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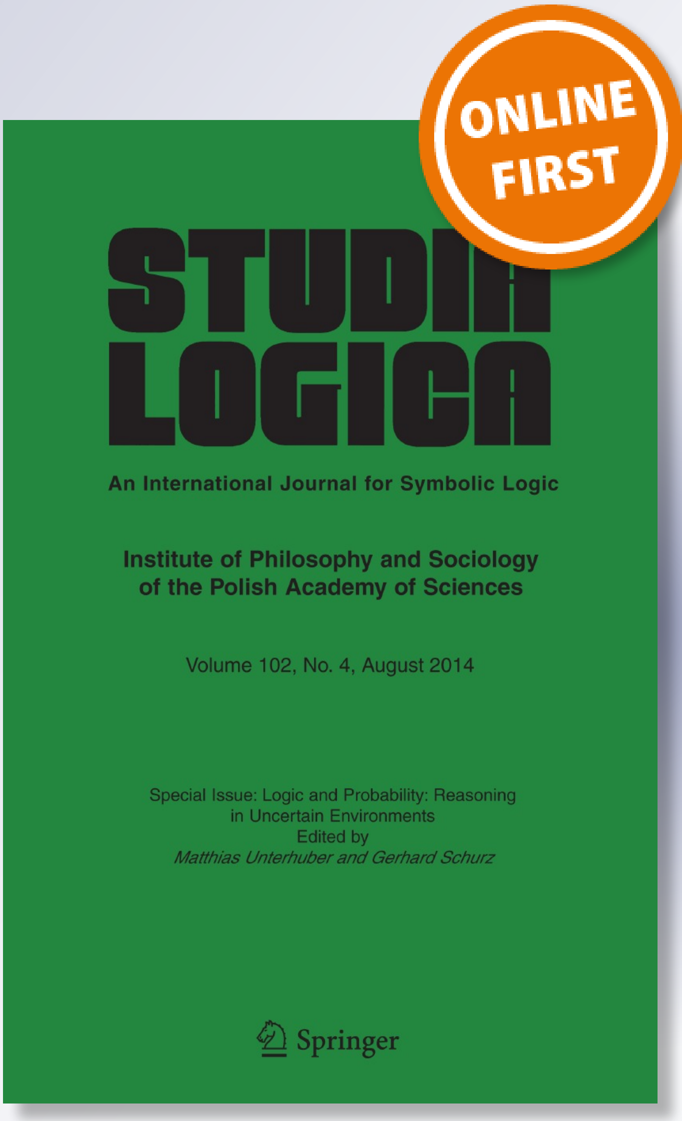

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Abstract. In this paper we introduce a logic that we name *semi Heyting–Brouwer logic*, $\mathcal{SHB}$, in such a way that the variety of double semi-Heyting algebras is its algebraic counterpart. We prove that, up to equivalences by translations, the Heyting–Brouwer logic $\mathcal{HB}$ is an axiomatic extension of $\mathcal{SHB}$ and that the propositional calculi of intuitionistic logic $\mathcal{I}$ and semi-intuitionistic logic $\mathcal{SI}$ turn out to be fragments of $\mathcal{SHB}$.

Keywords: Semi Heyting–Brouwer logic, Semi-Heyting algebras, Heyting–Brouwer logic, Heyting algebras.

1. Introduction

The *Heyting–Brouwer logic*, $\mathcal{HB}$, was introduced by Rauszer [17,18] by means of a Hilbert-style propositional calculus. It has been also studied by other authors under different names, such as dual intuitionistic logic [7,9], subtractive logic [5,6] or bi-intuitionistic logic [10,11], and even with different approaches (algebraic, relational, axiomatic, by sequents, etc.). Its algebraic counterpart is the variety **DblH** of *double Heyting algebras* [2,12], also known as Heyting–Brouwer algebras [21], biHeyting-algebras [13] and semi-Boolean algebras [17,18].

In [19], Sankappanavar introduced the variety **SH** of semi-Heyting algebras, an abstraction of Heyting algebras, and in [3] we defined a new logic $\mathcal{SI}$ called semi-intuitionistic logic such that the semi-Heyting algebras are the semantics for $\mathcal{SI}$. Besides, the intuitionistic logic is an axiomatic extension of $\mathcal{SI}$.

Several expansions of semi-Heyting algebras were later studied in [20], the most important of which is perhaps the variety **DblSH** of *double semi-Heyting algebras*, an abstraction of **DblH**.

In this paper we introduce the *semi Heyting–Brouwer logic* $\mathcal{SHB}$ by means of a Hilbert-style propositional calculus. We prove that, up to equivalences by translations, $\mathcal{HB}$ is an axiomatic extension of $\mathcal{SHB}$ and that the

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propositional calculi of intuitionistic logic $\mathcal{I}$ [16] and semi-intuitionistic logic $\mathcal{SI}$ [3,4] turn out to be fragments of $\mathcal{SHB}$. We prove that the algebraic semantics of this logic is the variety **DblSH** and we study several properties of it.

2. Preliminaries

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \perp, \top \rangle$ is said to be a *semi-Heyting algebra*, **SH**-algebra for short, if it satisfies the following conditions:

(SH1) $\langle A; \vee, \wedge, \perp, \top \rangle$ is a bounded lattice,

(SH2) $x \wedge (x \to y) \approx x \wedge y$,

(SH3) $x \wedge (y \to z) \approx x \wedge ((x \wedge y) \to (x \wedge z))$,

(SH4) $x \to x \approx \top$.

If instead of axiom (SH4) we put

(SH5) $(x \wedge y) \to y \approx \top$,

then $\mathbf{A}$ is a Heyting algebra [19] (**H**-algebra, for short).

An algebra $\mathbf{A} = \langle A; \vee, \wedge, \to, \leftarrow, \perp, \top \rangle$ is said to be a *double semi-Heyting algebra*, **DblSH**-algebra for short, if it satisfies the following conditions:

(dSH1) $\langle A; \vee, \wedge, \to, \perp, \top \rangle$ is an **SH**-algebra,

(dSH2) $x \vee (x \leftarrow y) \approx x \vee y$,

(dSH3) $x \vee (y \leftarrow z) \approx x \vee ((x \vee y) \leftarrow (x \vee z))$,

(dSH4) $x \leftarrow x \approx \perp$.

If a **DblSH**-algebra $\mathbf{A}$ is such that $\langle A; \vee, \wedge, \to, \perp, \top \rangle$ is a **H**-algebra and, we replace condition (dSH4), by

(dSH5) $(x \vee y) \leftarrow y \approx \perp$

then $\mathbf{A}$ is a double Heyting algebra (**DblH**-algebra) [12].

The class of **SH**-algebras, **H**-algebras, **DblSH**-algebras and **DblH**-algebras will be respectively denoted by $\mathbf{SH}, \mathbf{H}, \mathbf{DblSH}$ and $\mathbf{DblH}$.

It is known that if $\mathbf{A} \in \mathbf{H}$ then for $a, b, c \in A$,

(R) $\quad a \wedge b \leq c$ if and only if $a \leq b \to c$.

In a similar way, (see [17,21]) if $\mathbf{A} \in \mathbf{DblH}$ and $a, b, c \in A$ then

(dR) $\quad a \vee b \geq c$ if and only if $a \geq b \leftarrow c$.

From (R) and (dR) it follows that for $a, b \in A$, with $\mathbf{A} \in \mathbf{DblH}$,

$$a \leq b \text{ if and only if } a \to b = \top \quad \text{and} \quad a \leq b \text{ if and only if } b \leftarrow a = \bot.$$

Following [8], a *(logical) language* $\mathbf{L} = \langle L, ar \rangle$ will be a set L of connectives, each with a fixed arity $ar \geq 0$. For a countably infinite set Var of propositional variables, the *formulas* of the logical language $\mathbf{L}$ are inductively defined as usual.

A *logic* in the language $\mathbf{L}$ is a pair $\mathcal{L} = \langle Fm_{\mathbf{L}}, \vdash_{\mathcal{L}} \rangle$ where $Fm_{\mathbf{L}}$ is the set of formulas and $\vdash_{\mathcal{L}}$ is a substitution-invariant consequence relation on $Fm_{\mathbf{L}}$. The set $Fm_{\mathbf{L}}$ may also be endowed with an algebraic structure, by considering the connectives of the language as operation symbols. The resulting algebra is often called the *algebra of formulas* and denoted by $\mathsf{Fm}_{\mathbf{L}}$. We will present finitary logics by means of their "Hilbert style" sets of axioms and inferences rules.

An *expansion* of a language $\mathbf{L} = \langle L, ar \rangle$ is a language $\mathbf{L}' = \langle L', ar' \rangle$ such that $L \subseteq L'$ and $ar' \restriction L = ar$. A language $\mathbf{L}$ is a *fragment* of a language $\mathbf{L}'$ when $\mathbf{L}'$ is an expansion of $\mathbf{L}$.

