RIEMANNIAN GEOMETRY OF FINITE RANK POSITIVE OPERATORS*

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Abstract

A riemannian metric is introduced in the infinite dimensional manifold Σ_n of positive operators with rank $n < \infty$ on a Hilbert space H. The geometry of this manifold is studied and related to the geometry of the submanifolds Σ_p of positive operators with range equal to the range of a projection p (rank of $p = n$), and $\mathcal{P}_p$ of selfadjoint projections in the connected component of p. It is shown that these spaces are complete in the geodesic distance.

Keywords: positive operator, finite rank projection.

1 Introduction

The space $M_n^+(\mathbb{C})$ of positive definite (invertible) matrices is a differentiable manifold, in fact an open subset of the real euclidean space of hermitian matrices. Let x, y be hermitian matrices and a positive definite, the formula

$$< x, y >_a = tr(xa^{-1}ya^{-1})$$

endows $M_n^+(\mathbb{C})$ with a riemannian metric, which makes it a negatively curved, complete metric space. This fact is well known and has been used in a variety of contexts. For example, in interpolation theory of Banach and Hilbert spaces [9], [21], in partial differential equations [20], or in mathematical physics [18], [22], [11]. It has also been generalized to infinite dimensions, i.e. Hilbert spaces and operator algebras: [22], [7], [8], [4].

The purpose of this paper is to introduce a riemannian structure in the set Σ_n of positive operators of finite (fixed) rank n on an infinite dimensional Hilbert space H. Note that even though $n < \infty$, this set Σ_n is infinite dimensional. Corach et al. [4], [5] considered a Finsler structure for positive non invertible operators with fixed *range*. We go one step further fixing only the *rank*. The condition that the rank is fixed ensures that for all $a \in \Sigma_n$, the projections onto their ranges, which we denote by $\rho(a)$, are unitarily equivalent.

In particular, if p is a projection with rank n, then the connected component $\mathcal{P}_p$ of p in the space of projections, lies inside Σ_n. Also inside Σ_n lies Σ_p, the space of positive operators with range

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equal to the range of p. Apparently, Σ_p identifies with $M_n^+(\mathbb{C})$. We shall introduce a riemannian metric in Σ_n, which naturally generalizes the metric given above for $M_n^+(\mathbb{C})$, and which restricted to Σ_p makes the identification of this space with $M_n^+(\mathbb{C})$ isometric. Moreover, when restricted to $\mathcal{P}_p$, one obtains the trace inner product of this space. Our main result on Σ_n (6.9) states that, though we lose the negative curvature properties for positive operators (because $\mathcal{P}_p$ inside Σ_n is positively curved), Σ_n is a complete metric space for the geodesic distance.

Let us fix some notation. Let H be a Hilbert space, $\mathcal{U}(H)$ and $\mathcal{G}l(H)$ the Banach-Lie groups of, respectively, unitary and invertible operators of H. Throughout this paper $\|x\|$ will denote the usual operator norm of $x \in \mathcal{B}(H)$. Fix $n < \infty$, and let p a projection with rank n, and consider the following sets:

- Σ_n the set of positive operators with rank n.
- Σ_p the set of positive operators with range equal to the range of p.
- $\mathcal{I}_p$ the set of partial isometries with initial space equal to the range of p.

Clearly these sets $\Sigma_p \subset \Sigma_n$ and $\mathcal{I}_p$ are subsets of $\mathcal{B}_2(H)$, the class of Hilbert-Schmidt operators of H. Denote by $\mathcal{P}$ the set of projections acting on H, and by $\mathcal{P}_p$ the connected component (in the norm topology) of p in $\mathcal{P}$, which coincides with the unitary orbit of p, $\{upu^* : u \text{ unitary in } H\}$. The three sets of the above list and $\mathcal{P}_p$ will be considered with the inner product topology of $\mathcal{B}_2(H)$. Since these are sets of finite rank operators, this topology coincides there with the operator norm topology of $\mathcal{B}(H)$.

A relevant feature in this study is the map

$$\rho : \Sigma_n \to \mathcal{P}_p,$$

$\rho(a) =$ projection onto the range of a. This map is continuous due to the fact that $n < \infty$. Moreover, it was shown in [5] that it is differentiable. In this paper we revise the differentiable structure of Σ_n, $\mathcal{P}_p$ and $\mathcal{I}_p$. We introduce a riemannian metric in Σ_n, based on the trace of $\mathcal{B}_2(H)$, and consider geometric problems therein. When restricted to the submanifold Σ_p of positive operators with fixed range $p(H)$, one obtains the well studied non positive curvature connection for the set of positive invertible operators [7].

The contents of the paper are as follows. In section 2 we revise the riemannian geometry of $\mathcal{P}_p$. As it turns out, the connection looks formally identical to the reductive connection for the space of projections in an abstract C*-algebra [6], [16], [17]. Then one can profit from the computations done there: geodesics, curvature tensor, etc. Here we establish that $\mathcal{P}_p$ is complete. In section 3 we consider $\mathcal{I}_p$ with the metric given, as with $\mathcal{P}_p$, by the inner product of $\mathcal{B}_2(H)$. Here we are able to compute the geodesics, and prove that $\mathcal{I}_p$ is complete. Note that both $\mathcal{P}_p$ and $\mathcal{I}_p$ are infinite dimensional, so that even though they are geodesically complete, completeness in the geodesic distance needs proofs. In section 4 we introduce the riemannian metric in Σ_n, and compute the connection. We examine how the submanifolds $\mathcal{P}_p$ and Σ_p sit inside Σ_n. In section 5 we consider the homotopy type of Σ_n, by means of the map

$$\varpi : \Sigma_p \times \mathcal{I}_p \to \Sigma_n, \quad \varpi(b, x) = xbx^*,$$

which is a smooth principal bundle. In section 6 we prove our main result: completeness of Σ_n in the geodesic distance. This is done without knowing how the geodesics do actually look. A role here is played by the space Σ_∞ of positive definite infinite matrices [3].

2 Geometry of Projections

The differential geometry of the space of projections of a C*-algebra is the subject of several papers [6],[8], [4], [20]. Let us mention the book [23] by H. Upmeier which treats general symmetric spaces in the infinite dimensional setting. Let us recall some known facts. We shall be concerned only with the case of the full operator algebra $\mathcal{B}(H)$. The set $\mathcal{P}$ of selfadjoint projections is a submanifold of $\mathcal{B}(H)$ (in the norm topology), whose connected components are the unitary orbits $\{upu^* : u \in \mathcal{U}(H)\}$, which are parametrized by the possible ranks $k \in \mathbb{N} \cup \{\infty\}$. If p is an arbitrary projection, the tangent space $(T\mathcal{P})_p$ identifies with the selfadjoint elements x of $\mathcal{B}(H)$ which satisfy $x = xp + px$. If one represents the elements of $\mathcal{B}(H)$ as 2×2 block matrices in terms of p, then $(T\mathcal{P})_p$ consist of the selfadjoint codiagonal matrices. The unitary group $\mathcal{U}(H)$ acts on $\mathcal{P}$, as inner automorphisms: $u \cdot p = upu^*$. The isotropy group of this action consists of the unitaries which commute with p, i.e. the unitaries of the commutant of p, which is the C*-algebra of diagonal matrices in terms of p. This endows $\mathcal{P}$ (or rather, its connected components) with a homogeneous structure. The reason for this is that the action admits local smooth cross sections. There is more than one way to obtain local cross sections. For instance, if $\|q - p\| < 1$, then $\sigma_p(q) =$ unitary part of (the invertible element) $qp + (1-q)(1-p)$ is a unitary operator, defined locally around p, which conjugates p and q:

$$\sigma_p(q) p \sigma_p^*(q) = q.$$

Clearly σ_p is a smooth map. From now on, we make the asumption that the rank of p is $< \infty$. Therefore, $\mathcal{P}_p \subset \mathcal{B}_2(H)$, and the norm topology of $\mathcal{B}(H)$ and the inner product norm topology of $\mathcal{B}_2(H)$ coincide in $\mathcal{P}_p$. Indeed among operators a, b of rank $\leq n$ one has the trivial estimate

$$\|a - b\| \leq \|a - b\|_2 \leq \sqrt{n}\|a - b\|. \tag{2.1}$$

It is well known that $\mathcal{P}_p$ (for any p, not necesarilly of finite rank) is a C^∞ submanifold of $\mathcal{B}(H)$. In our case, $\mathcal{P}_p$ is a C^∞ submanifold of $\mathcal{B}_2(H)$. Let us state this result for the sake of completeness of the exposition.

Lemma 2.1 *Let p be a projection of rank $n < \infty$. Then the map*

$$\pi_p : \mathcal{U}(H) \to \mathcal{P}_p \subset \mathcal{B}_2(H), \quad \pi_p(u) = upu^*$$

is a C^∞ submersion, and induces on $\mathcal{P}_p$ a homogeneous structure. In particular $\mathcal{P}_p$ is a C^∞ submanifold of $\mathcal{B}_2(H)$.

Proof. In order to prove this result, we use an elegant consequence of the inverse function theorem in the context of Banach spaces, written by Raeburn in [19]. This states that in order to prove the above result it suffices to show that

1. the map π_p is C^∞, as a map from $\mathcal{U}(H)$ to $\mathcal{B}_2(H)$,
2. the map π_p is open, as a map from $\mathcal{U}(H)$ to $\mathcal{P}_p$,
3. the range of $d(\pi_p)_1$ is a complemented subspace of $\mathcal{B}_2(H)$, and
4. the kernel of $d(\pi_p)_1$ is a complemented subspace of $(T\mathcal{U}(H))_1 = \mathcal{B}(H)_{ah}$ (the space of anti-hermitic operators of H).

The first assertion is a consequence of the fact that the (real) bilinear map

$$\mathcal{B}(H) \times \mathcal{B}(H) \to \mathcal{B}_2(H), \quad (x, y) \mapsto xpy^*$$

is bounded, which is apparent. Note that therefore π_p is C^∞, being the restriction of this map to the submanifold $\{(u, u) : u \in \mathcal{U}(H)\}$ of $\mathcal{B}(H) \times \mathcal{B}(H)$.

The second assertion follows from the existence of continuous local cross sections for π_p, remarked above. Here continuity holds both for the operator and Hilbert-Schmidt norms.

The differential $d(\pi_p)_1$ equals the map δ_p,

$$\delta_p : \mathcal{B}(H)_{ah} \to \mathcal{B}_2(H), \quad \delta_p(x) = xp - px.$$

Its range consists of the selfadjoint operators of $\mathcal{B}_2(H)$, which have 2×2 codiagonal matrices with respect to p. Clearly, this real linear subspace of $\mathcal{B}_2(H)$ is closed, and therefore complemented. Its orthogonal complement (with respect to the trace inner product) consists of 2×2 diagonal matrices with respect to p.

