3^5 + V_{17}(\hat{\omega}_3)\hat{\theta}_3^7\,,$$
$$E^2 = V_{22}(\hat{\omega}_3)\hat{\theta}_3^2 + V_{24}(\hat{\omega}_3)\hat{\theta}_3^4 + V_{26}(\hat{\omega}_3)\hat{\theta}_3^6\,,$$
$$E^3 = V_{33}(\hat{\omega}_3)\hat{\theta}_3^3 + V_{35}(\hat{\omega}_3)\hat{\theta}_3^5\,,$$
$$E^4 = V_{44}(\hat{\omega}_3)\hat{\theta}_3^4 + V_{46}(\hat{\omega}_3)\hat{\theta}_3^6\,,$$
$$E^5 = V_{55}(\hat{\omega}_3)\hat{\theta}_3^5\,,$$
$$E^6 = V_{66}(\hat{\omega}_3)\hat{\theta}_3^6\,.$$

Vectors v, V_{02} and V_{22} are obtained in Sec. 4.1 for the Genesio and Tesi system and in Sec. 4.2 for the modified Rössler system with a quadratic nonlinearity. The formulas for the remaining vectors are given in [Moiola & Chen, 1996] and for both examples the results are

$$V_{13} = V_{15} = V_{17} = 0\,,$$
$$V_{33} = -\frac{1}{2}H(i3\hat{\omega}_3)D_2 v V_{22}\,,$$
$$V_{04} = -\frac{1}{4}H(0)D_2(2V_{02}^2 + V_{22}\overline{V}_{22})\,,$$
$$V_{24} = -\frac{1}{2}H(i2\hat{\omega}_3)D_2(2V_{02}V_{22} + \overline{v}V_{33})\,,$$
$$V_{44} = -\frac{1}{4}H(i4\hat{\omega}_3)D_2(V_{22}^2 + 2vV_{33})\,,$$
$$V_{06} = -\frac{1}{4}H(0)D_2(4V_{02}V_{04} + V_{22}\overline{V}_{24} + \overline{V}_{22}V_{24} + V_{33}\overline{V}_{33})$$
$$V_{35} = -\frac{1}{2}H(i3\hat{\omega}_3)D_2(vV_{24} + 2V_{02}V_{33} + \overline{v}V_{44})\,,$$
$$V_{26} = -\frac{1}{2}H(i2\hat{\omega}_3)D_2 2V_{02}V_{24} + 2V_{04}V_{22} + \overline{v}V_{35} + \overline{V}_{22}V_{44}\,,$$
$$V_{55} = -\frac{1}{2}H(i5\hat{\omega}_3)(D_2 vV_{44} + V_{22}V_{33})\,,$$
$$V_{46} = -\frac{1}{2}H(i4\hat{\omega}_3)D_2(vV_{35} + V_{22}V_{24} + 2V_{02}V_{44} + \overline{v}V_{55})\,,$$
$$V_{66} = -\frac{1}{2}H(i6\hat{\omega}_3)D_2\left(vV_{55} + V_{22}V_{44} + \frac{1}{2}V_{33}^2\right)\,,$$

where $D_2 = D_e^2 f(\hat{e}) = 2$ and $H(ik\hat{\omega}_3)$ depends on the example (see the expressions in Sec. 4).

In order to obtain the approximations of $\hat{\omega}$ and $\hat{\theta}$ the complex numbers $\xi_2(\hat{\omega}_3)$ and $\xi_3(\hat{\omega}_3)$ are computed

$$\xi_2(\hat{\omega}_3) = -G(i\hat{\omega}_3)p_2(\hat{\omega}_3;\,\mu)\,,$$
$$\xi_3(\hat{\omega}_3) = -G(i\hat{\omega}_3)p_3(\hat{\omega}_3;\,\mu)\,,$$

where $G(i\hat{\omega}_3)$ is given in Secs. 4.1 and 4.2 for each example, and

$$p_2(\hat{\omega}_3;\,\mu) = \frac{1}{2}D_2(2vV_{04} + \overline{V}_{22}V_{33} + \overline{v}V_{24})\,,$$
$$p_3(\hat{\omega}_3;\,\mu) = \frac{1}{2}D_2 2vV_{06} + \overline{v}V_{26} + \overline{V}_{22}V_{35} + \overline{V}_{24}V_{33} + \overline{V}_{33}V_{44}\,.$$