The Heyting–Brouwer logic $\mathcal{HB}$, over a language $\{\vee, \wedge, \to, \dot{-}, \neg, \urcorner\}$, is defined in [17] in terms of the following set of axioms:

(B1) $(\alpha \to \beta) \to [(\beta \to \gamma) \to (\alpha \to \gamma)]$,

(B2) $\alpha \to (\alpha \vee \beta)$,

(B3) $\beta \to (\alpha \vee \beta)$,

(B4) $(\alpha \to \gamma) \to [(\beta \to \gamma) \to ((\alpha \vee \beta) \to \gamma)]$,

(B5) $(\alpha \wedge \beta) \to \alpha$,

(B6) $(\alpha \wedge \beta) \to \beta$,

(B7) $(\gamma \to \alpha) \to [(\gamma \to \beta) \to (\gamma \to (\alpha \wedge \beta))]$,

(B8) $((\alpha \wedge \beta) \to \gamma) \to (\alpha \to (\beta \to \gamma))$,

(B9) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \wedge \beta) \to \gamma)$,

(B10) $\alpha \to (\beta \vee (\alpha \dot{-} \beta))$,

(B11) $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$,

(B12) $(\alpha \dot{-} \beta) \to (\urcorner\, (\alpha \to \beta))$,

(B13) $(\gamma \dot{-} (\alpha \dot{-} \beta)) \to (\alpha \dot{-} (\gamma \vee \beta))$,

(B14) $(\neg(\alpha \dot{-} \beta)) \to (\alpha \to \beta)$,

(B15) $(\alpha \to (\gamma \dot{-} \gamma)) \to (\neg\alpha)$,

(B16) $(\neg\alpha) \to (\alpha \to (\gamma \dot{-} \gamma))$, (B17) $((\gamma \to \gamma)\dot{-}\alpha) \to (\neg\, \alpha)$,

(B18) $(\neg\, \alpha) \to ((\gamma \to \gamma)\dot{-}\alpha)$,

whose inference rules are:

(MP) $\Gamma \vdash_{\mathcal{HB}} \phi$ and $\Gamma \vdash_{\mathcal{HB}} \phi \to \gamma$ yield $\Gamma \vdash_{\mathcal{HB}} \gamma$ (Modus Ponens),

(r) $\Gamma \vdash_{\mathcal{HB}} \phi$ yields $\Gamma \vdash_{\mathcal{HB}} \neg(\neg\, \phi)$.

In [17] the author shows that the algebraic semantics associated to $\mathcal{HB}$ is the class of double Heyting algebras (named *semi-Boolean algebras* in [17]).

3. Semi Heyting–Brouwer Logic

This section is devoted to define a logic whose propositional calculus contains an axiomatic extension equivalent to the logic $\mathcal{HB}$ and whose algebraic counterpart is the variety of double semi-Heyting algebras [20].

We define the *semi Heyting–Brouwer logic* $\mathcal{SHB}$ over the language $\mathbf{L} = \{\vee, \wedge, \to, \leftarrow, \bot, \top\}$ in terms of the following set of axioms, in which we respectively denote $\alpha \to_H \beta$ and $\alpha \leftarrow_H \beta$ to represent the formulas $\alpha \to (\alpha \wedge \beta)$ and $\alpha \leftarrow (\alpha \vee \beta)$.

(S1) $\alpha \to_H (\alpha \vee \beta)$,

(S2) $\beta \to_H (\alpha \vee \beta)$,

(S3) $(\alpha \to_H \gamma) \to_H [(\beta \to_H \gamma) \to_H ((\alpha \vee \beta) \to_H \gamma)]$,

(S4) $(\alpha \wedge \beta) \to_H \alpha$,

(S5) $(\gamma \to_H \alpha) \to_H [(\gamma \to_H \beta) \to_H (\gamma \to_H (\alpha \wedge \beta))]$,

(S6) $\top$,

(S7) $\bot \to_H \alpha$,

(S8) $((\alpha \wedge \beta) \to_H \gamma) \to_H (\alpha \to_H (\beta \to_H \gamma))$,

(S9) $(\alpha \to_H (\beta \to_H \gamma)) \to_H ((\alpha \wedge \beta) \to_H \gamma)$,

(S10) $[\alpha \to_H (\beta \vee \gamma)] \to_H [(\beta \leftarrow_H \alpha) \to_H \gamma]$,

(S11) $[(\beta \leftarrow_H \alpha) \to_H \gamma] \to_H [\alpha \to_H (\beta \vee \gamma)]$,

(S12) $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\alpha \to \gamma) \to_H (\beta \to \gamma)))$,

(S13) $(\alpha \to_H \beta) \to_H ((\beta \to_H \alpha) \to_H ((\gamma \to \beta) \to_H (\gamma \to \alpha)))$,

(S14) $((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow_H \alpha) \leftarrow_H ((\alpha \leftarrow \gamma) \leftarrow_H (\beta \leftarrow \gamma)))) \to_H \bot$,

(S15) $((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow_H \alpha) \leftarrow_H ((\gamma \leftarrow \beta) \leftarrow_H (\gamma \leftarrow \alpha)))) \to_H \bot$, (S16) $((\alpha \vee (\beta \leftarrow \gamma)) \leftarrow_H (\alpha \vee ((\alpha \vee \beta) \leftarrow (\alpha \vee \gamma)))) \rightarrow_H \perp$,

(S17) $((\alpha \vee ((\alpha \vee \beta) \leftarrow (\alpha \vee \gamma))) \leftarrow_H (\alpha \vee (\beta \leftarrow \gamma))) \rightarrow_H \perp$,

(S18) $((\alpha \vee (\alpha \leftarrow \beta)) \leftarrow_H (\alpha \vee \beta)) \rightarrow_H \perp$,

(S19) $((\alpha \vee \beta) \leftarrow_H (\alpha \vee (\alpha \leftarrow \beta))) \rightarrow_H \perp$,

(S20) $(\alpha \leftarrow \alpha) \rightarrow_H \perp$.

The inference rule is *SHB-Modus Ponens* (sHB-MP): $\Gamma \vdash_{\mathcal{SHB}} \phi$ and $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \gamma$ yield $\Gamma \vdash_{\mathcal{SHB}} \gamma$. As we will see in Lemma 4.8, sHB-MP implies *Modus Ponens*.

Let us show now some elemental properties of the logic $\mathcal{SHB}$ that will be useful in what follows. Consider the following formulas:

(S′1) $(\alpha \wedge \beta) \rightarrow_H \beta$,

(S′2) $(\alpha \rightarrow_H \beta) \rightarrow_H [(\beta \rightarrow_H \gamma) \rightarrow_H (\alpha \rightarrow_H \gamma)]$,

(S′3) $(\alpha \wedge (\beta \rightarrow \gamma)) \rightarrow_H (\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma)))$,

(S′4) $(\alpha \wedge ((\alpha \wedge \beta) \rightarrow (\alpha \wedge \gamma))) \rightarrow_H (\alpha \wedge (\beta \rightarrow \gamma))$,

(S′5) $(\alpha \wedge \beta) \rightarrow_H (\alpha \wedge (\alpha \rightarrow \beta))$,

(S′6) $(\alpha \wedge (\alpha \rightarrow \beta)) \rightarrow_H (\alpha \wedge \beta)$.

The following lemma can be proved as in [4].

LEMMA 3.1. *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *In* $\mathcal{SHB}$, *the following conditions hold:*

(a) *If* $\Gamma \vdash_{\mathcal{SHB}} \alpha$ *then* $\Gamma \vdash_{\mathcal{SHB}} \beta \rightarrow_H \alpha$.

(b) $\Gamma \vdash_{\mathcal{SHB}} \alpha \rightarrow_H \alpha$.

(c) $\Gamma \vdash_{\mathcal{SHB}} \alpha$ *where* α *is one of the axioms (S′1) to (S′6).*

LEMMA 3.2. *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *Then*

(a) $\Gamma, \alpha \rightarrow_H \beta, \beta \rightarrow_H \gamma \vdash_{\mathcal{SHB}} \alpha \rightarrow_H \gamma$.

(b) $\Gamma, \alpha, \alpha \rightarrow_H \beta \vdash_{\mathcal{SHB}} \beta$.

(c) $\Gamma, \alpha \vdash_{\mathcal{SHB}} \beta \rightarrow_H \alpha$.

(d) $\Gamma, \alpha \rightarrow_H \beta, \gamma \rightarrow_H \delta, \beta \rightarrow_H \alpha, \delta \rightarrow_H \gamma \vdash_{\mathcal{SHB}} (\alpha \vee \gamma) \rightarrow_H (\beta \vee \delta)$.

(e) $\Gamma, \alpha \rightarrow_H \beta, \gamma \rightarrow_H \delta, \beta \rightarrow_H \alpha, \delta \rightarrow_H \gamma \vdash_{\mathcal{SHB}} (\alpha \wedge \gamma) \rightarrow_H (\beta \wedge \delta)$.

(f) $\Gamma, \alpha \rightarrow_H \beta, \gamma \rightarrow_H \delta, \beta \rightarrow_H \alpha, \delta \rightarrow_H \gamma \vdash_{\mathcal{SHB}} (\alpha \rightarrow \gamma) \rightarrow_H (\beta \rightarrow \delta)$.

(g) $\Gamma, \alpha, \beta \vdash_{\mathcal{SHB}} \alpha \wedge \beta$.

(h) $\Gamma \vdash_{\mathcal{SHB}} ((\alpha \rightarrow_H \perp) \wedge (\beta \rightarrow_H \perp)) \rightarrow_H ((\alpha \vee \beta) \rightarrow_H \perp)$.

(i) $\Gamma \vdash_{\mathcal{SHB}} (\alpha \vee \perp) \rightarrow_H \alpha$

(j) $\Gamma \vdash_{\mathcal{SHB}} ((\beta \leftarrow_H \alpha) \to_H \bot) \to_H (\alpha \to_H \beta)$.

(k) $\Gamma, (\alpha \leftarrow_H \beta) \to_H \bot, \alpha \to_H \bot \vdash_{\mathcal{SHB}} \beta \to_H \bot$.

(l) $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \to_H \bot \vdash_{\mathcal{SHB}} \beta$.

Proof. Items (a) to (g) can be proved as in [4].

(h)
1. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \to_H \bot) \to_H [(\beta \to_H \bot) \to_H ((\alpha \vee \beta) \to_H \bot)]$ by axiom (S3).
2. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H \bot) \to_H [(\beta \to_H \bot) \to_H ((\alpha \vee \beta) \to_H \bot)]] \to_H [[(\alpha \to_H \bot) \wedge (\beta \to_H \bot)] \to_H [((\alpha \vee \beta) \to_H \bot)]]$ by axiom (S9).
3. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H \bot) \wedge (\beta \to_H \bot)] \to_H [((\alpha \vee \beta) \to_H \bot)]$ from sHB-MP in 1 and 2.

