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CHARACTERIZATION OF DYNAMIC BIFURCATIONS IN THE FREQUENCY DOMAIN

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In this paper dynamical systems with certain degenerate Hopf bifurcations are considered. An analysis of the bifurcation behavior is proposed using several tools from the frequency domain approach. The analyzed bifurcations are the building blocks to understand the multiplicity of Hopf bifurcation points and to propose certain strategies in the future for controlling the bifurcation behavior in nonlinear systems.

1. Introduction

The bifurcation structure of dynamical systems can be described qualitatively by using the variation of several control parameters. The defining conditions and nondegeneracy conditions which typify the singularities are not only important for the proper design of engineering systems but also for their control. In this regard, it is interesting to mention the classification of static singularities in chemical reacting systems by using the singularity theory by Alhumaizi and Aris [1995], Balakotaiah and Luss [1984], and Farr [1986], and later extended to cover dynamic (or Hopf) bifurcation degeneracies in [Alhumaizi & Aris, 1995; Byeon & Chung, 1989; Farr, 1986; Planeaux, 1993]. In the field of biology, the applications of the singularity theory in dynamic bifurcations are recognized in the early works of Hassard and Shiau [1989] and Shiau and Hassard [1991]. Other more recent applications dealing with degenerate Hopf bifurcations are the contributions of Fukai *et al.* [2000a, 2000b] and Xu *et al.* [1998], to mention some of the most sophisticated ones.

In this article, the applied methodology comes from the theory of multivariable control systems, known as the frequency domain method, and intends to complement previous results [Itovich & Moiola, 2001; Moiola & Chen, 1993] regarding dynamic bifurcation degeneracies, as well as other related bifurcations analyzed by other researchers in the frequency domain [Aracil *et al.*, 2000; Llibre & Ponce, 1996; Llibre & Sotomayor, 1996]. These results are the preliminary steps to locate degenerate bifurcations which act as organizing centers for dynamics and/or to propose strategies to delay, modify, avoid or control the degeneracies [Moiola *et al.*, 1999]. This latter issue has been explored extensively since the pioneering works of Abed and Fu [1987] and later by Fu [2000a], Kang and Krener [1992], and Kang [2000] regarding static singularities, and by Abed and Fu [1986], Berglund [2000], and Fu [2000b] concerning dynamic degeneracies. Recent developments have shown that bifurcation methods are very effective and provide nonclassical and useful new strategies in the control of nonlinear systems [Chen & Dong, 1998; Chen *et al.*, 2000].

2. Preliminaries

Consider an n-dimensional system of ordinary differential equations, given by

$$\begin{aligned} \dot{x} &= f(x;\, \mu)\,, \\ x(0) &= 0\,, \end{aligned} \tag{1}$$

where μ is a real parameter and f satisfies adequate conditions to guarantee the existence and uniqueness of the established initial value problem, for each fixed value of μ.

If Eq. (1) is written in a state-variable form, the following is attained

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)g(y;\, \mu)\,, \\ y &= -C(\mu)x\,, \\ x(0) &= 0\,, \end{aligned} \tag{2}$$

where A is an $n \times n$ matrix, which can be arbitrarily chosen for convenience (invertible and stable for all values of μ), B and C are $n \times p$ and $m \times n$ matrices respectively, and $g(y;\, \mu)$ is a $p \times 1$ nonlinear vectorial function which belongs to C^4.

Introducing in the system (2) a state-feedback control $u = g(y;\, \mu)$, a linear system with a nonlinear control variable is obtained as follows

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u\,, \\ y &= -C(\mu)x\,, \\ u &= g(y;\, \mu)\,. \end{aligned} \tag{3}$$

Taking Laplace transform in (3), with zero-initial condition, results in

$$\mathcal{L}(y) = -G(s;\, \mu) \cdot \mathcal{L}(g(y;\, \mu))\,,$$

where $G(s;\, \mu) = C(\mu)[sI - A(\mu)]^{-1}B(\mu)$ is the usual transfer matrix of the linear part of (3). From the last equation, the original problem can be solved for the variable y (contained in $\mathcal{L}(\cdot)$) in the so-called frequency domain. Thus, if $\hat{x}(t;\, \mu)$ is an equilibrium solution of (3), $\hat{y}(t;\, \mu) = -C(\mu)\hat{x}(t;\, \mu)$ can be considered as an equilibrium solution in the frequency domain. Taking into account the last observation, it can be deduced that the equilibrium points are the solutions of the following equation

$$\hat{y}(t;\, \mu) = -G(0;\, \mu)g(\hat{y}(t;\, \mu);\, \mu)\,, \tag{4}$$

where $-G(0;\, \mu) = C(\mu)[A(\mu)]^{-1}B(\mu)$.

Linearizing (3) about the equilibrium $\hat{y}(t;\, \mu)$, which solves (4), a system with the following transfer matrix is obtained

$$G(s;\, \mu)J(\mu)\,, \tag{5}$$

where $J(\mu) = \partial g/\partial y|_{y=\hat{y}}$.