For the vectors involved up to a fourth-order approximation, compact expressions (in terms of the bifurcation parameter) are obtained:

Genesio and Tesi system

$$V_{33} = \frac{1}{2}H(i3\hat{\omega}_2)H(i2\hat{\omega}_2)\,,$$
$$V_{04} = -\frac{1}{8}[2 + H(i2\hat{\omega}_2)\overline{H}(i2\hat{\omega}_2)]\,,$$
$$V_{24} = -\frac{1}{2}H(i2\hat{\omega}_2)H(i2\hat{\omega}_2)[1 + H(i3\hat{\omega}_2)]\,,$$
$$V_{44} = -\frac{1}{2}H(i4\hat{\omega}_2)H(i2\hat{\omega}_2)\left[\frac{1}{4}H(i2\hat{\omega}_2) + H(i3\hat{\omega}_2)\right]\,,$$

where

$$H(ik\hat{\omega}_2) = \frac{1}{1 + \mu k^2\hat{\omega}_2^2 + ik\hat{\omega}_2(1.2 - k^2\hat{\omega}_2^2)}\,.$$

Modified Rössler system with a quadratic nonlinearity

$$V_{33} = \frac{1}{2}H(i3\hat{\omega}_2)H(i2\hat{\omega}_2),$$

$$V_{04} = -\frac{1}{8}H(0)\left[2\left(\frac{\mu}{c}\right)^2 + H(i2\hat{\omega}_2)\overline{H}(i2\hat{\omega}_2)\right],$$

$$V_{24} = -\frac{1}{4}H(i2\hat{\omega}_2)\left[2\frac{\mu}{c}H(i2\hat{\omega}_2) + 2H(i3\hat{\omega}_2)H(i2\hat{\omega}_2)\right],$$

$$V_{44} = -\frac{1}{4}H(i4\hat{\omega}_2)\left[\frac{1}{2}H^2(i2\hat{\omega}_2) + 2H(i3\hat{\omega}_2)H(i2\hat{\omega}_2)\right],$$

where

$$H(ik\hat{\omega}_2) = \frac{-(ik\hat{\omega}_2 - \mu)}{c - k^2\hat{\omega}_2^2(c-\mu) + ik\hat{\omega}_2(1 - \mu c - k^2\hat{\omega}_2^2)}.$$

A.2. *Eighth-order approximation formulas for the example of Sec. 4.3*

The Fourier coefficients E^k needed in the calculation of an eighth-order HBA are

$$\begin{aligned}
E^0 &= V_{02}(\hat{\omega}_4)\hat{\theta}_4^2 + V_{04}(\hat{\omega}_4)\hat{\theta}_4^4 \\
&\quad + V_{06}(\hat{\omega}_4)\hat{\theta}_4^6 + V_{08}(\hat{\omega}_4)\hat{\theta}_4^8, \\
E^1 &= v(\hat{\omega}_4)\hat{\theta}_4 + V_{13}(\hat{\omega}_4)\hat{\theta}_4^3 + V_{15}(\hat{\omega}_4)\hat{\theta}_4^5 \\
&\quad + V_{17}(\hat{\omega}_4)\hat{\theta}_4^7 + V_{19}(\hat{\omega}_4)\hat{\theta}_4^9, \\
E^2 &= V_{22}(\hat{\omega}_4)\hat{\theta}_4^2 + V_{24}(\hat{\omega}_4)\hat{\theta}_4^4 \\
&\quad + V_{26}(\hat{\omega}_4)\hat{\theta}_4^6 + V_{28}(\hat{\omega}_4)\hat{\theta}_4^8, \\
E^3 &= V_{33}(\hat{\omega}_4)\hat{\theta}_4^3 + V_{35}(\hat{\omega}_4)\hat{\theta}_4^5 + V_{37}(\hat{\omega}_4)\hat{\theta}_4^7, \\
E^4 &= V_{44}(\hat{\omega}_4)\hat{\theta}_4^4 + V_{46}(\hat{\omega}_4)\hat{\theta}_4^6 + V_{48}(\hat{\omega}_4)\hat{\theta}_4^8, \\
E^5 &= V_{55}(\hat{\omega}_4)\hat{\theta}_4^5 + V_{57}(\hat{\omega}_4)\hat{\theta}_4^7, \\
E^6 &= V_{66}(\hat{\omega}_4)\hat{\theta}_4^6 + V_{68}(\hat{\omega}_4)\hat{\theta}_4^8, \\
E^7 &= V_{77}(\hat{\omega}_4)\hat{\theta}_4^7, \\
E^8 &= V_{88}(\hat{\omega}_4)\hat{\theta}_4^8.
\end{aligned}$$