(i)
1. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \to_H \alpha) \to_H [(\bot \to_H \alpha) \to_H ((\alpha \vee \bot) \to_H \alpha)]$ by (S3).
2. $\Gamma \vdash_{\mathcal{SHB}} \alpha \to_H \alpha$ from Lemma 3.1 (b).
3. $\Gamma \vdash_{\mathcal{SHB}} (\bot \to_H \alpha) \to_H ((\alpha \vee \bot) \to_H \alpha)$ by sHB-MP in 1 and 2.
4. $\Gamma \vdash_{\mathcal{SHB}} \bot \to_H \alpha$ by axiom (S7).
5. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \vee \bot) \to_H \alpha$ by sHB-MP in 3 and 4.

(j)
1. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H (\beta \vee \bot)) \wedge \alpha] \to_H [\alpha \wedge (\alpha \to_H (\beta \vee \bot))]$ from [4, Lemma 2.1 (e)].
2. $\Gamma \vdash_{\mathcal{SHB}} [\alpha \wedge (\alpha \to_H (\beta \vee \bot))] \to_H (\alpha \wedge (\alpha \wedge (\beta \vee \bot)))$ from Lemma 3.1 (c).
3. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H (\beta \vee \bot)) \wedge \alpha] \to_H (\alpha \wedge (\alpha \wedge (\beta \vee \bot)))$ from Lemma 3.1 (c) and sHB-MP in 1 and 2.
4. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \wedge (\alpha \wedge (\beta \vee \bot))) \to_H (\alpha \wedge (\beta \vee \bot))$ from Lemma 3.1 (c).
5. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H (\beta \vee \bot)) \wedge \alpha] \to_H (\alpha \wedge (\beta \vee \bot))$ from Lemma 3.1 (c) and sHB-MP in 3 and 4.
6. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \wedge (\beta \vee \bot)) \to_H (\beta \vee \bot)$ from Lemma 3.1 (c).
7. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \wedge (\alpha \wedge (\beta \vee \bot))) \to_H (\beta \vee \bot)$ from Lemma 3.1 (c) and sHB-MP in 5 and 6.
8. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H (\beta \vee \bot)) \wedge \alpha] \to_H (\beta \vee \bot)$ from Lemma 3.1 (c) and sHB-MP in 3 and 7.
9. $\Gamma \vdash_{\mathcal{SHB}} (\beta \vee \bot) \to_H \beta$ by item (i).
10. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \to_H (\beta \vee \bot)) \wedge \alpha] \to_H \beta$ from Lemma 3.1 (c) and sHB-MP in 8 and 9.
11. $\Gamma \vdash_{\mathcal{SHB}} [(\beta \leftarrow_H \alpha) \to_H \bot] \to_H [\alpha \to_H (\beta \vee \bot)]$ by axiom (S11).
12. $\Gamma \vdash_{\mathcal{SHB}} [\alpha \to_H (\beta \vee \bot)] \to_H (\alpha \to_H \beta)$ by (S8) and sHB-MP in 10.
13. $\Gamma \vdash_{\mathcal{SHB}} [(\beta \leftarrow_H \alpha) \to_H \bot] \to_H (\alpha \to_H \beta)$ from Lemma 3.1 (c) and sHB-MP in 11 and 12.

(k) 1. $\Gamma, (\alpha \leftarrow_H \beta) \rightarrow_H \perp, \alpha \rightarrow_H \perp \vdash_{\mathcal{SHB}} (\alpha \leftarrow_H \beta) \rightarrow_H \perp$.
2. $\Gamma, (\alpha \leftarrow_H \beta) \rightarrow_H \perp, \alpha \rightarrow_H \perp \vdash_{\mathcal{SHB}} ((\alpha \leftarrow_H \beta) \rightarrow_H \perp) \rightarrow_H (\beta \rightarrow_H \alpha)$ by item (j) .
3. $\Gamma, (\alpha \leftarrow_H \beta) \rightarrow_H \perp, \alpha \rightarrow_H \perp \vdash_{\mathcal{SHB}} \beta \rightarrow_H \alpha$ by sHB-MP in 1 and 2.
4. $\Gamma, (\alpha \leftarrow_H \beta) \rightarrow_H \perp, \alpha \rightarrow_H \perp \vdash_{\mathcal{SHB}} \alpha \rightarrow_H \perp$.
5. $\Gamma, (\alpha \leftarrow_H \beta) \rightarrow_H \perp, \alpha \rightarrow_H \perp \vdash_{\mathcal{SHB}} \beta \rightarrow_H \perp$ from Lemma 3.1 (c) and sHB-MP in 3 and 4.

(l) 1. $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \rightarrow_H \perp \vdash_{\mathcal{SHB}} (\beta \leftarrow_H \alpha) \rightarrow_H \perp$.
2. $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \rightarrow_H \perp \vdash_{\mathcal{SHB}} ((\beta \leftarrow_H \alpha) \rightarrow_H \perp) \rightarrow_H (\alpha \rightarrow_H \beta)$ by item (j).
3. $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \rightarrow_H \perp \vdash_{\mathcal{SHB}} \alpha \rightarrow_H \beta$ by sHB-MP in 1 and 2.
4. $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \rightarrow_H \perp \vdash_{\mathcal{SHB}} \alpha$.
5. $\Gamma, \alpha, (\beta \leftarrow_H \alpha) \rightarrow_H \perp \vdash_{\mathcal{SHB}} \beta$ by sHB-MP in 3 and 4. ■

The following is a version of the deduction theorem for this logic.

THEOREM 3.3. (Deduction) *Let* $\Gamma \cup \{\phi, \psi\} \subseteq Fm_{\mathbf{L}}$. *Then*

$$\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \psi \textit{ if and only if } \Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi.$$

PROOF. Suppose that $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \psi$. We are going to prove that $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi$ by induction on the length of the proof of the formula ψ. If ψ is an axiom of the logic $\mathcal{SHB}$, then $\Gamma \vdash_{\mathcal{SHB}} \psi$, and by Lemma 3.1 (a), $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi$. Suppose that there exists $\alpha \in Fm_{\mathbf{L}}$ such that $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \alpha$ and $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \alpha \rightarrow_H \psi$. Then, by inductive hypothesis,

1. $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \alpha$ and
2. $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H (\alpha \rightarrow_H \psi)$.
3. $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \phi$ by Lemma 3.1 (b).
4. $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H (\phi \wedge \alpha)$ by (S5) and sHB-MP in 1 and 3.
5. $\Gamma \vdash_{\mathcal{SHB}} (\phi \wedge \alpha) \rightarrow_H \psi$ by (S9) and sHB-MP in 2.
6. $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi$ from Lemma 3.1 (c) and sHB-MP in 4 and in 5.

For the converse, assume that $\Gamma \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi$. Then $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \phi \rightarrow_H \psi$. Since $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \phi$, it follows that $\Gamma \cup \{\phi\} \vdash_{\mathcal{SHB}} \psi$ by sHB-MP. ■

The following lemma will be used to prove that $\mathcal{SHB}$ is an implicative logic in the sense of [15].

LEMMA 3.4. *Let* $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. *Then*

$$\Gamma, \alpha \rightarrow_H \beta, \gamma \rightarrow_H \delta, \beta \rightarrow_H \alpha, \delta \rightarrow_H \gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow \gamma) \rightarrow_H (\beta \leftarrow \delta).$$

PROOF.

1. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} \beta \to_H \alpha$.
2. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} \alpha \to_H (\alpha \vee \bot)$ by axiom (S1).
3. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} \beta \to_H (\alpha \vee \bot)$ by Lemma 3.1 (c) and sHB-MP in 1 and 2.
4. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} [\beta \to_H (\alpha \vee \bot)] \to_H [(\alpha \leftarrow_H \beta) \to_H \bot]$ by (S10).
5. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow_H \beta) \to_H \bot$ by sHB-MP in 3 and 4.

   Similarly,

6. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\beta \leftarrow_H \alpha) \to_H \bot$.

   Denote $\epsilon = (\beta \leftarrow_H \alpha)$ and $\theta = ((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow \gamma) \leftarrow_H (\alpha \leftarrow \gamma)))$.

7. $\Gamma, (\epsilon \leftarrow_H \theta) \to_H \bot, \epsilon \to_H \bot \vdash_{\mathcal{SHB}} \theta \to_H \bot$ by Lemma 3.2 (k).
8. $\Gamma, (\epsilon \leftarrow_H \theta) \to_H \bot \vdash_{\mathcal{SHB}} (\epsilon \to_H \bot) \to_H (\theta \to_H \bot)$ by Theorem 3.3.
9. $\Gamma \vdash_{\mathcal{SHB}} ((\epsilon \leftarrow_H \theta) \to_H \bot) \to_H ((\epsilon \to_H \bot) \to_H (\theta \to_H \bot))$ by Theorem 3.3.
10. $\Gamma \vdash_{\mathcal{SHB}} (\epsilon \leftarrow_H \theta) \to_H \bot$ by axiom (S14).
11. $\Gamma \vdash_{\mathcal{SHB}} (\epsilon \to_H \bot) \to_H (\theta \to_H \bot)$ by sHB-MP in 9 and 10.
12. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\epsilon \to_H \bot) \to_H (\theta \to_H \bot)$.
13. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\beta \leftarrow_H \alpha) \to_H \bot) \to_H (\theta \to_H \bot)$ by definition of ϵ.
14. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} \theta \to_H \bot$ by sHB-MP in 6 and 13.
15. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow \gamma) \leftarrow_H (\alpha \leftarrow \gamma))) \to_H \bot$ by definition of θ.

   In the rest of the proof, ϵ and θ will respectively denote the formulas

   $$(\alpha \leftarrow_H \beta) \text{ and } ((\beta \leftarrow \gamma) \leftarrow_H (\alpha \leftarrow \gamma)).$$

   Then,

16. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\epsilon \leftarrow_H \theta) \to_H \bot$.