An application of the generalized Nyquist stability criterion [MacFarlane & Postlethwaite, 1977], where $s = i\omega$ gives the following result:

Lemma 1. *If an eigenvalue of the Jacobian of the system (3), in the time domain, takes a purely imaginary value $i\omega_0$ at a particular value $\mu = \mu_0$, then the corresponding eigenvalue of the matrix $G(i\omega_0;\, \mu_0)J(\mu_0)$ in the frequency domain must take the value $-1 + i0$ at $\mu = \mu_0$.*

Let $\hat{\lambda} = \hat{\lambda}(i\omega;\, \mu)$ be the unique eigenvalue of the matrix $G(i\omega;\, \mu)J(\mu)$ which satisfies $\hat{\lambda}(i\omega_0;\, \mu_0) = -1 + i0$. Fixing $\mu = \tilde{\mu}$ and varying ω, the locus of the eigenvalue or "eigenlocus" is obtained. From the frequency analysis viewpoint, the appearance of a dynamic bifurcation fits the analysis of this locus for a certain value $\omega_0 \neq 0$ and this is precisely formulated below.

In general, the eigenvalues $\lambda(s;\, \mu)$ of (5) are the solutions of the following algebraic equation

$$h(\lambda,\, s;\, \mu) = \det(\lambda I - G(s;\, \mu)J(\mu)) = 0\,. \tag{6}$$

Considering Lemma 1 and imposing the condition $\lambda = -1$ in (6), a necessary relationship between $s = i\omega$ and μ to find a bifurcation point is obtained. Thereby, the resulting equation is

$$h(-1,\, i\omega;\, \mu) = 0\,. \tag{7}$$

By separating (7) into real ($\Re$) and imaginary ($\Im$) parts, the following system is attained

$$\begin{cases} F_1(\omega,\, \mu) = \Re\{h(-1,\, i\omega;\, \mu)\} = 0\,, \\ F_2(\omega,\, \mu) = \Im\{h(-1,\, i\omega;\, \mu)\} = 0\,. \end{cases} \tag{8}$$

Definition 2.1. A singular point (λ, s) is a solution of any of the following systems:

$$\text{(i)}\ \begin{cases} h(\lambda, s, \cdot) = 0\,, \\ \dfrac{\partial h}{\partial \lambda}(\lambda,\, s, \cdot) = 0, \end{cases} \quad \text{or} \quad \text{(ii)}\ \begin{cases} h(\lambda,\, s, \cdot) = 0\,, \\ \dfrac{\partial h}{\partial s}(\lambda,\, s, \cdot) = 0\,. \end{cases}$$

It is important to point out that the functions F_1 and F_2 given in (8) and their partial derivatives allow to find and classify the singular points.

Proposition 2.2. *A necessary condition for* (ω_0, μ_0) *to be a singular point is obtained by satisfying (8) [Moiola & Chen, 1996].*

Definition 2.3. A dynamic or Hopf bifurcation condition is obtained satisfying (8) with $\omega_0 \neq 0$.

It is convenient to keep in mind that, if the original problem is formulated in the time domain, a dynamic bifurcation condition goes together with the existence of a simple pair of pure imaginary eigenvalues. This result is established in the following classic theorem about existence and stability of limit cycles for a nonlinear system as (1) [Arrowsmith & Place, 1990]:

Theorem (Poincaré–Andronov–Hopf). *Suppose that the nonlinear autonomous system (1) has an equilibrium point at the origin:* $\hat{x} = 0$, *and the associated Jacobian* $\hat{J} = \partial f/\partial x|_{x=\hat{x}=0}$ *has a simple pair of purely imaginary eigenvalues* $\lambda_t(\mu)$ *and* $\overline{\lambda_t(\mu)}$. *If* $d\Re[\lambda_t(\mu)]/d\mu > 0$ *for some* μ_0, *where* $\Re\{\cdot\}$ *denotes the real part of the complex eigenvalues, then*

(1) $\mu = \mu_0$ *is a bifurcation point of the system;*
(2) *for close enough values* $\mu < \mu_0$, *the origin* $\hat{x} = 0$ *is an asymptotically stable equilibrium solution of the system;*
(3) *for close enough values* $\mu > \mu_0$, *the origin* $\hat{x} = 0$ *is an unstable equilibrium solution of the system; and*
(4) *for close enough values* $\mu \neq \mu_0$, *the origin* $\hat{x} = 0$ *is surrounded by a limit cycle of magnitude* $O(\sqrt{\mu})$.