The kernel of δ_p consists of antihermitic operators in $\mathcal{B}(H)$ which commute with p. This space is complemented in $\mathcal{B}(H)_{ah}$, by the space of antihermitic operators which have 2×2 codiagonal matrices with respect to p. $\square$

Let us consider the following riemannian metric on $\mathcal{P}_p$:

$$< x,y >_q = tr(xy), \ \ q\in \mathcal{P}_p, x,y\in (T\mathcal{P}_p)_q,$$

i.e. the usual inner product of $\mathcal{B}_2(H)$ at every point of $\mathcal{P}_p$. Let us compute the riemannian connection of this metric:

$$\nabla_x y_q = P_{(T\mathcal{P}_p)_q}(x\{y\}),$$

for $x\in (T\mathcal{P}_p)_q$ and y a tangent vector field. Here $P_{(T\mathcal{P}_p)_q}$ stands for the orthogonal projection onto $(T\mathcal{P}_p)_q$, and $x\{y\}$ is the derivative of y in the x-direction, performed in $\mathcal{B}_2(H)$. Note that if $a\in \mathcal{B}(\mathcal{H})_h$, $P_{(T\mathcal{P}_p)_q}(a) = pa(1-p)+(1-p)ap$. Therefore

$$\nabla_x y_q = qx\{y\}(1-q)+(1-q)x\{y\}q.$$

Remarkably, this is the same connection as the reductive connection for the space of projections in an abstract C*-algebra [6]. Then one has the explicit form for the geodesics, the exponential map and the curvature tensor. Moreover the results of existence of geodesics joining two given endpoints, as well as the minimality results, can be derived from previous work (see also [16], [17]). Let us list this facts.

Remark 2.2 *1. The unique geodesic $\rho(t)$ with $\rho(0)=q$ and $\dot{\rho}(0)=x$ is given by*

$$\rho(t) = e^{t\delta_q(x)}qe^{-t\delta_q(x)}, \ \ t\in \mathbb{R}. \tag{2.2}$$

2. The curvature tensor is given by

$$R(x,y)z = [[x,y],z], \ \ x,y,z\in (T\mathcal{P}_p)_q,$$

where $[a,b]=ab-ba$.

3. Two projections p_0 and p_1 in $\mathcal{P}_p$ such that the geodesic distance $d_g(p_0,p_1) < \frac{\pi}{2}$ are joined by a unique geodesic whose length equals the geodesic distance.

The exponential map suggests the definition of a different local cross section for π_p [6]. Namely,

$$\vartheta_p : \{q\in \mathcal{P}_p : \|q-p\|<1\} \to \mathcal{U}(H), \ \ \vartheta_p(q) = e^x, \tag{2.3}$$

where $x\in (T\mathcal{P}_p)_p$ is the unique p-codiagonal antihermitic element of $\mathcal{B}(H)$ such that $e^x pe^{-x}=q$.

Remark 2.3 *$\mathcal{P}_p$ has non negative sectional curvature. Indeed,*

$$R(x,y)y = xy^2 - 2yxy - y^2x,$$

and therefore

$$< R(x,y)y, x >_q = 2(tr(x^2y^2) - tr(xyxy)).$$

Now, by the Cauchy-Schwarz inequality, $tr(xyxy) = tr((yx)^*xy) \le tr((yx)^*yx)^{1/2}tr((xy)^*xy)^{1/2} = tr(xy^2x)^{1/2}tr(yx^2y)^{1/2} = tr(x^2y^2)$.

Note (2.2) that $\mathcal{P}_p$ is geodesically complete. Let us show that it is complete with the geodesic distance.

Proposition 2.4 *$\mathcal{P}_p$ is complete with the geodesic distance.*

Proof. From (2.1) it follows that the geodesic riemannian distance is equivalent to the geodesic distance of the Finsler structure obtained by considering the usual operator norm in each tangent space, with the same geodesic curves as minimal curves. This was studied in the general context of abstract C*-algebras [17], [6]. Let us denote by d_f the distance induced in $\mathcal{P}_p$ by the Finsler structure. It is known [2], [15] that if $\|q_1 - q_2\| < 1$ (which is equivalent to $d_f(q_1, q_2) < \pi/2$), then $d_f(q_1, q_2) = arcsin(\|q_1 - q_2\|)$. If one further requires that $\|q_1 - q_2\| < 0.9$, then

$$\frac{3}{2} d_f(q_0, q_1) = \frac{3}{2} arcsin(\|q_0 - q_1\|) \leq \|q_0 - q_1\| \leq arcsin(\|q_0 - q_1\|) = d_f(q_0, q_1).$$

If follows that if $\{p_k\}$ is a Cauchy sequence in $\mathcal{P}_p$ for the geodesic riemannian metric, then it also a Cauchy sequence for d_f, and the inequalities above show that it is a Cauchy sequence for the usual norm of operators. Since $\mathcal{P}$ is closed in $\mathcal{B}(H)$ in the norm topology, and therefore complete for the norm metric, the result follows. □

3 Partial isometries with initial space p

In this section we consider the set $\mathcal{I}_p = \{u \in \mathcal{B}(H) : u^*u = p\}$ of partial isometries with initial space p. Note that since $dim\ p(H) < \infty$, $\mathcal{I}_p \subset \mathcal{B}_2(H)$. We shall consider this set endowed with the inner product topology. This set was shown to be a C^∞ submanifold of $\mathcal{B}(H)$ (in the norm topology) [1], in the abstract setting of arbitrary C*-algebras. Here we shall see that in our context, $\mathcal{I}_p$ is a C^∞ submanifold of $\mathcal{B}_2(H)$. As with the set of projections, this will be done by considering the appropriate action from the group $\mathcal{U}(H)$. Namely,

$$\mathcal{U}(H) \times \mathcal{I}_p \to \mathcal{I}_p, \quad (w, u) \mapsto wu.$$

First note that the metrics given by the operator norm and the inner product norm are also equivalent in $\mathcal{I}_p$. Indeed, operators in $\mathcal{I}_p$ have rank $n = rank(p)$. In [1] it was shown that this action is locally transitive, and that if p is of finite rank (more generally, if it is a finite projection), then $\mathcal{I}_p$ is connected, i.e., the action is transitive.

Proposition 3.1 *The map*

$$\mu_p : \mathcal{U}(H) \to \mathcal{I}_p \subset \mathcal{B}_2(H), \quad \mu_p(w) = wp$$

is a C^∞ submersion, and defines on $\mathcal{I}_p$ a homogeneous structure. In particular, $\mathcal{I}_p$ is a C^∞ submanifold of $\mathcal{B}_2(H)$.

Proof. This map is clearly C^∞, since it is the restriction of the bounded linear map $\mathcal{B}(H) \to \mathcal{B}_2(H)$, $x \mapsto xp$. As in the analogous result in the preceeding section, it suffices to prove that μ_p is open, and that its differential has closed range and complemented kernel. That it is open follows from the fact that it has local cross sections in which are continuous in the norm (equivalent to the Hilbert-Schmidt) metric. The differential $d(\mu_p)_1 : \mathcal{B}(H)_{ah} \to \mathcal{B}_2(H)$ is $d(\mu_p)_1(x) = xp$. It is apparent that $\ker d(\mu_p)_1 \subset \mathcal{B}(H)_{ah}$ and $Im\ d(\mu_p)_1 \subset \mathcal{B}_2(H)$ are complemented subspaces. □

Let us characterize the tangent spaces.

Lemma 3.2 *The tangent space* $(T\mathcal{I}_p)_u$ *equals*

$$(T\mathcal{I}_p)_u = \{xu : x^* = -x\} = \{zp \in \mathcal{B}_2(H).p : z^*u + u^*z = 0\}.$$

Proof. Let $w \in \mathcal{U}(H)$ such that $u = wp$. The fact that μ_p above is a submersion implies that the tangent space $(T\mathcal{I}_p)_u$ equals $d(\mu_p)_w((T\mathcal{U}(H))_w)$. This $(T\mathcal{U}(H))_w$ equals $\mathcal{B}(H)_{ah}w$, and therefore $(T\mathcal{I}_p)_u = \{xwp : x^* = -x\} = \{xu : x^* = -x\}$. Let us prove now that $(T\mathcal{I}_p)_u = \{z \in \mathcal{B}_2(H).p : z^*u+u^*z = 0\}$. If $z = xu$ with $x^* = -x$, then $zp = xup = xu = z$ and $z^*u+u^*z = -u^*xu+u^*xu = 0$. Conversely, if $z = zp$ verifies $z^*u+u^*z = 0$. Consider $y = \frac{1}{2}uu^*zw^* - \frac{1}{2}wz^*uu^* + (1-uu^*)zw^* - wz^*(1-uu^*)$. Clearly this element verifies $y^* = -y$. Now compute

$$yu = \frac{1}{2}uu^*zw^*u - \frac{1}{2}wz^*uu^*u + (1-uu^*)zw^*u - wz^*(1-uu^*)u = \frac{1}{2}uu^*zp - \frac{1}{2}wz^*u + (1-uu^*)zp.$$

Using that $z^*u = -u^*z$ and $zp = z$, one obtains

$$yu = \frac{1}{2}uu^*z + \frac{1}{2}wu^*z + (1-uu^*)z = \frac{1}{2}uu^*z + \frac{1}{2}wpw^*z + (1-uu^*)z = z.$$

$\square$

As before, we introduce a riemannian metric in $\mathcal{I}_p$, by means of the inner product of the ambient space $\mathcal{B}_2(H)$.