Since $D_2 = D_e^2 f(\hat{e}) = 0$ and $D_3 = D_e^3 f(\hat{e}) = -6$, these formulas can be reduced to

$$\begin{aligned}
E^0 &= E^2 = E^4 = E^6 = E^8 = 0, \\
E^1 &= \hat{\theta}v, \\
E^3 &= \hat{\theta}_4^3 V_{33} + \hat{\theta}_4^5 V_{35} + \hat{\theta}_4^7 V_{37}, \\
E^5 &= \hat{\theta}_4^5 V_{55} + \hat{\theta}_4^7 V_{57}, \\
E^7 &= \hat{\theta}_4^7 V_{77},
\end{aligned}$$

where

$$V_{33} = \frac{1}{4}H(i3\hat{\omega}_4),$$

$$V_{35} = \frac{3}{8}[H(i3\hat{\omega}_4)]^2,$$

$$V_{55} = \frac{3}{16}H(i5\hat{\omega}_4)H(i3\hat{\omega}_4),$$

$$V_{37} = \frac{9}{16}[H(i3\hat{\omega}_4)]^2\left[H(i3\hat{\omega}_4) + \frac{1}{4}H(i5\hat{\omega}_4)\right],$$

$$V_{57} = \frac{1}{32}H(i5\hat{\omega}_4)H(i3\hat{\omega}_4)\left[\frac{21}{2}H(i3\hat{\omega}_4) + 9H(i5\hat{\omega}_4)\right],$$

$$V_{77} = \frac{3}{64}H(i7\hat{\omega}_4)H(i3\hat{\omega}_4)[3H(i5\hat{\omega}_4) + H(i3\hat{\omega}_4)],$$

and

$$H(ik\hat{\omega}_4) = \frac{-(ik\hat{\omega}_4 - \mu)}{c - k^2\hat{\omega}_4^2(c-\mu) + ik\hat{\omega}_4(1 - \mu c - k^2\hat{\omega}_4^2)}.$$

The complex numbers $\xi_2(\hat{\omega}_4)$, $\xi_3(\hat{\omega}_4)$ and $\xi_4(\hat{\omega}_4)$ required for computing the eighth-order approximation are given by

$$\begin{aligned}
\xi_2(\hat{\omega}_4) &= -G(i\hat{\omega}_4)p_2(\hat{\omega}_4;\, \mu), \\
\xi_3(\hat{\omega}_4) &= -G(i\hat{\omega}_4)p_3(\hat{\omega}_4;\, \mu), \\
\xi_4(\hat{\omega}_4) &= -G(i\hat{\omega}_4)p_4(\hat{\omega}_4;\, \mu),
\end{aligned}$$

where

$$G(i\hat{\omega}_4) = -\frac{(i\hat{\omega}_4 - \mu)}{\mu + c - \hat{\omega}_4^2(c-\mu) - i\hat{\omega}_4(\mu c - \hat{\omega}_4^2)},$$

and

$$p_2(\hat{\omega}_4;\, \mu) = -\frac{3}{16}H(i3\hat{\omega}_4),$$

$$p_3(\hat{\omega}_4;\, \mu) = -\frac{3}{32}\{3[H(i3\hat{\omega}_4)]^2 + H(i3\hat{\omega}_4)\overline{H}(i3\hat{\omega}_4)\},$$

$$p_4(\hat{\omega}_4;\, \mu) = -\frac{3}{4}V_{37} - \frac{3}{2}V_{33}\overline{V}_{35} - \frac{3}{2}\overline{V}_{33}V_{35} - \frac{3}{2}\overline{V}_{33}V_{55}.$$