17. $\Gamma \vdash_{\mathcal{SHB}} ((\epsilon \leftarrow_H \theta) \to_H \bot) \to_H ((\epsilon \to_H \bot) \to_H (\theta \to_H \bot))$ By Lemma 3.2 (k) and Theorem 3.3.
18. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\epsilon \leftarrow_H \theta) \to_H \bot) \to_H ((\epsilon \to_H \bot) \to_H (\theta \to_H \bot))$.
19. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\epsilon \to_H \bot) \to_H (\theta \to_H \bot)$ by sHB-MP in 16 and 18.
20. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\alpha \leftarrow_H \beta) \to_H \bot) \to_H (\theta \to_H \bot)$ by the definition of ϵ.
21. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} \theta \to_H \bot$ by sHB-MP in 5 and 20.
22. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\beta \leftarrow \gamma) \leftarrow_H (\alpha \leftarrow \gamma)) \to_H \bot$ by the definition of θ.
23. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} [((\beta \leftarrow \gamma) \leftarrow_H (\alpha \leftarrow \gamma)) \to_H \bot] \to_H [(\alpha \leftarrow \gamma) \to_H ((\beta \leftarrow \gamma) \vee \bot)]$ by the axiom (S11).
24. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow \gamma) \to_H ((\beta \leftarrow \gamma) \vee \bot)$ by sHB-MP in 22 and 23.
25. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} ((\beta \leftarrow \gamma) \vee \bot) \to_H (\beta \leftarrow \gamma)$ from Lemma 3.2 (i).
26. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow \gamma) \to_H (\beta \leftarrow \gamma)$ by Lemma 3.1 (c) and sHB-MP in 24 and 25.

    In a similar way, by axiom (S15) we have that
27. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\beta \leftarrow \gamma) \to_H (\beta \leftarrow \delta)$.
28. $\Gamma, \alpha \to_H \beta, \gamma \to_H \delta, \beta \to_H \alpha, \delta \to_H \gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow \gamma) \to_H (\beta \leftarrow \delta)$ by Lemma 3.1 (c) and sHB-MP in 26 and 27. ∎

Definition 3.5. [15] An implicative logic is a logic $\mathcal{L}$ in a language $\mathbf{L}$ with a binary connective $\to$ (either primitive or defined by an algebraic term in exactly two variables) such that the following conditions are satisfied (for all formulas appearing in them):

(IL1) $\vdash_{\mathcal{L}} \alpha \to \alpha$.

(IL2) $\alpha \to \beta, \beta \to \gamma \vdash_{\mathcal{L}} \alpha \to \gamma$.

(IL3) For each $\lambda \in \mathbf{L}$, of arity $n > 0$,

$$\left\{ \begin{array}{l} \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n \\ \beta_1 \to \alpha_1, \ldots, \beta_n \to \alpha_n \end{array} \right\} \vdash_{\mathcal{L}} \lambda\alpha_1 \ldots \alpha_n \to \lambda\beta_1 \ldots \beta_n.$$

(IL4) $\alpha, \alpha \to \beta \vdash_{\mathcal{L}} \beta$.

(IL5) $\alpha \vdash_{\mathcal{L}} \beta \to \alpha$.

DEFINITION 3.6. [15, Definition 6, p. 181] Let $\mathcal{L}$ be an implicative logic on the language $\mathbf{L}$. An $\mathcal{L}$-algebra is an algebra $\mathbf{A}$ of similarity type $\mathbf{L}$ that has an element $\top$ with the following properties:

(LALG1) For all $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ and all $h \in Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$, if $\Gamma \vdash_{\mathcal{L}} \phi$ and $h\Gamma \subseteq \{\top\}$ then $h\phi = \top$

(LALG2) For all $a, b \in A$, if $a \to b = \top$ and $b \to a = \top$ then $a = b$.

The class of $\mathcal{L}$-algebras will be denoted $\mathsf{Alg}^*\mathcal{L}$.

The following theorem is now an immediate consequence of Lemma 3.1 (b), Lemma 3.2 (d), (e), (f) and Lemma 3.4.

THEOREM 3.7. *The logic $\mathcal{SHB}$ is implicative with respect to the connective $\to_H$.*

4. Completeness

In this section we show that in any double semi-Heyting algebra it is possible to define a new implication operation in such a way that the resulting algebra becomes a double Heyting algebra. Note that $x \to_H y$ and $x \leftarrow_H y$ respectively denote $x \to (x \wedge y)$ and $x \leftarrow (x \vee y)$.

It is known that if $\langle A; \vee, \wedge, \to, \bot, \top\rangle$ is a semi-Heyting algebra, then $\langle A; \vee, \wedge, \to_H, \bot, \top\rangle \in \mathbf{H}$ [1, Lemma 4.1]. In a similar way we are going to check that if $\langle A; \vee, \wedge, \to, \leftarrow, \bot, \top\rangle \in \mathbf{DblSH}$ then$\langle A; \vee, \wedge, \to_H, \leftarrow_H, \bot, \top\rangle$ is a $\mathbf{DblH}$-algebra.

LEMMA 4.1. *If $\langle A; \vee, \wedge, \to, \leftarrow, \bot, \top\rangle$ is a $\mathbf{DblSH}$-algebra then $\langle A; \vee, \wedge, \to_H, \leftarrow_H, \bot, \top\rangle \in \mathbf{DblH}$. In addition, if $a, b \in A$ then $a \to b \leq a \to_H b$ and $a \leftarrow_H b \leq a \leftarrow b$.*

PROOF. From the previous remark, $\langle A; \vee, \wedge, \to_H, \bot, \top\rangle$ is an $\mathbf{H}$-algebra. Let $a, b \in A$. Then $a \vee (a \leftarrow_H b) = a \vee (a \leftarrow (a \vee b)) = a \vee (a \vee b) = a \vee b$ by (dSH2) and $a \vee (b \leftarrow_H c) = a \vee (b \leftarrow (b \vee c)) = a \vee ((a \vee b) \leftarrow (a \vee (b \vee c))) = a \vee ((a \vee b) \leftarrow ((a \vee b) \vee (a \vee c))) = a \vee ((a \vee b) \leftarrow_H (a \vee c))$ by (dSH3). Besides, $a \leftarrow_H a = a \leftarrow (a \vee a) = a \leftarrow a = \bot$ by using condition (dSH4)

Since $(a \vee b) \leftarrow_H b = (a \vee b) \leftarrow ((a \vee b) \vee b) = (a \vee b) \leftarrow (a \vee b) = \bot$ it follows that
$\langle A; \vee, \wedge, \to_H, \leftarrow_H, \bot, \top\rangle \in \mathbf{DblH}$ [20].

In [1, Lemma 4.1] it is proved that if $a, b \in A$ then $a \to b \leq a \to_H b$. From conditions (dSH3) and (dSH4), the inequality $a \leftarrow_H b \leq a \leftarrow b$ follows from $(a \leftarrow_H b) \vee (a \leftarrow b) = [a \leftarrow (a \vee b)] \vee (a \leftarrow b) = [(a \vee (a \leftarrow b)) \leftarrow ((a \vee b) \vee (a \leftarrow b))] \vee (a \leftarrow b) = ((a \vee b) \leftarrow (a \vee b)) \vee (a \leftarrow b) = \bot \vee (a \leftarrow b) = a \leftarrow b$. ■

COROLLARY 4.2. *If $\langle A; \vee, \wedge, \to, \leftarrow, \bot, \top \rangle \in$ **DblH** and $a, b \in A$ then $a \to b = a \to_H b$ and $a \leftarrow_H b = a \leftarrow b$.*

PROOF. For $a, b \in A$, $(a \to b) \wedge (a \to_H b) = [(a \wedge (a \to_H b)) \to b)] \wedge (a \to_H b) = (a \wedge b \to b) \wedge (a \to_H b) = \top \wedge (a \to_H b) = a \to_H b$, and, similarly, $(a \leftarrow b) \vee (a \leftarrow_H b) = [(a \vee (a \leftarrow_H b)) \leftarrow b)] \vee (a \leftarrow_H b) = (a \vee b \leftarrow b) \vee (a \leftarrow_H b) = \bot \vee (a \leftarrow_H b) = a \leftarrow_H b$. Now we apply Lemma 4.1. ■

DEFINITION 4.3. [15] A logic $\mathcal{L}$ defined over a language **L** is said to be complete with respect to a class of algebras **K** of the same type if it verifies the following condition: For all $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}, \Gamma \vdash_{\mathcal{L}} \phi$ if and only if

$$h\Gamma \subseteq \{\top\} \text{ implies } h\phi = \top \text{ for all } h \in Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A}) \text{ and all } \mathbf{A} \in \mathbf{K},$$

where $Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$ represents the set of the homomorphisms from the formula algebra $\mathsf{Fm}_{\mathbf{L}}$ into $\mathbf{A}$.

We are going to prove that the semi Heyting–Brouwer logic $\mathcal{SHB}$ is complete with respect to the variety **DblSH**.

Since $\mathcal{SHB}$ is an implicative logic with respect to the binary connective $\to_H$, by Theorem 3.7, we have the next result using [15, Theorem 7.1, p. 222].

LEMMA 4.4. *The $\mathcal{SHB}$ logic is complete with respect to the class $\mathsf{Alg}^*\mathcal{SHB}$ in the sense of Definition 4.3.*

In order to prove that the logic $\mathcal{SHB}$ is complete with respect to the variety **DblSH** of double semi-Heyting algebras, by Lemma 4.4, it is enough to prove that $\mathsf{Alg}^*\mathcal{SHB} = \mathbf{DblSH}$. We need first the following lemma.