The inner product $tr((yu)^*xu) = -tr(u^*yxu)$ may take complex values. Therefore we define:

$$< xu, yu >_u = Re(tr((yu)^*xu)).$$

Let us compute the riemannian connection corresponding to this metric. First we must compute the orthogonal projection $P_{(T\mathcal{I}_p)_u} : \mathcal{B}_2(H) \to (T\mathcal{I}_p)_u$. This is given next

Lemma 3.3 *The projection* $P_{(T\mathcal{I}_p)_u}$ *equals* P_u*,*

$$P_u(x) = xp - \frac{1}{2}uu^*xp - \frac{1}{2}ux^*u, x \in \mathcal{B}_2(H).$$

Proof. First note that if $x = yu \in (T\mathcal{I}_p)_u$ with $y^* = -y$, then $P_u(x) = yup - \frac{1}{2}uu^*yup - \frac{1}{2}uu^*y^*u = yu + \frac{1}{2}uu^*yu + \frac{1}{2}uu^*yu = yu = x$. Next, if $z = P_u(x)$, clearly $zp = z$ and

$$z^*u + u^*z = (px^* - \frac{1}{2}px^*uu^* - \frac{1}{2}u^*xu^*)u + u^*(xp - \frac{1}{2}uu^*xp - \frac{1}{2}ux^*u)$$

$$= px^*u - \frac{1}{2}px^*u - \frac{1}{2}u^*xp + u^*xp - \frac{1}{2}u^*xp - \frac{1}{2}px^*u = 0.$$

This shows that P_u is a projection with range equal to $(T\mathcal{I}_p)_u$. Let us see that it is orthogonal for the (real) inner product of $\mathcal{B}_2(H)$. If $x, y \in \mathcal{B}_2(H)$,

$$< P_u(x), y >= Re\ tr(y^*xp) - \frac{1}{2}Re\ tr(y^*uu^*xp) - \frac{1}{2}Re\ tr(y^*ux^*u).$$

Let us examine these terms. In the first, one has $tr(y^*xp) = tr(py^*x) = tr((yp)^*x)$. In the second, $tr(y^*uu^*xp) = tr(xpy^*uu^*) = tr(x(uu^*yp)^*) = tr((uu^*yp)^*x)$. In the third, $Re\ tr(y^*ux^*u) = Re\ tr(uy^*ux^*) = Re\ tr((uy^*ux^*)^*) = Re\ tr(xu^*yu^*) = Re\ tr((uy^*u)^*x)$. It follows that

$$< P_u(x), y >= Re\ tr((yp)^*x) - \frac{1}{2}Re\ tr((uu^*yp)^*x) - \frac{1}{2}Re\ tr((uy^*u)^*x) =< x, P_u(y) > .$$

Therefore P_u is the orthogonal projection onto the tangent space of $\mathcal{I}_p$ at u. $\square$

Now suppose that x is a vector field which is tangent along the curve γ in $\mathcal{I}_p$. The covariant derivative of the riemannian connection of the metric defined above gives

$$\frac{Dx}{dt} = P_{(T\mathcal{I}_p)_\gamma}(\dot{x}) = \dot{x}p - \frac{\gamma\gamma^*\dot{x}p + \gamma\dot{x}^*\gamma}{2} = \dot{x} - \frac{\gamma\gamma^*\dot{x} + \gamma\dot{x}^*\gamma}{2},$$

where $\dot{x}p = \dot{x}$ because x takes values in the subspace $\mathcal{B}_2(H)p$ (which contains $\mathcal{I}_p$ and all its tangent spaces). This expression can be simplified, if one uses (3.2) that any field x which is tangent along γ satisfies $x^*\gamma + \gamma^*x = 0$. Differentiating this, one obtains $x^*\dot{\gamma} + \dot{\gamma}^*x = -\dot{x}^*\gamma - \gamma^*\dot{x}$. Therefore

$$\frac{Dx}{dt} = \dot{x} + \frac{\gamma\dot{\gamma}^* + \gamma x^*\dot{\gamma}}{2}.$$

The riemannian connection is therefore given by

$$(\nabla_x y)_u = x(y) + \frac{ux^*y + uy^*x}{2}.$$

In particular, a curve δ is a geodesic of the riemannian connection if it satisfies the equation

$$\ddot{\delta} + \delta\dot{\delta}^*\dot{\delta} = 0. \tag{3.4}$$

The action of $\mathcal{U}(H)$ on $\mathcal{I}_p$, is clearly isometric for the riemannian metric. It follows that if δ is a geodesic and $w \in \mathcal{U}(H)$, then $w\delta$ is also a geodesic. Therefore it suffices to compute the geodesics which start at p. This is done in the following:

Theorem 3.4 *Let $x \in (T\mathcal{I}_p)_p$. The unique geodesic δ with $\delta(0) = p$ and $\dot{\delta}(0) = x$ is given by*

$$\delta(t) = e^{tz}e^{ty}p, \ t \in \mathbb{R},$$

where $z = 2pxp + (1-p)xp - px^*(1-p)$ and $y = -pxp$.

Proof. Note that $x = x_0p$, with $x_0^* = -x_0$. This implies that both z and y are antihermitic: $z^* = 2px_0^*p + px_0^*(1-p) - (1-p)x_0^*p = -2px_0p - px_0(1-p) + (1-p)x_0p = -z$, and analogously with y. It follows that $e^{tz}e^{ty}$ is a curve of unitaries, and therefore $\delta(t) \in \mathcal{I}_p$, satisfies $\delta(0) = p$ and

$$\dot{\delta}(0) = zp + yp = xp = x.$$

Let us show that δ satisfies the equation 3.4. Note that y commutes with p. Compute

$$\dot{\delta}(t) = e^{tz}ze^{ty}p + e^{tz}e^{ty}yp = e^{tz}(zp + yp)e^{ty}$$

and

$$\ddot{\delta}(t) = e^{tz}z^2e^{ty}p + 2e^{tz}zye^{ty}p + etze^{ty}y^2p = e^{tz}(z^2p + 2zyp + y^2p)e^{ty}.$$

Then $\dot{\delta}^*(t) = -e^{-ty}(pz + yp)e^{-tz}$. It follows that $\delta(t)\dot{\delta}^*(t)\dot{\delta}(t) = -e^{tz}p(pz + py)(zp + yp)e^{ty}$. Replacing these expressions in equation 3.4 one obtains

$$e^{tz}(z^2p + 2zp + y^2p - pz^2p - pzyp - pyzp - y^2p)e^{ty}.$$

Recall that $xp = x$. Then $z^2p = (2pxp + (1-p)xp - px^*(1-p))^2p = 4px^2p + 2(1-p)x^2p$. Also $zyp = (2pxp + (1-p)xp - px^*(1-p))(-pxp) = -2px^2p - (1-p)x^2p$ and $pyzp = -pxp(2pxp + (1-p)xp - px^*(1-p)) = -2px^2p$. These facts imply that

$$z^2 + 2zyp = 0,$$

and

$$pz^2p + pzyp + pyzp = 0.$$

Therefore δ verifies equation 3.4. $\square$

Remark 3.5 *$\mathcal{I}_p$ is geodesically complete.*

Next we show that $\mathcal{I}_p$ is complete (with the geodesic distance). Let $R > 0$ be the radius of a normal neighbourhood of $p \in \mathcal{I}_p$. Since the action of $\mathcal{U}(H)$ on $\mathcal{I}_p$ is transitive and isometric, it follows that the same radius works for every element in $\mathcal{I}_p$.

Lemma 3.6 *There exists a number $R_0 \leq R$ such that if $x \in (T\mathcal{I}_p)_p$, $x = (z + y)p$ as in 3.4, with $\|x\|_2 < R_0$ then*

$$\frac{\sqrt{2}}{2}\|x\|_2 \leq \|p - e^z pe^y\|_2 \leq \frac{\sqrt{6}}{2}\|x\|_2. \tag{3.5}$$

Proof. Note that

$$\|p - e^z pe^y\|_2^2 = tr(p) + tr(e^{-y}pe^y) - 2Re\ tr(pe^z pe^y) = 2n - 2Re\ tr(pe^z pe^y).$$

Recall that z, y are antihermitic, and y commutes with p. Compute

$$tr(pe^z pe^y) = tr(e^z pe^y) = tr(p + zp + py + \frac{1}{2}z^2 p + \frac{1}{2}py^2 + zpy + \ldots)$$

Since z, y are antihermitic, the trace of terms of odd degree in the above expansion give pure imaginary numbers. Since we are taking real part of the trace, only terms of even degree remain. Also it is clear that the terms of degree 2 give

$$tr(\frac{1}{2}z^2 p + \frac{1}{2}py^2 + zpy) = \frac{1}{2}\{tr(z^2 p) + tr(y^2 p) + tr(zpy) + tr(yzp)\} = \frac{1}{2}\|x\|_2^2.$$

Next note that $|tr(z^j py^i)| \leq \|z^j py^i\|_1 \leq n\|z^j py^i\| \leq n\|z\|^j\|y\|^i$. Then

$$\|p - e^z pe^y\|^2 = \|x\|_2^2 - 2tr\{\text{terms of even degree} \geq 4\},$$

and

$$|2tr\{\text{terms of even degree} \geq 4\}| \leq 2n\{\frac{1}{4!}\|z\|^4 + \frac{1}{3!}\|z\|^3\|y\| + \frac{1}{2}\frac{1}{2}\|z\|^2\|y\|^2 + \frac{1}{3!}\|z\|\|y\|^3 + \frac{1}{4!}\|y\|^4 + \ldots\}$$

$$= 2n\{cosh(\|z\| + \|y\|) - 1 - \frac{1}{2}(\|z\| + \|y\|)^2\}.$$

Recall that $z = 2pxp + (1 - p)xp - px^*(1 - p)$ and $y = -pxp$. Then $\|z\| \leq 4\|x\|_2$ and $\|y\| \leq \|x\|_2$. Since $cosh(t) - 1 - \frac{t^2}{2}$ is increasing for $t \geq 0$, this implies that

$$|2tr\{\text{terms of even degree} \geq 4\}| \leq 10n\|x\|_2^2\{\frac{1}{4!}(5\|x\|_2)^2 + \frac{1}{6!}(5\|x\|_2)^4 + \ldots\}.$$

The function $f(t) = \frac{1}{4!}(5t)^2 + \frac{1}{6!}(5t)^4 + \ldots = \frac{1}{t^2}\big(cosh(5t) - 1 - \frac{(5t)^2}{2}\big)$ is also increasing for $t \geq 0$ and verifies $f(0) = 0$. It follows that there exists $R_0 > 0$ (chosen $R_0 \leq R$) such that $f(t) < \frac{1}{20n}$ if $t < R_0$. Puting these facts together yields the following: if $\|x\|_2 < R_0$, then

$$|2tr\{\text{terms of even degree} \geq 4\}| \leq \frac{1}{2}\|x\|_2^2.$$

Therefore , if $\|x\|_2 < R_0$, we have

$$\frac{1}{2}\|x\|_2^2 \leq \|p - e^z pe^y\|_2^2 \leq \frac{3}{2}\|x\|_2^2,$$

and the statement follows. □

Corollary 3.7 *The space $\mathcal{I}_p$ is complete with the geodesic distance.*

Proof. Let $\{u_k\}$ be a Cauchy sequence in $\mathcal{I}_p$ for the metric d_g. There exists k_0 such that $d_g(u_k, u_l) < R_0$ if $k, l \geq k_0$. The above lemma states that for such elements, the geodesic distance is equivalent to the metric given by the Hilbert Schmidt norm $\| \ \|_2$. Indeed, if $d_g(u,v) < R_0 \leq R$ there exists a (unique) geodesic $\gamma(t) = we^{tz}pe^{ty}$ with $\gamma(0) = wp = u$ and $\gamma(1) = v$ for an appropriate unitary operator w. Then $d_g(u,v) = \|w(z+y)p\|_2 = \|(z+y)p\|_2 < R_0$. On the other hand $\|u - v\|_2 = \|wp - we^z pe^y\|_2 = \|p - e^z pe^y\|_2$, and the claim follows. Therefore $\{u_k\}$ is a Cauchy sequence in the $\| \ \|_2$ metric, or equivalently, in the usual operator norm $\| \ \|$ metric. It is apparent that $\mathcal{I}_p$ is closed in norm in $\mathcal{B}(H)$, which is complete. □

4 A riemannian metric for the space Σ_n

In [5] it was proven that the space Σ_n of positive operators with rank $n < \infty$ is a differential manifold. It coincides with the orbit of the action $g \cdot a = gag^*$ of $\mathcal{G}l(H)$ on any $a \geq 0$ with rank n, for instance

$$\Sigma_n = \{gpg^* : g \in \mathcal{G}l(H)\}.$$

Moreover, on Σ_n the maps $a \mapsto \rho(a)$ and $a \mapsto a^\dagger$ (=inverse of a as an operator in $a(H)$, also called Moore-Penrose inverse of a) are differentiable maps. Note that $a^\dagger a = aa^\dagger = \rho(a)$. First we shall verify that in fact Σ_n is also a submanifold of $\mathcal{B}_2(H)$.

Proposition 4.1 *The set Σ_n is a C^∞ submanifold of $\mathcal{B}_2(H)$. The map*

$$\mathcal{G}l(H) \to \Sigma_n, \quad g \mapsto gpg^*$$

is a C^∞ submersion.

Proof. We use the same argument as before. The map is C^∞ because it is the restriction of a (real) bilinear bounded map as in 2.1. The differential at 1 is the map $\mathcal{B}(H) \to \mathcal{B}_2(H)$, $x \mapsto xp + px^*$. If one represents the elements of $\mathcal{B}(H)$ as 2×2 matrices in terms of p, the kernel of this map consists of elements $x \in \mathcal{B}(H)$ such that

$$pxp + px^*p = 0 \text{ and } (1-p)xp = 0.$$

A supplement for this space is, for example, $\{y \in \mathcal{B}(H) : pyp = py^*p, y(1-p) = 0\}$. The range is $\{xp + px^* : x \in \mathcal{B}(H)\}$, which coincides with $\{z \in \mathcal{B}_2(H) : z^* = z, (1-p)z(1-p) = 0\}$. Indeed, if $z = xp + px^*$, it is apparent that $z = z^*$ and $(1-p)z(1-p) = 0$. Conversely suppose that $z \in \mathcal{B}_2(H)$ is selfadjoint and verifies $(1-p)z(1-p) = 0$. Put $x = pz + zp - \frac{3}{2}pzp$. Then $xp + px^* = zp + pz - pzp = z$ because $(1-p)z(1-p) = 0$. Finally, the map $g \mapsto gpg^*$ is open because it has continuous local cross sections [5] in the norm topology (which coincides with the Hilbert-Schmidt topology) in Σ_n. □

Let us compute the tangent spaces $(T\Sigma_n)_a$. There is $g \in \mathcal{G}l(H)$ such that $gpg^* = a$. Therefore $(T\Sigma_n)_a = g(T\Sigma_n)_p g^*$ and by the proposition above, $(T\Sigma_n)_p = \{z \in \mathcal{B}_2(H) : z^* = z, (1-p)z(1-p) = 0\}$. Then $(T\Sigma_n)_a = \{gzg^* \in \mathcal{B}_2(H) : z^* = z, (1-p)z(1-p) = 0\}$. Note that gzg^* is selfadjoint. Suppose that $\xi \in \ker(a) = \ker(gpg^*)$, then $g^*\xi \in \ker(gp) = \ker(p)$. Since $(1-p)z(1-p) = 0$, then z sends the kernel of p inside the range of p, i.e. $zg^*\xi \in Im(p)$. Then $gzg^*\xi \in Im(gp) = Im(gpg^*) = Im(a)$, it follows that gzg^* (which is an arbitrary element in $(T\Sigma_n)_a$) sends the kernel of a inside the range of a, i.e. $(1-\rho(a))gzg^*(1-\rho(a)) = 0$. Therefore we have

$$(T\Sigma_n)_a = \{z \in \mathcal{B}_2(H) : z^* = z, (1-\rho(a))z(1-\rho(a)) = 0\},$$

or in other words, it consists of the selfadjoint elements of $\mathcal{B}_2(H)$ whose 2×2 matrices in terms of $\rho(a)$ have 0 in the 2, 2 coordinate.

Remark 4.2 *1. The space $\mathcal{P}_p$ lies inside Σ_n, and it is a complemented submanifold. The inclusion $i : \mathcal{P}_p \hookrightarrow \Sigma_n$ induces an injection between the tangent spaces with complemented range. Indeed, if $q \in \mathcal{P}_p$, the image of di_q is $\{z \in \mathcal{B}_2(H) : z^* = z, pzp = (1-p)z(1-p) = 0\}$ which is complemented in $(T\Sigma_n)_q = \{z \in \mathcal{B}_2(H) : z^* = z,\ (1-p)z(1-p) = 0\}$.*

2. For a fixed $a \in \Sigma_n$, recall that $\Sigma_{\rho(a)}$ consist of all elements of Σ_n with range equal to the range of a. These are submanifolds of Σ_n: the tangent spaces are complemented subspaces of $(T\Sigma_n)_a$. If $b \in \Sigma_{\rho(a)}$, then $(T\Sigma_{\rho(a)})_b = \{z \in \mathcal{B}_2(H) : z^* = z,\ z = \rho(a)z\rho(a)\}$. In fact $\Sigma_{\rho(a)}$ lies inside $\rho(a)\mathcal{B}_2(H)\rho(a)$, therefore it is a finite dimensional manifold, which identifies with the space of complex positive invertible $n \times n$ matrices. Note that $\Sigma_{\rho(a)} \cap \mathcal{P}_p = \{\rho(a)\}$. We shall see that these two submanifolds are orthogonal in a natural sense.

Let us introduce a riemannian metric in Σ_n. If $a \in \Sigma_n$, denote by $\kappa(a) = 1 - \rho(a)$ (=projection onto the kernel of a). If $a \in \Sigma_n$ and $z_1, z_2 \in (T\Sigma_n)_a$,

$$< z_1, z_2 >_a = tr(z_1(a + \kappa(a))^{-1} z_2 (a + \kappa(a))^{-1}) = tr(z_1(a^\dagger + \kappa(a)) z_2 (a^\dagger + \kappa(a))). \tag{4.6}$$

Since $a \mapsto \rho(a)$ is a C^∞ mapping, it follows that the distribution 4.6 is smooth. It is a metric, i.e. it is positive definite: if $0 =< z, z >_a = tr(z(a + \kappa(a))^{-1} z(a + \kappa(a))^{-1}) = tr((a + \kappa(a))^{-1/2} z(a + \kappa(a))^{-1} z(a + \kappa(a))^{-1/2})$, then $0 = z(a + \kappa(a))^{-1} z = z(a + \kappa(a))^{-1/2}(a + \kappa(a))^{-1/2} z$, and therefore $(a + \kappa(a))^{-1/2} z = 0$, i.e. $z = 0$.

Remark 4.3 *It is straightforward to verify that if $u \in \mathcal{U}(H)$, then the map*

$$ad(u) : \Sigma_n \to \Sigma_n, \quad ad(u)(a) = uau^*$$

is an isometric diffeomorphism for this metric.

Note that if we regard $q \in \mathcal{P}_p$ as an element in Σ_n, then $q + \kappa(q) = 1$ and therefore if $z_1, z_2 \in (T\Sigma_n)_q$,

$$< z_1, z_2 >_q = tr(z_1 z_2).$$

In particular this inner product, restricted to the tangent spaces of $\mathcal{P}_p$, yields the same metric introduced in section 2 for $\mathcal{P}_p$.

On the other hand, if one restricts it to measure vectors z_1, z_2 in $(T\Sigma_{\rho(a)})_b$, this computation takes place in $\rho(a)\mathcal{B}_2(H)\rho(a) \simeq \mathcal{B}(\rho(a)(H))$. If we use the same letters to denote the operators b, z_1, z_2 in $\rho(a)(H)$, this measurement gives

$$< z_1, z_2 >_b = tr(z_1 b^{-1} z_2 b^{-1})$$

which is the well studied non positively curved metric of $M_n(\mathbb{C})$ [7], [20], [21].

Finally, note that with this metric, $\mathcal{P}_p$ is the normal submanifold of $\Sigma_{\rho(a)}$ at $\rho(a)$. That is, if $z_1 \in (T\mathcal{P}_p)_{\rho(a)}$ and $z_2 \in (T\Sigma_{\rho(a)})_{\rho(a)}$, then

$$< z_1, z_2 >_{\rho(a)} = tr(z_1 z_2) = 0$$

and

$$(T\mathcal{P}_p)_{\rho(a)} \oplus (T\Sigma_{\rho(a)})_{\rho(a)} = (T\Sigma_n)_{\rho(a)}.$$

The first assertion: $z_1 = \rho(a)z_1(1 - \rho(a)) + (1 - \rho(a))z_1\rho(a)$ and $z_2 = \rho(a)z_2\rho(a)$, then $tr(z_1 z_2) = tr((\rho(a)z_1(1 - \rho(a)) + (1 - \rho(a))z_1\rho(a))\rho(a)z_2\rho(a)) = tr((1 - \rho(a))z_1\rho(a))\rho(a)z_2\rho(a)) = 0$.

The second assertion: note that, regarded as 2×2 matrices in terms of $\rho(a)$, $(T\mathcal{P}_p)_{\rho(a)}$ consists of selfadjoint codiagonal matrices, $(T\Sigma_{\rho(a)})_{\rho(a)}$ consists of selfadjoint matrices with zeros except in the $1,1$ entry, and therefore these add all selfadjoint matrices with zero in the $2,2$ entry, which is precisely the space $(T\Sigma_n)_{\rho(a)}$.