LEMMA 4.5. *If $\mathbf{A} \in$ **DblSH** and $a, b, c \in A$ then the following conditions hold:*

(a) $((a \leftarrow_H b) \leftarrow_H ((b \leftarrow_H a) \leftarrow_H ((a \leftarrow c) \leftarrow_H (b \leftarrow c)))) \to_H \bot = \top$

(b) $((a \leftarrow_H b) \leftarrow_H ((b \leftarrow_H a) \leftarrow_H ((c \leftarrow b) \leftarrow_H (c \leftarrow a)))) \to_H \bot = \top$

PROOF. Observe that

$$
\begin{aligned}
&(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee ((a \leftarrow c) \leftarrow_H (b \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee (([a \vee (a \leftarrow_H b)] \leftarrow c) \leftarrow_H (b \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee (((a \vee b) \leftarrow c) \leftarrow_H (b \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee (((a \vee b) \leftarrow c) \leftarrow_H ([b \vee (b \leftarrow_H a)] \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee (((a \vee b) \leftarrow c) \leftarrow_H ((b \vee a) \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee (((a \vee b) \leftarrow c) \leftarrow_H ((a \vee b) \leftarrow c)) \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a) \vee \perp \\
= &(a \leftarrow_H b) \vee (b \leftarrow_H a).
\end{aligned}
$$

Consequently, $((a \leftarrow c) \leftarrow_H (b \leftarrow c)) \leq (a \leftarrow_H b) \vee (b \leftarrow_H a)$. By Lemma 4.1, the condition (dR) for the connective $\leftarrow_H$ holds. So $(b \leftarrow_H a) \leftarrow_H ((a \leftarrow c) \leftarrow_H (b \leftarrow c)) \leq (a \leftarrow_H b)$. Therefore

(a) $(a \leftarrow_H b) \leftarrow_H [(b \leftarrow_H a) \leftarrow_H ((a \leftarrow c) \leftarrow_H (b \leftarrow c))] = \perp.$

Item (b) can be proved in a similar way. ■

So we have the following theorem.

THEOREM 4.6. $\mathsf{Alg}^*\mathcal{SHB} = \mathbf{DblSH}$.

PROOF. For $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \leftarrow, \perp, \top \rangle \in \mathbf{DblSH}$, consider $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}}$ and $h \in Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$ such that $\Gamma \vdash_{\mathcal{SHB}} \phi$ and $h\Gamma \subseteq \{\top\}$. We are going to prove that $h\phi = \top$ by induction on the proof of the formula ϕ.

If ϕ is an axiom of (S1) to (S9), (S12) or (S13) then $h\phi = \top$ [3,4]. If ϕ is either the axiom (S14) or (S15) we also have $h\phi = \top$ by Lemma 4.5. The same conclusion can be obtained if ϕ is the axiom (S10), (S11) or from (S16) to (S20) by using conditions (dSH1) to (dSH4) of the definition of a **DblSH**-algebra [17].

Suppose now that there exists $\psi \in Fm_{\mathbf{L}}$ such that $\Gamma \vdash_{\mathcal{SHB}} \psi$ and $\Gamma \vdash_{\mathcal{SHB}} \psi \rightarrow_H \phi$. By inductive hypothesis, $\top = h(\psi \rightarrow_H \phi) = h(\psi) \rightarrow_H h(\phi) = \top \rightarrow_H h(\phi) = h(\phi)$. Consequently, we have (LALG1).

Condition (LALG2) is immediate, since if $a \rightarrow b = \top$ then $a \leq b$. Thus $\mathbf{A} \in \mathsf{Alg}^*\mathcal{SHB}$.

Consider now $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \leftarrow, \perp, \top \rangle \in \mathsf{Alg}^*\mathcal{SHB}$. Let $a, b, c \in A$ and $h \in Hom(\mathsf{Fm}_{\mathbf{L}}, \mathbf{A})$ such that $h(x) = a, h(y) = b$ and $h(z) = c$ with $x, y, z \in Var$. By axioms (S1) to (S9), (S12), (S13) we have that $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \perp, \top \rangle \in \mathbf{SH}$ [3,4]. Besides, by axioms (S14), (S15), (S10), (S11) and (S16) to (S20) the identities (dSH1) to (dSH4) hold for any a, b, c. Thus $\mathbf{A} = \langle A; \vee, \wedge, \rightarrow, \leftarrow, \perp, \top \rangle \in \mathbf{DblSH}$. ■

We are now ready to prove the following theorem.

Theorem 4.7. *The $\mathcal{SHB}$ logic is complete with respect to the class* **DblSH**.

The following lemma proves that in $\mathcal{SHB}$, (sHB-MP) implies (MP). It also proves the inference rule (r) for the logic $\mathcal{SHB}$.

Lemma 4.8. *In $\mathcal{SHB}$,*

(a) $\Gamma \vdash_{\mathcal{SHB}} \phi$ *and* $\Gamma \vdash_{\mathcal{SHB}} \phi \to \gamma$ *imply* $\Gamma \vdash_{\mathcal{SHB}} \gamma$.

(b) *If we define for* $\alpha \in Fm_{\mathbf{L}}$ *the operations* $\neg\alpha = \alpha \to \bot$ *and* $\ulcorner \alpha = \alpha \leftarrow \top$ *we have that* $\Gamma \vdash_{\mathcal{SHB}} \phi$ *yields* $\Gamma \vdash_{\mathcal{SHB}} \neg(\ulcorner \phi)$.

Proof. Let us check condition (a).

1. $\Gamma \vdash_{\mathcal{SHB}} \phi$ by hypothesis.
2. $\Gamma \vdash_{\mathcal{SHB}} \phi \to \gamma$ by hypothesis.
3. $\Gamma, \phi, \phi \to \gamma \vdash_{\mathcal{SHB}} \phi \wedge (\phi \to \gamma)$ by Lemma 3.2 (g).
4. $\Gamma \vdash_{\mathcal{SHB}} \phi \to_H ((\phi \to \gamma) \to_H (\phi \wedge (\phi \to \gamma)))$ by Theorem 3.3.
5. $\Gamma \vdash_{\mathcal{SHB}} \phi \wedge (\phi \to \gamma)$ by sHB-MP in 1, 2 and 4.
6. $\Gamma \vdash_{\mathcal{SHB}} [\phi \wedge (\phi \to \gamma)] \to_H (\phi \wedge \gamma)$ by Lemma 3.1 (c).
7. $\Gamma \vdash_{\mathcal{SHB}} \phi \wedge \gamma$ by sHB-MP in 5 and 6.
8. $\Gamma \vdash_{\mathcal{SHB}} (\phi \wedge \gamma) \to_H \gamma$ by Lemma 3.1 (c).
9. $\Gamma \vdash_{\mathcal{SHB}} \gamma$ by sHB-MP in 7 and 8.

Now we prove (b).

1. $\Gamma, \phi \vdash_{\mathcal{SHB}} \top \to_H (\phi \vee \top)$ by (S2).
2. $\Gamma, \phi \vdash_{\mathcal{SHB}} [\top \to_H (\phi \vee \top)] \to_H [(\alpha \leftarrow_H \top) \to_H \bot]$ by (S10).
3. $\Gamma, \phi \vdash_{\mathcal{SHB}} (\alpha \leftarrow_H \top) \to_H \bot$ by sHB-MP in 1 and 2.
4. $\Gamma \vdash_{\mathcal{SHB}} \phi \to_H [(\alpha \leftarrow_H \top) \to_H \bot]$ by Theorem 3.3.
5. $\Gamma \vdash_{\mathcal{SHB}} \phi$ by hypothesis.
6. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow_H \top) \to_H \bot$ by sHB-MP in 4 and 5.

   Let $\mathbf{A} \in$ **DblSH** and $a \in A$. Since $(a \leftarrow_H \top) \to_H \bot = (a \leftarrow (a \vee \top)) \to_H \bot = (a \leftarrow \top) \to_H \bot = (a \leftarrow \top) \to [(a \leftarrow \top) \wedge \bot] = (a \leftarrow \top) \to \bot$, by Theorem 4.7.

7. $\Gamma \vdash_{\mathcal{SHB}} [(\alpha \leftarrow_H \top) \to_H \bot] \to_H [(\alpha \leftarrow \top) \to \bot]$.
8. $\Gamma \vdash_{\mathcal{SHB}} (\alpha \leftarrow \top) \to \bot$ by sHB-MP in 6 and 7. ∎

Following [4], the following example shows that Modus Ponens does not imply sHB-MP. Consider the logic $\mathcal{D}$ defined by axioms (S1) to (S20), with

MP as its only inference rule. We next present an algebraic model for the logic $\mathcal{D}$ that is not a double semi-Heyting algebra.

Consider the algebra $\mathbf{A}$ with universe $\{\bot, \top\}$ and the operations $\wedge, \vee, \to, \leftarrow$ defined by:

$\wedge$	$\bot$	$\top$
$\bot$	$\top$	$\top$
$\top$	$\top$	$\top$

$\vee$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\top$	$\top$

$\to$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$

$\leftarrow$	$\bot$	$\top$
$\bot$	$\bot$	$\top$
$\top$	$\bot$	$\top$

This algebra $\mathbf{A}$ satisfies $x \to_H y = x \to (x \wedge y) \approx \top$ for any election of x and y, and all the axioms except (S6) are of that form. Therefore, $\mathbf{A}$ is a model for the logic $\mathcal{D}$. The rule MP is satisfied and $\mathbf{A}$ is not a double semi-Heyting algebra.

5. Relationships with the Logics $\mathcal{HB}, \mathcal{I}$ and $\mathcal{SI}$

In this section we prove that the logic $\mathcal{HB}$ is, up to equivalences by translations (see Definition 5.3), an axiomatic extension of the logic $\mathcal{SHB}$. At the end of this section we check that the propositional calculi of intuitionistic logic $\mathcal{I}$ and semi-intuitionistic logic $\mathcal{SI}$ turn out to be fragments of $\mathcal{SHB}$.

Let us recall the following definitions [14].

Definition 5.1. Given two languages $\mathbf{L}$ and $\mathbf{L}'$, a function $h : Fm_{\mathbf{L}} \to Fm_{\mathbf{L}'}$ is a *translation*, where $Fm_{\mathbf{L}}$ and $Fm_{\mathbf{L}'}$ are built using the same set of variables, if satisfies the following conditions:

1. If x_i is a propositional variable in $\mathbf{L}$, then $h(x_i) = y_i$ where y_i is a propositional variable in $\mathbf{L}$';
2. Let f be a k-place connective of $\mathbf{L}$, for any subset $\{x_1, \ldots, x_k\} \subseteq X$. To the formula $f(x_1, \ldots, x_k)$ we assign a formula β_f of $Fm_{\mathbf{L}'}$, where β_f contains only variables from $\{x_1, \ldots, x_k\}$. Then
$$h(f(\alpha_1, \ldots, \alpha_k)) = \beta_f(h(\alpha_1), \ldots, h(\alpha_k))$$

Definition 5.2. If $\mathcal{A}$ and $\mathcal{B}$ are logics in the languages $\mathbf{L}$ and $\mathbf{L}'$, and h is a translation from $\mathbf{L}$ into $\mathbf{L}'$, then the translation is *sound* if $h(\phi)$ is provable in $\mathcal{B}$ whenever ϕ is provable in $\mathcal{A}$. That is,

$$\text{If } \vdash_{\mathcal{A}} \phi \text{ then } \vdash_{\mathcal{B}} h(\phi).$$

Furthermore, we assume that in the logics there is a connective $\leftrightarrow$ such that the following schematic axioms and inference rules governing this connective are valid:

(T1) $\vdash \phi \leftrightarrow \phi$,

(T2) $\phi \leftrightarrow \psi \vdash \psi \leftrightarrow \phi$,

(T3) $\phi \leftrightarrow \psi, \psi \leftrightarrow \gamma \vdash \phi \leftrightarrow \gamma$,

(T4) $\alpha_1 \leftrightarrow \beta_1, \ldots, \alpha_k \leftrightarrow \beta_k \vdash f(\alpha_1, \ldots, \alpha_k) \leftrightarrow f(\beta_1, \ldots, \beta_k)$, where f is any k-place connective in the system.

DEFINITION 5.3. [14] We say that the logics $\mathcal{A}$ and $\mathcal{B}$ are *translationally equivalent* if there are translations h_1 and h_2 so that

1. Both h_1 and h_2 are sound;
2. For any formula ϕ in $Fm_{\mathbf{L}}, \vdash_{\mathcal{A}} \phi \leftrightarrow h_2(h_1(\phi))$,
3. For any formula ϕ in $Fm_{\mathbf{L}'}, \vdash_{\mathcal{B}} \phi \leftrightarrow h_1(h_2(\phi))$.

For the rest of this sections, we fix the languages $\mathbf{L} = \{\vee, \wedge, \rightarrow, \leftarrow, \perp, \top\}$ and $\mathbf{L}' = \{\vee, \wedge, \rightarrow, \dot{-}, \neg, \urcorner\}$. For $\alpha, \beta \in Fm_{\mathbf{L}}$ or $Fm_{\mathbf{L}'}$, we write $\alpha \leftrightarrow_H \beta$ by $(\alpha \rightarrow_H \beta) \wedge (\beta \rightarrow_H \alpha)$. We check that this connective satisfies the conditions (T1) to (T4) in the logics $\mathcal{SHB}$ and $\mathcal{HB}$.

In [15] the author proves the following result:

THEOREM 5.4. *$\mathcal{HB}$ is complete with respect to the variety* **DblH**, *that is, for $\Gamma \cup \{\phi\} \subseteq Fm_{\mathbf{L}'}, \Gamma \vdash_{\mathcal{HB}} \phi$ if and only if $h\Gamma \subseteq \{\top\}$ implies $h\phi = \top$ for all $h \in Hom(\mathsf{Fm}_{\mathbf{L}'}, \mathbf{A})$ and all $\mathbf{A} \in$* **DblH**.

Observe that if $\mathbf{A} \in$ **DblH** and $a \rightarrow_H b = \top$ with $a, b \in A$ then $a \leq b$. So it is easy to prove, by Theorem 5.4, the following result.

LEMMA 5.5. *In the logic $\mathcal{HB}$ the following conditions hold:*

(a) $\vdash_{\mathcal{HB}} \phi \leftrightarrow_H \phi$,

(b) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{HB}} \psi \leftrightarrow_H \phi$,

(c) $\phi \leftrightarrow_H \psi, \psi \leftrightarrow_H \gamma \vdash_{\mathcal{HB}} \phi \leftrightarrow_H \gamma$,

(d) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{HB}} (\neg\phi) \leftrightarrow_H (\neg\psi)$,

(e) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{HB}} (\urcorner\, \phi) \leftrightarrow_H (\urcorner\, \psi)$,

(f) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{HB}} (\phi \vee \alpha) \leftrightarrow_H (\psi \vee \beta)$,

(g) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{HB}} (\phi \wedge \alpha) \leftrightarrow_H (\psi \wedge \beta)$,

(h) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{HB}} (\phi \rightarrow \alpha) \leftrightarrow_H (\psi \rightarrow \beta)$,

(i) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{HB}} (\alpha \dot{-} \phi) \leftrightarrow_H (\beta \dot{-} \psi)$.

In a similar way, by Theorem 4.7 we have the following result.

LEMMA 5.6. *In the logic $\mathcal{SHB}$ the following statements hold:*

(a) $\vdash_{\mathcal{SHB}} \phi \leftrightarrow_H \phi$,

(b) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{SHB}} \psi \leftrightarrow_H \phi$,

(c) $\phi \leftrightarrow_H \psi, \psi \leftrightarrow_H \gamma \vdash_{\mathcal{SHB}} \phi \leftrightarrow_H \gamma$,

(d) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{SHB}} \bot \leftrightarrow_H \bot$,

(e) $\phi \leftrightarrow_H \psi \vdash_{\mathcal{SHB}} \top \leftrightarrow_H \top$,

(f) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{SHB}} (\phi \vee \alpha) \leftrightarrow_H (\psi \vee \beta)$,

(g) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{SHB}} (\phi \wedge \alpha) \leftrightarrow_H (\psi \wedge \beta)$,

(h) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{SHB}} (\phi \to \alpha) \leftrightarrow_H (\psi \to \beta)$,

(i) $\phi \leftrightarrow_H \psi, \alpha \leftrightarrow_H \beta \vdash_{\mathcal{SHB}} (\alpha \leftarrow \phi) \leftrightarrow_H (\beta \leftarrow \psi)$.

In Lemmas 5.5 and 5.6 we checked conditions (T1) to (T4) for the logics $\mathcal{HB}$ and $\mathcal{SHB}$. Next, we will introduce an axiomatic extension that will be equivalent by translations to the Heyting–Brouwer Logic.

Definition 5.7. Let $\mathcal{A}$ be the axiomatic extension of $\mathcal{SHB}$ defined by the axioms:

(A1) $(\alpha \to_H \beta) \to_H (\alpha \to \beta)$,

(A2) $(\alpha \leftarrow \beta) \to_H (\alpha \leftarrow_H \beta)$

By Theorem 3.7, $\mathcal{A}$ is implicative with respect to the connective $\to_H$. Hence, by [15, Theorem 7.1, p. 222] we have that:

Lemma 5.8. *The logic $\mathcal{A}$ is complete with respect to the class $\mathsf{Alg}^*\mathcal{A}$ in the sense of definition 4.3.*

In what follows we define translations between the logics $\mathcal{A}$ and $\mathcal{HB}$ in order to verify that they are equivalent by translations.

Lemma 5.9. *Consider the translation scheme $\tilde{h}_1 : Fm_{\mathbf{L}} \to Fm_{\mathbf{L}'}$ in the logics $\mathcal{A}$ and $\mathcal{HB}$ defined by*

- $\tilde{h}_1(x) = x$ *if x is a variable,*
- $\tilde{h}_1(\alpha \wedge \beta) = \tilde{h}_1(\alpha) \wedge \tilde{h}_1(\beta)$,
- $\tilde{h}_1(\alpha \vee \beta) = \tilde{h}_1(\alpha) \vee \tilde{h}_1(\beta)$,
- $\tilde{h}_1(\alpha \to \beta) = \tilde{h}_1(\alpha) \to \tilde{h}_1(\beta)$,
- $\tilde{h}_1(\alpha \leftarrow \beta) = \tilde{h}_1(\beta) \dot{-} \tilde{h}_1(\alpha)$,
- $\tilde{h}_1(\top) = \alpha \to \alpha$,
- $\tilde{h}_1(\bot) = \alpha \wedge \neg\alpha$.

Then the translation h_1 is sound.

Proof. We want to check that for $\phi \in Fm_{\mathbf{L}}$, if $\vdash_{\mathcal{A}} \phi$ then $\vdash_{\mathcal{HB}} h_1(\phi)$. Assume that ϕ is an axiom. Observe that the assertion is trivial if ϕ is (S1), (S2), (S3), (S4), (S5), (S8), (S9), (S10), (S11), (S12) or (S13). Let us prove the following sentences.

By Theorem 5.4,

1. $\vdash_{\mathcal{HB}} \alpha \to \alpha$ and
2. $\vdash_{\mathcal{HB}} (\beta \wedge \neg\beta) \to_H h_1(\alpha)$.

   From the definition of h_1,

3. $\vdash_{\mathcal{HB}} h_1(\top)$ and
4. $\vdash_{\mathcal{HB}} h_1(\bot \to_H \alpha)$.

   As in Lemma 4.5, $(a \leftarrow_H b) \leftarrow_H [(b \leftarrow_H a) \leftarrow_H ((a \leftarrow c) \leftarrow_H (b \leftarrow c))] = \bot$. for $a, b, c \in A$ and $A \in \mathbf{DblH}$. So $[(a \leftarrow_H b) \leftarrow_H [(b \leftarrow_H a) \leftarrow_H ((a \leftarrow c) \leftarrow_H (b \leftarrow c))]] \to_H \bot = \top$.

   By completeness between **DblH** and $\mathcal{HB}$ [17],

5. $\vdash_{\mathcal{HB}} ((h_1\alpha \leftarrow_H h_1\beta) \leftarrow_H ((h_1\beta \leftarrow_H h_1\alpha) \leftarrow_H (h_1(\alpha \leftarrow \gamma) \leftarrow_H h_1(\beta \leftarrow \gamma)))) \to_H (\psi \wedge \neg\psi)$.
6. $\vdash_{\mathcal{HB}} h_1(((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow_H \alpha) \leftarrow_H ((\alpha \leftarrow \gamma) \leftarrow_H (\beta \leftarrow \gamma)))) \to_H \bot)$ by the definition of h_1.