We shall next study the riemannian connection induced by this metric. We shall see that $\mathcal{P}_p$ is curved in Σ_n, specifically, the geodesics of $\mathcal{P}_p$ are never geodesics in the ambient Σ_n (least they are constant curves). On the other hand $\Sigma_{\rho(a)}$ lies flat in Σ_n: the connection of Σ_n, when applied to tangent vector fields which lie in $T\Sigma_{\rho(a)}$ (a fixed), gives the connection of $\Sigma_{\rho(a)}$. In particular, the geodesic curves of $\Sigma_{\rho(a)}$ are geodesics in the ambient space Σ_n. The reason for this is that the orthogonal projection from $T\Sigma_n$ onto $T\Sigma_{\rho(a)}$ factors through the linear space $\rho(a)\mathcal{B}_2(H)\rho(a)$.

In order to obtain a formula for the connection, we need to compute the differential of the map $a\mapsto\rho(a)$. In what follows, if $x\in(T\Sigma_n)_a$, then

$$x = x_d + x_c$$

denotes the diagonal+codiagonal decomposition of x in terms of the projection $\rho(a)$, i.e. $x_d = \rho(a)x\rho(a)$ and $x_c = \rho(a)x\kappa(a) + \kappa(a)x\rho(a)$.

Proposition 4.4 *The differential of the map $a\mapsto\rho(a)$ is*

$$d\rho_a(x) = x_c a^\dagger + a^\dagger x_c, \quad x\in(T\Sigma_n)_a.$$

Proof. Decompose $x = x_d + x_c$. Note that $x_d\in(T\Sigma_{\rho(a)})_a$, therefore it can be realized as the tangent vector to a curve $\gamma(t)\in\Sigma_{\rho(a)}$, with $\gamma(0)=a$ and $\dot\gamma(0)=x_d$. It follows that $\rho(\gamma(t))=\rho(a)$, and therefore $0 = \dot{\rho(\gamma)}(t)|_{t=0} = d\rho_a(x_d)$. Therefore $d\rho_a(x) = d\rho_a(x_c)$. Consider the curve $\delta(t) = e^{tz}ae^{-tz}$, with $z = x_c a^\dagger - a^\dagger x_c$. Note that $z^* = -z$ and therefore $\delta(t)\in\Sigma_n$. Also $\delta(0)=a$ and

$$\dot\delta(0) = za - az = x_c a^\dagger a - a^\dagger x_c a - a x_c a^\dagger + a a^\dagger x_c = x_c\rho(a) + \rho(a)x_c = x_c,$$

which holds because x_c, being codiagonal with respect to $\rho(a)$, verifies $a^\dagger x_c a = a^\dagger\rho(a)x_c\rho(a)a = 0$, and $a x_c a^\dagger = 0$. Then $d\rho_a(x_c) = \dot{\rho(\delta)}(t)|_{t=0}$. If $w\in\mathcal{U}(H)$, then $\rho(waw^*) = w\rho(a)w^*$, therefore $\rho(\delta(t)) = \rho(e^{tz}ae^{-tz}) = e^{tz}\rho(a)e^{-tz}$. Then

$$\dot{\rho(\delta)}(t)|_{t=0} = z\rho(a) - \rho(a)z = x_c a^\dagger\rho(a) - a^\dagger x_c\rho(a) - \rho(a)x_c a^\dagger + \rho(a)a^\dagger x_c = x_c a^\dagger + a^\dagger x_c,$$

i.e. $d\rho_a(x) = x_c a^\dagger + a^\dagger x_c$ as claimed. □

It shall also be useful to compute the adjoint of

$$d\rho_a : (T\Sigma_n)_a \to (T\mathcal{P}_p)_{\rho(a)},$$

with respect to the corresponding inner products.

Remark 4.5 *The adjoint $d\rho_a^* : (T\mathcal{P}_p)_{\rho(a)} \to (T\Sigma_n)_a$ is the inclusion map*

$$d\rho_a^*(x) = x, \ x\in(T\mathcal{P}_p)_{\rho(a)}.$$

Indeed, if $y\in(T\Sigma_n)_a$, because x is codiagonal with respect to $\rho(a)$, one has $<x,y>_a = tr(y(a^\dagger+\kappa(a))x(a^\dagger+\kappa(a))) = tr(y(a^\dagger x\kappa(a)+\kappa(a)xa^\dagger))$. This equals $tr(y(a^\dagger x + xa^\dagger))$, because $a^\dagger x\kappa(a) + \kappa(a)xa^\dagger = a^\dagger x + xa^\dagger$. Then $<x,y>_a = tr(ya^\dagger x) + tr(yxa^\dagger) = tr(x(ya^\dagger + a^\dagger y) = <x, d\rho_a(y)>_{\rho(a)}$.

We use Koszul's formula to compute $\nabla_x y$. Recall that if x,y,z are vector fields then

$$2<\nabla_x y, z> = x<y,z> + y<z,x> - z<x,y> + <[x,y],z> + <[z,x],y> - <[y,z],x>.$$

Let us compute $x < y, z >$. In order to distinguish *derivation* of a field y with respect to x from the usual product of the operator valued functions x and y, we denote by $x\{y\}$ the former and by xy the latter. Derivations are performed at the point $a \in \Sigma_n$, which is omitted.

$$x < y, z >= tr(x\{y(a^\dagger + \kappa(a))z(a^\dagger + \kappa(a))\}) =$$

$$= tr(x\{y\}(a^\dagger + \kappa(a))z(a^\dagger + \kappa(a)) + yx\{a^\dagger + \kappa(a)\}z(a^\dagger + \kappa(a))+$$
$$+y(a^\dagger + \kappa(a))x\{z\} + y(a^\dagger + \kappa(a))zx\{a^\dagger + \kappa(a)\}).$$

Note that $x\{a^\dagger + \kappa(a)\} = x\{(a+\kappa(a))^{-1}\} = -(a^\dagger + \kappa(a))x\{a+\kappa(a)\}(a^\dagger + \kappa(a))$, and $x\{a+\kappa(a)\} = x\{a\} + x\{\kappa(a)\} = x + x\{1-\rho(a)\} = x - d\rho_a(x)$. The second and third term in Koszul's formula are dealt analogously. The other terms, e.g. $< [x, y], z >$, give

$$< [x, y], z >= tr(z(a^\dagger + \kappa(a))x\{y\}(a^\dagger + \kappa(a))) - tr(z(a^\dagger + \kappa(a))y\{x\}(a^\dagger + \kappa(a))).$$

After adding up all these formulas one gets:

$$2 < \nabla_x y, z >=< P_{\Sigma_n,a}\big(2x\{y\} - x(a^\dagger + \kappa(a))y - y(a^\dagger + \kappa(a))x + d\rho_a(x)(a^\dagger + \kappa(a))y+$$

$$+y(a^\dagger + \kappa(a))d\rho_a(x) + +d\rho_a(y)(a^\dagger + \kappa(a))x + x(a^\dagger + \kappa(a))d\rho_a(y)\big)-$$
$$-d\rho_a^*\big(P_{\mathcal{P}_p,q}[x(a^\dagger + \kappa(a))y + y(a^\dagger + \kappa(a))x]\big), z > . \tag{4.7}$$

Here $P_{\Sigma_n,a}$ denotes the projection from the space $\mathcal{B}_2(H)_h$ of hermitic operators in $\mathcal{B}_2(H)$, onto the tangent space $(T\Sigma_n)_a$, which is given by

$$P_{\Sigma_n,a}(x) = x - (1-\rho(a))x(1-\rho(a)).$$

We by $P_{\mathcal{P}_p,q}$ the projection from $\mathcal{B}_2(H)_h$ onto the tangent space $(T\mathcal{P}_p)_q$, given by

$$P_{\mathcal{P}_p,q}(x) = qx(1-q) + (1-q)xq.$$

These two projections are orthogonal with respect to the inner product $< \, , \, >_a$, when this form is extended to the whole $\mathcal{B}_2(H)$. Using these abreviations, and the computations above, one obtains

$$2\nabla_x y = P_{\Sigma_n,a}\big(2x\{y\} - x(a^\dagger + \kappa(a))y - y(a^\dagger + \kappa(a))x + (x_c a^\dagger + a^\dagger x_c)(a^\dagger + \kappa(a))y+$$

$$+y(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c) + x(a^\dagger + \kappa(a))(y_c a^\dagger + a^\dagger y_c) + (y_c a^\dagger + a^\dagger y_c)(a^\dagger + \kappa(a))x\big)-$$
$$-P_{\mathcal{P}_p,q}\big(x(a^\dagger + \kappa(a))y + y(a^\dagger + \kappa(a))x\big). \tag{4.8}$$

In particular, the equation for the geodesic curves of Σ_n is

$$0 = P_{\Sigma_n,\gamma}\big(\ddot\gamma - \dot\gamma(\gamma^\dagger + \kappa(\gamma))\dot\gamma + (\dot\gamma_c\gamma^\dagger + \gamma^\dagger\dot\gamma_c)(\gamma^\dagger + \kappa(\gamma))\dot\gamma + \dot\gamma(\gamma^\dagger + \kappa(\gamma))(\dot\gamma_c\gamma^\dagger + \gamma^\dagger\dot\gamma_c)\big)-$$
$$-P_{\mathcal{P}_p,\rho(\gamma)}\big(\dot\gamma(\gamma^\dagger + \kappa(\gamma))\dot\gamma\big). \tag{4.9}$$

We were not able to compute the geodesics of this connection. As was remarked before, this metric induces on the submanifolds $\mathcal{P}_p$ and Σ_q the usual connections on these spaces. The properties of the connection on $\mathcal{P}_p$ were recalled in section 2. If one identifies Σ_q with the space of positive invertible operators in the range of q, i.e. $n \times n$ positive definite matrices, the connection induced on Σ_q is well known in differential geometry [7], [20]. It has non positive sectional curvature, and the geodesics, as well as the geodesic distance, have been characterized. If $a, b \in \Sigma_q$, then there is a unique minimizing geodesic which joins them, given by

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}, \ t \in \mathbb{R}.$$

Here inverses are taken as operators in the range of q. The geodesic distance is given by

$$d_g(a,b) = (tr\big((log(a^{-1/2}ba^{-1/2})^2)\big))^{1/2}.$$

Note that Σ_q (being finite dimensional) is a complete metric space.