   Similarly,

7. $\vdash_{\mathcal{HB}} h_1(((\alpha \leftarrow_H \beta) \leftarrow_H ((\beta \leftarrow_H \alpha) \leftarrow_H ((\gamma \leftarrow \beta) \leftarrow_H (\gamma \leftarrow \alpha)))) \to_H \bot)$.

   From conditions (dSH2) and (dSH3), by Theorem 5.4, it follows that:

8. $\vdash_{\mathcal{HB}} h_1(((\alpha \vee (\beta \leftarrow \gamma)) \leftarrow_H (\alpha \vee ((\alpha \vee \beta) \leftarrow (\alpha \vee \gamma)))) \to_H \bot)$.
9. $\vdash_{\mathcal{HB}} h_1(((\alpha \vee ((\alpha \vee \beta) \leftarrow (\alpha \vee \gamma))) \leftarrow_H (\alpha \vee (\beta \leftarrow \gamma))) \to_H \bot)$.
10. $\vdash_{\mathcal{HB}} h_1(((\alpha \vee (\alpha \leftarrow \beta)) \leftarrow_H (\alpha \vee \beta)) \to_H \bot)$.
11. $\vdash_{\mathcal{HB}} h_1(((\alpha \vee \beta) \leftarrow_H (\alpha \vee (\alpha \leftarrow \beta))) \to_H \bot)$.

    In a similar way, by condition (dSH4), we have that

12. $\vdash_{\mathcal{HB}} h_1((\alpha \leftarrow \alpha) \to_H \bot)$.

    By conditions 3, 4, 6, 7, 8, 9, 10, 11, 12 we have that if ϕ is (S6), (S7), (S14), (S15), (S16), (S17), (S18), (S19) or (S20) then $\vdash_{\mathcal{HB}} h_1(\phi)$. If ϕ is (A1) or (A2), by Corollary 4.2 and completeness of $\mathcal{HB}$, it is immediate. Suppose now that there exists $\gamma \in Fm_{\mathbf{L}}$ such that $\Gamma \vdash_{\mathcal{A}} \beta$

and $\Gamma \vdash_{\mathcal{A}} \beta \to_H \phi$. Then, by inductive hypothesis, $h_1(\Gamma) \vdash_{\mathcal{HB}} h_1(\beta)$ and $h_1(\Gamma) \vdash_{\mathcal{HB}} h_1(\beta \to_H \phi)$. Hence $h_1(\Gamma) \vdash_{\mathcal{HB}} h_1(\beta) \to_H h_1(\phi)$ and, thus, by (sHB-MP), $h_1(\Gamma) \vdash_{\mathcal{HB}} h_1(\phi)$. ■

We will need the following result in order to prove the soundness of the function defined in Lemma 5.11.

LEMMA 5.10. *If* $\mathbf{A} \in \mathsf{Alg}^*\mathcal{A}$ *and* $a, b, c \in A$ *then*

(a) $(a \to b) \to [(b \to \bot) \to (a \to \bot)] = \top$,

(b) $(b \leftarrow a) \to ((a \to b) \leftarrow \top) = \top$,

(c) $(c \leftarrow (b \leftarrow a)) \to ((b \vee c) \leftarrow a) = \top$,

(d) $[(b \leftarrow a) \to \bot] \to [(a \to b)] = \top$.

PROOF.

(a) Recall that, by Lemma 4.1, $\to_H$ is a Heyting implication. Since

$$a \wedge (a \to b) \wedge (b \to \bot) = a \wedge b \wedge (b \to \bot) = a \wedge b \wedge \bot$$

it follows that $(a \to b) \wedge (b \to \bot) \leq a \to_H \bot$. Consequently, $a \to b \leq (b \to \bot) \to_H (a \to_H \bot)$ and, thus, $(a \to b) \to_H [(b \to \bot) \to_H (a \to_H \bot)] = \top$.

Since $\vdash_{\mathcal{A}} (\alpha \to_H \beta) \to_H (\alpha \to \beta)$, by Lemma 5.8, $a \to_H b \leq a \to b$. Hence, by Lemma 4.1,

$$a \to_H b = a \to b. \tag{I}$$

Then $(a \to b) \to [(b \to \bot) \to (a \to \bot)] = \top$.

(b) Observe that $a \wedge [((a \to b) \leftarrow \top) \vee b] = [a \wedge ((a \to b) \leftarrow \top)] \vee [a \wedge b] = [a \wedge ((a \to b) \leftarrow \top)] \vee [a \wedge (a \to b)] = a \wedge [((a \to b) \leftarrow \top) \vee (a \to b)] = a \wedge ((a \to b) \vee \top) = a \wedge \top = a$. Then $a \leq ((a \to b) \leftarrow \top) \vee b$. By condition (dR), $b \leftarrow_H a \leq (a \to b) \leftarrow \top$, and consequently,

$$[b \leftarrow_H a] \to_H [(a \to b) \leftarrow \top] = \top.$$

Since $\vdash_{\mathcal{A}} (\alpha \leftarrow \beta) \to_H (\alpha \leftarrow_H \beta)$ (axiom A2), by Lemma 5.8, we have that $a \leftarrow b \leq a \leftarrow_H b$. Thus, by Lemma 4.1,

$$a \leftarrow_H b = a \leftarrow b. \tag{II}$$

So, from (I) and (II), $[b \leftarrow a] \to [(a \to b) \leftarrow \top] = \top$

(c) Since $(c \leftarrow (b \leftarrow a)) \vee ((b \vee c) \leftarrow a) = ((c \vee ((b \vee c) \leftarrow a)) \leftarrow (b \leftarrow a)) \vee ((b \vee c) \leftarrow a) = ((c \vee (b \leftarrow a)) \leftarrow (b \leftarrow a)) \vee ((b \vee c) \leftarrow a) = ((c \vee (b \leftarrow a)) \leftarrow_H (b \leftarrow a)) \vee ((b \vee c) \leftarrow a)$ by (I) and (II), then $(c \leftarrow$

$(b \leftarrow a)) \vee ((b \vee c) \leftarrow a) = ((c \vee (b \leftarrow a)) \leftarrow_H (b \leftarrow a)) \vee ((b \vee c) \leftarrow a) = \bot \vee ((b \vee c) \leftarrow a) = (b \vee c) \leftarrow a$. So $(c \leftarrow (b \leftarrow a)) \rightarrow_H ((b \vee c) \leftarrow a) = \top$. Therefore $(c \leftarrow (b \leftarrow a)) \rightarrow ((b \vee c) \leftarrow a) = \top$.

(d) By axiom (S11), $\vdash_{\mathcal{A}} [(\beta \leftarrow_H \alpha) \rightarrow_H \gamma] \rightarrow_H [\alpha \rightarrow_H (\beta \vee \gamma)]$. Then, by Lemma 5.8 we have that $[(b \leftarrow_H a) \rightarrow_H \bot] \rightarrow_H [a \rightarrow_H (b \vee \bot)] = \top$ and consequently, $[(b \leftarrow a) \rightarrow_H \bot] \rightarrow_H [(a \rightarrow_H b)] = \top$. The conclusion follows now from (I) y (II). ■

LEMMA 5.11. *Let the translation $\tilde{h}_2 : Fm_{\mathbf{L}'} \rightarrow Fm_{\mathbf{L}}$ in the logics $\mathcal{HB}$ and $\mathcal{A}$ be defined by*

- $\tilde{h}_2(x) = x$ *if x is a variable,*
- $\tilde{h}_2(\alpha \wedge \beta) = \tilde{h}_2(\alpha) \wedge \tilde{h}_2(\beta)$,
- $\tilde{h}_2(\alpha \vee \beta) = \tilde{h}_2(\alpha) \vee \tilde{h}_2(\beta)$,
- $\tilde{h}_2(\alpha \rightarrow \beta) = \tilde{h}_2(\alpha) \rightarrow \tilde{h}_2(\beta)$,
- $\tilde{h}_2(\alpha \dot{-} \beta) = \tilde{h}_2(\beta) \leftarrow \tilde{h}_2(\alpha)$,
- $\tilde{h}_2(\neg \alpha) = \alpha \rightarrow \bot$,
- $\tilde{h}_2(\ulcorner \alpha) = \alpha \leftarrow \top$.

Then the translation h_2 is sound.

PROOF. Suppose that ϕ is an axiom. We want to prove that $\vdash_{\mathcal{A}} h_2(\phi)$. This is clear if ϕ is one of the axioms (B1) to (B9), being that the connectives $\neg, \ulcorner\ \dot{-}$ do not appear in ϕ.

1. $\vdash_{\mathcal{A}} ((h_2(\beta) \vee (h_2(\beta) \leftarrow h_2(\alpha))) \leftarrow_H (h_2(\beta) \vee h_2(\alpha))) \rightarrow_H \bot$ by (S18).
2. $\vdash_{\mathcal{A}} (((h_2(\beta) \vee (h_2(\beta) \leftarrow h_2(\alpha))) \leftarrow_H (h_2(\beta) \vee h_2(\alpha))) \rightarrow_H \bot) \rightarrow_H ((h_2(\beta) \vee h_2(\alpha)) \rightarrow_H (h_2(\beta) \vee (h_2(\beta) \leftarrow h_2(\alpha))))$ by Lemma 3.2 (j).
3. $\vdash_{\mathcal{A}} (h_2(\beta) \vee h_2(\alpha)) \rightarrow_H (h_2(\beta) \vee (h_2(\beta) \leftarrow h_2(\alpha)))$ by sHB-MP in 1 and 2.
4. $\vdash_{\mathcal{A}} h_2(\alpha \rightarrow (\beta \vee (\alpha \dot{-} \beta)))$ by definition of h_2.

   From Lemmas 5.8 and 5.10,
5. $\vdash_{\mathcal{A}} (h_2(\alpha) \rightarrow h_2(\beta)) \rightarrow [(h_2(\beta) \rightarrow \bot) \rightarrow (h_2(\alpha) \rightarrow \bot)]$.
6. $\vdash_{\mathcal{A}} h_2((\alpha \rightarrow \beta) \rightarrow (\neg \beta \rightarrow \neg \alpha))$ by definition of h_2.

   In a similar way, the following conditions can be checked:
7. $\vdash_{\mathcal{A}} h_2((\alpha \dot{-} \beta) \rightarrow (\ulcorner (\alpha \rightarrow \beta)))$,
8. $\vdash_{\mathcal{A}} h_2((\gamma \dot{-} (\alpha \dot{-} \beta)) \rightarrow (\alpha \dot{-} (\gamma \vee \beta)))$ and

9. $\vdash_{\mathcal{A}} h_2((\neg(\alpha \dot{-} \beta)) \to (\alpha \to \beta))$.

   By completeness it is immediate that

10. $\vdash_{\mathcal{A}} [h_2(\alpha) \to (h_2(\gamma) \leftarrow h_2(\gamma))] \to [h_2(\alpha) \to \bot]$.
11. $\vdash_{\mathcal{A}} h_2((\alpha \to (\gamma \dot{-} \gamma)) \to (\neg \alpha))$.

   Similarly,

12. $\vdash_{\mathcal{A}} h_2((\neg \alpha) \to (\alpha \to (\gamma \dot{-} \gamma)))$.

   As in the items 11 and 12 we have that

13. $\vdash_{\mathcal{A}} h_2(((\gamma \to \gamma) \dot{-} \alpha) \to (\neg\!\!\!\!\!\!\!\!\!\!\! \ulcorner \alpha))$ and
14. $\vdash_{\mathcal{A}} h_2((\ulcorner \alpha) \to ((\gamma \to \gamma) \dot{-} \alpha))$.

By conditions 4, 6, 7, 8, 9, 11, 12, 13 and 14 we have that if ϕ is (B10), (B11), (B12), (B13), (B14), (B15), (B16), (B17) or (B18) then $\vdash_{\mathcal{A}} h_2(\phi)$.

Suppose now that there exists $\gamma \in Fm_{\mathbf{L}'}$ such that $\Gamma \vdash_{\mathcal{HB}} \beta$ and $\Gamma \vdash_{\mathcal{HB}} \beta \to_H \phi$. Then, by inductive hypothesis, $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\beta)$ and $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\beta \to_H \phi)$. Hence $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\beta) \to_H h_2(\phi)$ and, consequently, by (sHB-MP), $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\phi)$.

If $\phi = \neg(\ulcorner \psi)$ and $\Gamma \vdash_{\mathcal{HB}} \psi$ then, by inductive hypothesis, $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\psi)$. By Lema 4.8, $h_2(\Gamma) \vdash_{\mathcal{A}} h_2(\neg(\ulcorner \psi))$. ■

The following lemma follows immediately from Theorem 4.7.

Corollary 5.12. *Let $\Gamma \cup \{\alpha, \beta\} \subseteq Fm_{\mathbf{L}}$. Then the following conditions are equivalent:*

(a) $\Gamma \vdash_{\mathcal{SHB}} \alpha \leftrightarrow_H \beta$,

(b) $\Gamma \vdash_{\mathcal{SHB}} \alpha \to_H \beta$ *y* $\Gamma \vdash_{\mathcal{SHB}} \beta \to_H \alpha$.

Theorem 5.13. *The logics $\mathcal{A}$ and $\mathcal{HB}$ are translationally equivalent.*

Proof. Consider the translations h_1, h_2 associated to the functions $\tilde{h}_1$ and $\tilde{h}_2$ introduced in Lemmas 5.9 and 5.11.

For a given $\phi \in Fm_{\mathbf{L}}$, let us prove that

$$\vdash_{\mathcal{A}} \phi \leftrightarrow_H h_2(h_1(\phi)) \tag{III}$$

by induction on the construction of ϕ. If $\phi = x$, with $x \in Var$ it is immediate by Lemma 5.6. Suppose that $\phi = \phi_1 \vee \phi_2$ with $\phi_1, \phi_2 \in Fm_{\mathbf{L}}$.

1. $\vdash_{\mathcal{A}} \phi_1 \leftrightarrow_H h_2(h_1(\phi_1))$ by inductive hypothesis.
2. $\vdash_{\mathcal{A}} \phi_2 \leftrightarrow_H h_2(h_1(\phi_2))$ by inductive hypothesis.
3. $\phi_1 \leftrightarrow_H h_2(h_1(\phi_1)), \phi_2 \leftrightarrow_H h_2(h_1(\phi_2)) \vdash_{\mathcal{A}} (\phi_1 \vee \phi_2) \leftrightarrow_H$

$(h_2(h_1(\phi_1)) \vee h_2(h_1(\phi_2)))$ by Lemma 5.6.

4. $\vdash_{\mathcal{A}} (\phi_1 \leftrightarrow_H h_2(h_1(\phi_1))) \to_H [(\phi_2 \leftrightarrow_H h_2(h_1(\phi_2))) \to_H ((\phi_1 \vee \phi_2) \leftrightarrow_H (h_2(h_1(\phi_1)) \vee h_2(h_1(\phi_2))))]$ by Theorem 3.3.
5. $\vdash_{\mathcal{A}} (\phi_1 \vee \phi_2) \leftrightarrow_H (h_2(h_1(\phi_1)) \vee h_2(h_1(\phi_2)))$ by sHB-MP in 1, 2 y 4.
6. $\vdash_{\mathcal{A}} (\phi_1 \vee \phi_2) \leftrightarrow_H h_2(h_1(\phi_1 \vee \phi_2))$ by definition of h_1 and h_2.

The cases in which $\phi = \phi_1 \to \phi_2, \phi = \phi_1 \leftarrow \phi_2$ or $\phi = \phi_1 \wedge \phi_2$ are similar. Assume now that $\phi = \bot$. Let $x \in Var$.

1. $\vdash_{\mathcal{A}} \bot \to_H (x \wedge (x \to \bot))$ by (S7).
2. $\vdash_{\mathcal{A}} (x \wedge (x \to \bot)) \to_H (x \wedge \bot)$ by Lemma 3.1 (c).
3. $\vdash_{\mathcal{A}} (x \wedge \bot) \to_H \bot$ by Lemma 3.1 (c).
4. $\vdash_{\mathcal{A}} (x \wedge (x \to \bot)) \to_H \bot$ by Lemma 3.1 (c) and sHB-MP in 2 and 3.
5. $\vdash_{\mathcal{A}} \bot \leftrightarrow_H (x \wedge (x \to \bot))$ by Corollary 5.12 in 1 and 4.
6. $\vdash_{\mathcal{A}} \bot \leftrightarrow_H h_2(h_1(\bot))$ by definition of h_1 and h_2.

In order to prove condition (III) it is enough to consider $\phi = \top$. By completeness between **DblH** and $\mathcal{HB}$ [17],

1. $\vdash_{\mathcal{A}} \top \leftrightarrow_H (x \to x)$.
2. $\vdash_{\mathcal{A}} \top \leftrightarrow_H h_2(h_1(\top))$ by definition of h_1 and h_2.

Let us see now that

$$\vdash_{\mathcal{HB}} \psi \leftrightarrow_H h_1(h_2(\psi)) \tag{IV}$$

for $\psi \in Fm_{\mathbf{L}'}$. The cases in which $\psi = x$, with $x \in Var, \psi = \psi_1 \vee \psi_2, \psi = \psi_1 \to \psi_2, \psi = \psi_1 \leftarrow \psi_2$ and $\psi = \psi_1 \wedge \psi_2$ with $\psi_1, \psi_2 \in Fm_{\mathbf{L}'}$ can be checked as before. It remains to be seen that

$$\vdash_{\mathcal{HB}} \neg\alpha \leftrightarrow_H h_1(h_2(\neg\alpha)) \quad \text{and} \quad \vdash_{\mathcal{HB}} \urcorner\, \alpha \leftrightarrow_H h_1(h_2(\urcorner\, \alpha))$$

with $\alpha \in Fm_{\mathbf{L}'}$.

1. $\vdash_{\mathcal{HB}} \alpha \leftrightarrow_H h_1(h_2(\alpha))$ by inductive hypothesis.

   By completeness between **DblH** and $\mathcal{HB}$ [17],
2. $\vdash_{\mathcal{HB}} \neg\alpha \to_H (\alpha \to (x \wedge \neg x))$ and
3. $\vdash_{\mathcal{HB}} [\alpha \to (x \wedge \neg x)] \to_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$.
4. $\vdash_{\mathcal{HB}} (\neg\alpha) \to_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by (S1) and sHB-MP in 2 and 3.

   In a similar way,

5. $\vdash_{\mathcal{HB}} [h_1(h_2(\alpha)) \to (x \wedge \neg x)] \to_H (\neg\alpha)$.
6. $\vdash_{\mathcal{HB}} (\neg\alpha) \leftrightarrow_H [h_1(h_2(\alpha)) \to (x \wedge \neg x)]$ by Corollary 5.12.
7. $\vdash_{\mathcal{HB}} (\neg\alpha) \leftrightarrow_H [h_1(h_2(\neg\alpha))]$ by the definition of h_2.

   In a similar way as item 7 it can be proved that :

8. $\vdash_{\mathcal{HB}} (\frown \alpha) \leftrightarrow_H h_1(h_2(\frown \alpha))$. ■

Theorem 5.13 allows us to state that the Heyting–Brouwer logic is, up to equivalence by translations, an axiomatic extension of the semi Heyting–Brouwer logic.

In [3] we introduced a semi-intuitionistic logic which is the algebraic counterpart of the Sankappanavar's semi-Heyting algebras [19] and it has the intuitionistic propositional calculus [16] as an axiomatic extension. In [4] it is proved that the logic defined by the axioms (S1), (S2), (S3), (S4), (S5), (S6), (S7), (S8), (S9), (S12) and (S13) is equivalent by translations to the one introduced in [3]. Thus $\mathcal{SI}$ (and consequently also $\mathcal{I}$ [3]) turns out to be a fragment of the logic $\mathcal{SHB}$.

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