These submanifolds $\mathcal{P}_p$ and Σ_q ($q \in \mathcal{P}_p$) sit inside Σ_n in quite different manners:

Proposition 4.6 *The geodesic curves of $\mathcal{P}_p$ are never geodesic curves of Σ_n (except if they are constant curves). The geodesic curves of Σ_q are geodesics of Σ_n, and in particular Σ_q is totally geodesic in Σ_n.*

Proof. The second assertion follows from the fact that $\Sigma_q \subset q\mathcal{B}_2(H)q$, and $\Sigma_q = \Sigma_n \cap q\mathcal{B}_2(H)q$. Note that $q\mathcal{B}_2(H)q$ is a linear and complemented subspace of $\mathcal{B}_2(H)$. Indeed, if x and y are vector fields in Σ_n which happen to take values in $T\Sigma_q$, then $x\{y\}$ as well as all terms in 4.8 take values in $q\mathcal{B}_2(H)q$. It follows that the riemannian connection $\nabla_x^{\Sigma_q} y$ induced by the metric on Σ_q coincides with $\nabla_x y$ of the ambient space Σ_n.

With respect to the first assertion, let x be a q-codiagonal (antihermitian) vector, and consider the geodesic $\delta(t) = e^{tx}qe^{-tx}$ of $\mathcal{P}_p$. Straightforward computations show that

1. $\dot{\gamma} = e^{tx}(xq - qx)e^{-tx}$.
2. $\ddot{\gamma} = e^{tx}(x^2q - 2xqx + qx^2)e^{tx}$.
3. $\dot{\gamma}_c = \dot{\gamma}$.
4. $\gamma^\dagger + \kappa(\gamma) = 1$.

Note that x^2 and $(xq - qx)^2$ commute with q. Note also that $P_{\Sigma_n, waw^*}(wyw^*) = wP_{\Sigma_n,a}(y)w^*$. Lets replace these relations in the geodesic equation 4.9. First,

$$\dot{\gamma}(\gamma^\dagger + \kappa(\gamma))\dot{\gamma} = e^{tx}(xq - qx)^2 e^{-tx}$$

which commutes with $\gamma = e^{tx}qe^{-tx}$. It follows that

$$-P_{\mathcal{P}_p,\rho(\gamma)}\big(\dot{\gamma}(\gamma^\dagger + \kappa(\gamma))\dot{\gamma}\big) = 0.$$

Using the covariance of the projections $P_{\Sigma_n,a}$ with respect to inner automorphisms remarked above, in order that γ be a geodesic, the projection onto $(T\Sigma_n)_q$ of the term

$$x^2q - 2xqx + qx^2 + (xq - qx)^2 = x^2q - 3xqx$$

must vanish. That is

$$P_{\mathcal{P}_p,q}(x^2q - 3xqx) = x^2q - 3xqx - (1-q)x^2q(1-q) + 3(1-q)xqx(1-q) = qx^2q = 0.$$

That is, $qx = xq = 0$. Since x is q-codiagonal, $x = xq + qx$, and therefore $x = 0$, i.e. γ is constant. $\square$

5 Homotopy of Σ_n

Consider the following map

$$\varpi : \Sigma_p \times \mathcal{I}_p \to \Sigma_n, \quad \varpi(b,x) = xbx^*.$$

Clearly this map is smooth and surjective. Let us compute the fibre over p. If $xbx^* = p$, then the range of xbx^* is contained in $p(H)$. Since xbx^* is one to one when restricted to $p(H)$, this implies that the range of x is $p(H)$, i.e. x is a "unitary" operator in $p(H)$. Therefore $\varpi^{-1}(p) = \{(v,p) : v^*v = vv^* = p\} \simeq \mathcal{U}(p(H))$. We identify $\mathcal{U}(p(H))$ with this fibre. Note that $\mathcal{U}(p(H))$ acts both on Σ_p and $\mathcal{I}_p$ with right actions: $b \cdot v = v^*bv$ and $x \cdot v = xv$. We may consider the diagonal action on $\Sigma_p \times \mathcal{I}_p$: $(b,x) \cdot v = (v^*bv, xv)$. With this action one has $\varpi((b,x)\cdot v) = \varpi(b,x)$. Moreover,

$$\varpi(b,x) = \varpi(c,y)$$

only if there exists $v \in \mathcal{U}(p(H))$ such that

$$(b,x) \cdot v = (c,y).$$

Indeed, $xbx^* = ycy^*$ implies that $c = pcp = y^*ycy^*y = y^*xbx^*y = (x^*y)^*bx^*y$. Clearly, as above, from $xbx^* = ycy^*$ it follows that x and y have the same range, and since they are partial isometries, this implies that x^*y is a partial isometry with initial and final space p, i.e. an element in $\mathcal{U}(p(H))$. Therefore, if we denote $v = x^*y$, $(b,x)\cdot v = (v^*bv, xv) = (c, xx^*y) = (c,y)$, where $xx^*y = y$ because xx^* equals the projection onto the range of x (which is equal to the range of y).

Proposition 5.1 *The map ϖ is a principal bundle with structure group $\mathcal{U}(p(H))$.*

Proof. Let us show that ϖ has local cross sections. Fix $a_0 = x_0b_0x_0^* \in \Sigma_n$. Let $\rho(a_0) = p_0$ and $u_0 \in \mathcal{U}(H)$ such that $u_0pu_0^* = p_0$. Since ρ is continuous, it follows that the set $\mathcal{D}_{a_0} = \{a \in \Sigma_n : \|\rho(a) - p_0\| < 1\}$ is open in Σ_n. It follows that $\rho(a)$ lies in the domain of the local cross section σ_{p_0} of the unitary orbit of p_0 (see section 2). Therefore $u_0\sigma_{p_0}(\rho(a))$ satisfies that

$$(u_0\sigma_{p_0}(\rho(a)))^*\rho(a)u_0\sigma_{p_0}(\rho(a)) = p,$$

in other words, $(u_0\sigma_{p_0}(\rho(a)))^*au_0\sigma_{p_0}(\rho(a))$ lies in Σ_p. Let us define

$$s_{a_0} : \mathcal{D}_{a_0} \to \Sigma_p \times \mathcal{I}_p, \quad s_{a_0}(a) = ((u_0\sigma_{p_0}(\rho(a)))^*au_0\sigma_{p_0}(\rho(a)), u_0\sigma_{p_0}(\rho(a))p).$$

It is straightforward to verify that $s_{a_0}(a_0) = (a_0, p_0)$ and that $\varpi(s_{a_0}(a)) = a$. Clearly s_{a_0} is smooth. Now if $(b,x) \in \varpi^{-1}(\mathcal{D}_{a_0})$, then $s_{a_0}(\varpi(xbx^*)) = (\alpha(xbx^*), \chi(xbx^*))$. Note that $\chi(xbx^*)^*x$ lies in $\mathcal{U}(p(H))$. Put

$$\Phi : \varpi^{-1}(\mathcal{D}_{a_0}) \to \mathcal{D}_{a_0} \times \mathcal{U}(p(H)), \quad \Phi(b,x) = (xbx^*, \chi(xbx^*)^*x).$$

Then Φ is a smooth local trivialization for the map ϖ, which additionally satisfies that if $v \in \mathcal{U}(p(H))$, then $\Phi((b,x)\cdot v) = (xbx^*, \chi(xbx^*)^*xv)$, i.e. it is an equivariant trivialization. $\square$

Remark 5.2 *In* [1] *it was proven that the map*

$$\mathcal{U}(H) \to \mathcal{I}_p, \quad u \mapsto up$$

is a principal bundle (in fact, a homogeneous space), with fibre equal to $\mathcal{U}(p(H)^\perp)$. Since $p(H)$ is finite dimensional, and $\mathcal{I}_p$ is a connected differentiable manifold [1]*, these facts imply* [12], [14] *that $\mathcal{I}_p$ is contractible.*

Corollary 5.3 *For $k \geq 1$, $\pi_k(\Sigma_n) \simeq \pi_{k-1}(\mathcal{U}(n))$, where $\mathcal{U}(n)$ is the unitary group of $\mathbb{C}^n$.*

Proof This follows from the fact that $\varpi : \Sigma_p \times \mathcal{I}_p \to \Sigma_n$ is a fibration, with fibre $\mathcal{U}(p(H))$, where $\mathcal{I}_p$ is contractible and Σ_p is convex. $\square$

Moreover, Σ_n is $B(\mathcal{U}(n))$, the classifying space of $\mathcal{U}(n)$ which is unique up to homotopy equivalence. Thus it is homotopy equivalent to the Grassmann manifold of n-planes in H.

There are another interesting subsets of Σ_n, the "tubes"

$$\mathcal{T}_q = \{a \in \Sigma_n : \|\rho(a) - q\| < 1\}.$$

Note that $\mathcal{T}_q$ is an open neighbourhood of Σ_q. Let us define the following map

$$\tau = \tau_q : \mathcal{T}_q \to \Sigma_q, \quad \tau(b) = \vartheta_q(\rho(b))^* b \vartheta_q(\rho(b)),$$

where ϑ_q is the cross section for the unitary orbit of q, defined in 2.3. Note that $\vartheta_q(\rho(b))^*\rho(b)\vartheta_q(\rho(b)) = q$, and therefore τ is well defined. Clearly it is also smooth.

Proposition 5.4 *The map τ is an homotopy equivalence between $\mathcal{T}_q$ and Σ_q. In particular this implies that $\mathcal{T}_q$ is contractible.*

Proof. It is in fact a deformation retract. The cross section ϑ_q takes values in the domain of a smooth logarithm [6]: $\sigma_q(r) = e^{\zeta(r)}$, where ζ is a smooth function with values in $\mathcal{B}(H)_{ah}$, defined on the neighbourhood of q in $\mathcal{P}_p$ given by $\|q - r\| < 1$, with $\zeta(q) = 0$. For $t \in [0, 1]$, let $F_t(b) = e^{-t\zeta(\rho(b))} b e^{t\zeta(\rho(b))}$. Then $F_0 = id$, $F_1 = \tau$ and if $b \in \Sigma_q$, $\rho(b) = q$ and then $F_t(b) = b$. □

Remark 5.5 *In particular $\tau_q : \mathcal{T}_q \to \Sigma_q$ is a smooth retraction. Therefore for each $a \in \Sigma_q$, the set $\mathcal{N}_a = \tau^{-1}(\{a\})$ is a submanifold of Σ_n.*

Proposition 5.6 *For each $a \in \Sigma_q$, $\mathcal{N}_a$ and Σ_q are normal at a.*

Proof. We claim that if $x \in (T\mathcal{N}_a)_a$, then x is codiagonal with respect to q. Let $\gamma(t)$ be a curve in $\mathcal{N}_a$ with $\gamma(0) = a$ and $\dot\gamma(0) = x$. Then $\tau(\gamma(t)) = q$, so that $d\tau_a(x) = 0$. Now $d\tau_a(x) = [d(\vartheta_q))_q(d\rho_a(x))]^* a + x + ad(\vartheta_q))_q(d\rho_a(x))$. Note that since ϑ_q takes unitary values, then $[d(\vartheta_q))_q(d\rho_a(x))]^* = -d(\vartheta_q))_q(d\rho_a(x))$. Moreover (see [6]), if z is a tangent vector of $\mathcal{P}_p$ at q, then $d(\vartheta_q)_q(z) = zq - qz$. Therefore

$$d\tau_a(x) = -[d(\vartheta_q)_q(x_c a^\dagger + a^\dagger x_c)]a + x + a[d(\vartheta_q)_q(x_c a^\dagger + a^\dagger x_c)] = -(x_c a^\dagger - a^\dagger x_c)a + x + (x_c a^\dagger - a^\dagger x_c)a.$$

Recall that $a^\dagger x_c a = 0$. Therefore,

$$0 = d\tau_a(x) = -x_c q + x - q x_c = x - x_c,$$

i.e. $x = x_c$ is q-codiagonal. A tangent vector $y \in (T\Sigma_q)_q$ verifies $y = qyq$. It follows that

$$< x, y >_a = tr(x(a^\dagger + \kappa(a))y(a^\dagger + \kappa(a))) = 0$$

because x is q-codiagonal and $(a^\dagger + \kappa(a))y(a^\dagger + \kappa(a))$ is q-diagonal. □

If q_0, q_1 are projections in Σ_n such that $\|q_0 - q_1\| < 1$, then τ_{q_0} induces an isometric diffeomorphism between Σ_{q_1} and Σ_{q_0}. Indeed, if $b \in \Sigma_{q_1}$, $\tau_{q_0}(b) = w^* b w$, where $w = \vartheta_{q_0}(q_1)$ does not depend on b.

6 The embedding $\Sigma_q \to \Sigma_\infty$

In [3] we studied the space Σ_∞ of positive definite infinite matrices. Let us recall this space and its properties. Let $\mathcal{H}_{\mathbb{R}}$ be the real Hilbert space of operators in $\mathcal{B}(H)$ given by

$$\mathcal{H}_{\mathbb{R}} = \{\lambda + x \in \mathcal{B}(H) : \lambda \in \mathbb{R}, x = x^* \in \mathcal{B}_2(H)\},$$

with the inner product $< \lambda + x, \mu + y >= \lambda\mu + tr(xy)$. We define

$$\Sigma_\infty = \{a \in \mathcal{H}_{\mathbb{R}} : a \text{ is positive and invertible in } \mathcal{B}(H)\}.$$

Remark 6.1 [3]

1. *Σ_∞ is open in $\mathcal{H}_{\mathbb{R}}$.*
2. *If $a \in \Sigma_\infty$, and $x, y \in \mathcal{H}_{\mathbb{R}} \simeq (T\Sigma_\infty)_a$, then $< x, y >_a =< xa^{-1}, a^{-1}y >$ defines a riemannian metric for Σ_∞. With this metric, Σ_∞ has non positive curvature. In particular, any two points a, b in Σ_∞ are joined by a unique and minimizing geodesic, which is given by*

$$\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}, \ t \in \mathbb{R}.$$

3. *The geodesic metric is given by*

$$d_g(a, b) = \{< x, x >\}^{1/2},$$

where $x = log(a^{-1/2}ba^{-1/2})$. Σ_∞ is complete with this metric.

Let us consider the map

$$\jmath : \Sigma_n \to \Sigma_\infty, \ \jmath(a) = a + \kappa(a).$$

This map $\jmath$ is well defined, because $a + \kappa(a)$ is a finite rank perturbation of the identity, and also a positive and invertible operator. Clearly $\jmath$ is smooth. Indeed, Σ_n lies inside $\mathcal{H}_{\mathbb{R}}$, and the Hilbert space norm of $\mathcal{H}_{\mathbb{R}}$ restricted to Σ_n is the Hilbert Schmidt norm $\| \ \|_2$. The map $\rho : \Sigma_n \to \Sigma_n \subset \mathcal{H}_{\mathbb{R}}$ is therefore smooth as a map with values in $\mathcal{H}_{\mathbb{R}}$, and then so is $\kappa = 1 - \rho$.

Proposition 6.2 *Let $a \in \Sigma_n$. The differential*

$$d\jmath_a : (T\Sigma_n)_a \to (T\Sigma_\infty)_{\jmath(a)}$$

is contractive, i.e. verifies that $\|d\jmath_a(x)\|_{\jmath(a)} \leq \|x\|_a$ for all $x \in (T\Sigma_n)_a$, if and only if $\|a^\dagger\| \leq 2$ (or equivalently, $a \geq \frac{1}{2}\rho(a)$).

Proof. Let $x \in (T\Sigma_n)_a$. Compute $d\jmath_a(x) = x + d\kappa_a(x) = x - d\rho_a(x) = x - (x_c a^\dagger + a^\dagger x_c)$. Decompose $x = x_d + x_c$. Note that $(a + \kappa(a))^{-1}x_d$ is $\rho(a)$-diagonal while $(a + \kappa(a))^{-1}(x_c a^\dagger + a^\dagger x_c)$ is $\rho(a)$-codiagonal. It follows that they are orthogonal for the inner product of $\mathcal{H}_{\mathbb{R}}$ (which coincides with the trace inner product for these vectors). Then

$$\|x - (x_c a^\dagger + a^\dagger x_c)\|^2_{\jmath(a)} = \|x\|^2_{\jmath(a)} + \|x_c a^\dagger + a^\dagger x_c\|^2_{\jmath(a)} - 2tr((a^\dagger + \kappa(a))x_c(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c)).$$

The norm $\|x\|_{\jmath(a)}$ coincides with the norm of x as an element of $(T\Sigma_n)_a$. Therefore, in order to characterize when $d\jmath_a$ is contractive, it is necessary and sufficient to characterize when

$$\|x_c a^\dagger + a^\dagger x_c\|^2_{\jmath(a)} - 2Re \ tr((a^\dagger + \kappa(a))x_c(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c)) \leq 0. \tag{6.10}$$

Compute

$$(x_c a^\dagger + a^\dagger x_c)(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c)(a^\dagger + \kappa(a)) = x_c(a^\dagger)^3 x_c\kappa(a) + a^\dagger x_c\kappa(a)x_c(a^\dagger)^2.$$

This implies that

$$\|x_c a^\dagger + a^\dagger x_c\|^2_{\jmath(a)} = tr(x_c (a^\dagger)^3 x_c \kappa(a) + a^\dagger x_c \kappa(a) x_c (a^\dagger)^2) = 2tr(\kappa(a) x_c (a^\dagger)^3 x_c \kappa(a)).$$

On the other hand

$$x_c(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c)(a^\dagger + \kappa(a)) = x_c (a^\dagger)^2 x_c \kappa(a) + x_c \kappa(a) x_c (a^\dagger)^2.$$

Then

$$-2Re\ tr(x_c(a^\dagger + \kappa(a))(x_c a^\dagger + a^\dagger x_c)(a^\dagger + \kappa(a))) = -4tr(\kappa(a) x_c (a^\dagger)^2 x_c \kappa(a)).$$

Suppose now that $\|a^\dagger\| \le 2$. This implies that $a^\dagger \le 2\rho(a)$, and then $(a^\dagger)^3 \le 2(a^\dagger)^2$. This clearly implies

$$2\kappa(a) x_c (a^\dagger)^3 x_c \kappa(a) \le 4\kappa(a) x_c (a^\dagger)^2 x_c \kappa(a).$$

Taking traces on this inequality proves 6.10.

Conversely, suppose 6.10 holds. Note that $\rho(a) x_c (a^\dagger)^i = (a^\dagger)^i x_c \rho(a) = 0$, because x_c is $\rho(a)$-codiagonal. Then 6.10 implies that

$$tr(x_c (a^\dagger)^3 x_c) \le 2tr(x_c (a^\dagger)^2 x_c).$$

Let $x^* = x \in \mathcal{B}(H)$, then $(1-\rho(a))x\rho(a) + \rho(a)x(1-\rho(a))$ is $\rho(a)$-codiagonal and selfadjoint, if we put it in the place of x_c in the above inequality, we obtain that

$$tr((1-\rho(a))x(a^\dagger)^3 x(1-\rho(a))) \le 2tr((1-\rho(a))x(a^\dagger)^2 x(1-\rho(a))).$$

Since this happens for any selfadjoint operator x, it implies that $(a^\dagger)^3 \le 2(a^\dagger)^2$, which multiplying by a on both sides implies that $a^\dagger \le 2\rho(a)$, or equivalently, $\|a^\dagger\| \le 2$. $\square$

The map $\jmath$ is clearly not injective. For instance, for all $q \in \mathcal{P}_p$, $\jmath(q) = 1$. However, $\jmath$ is injective when restricted to the submanifolds Σ_q, in fact, in these cases it gives the natural way to embed Σ_q in Σ_∞.

Proposition 6.3 *For $q \in \mathcal{P}_p$, the map $\jmath|_{\Sigma_q}$ is an isometric embedding. In particular, if γ is a geodesic in Σ_q, then $\jmath(\gamma)$ is a geodesic in Σ_∞, and the length of curves is preserved under $\jmath$.*

Proof. If $x \in \Sigma_q$, $\jmath(x) = x + (1-q)$, and therefore $\jmath|_{\Sigma_q}$ is injective. Moreover, apparently $d\jmath_x(z) = z$ for all $z \in (T\Sigma_q)_x$. Note that $\{z : z \in (T\Sigma_q)_x\}$ is closed in $(T\Sigma_\infty)_{\jmath(x)} = \mathcal{H}_{\mathbb{R}}$. $\square$

Let us define, for $d > 0$,

$$\mathcal{W}_d = \{a \in \Sigma_n : \|a^\dagger\| < d\}$$

Clearly these sets are open in Σ_n. Also note that $\mathcal{P}_p \subset \mathcal{W}_d$ is $d > 1$.

Remark 6.4 *The sets $\mathcal{W}_d$ have also the following convexity properties:*

1. *If $a \in \mathcal{W}_d$ and $u \in \mathcal{U}(H)$, then $uau^* \in \mathcal{W}_d$. The proof is straightforward.*
2. *If $a, b \in \Sigma_q$ lie inside $\mathcal{W}_d$, then the unique geodesic $\gamma_{a,b}(t)$ in Σ_q joining them lies inside $\mathcal{W}_d$ for $t \in [0,1]$. This is a consequence of a geometric form of the Loewner-Heinz inequality* [8]*, namely: if g, h are positive invertible elements in a C^*-algebra, then for all $t \in [0,1]$,*

   $$\|g^{1/2}(g^{-1/2} h g^{-1/2})^t g^{1/2}\| \le \|g\|^{1-t} \|h\|^t.$$

   Using this inequality for $g = a^{-1}$ and $h = b^{-1}$ in the C^*-algebra $\mathcal{B}(q(H))$, and the fact that $a^\dagger$ (resp. $b^\dagger$) identifies with a^{-1} (resp. b^{-1}), one obtains that

   $$\|\gamma^\dagger_{a,b}(t)\| = \|\gamma_{a^{-1},b^{-1}}(t)\| \le \|a^\dagger\|^{1-t} \|b^\dagger\|^t < d^{1-t} d^t = d,$$

   for $t \in [0,1]$.

Corollary 6.5 *Let $a, b \in \Sigma_q \cap \mathcal{W}_2$. Then the (unique) geodesic $\gamma_{a,b}(t) = a^{1/2}(a^{-1/2}ba^{-1/2})^t a^{1/2}$ joining a, b in Σ_q (and in Σ_n) is shorter than any other curve in $\mathcal{W}_2$*

Proof. Let $\nu(t) \in \mathcal{W}_2$ be a (piecewise) smooth curve such that $\nu(0) = a$ and $\nu(1) = b$. By proposition 6.2, $\jmath(\nu)$ is shorter than ν. This curve $\jmath(\nu)$ joins $a + 1 - q$ with $b + 1 - q$ in Σ_∞. It is then longer than the geodesic in Σ_∞ joining the same endpoints, which is $\jmath(\gamma_{a,b})$ and has the same length as $\gamma_{a,b}$. □

Next we examine the behaviour of the map ρ in the sets $\mathcal{W}_d$.

Proposition 6.6 *If $a \in \mathcal{W}_d$, then for all $x \in (T\Sigma_n)_a$,*

$$\|d\rho_a(x)\|_{\rho(a)} \leq \sqrt{d}\ \|x\|_a.$$

Proof. Compute

$$\|d\rho_a(x)\|^2_{\rho(a)} = tr((x_c a^\dagger + a^\dagger x_c)^2) = 2tr(x_c (a^\dagger)^2 x_c).$$

On the other hand

$$\|x\|^2_a = tr(x_d a^\dagger x_d a^\dagger) + 2tr(\kappa(a) x_c a^\dagger x_c \kappa(a)) = tr(x_d a^\dagger x_d a^\dagger) + 2tr(x_c a^\dagger x_c),$$

where the last equality follows because $tr(\rho(a) x_c a^\dagger x_c \kappa(a)) = tr(\rho(a) x_c a^\dagger x_c \rho(a)) = 0$. Since $a \in \mathcal{W}_d$,

$$(a^\dagger)^2 \leq d a^\dagger,$$

and therefore (note that $tr(x_d a^\dagger x_d a^\dagger) \geq 0$)

$$\|d\rho_a(x)\|^2_{\rho(a)} \leq d\ \|x\|^2_a.$$

□

Corollary 6.7 *Let a and b in $\mathcal{W}_d$. then*

$$d_g(\rho(a), \rho(b)) \leq \sqrt{d}\ d_g(a, b),$$

where the term on the left denotes the geodesic distance in the submanifold $\mathcal{P}_p$ and the term on the right denotes the geodesic distance in Σ_n.

Lemma 6.8 *If $\{a_k\}$ is a Cauchy sequence in Σ_n, then the norms $\|a_k^\dagger\|$ are uniformly bounded.*

Proof. Suppose that the norms $\|a_k^\dagger\|$ are not bounded, then there exists a subsequence such that the norms tend to infinity. For simplicity of notation, let us suppose that $\|a_k^\dagger\| \to \infty$. Let u_k be unitaries such that $u_k p u_k^* = \rho(a_k)$. Then $b_k = u_k^* a_k u_k \in \Sigma_p$. Note that because unitary conjugation is isometric for this metric (Remark 4.3),

$$d_g(b_k, p) = d_g(a_k, \rho(a_k)) \leq d_g(a_k, p) + d_g(p, \rho(a_k)).$$

The terms $d_g(a_k, p)$ are bounded because $\{a_k\}$ is a Cauchy sequence. On the other hand $d_g(p, \rho(a_k))$ is bounded by the rectifiable diameter of $\mathcal{P}_p$, which is finite (see [15]). It follows that $\{b_k\}$ is a bounded sequence in Σ_p, in fact in $\mathcal{B}(p(H))$, which is finite dimensional. Then there exists a subsequence $\{b_{k_j}\}$ which converges to b in Σ_p. Then $b^\dagger_{k_j} \to b^\dagger$, and in particular $\|a^\dagger_{k_j}\| = \|b^\dagger_{k_j}\|$ is bounded, which is a contradiction. □

Our main result in this section is the following.

Theorem 6.9 *The space Σ_n is complete in the geodesic distance.*

Proof. Let $\{a_k\}$ be a Cauchy sequence in Σ_n for the geodesic distance. By virtue of the above lemma, we have that there exists $d > 0$ such that $a_k \in \mathcal{W}_d$ for all k. Let us suppose first that $d < 2$. The result (6.2) on $\mathcal{W}_2$ implies that $\{\jmath(a_k)\}$ is a Cauchy sequence for the geodesic distance in Σ_∞, which is complete. Therefore $\{\jmath(a_k)\}$ converges to $b \in \Sigma_\infty$ in the geodesic distance, as well as in the norm of $\mathcal{H}_{\mathbb{R}}$. On the other hand, corollary 6.7 implies that $\{\rho(a_k)\}$ is a Cauchy sequence in $\mathcal{P}_p$, which is also complete. Therefore $\rho(a_k) \to q \in \mathcal{P}_p$, in the geodesic distance, which is equivalent to the usual operator norm of $\mathcal{B}(H)$. It follows that there exists k_0 such that for all $k \geq k_0$, $\|\rho(a_k) - q\| < 1$. This implies, using one of the continuous local cross sections of π_q in section 2, that there exist unitary operators $u_k \to 1$ (in norm) such that $u_k q u_k^* = \rho(a_k)$. Denote by $a_k' = u_k^* a_k u_k$. Then $\rho(a_k') = q$. Note that $\rho(a_k) \to q$ in $\mathcal{P}_p$ implies that $\kappa(a_k) = 1 - \rho(a_k) \to 1 - q$ in Σ_∞. It follows that $a_k = \jmath(a_k) - \kappa(a_k) \to b - (1 - q) = b_0$ in Σ_∞. Since these elements belong to $\mathcal{B}_2(H)$, then $a_k \to b_0$ in $\| \ \|_2$. It follows that $a_k' \to b_0$, because $u_k \to 1$. We have shown above that $\rho(a_k') = q$, then $\rho(b_0) \leq q$. We claim that in fact $b_0 \in \Sigma_n$, i.e. $\rho(b_0) = q$. To this effect, note that $\{a_k'\}$ is a Cauchy sequence in Σ_n. Indeed

$$d_g(a_k', a_l') \leq d_g(a_k', a_k) + d_g(a_k, a_l) + d_g(a_l, a_l').$$

It suffices to prove that $d_g(a_k, a_k') \to 0$. Recall that $a_k' = u_k^* a_k u_k$, where $u_k = e^{x_k}$ for some antihermitic x_k of finite rank. Then $d_g(a_k, a_k') \leq length(\delta_k)$, where δ_k is the curve $\delta_k(t) = e^{-tx_k} a_k e^{tx_k}$, $t \in [0, 1]$. The length of this curve δ_k equals

$$\sqrt{2}\ tr(x_k a_k x_k^*)^{1/2} = \sqrt{2}\ tr(a_k^{1/2} x_k^* x_k a_k^{1/2})^{1/2} \leq \sqrt{2} \|x_k\| tr(a_k)^{1/2}.$$

Since $u_k \to 1$, the elements x_k converge to 0 in the usual operator norm. Then it suffices to show that the traces $tr(a_k)$ are bounded. We have shown that both $\{a_k\}$ and $\{\rho(a_k)\}$ are convergent in the trace norm of $\mathcal{B}_2(H)$. Then $tr(a_k) =< a_k, \rho(a_k) >$ is a bounded sequence, and our claim is proven. Then $\{a_k'\}$ is a Cauchy sequence in Σ_n, in fact it is a sequence in $\Sigma_q \cap \mathcal{W}_2$, and therefore, by 6.5, a Cauchy sequence in Σ_q. Since Σ_q is complete, it follows that $a_k' \to b_0$ in the geodesic distance of Σ_q, and therefore $b_0 \in \Sigma_q \subset \Sigma_n$. Then $a_k' \to b_0$ in the geodesic distance of Σ_n. Using

$$d_g(a_k, b_0) \leq d_g(a_k, a_k') + d_g(a_k', b_0),$$

and the computations above, one has that $a_k \to b_0$ in the geodesic distance of Σ_n.

It remains to prove the result for arbitrary $d \geq 2$. Let now $\{a_k\}$ be a Cauchy sequence (for the geodesic distance of Σ_n) lying in $\mathcal{W}_d$ with $d \geq 2$. It follows that $\{\frac{d}{2} a_k\}$ lies in $\mathcal{W}_2$. We claim that $\{\frac{d}{2} a_k\}$ is also a Cauchy sequence for the geodesic distance. If γ is a curve in Σ_n, and $\frac{d}{2} = c \geq 1$, then $c\gamma$ is another curve in Σ_n with $length(c\gamma) \leq \sqrt{c}\ length(\gamma)$. Let us prove this assertion, which implies our claim and finishes the proof. Note that

$$\|c\dot\gamma\|_{c\gamma^2} = tr\big(c\dot\gamma(\frac{\gamma^\dagger}{c} + \kappa(c\gamma))c\dot\gamma(\frac{\gamma^\dagger}{c} + \kappa(c\gamma))\big) = tr(\dot\gamma\gamma^\dagger\dot\gamma\gamma^\dagger) + 2c\ tr(\kappa(\gamma)\dot\gamma\gamma^\dagger\dot\gamma\kappa(\gamma))$$

$$\leq c\{tr(\dot\gamma\gamma^\dagger\dot\gamma\gamma^\dagger) + 2tr(\kappa(\gamma)\dot\gamma\gamma^\dagger\dot\gamma)\},$$

because $tr(\dot\gamma\gamma^\dagger\dot\gamma\gamma^\dagger) \geq 0$ and $c \geq 1$. That is,

$$\|c\dot\gamma\|_\gamma^2 \leq c\|\dot\gamma\|_\gamma^2,$$

which proves the assertion. $\square$

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