Observation. A system as the one described in the aforementioned theorem can be rewritten locally using polar coordinates and considering $\mu = 0$, through Taylor expansions up to grade three, leading to:

$$\begin{cases} \dot{r} = (d\mu + ar^2)r\,, \\ \dot{\theta} = \omega + c\mu + br^2\,. \end{cases} \tag{9}$$

It is appropriate to distinguish the following parameters: $d = d\Re[\lambda_t(\mu)]/d\mu$, which shows the way as the eigenvalues cross the imaginary axis; a, whose sign determines the stability of the emerging limit cycle and ω which is the approximate frequency of the bifurcating periodic solution [Guckenheimer & Holmes, 1983]. Complications in the analysis of the dynamics in (9) appear when $d = 0$ or $a = 0$. These situations give rise to the so-called degenerate Hopf bifurcations and specifically, the treatment of the first type (as well as its complications) will be the focus of the rest of this work.

3. Degenerate Hopf Bifurcations

Some of the degenerate cases introduced in the previous section are classified and analyzed in this section. The problems where $d = 0$ are known as the ones where the transversality condition in the crossing of the eigenvalues fails, while those with $a = 0$ are recognized as the cases where the curvature coefficient is zero and this hinders the straight determination of the stability of the existing periodic solution. The case where $d = 0$ but the rest of the hypotheses of the Hopf bifurcation theorem are satisfied will be noted as H_{01}, following the standard notation used by Alhumaizi and Aris [1995], Farr [1986] and Planeaux [1993]. This type of singularity has codimension 1 [Golubitsky & Schaeffer, 1985], which means that it is necessary to vary a unique auxiliary parameter together with the main one to locate and describe the current degenerate bifurcation in the parameter space.

In the following, some general results about degenerate Hopf bifurcations for the treatment of a system like (1) in the frequency domain are expounded.

Definition 3.1. It is said that in (ω_0, μ_0) occurs some kind of degenerate Hopf bifurcation if the following system is satisfied

$$F_1(\omega_0,\, \mu_0) = 0\,,$$
$$F_2(\omega_0,\, \mu_0) = 0\,,$$
$$M(\omega_0,\, \mu_0) = 0\,,$$

for a certain value $\omega_0 \neq 0$ and $(\partial F_1/\partial\omega)^2|_{(\omega_0,\mu_0)} + (\partial F_2/\partial\omega)^2|_{(\omega_0,\mu_0)} \neq 0$, where

$$M(\omega,\,\mu) = \begin{vmatrix} \dfrac{\partial F_1}{\partial\mu} & \dfrac{\partial F_1}{\partial\omega} \\ \dfrac{\partial F_2}{\partial\mu} & \dfrac{\partial F_2}{\partial\omega} \end{vmatrix}$$

is called the determinant of degeneracies [Moiola *et al.*, 1990].

It will be considered an auxiliary parameter in the description of the system (3) in the frequency domain setting. Thus, the unique eigenlocus which is closest to the critical point $P_c = -1 + i0$, according to Lemma 1, will be analyzed when the main and auxiliary parameters change simultaneously.

Let $\hat{\lambda} = G_1(\omega,\,\mu) + iG_2(\omega,\,\mu)$ where $G_1,\, G_2 \in C^1$, the above-mentioned eigenlocus and solution of (6) where $s = i\omega$. Given that (8) is satisfied at the point $(\omega_0,\,\mu_0)$,

$$G_1(\omega_0,\,\mu_0) = F_1(\omega_0,\,\mu_0) - 1 = -1\,,$$

$$G_2(\omega_0,\,\mu_0) = F_2(\omega_0,\,\mu_0) = 0\,,$$

and in agreement with the assumption of the uniqueness,

$$G_1(\omega,\,\mu) = F_1(\omega,\,\mu) - 1, \quad G_2(\omega,\,\mu) = F_2(\omega,\,\mu)\,.$$

Hence, the equalities of the partial derivatives

$$\begin{aligned} \frac{\partial G_1}{\partial\omega} &= \frac{\partial F_1}{\partial\omega}, \quad \frac{\partial G_1}{\partial\mu} = \frac{\partial F_1}{\partial\mu}, \\ \frac{\partial G_2}{\partial\omega} &= \frac{\partial F_2}{\partial\omega}, \quad \frac{\partial G_2}{\partial\mu} = \frac{\partial F_2}{\partial\mu}, \end{aligned} \tag{10}$$

are valid at $(\omega_0,\,\mu_0)$.

According with the objective of determining the extreme values of the eigenlocus $\hat{\lambda}$ with respect to the critical point P_c, the following proposition holds [Moiola *et al.*, 1990]:

Proposition 3.2. *Suppose that* $(\omega_0,\,\mu_0)$ *satisfies the dynamic bifurcation condition*

$$F_1(\omega_0,\,\mu_0) = F_2(\omega_0,\,\mu_0) = 0\,.$$

Then:

(a) *If* $G_1 = G_1(\omega,\,\mu)$ *has an extreme value at* $(\omega_0,\,\mu_0)$, *under the condition* $G_2 = G_2(\omega,\,\mu) = 0$, *and if* $\partial F_2/\partial\omega|_{(\omega_0,\,\mu_0)} \neq 0$, *then* $M(\omega_0,\,\mu_0) = 0$, *i.e. in* $(\omega_0,\,\mu_0)$ *occurs a degenerate dynamic bifurcation.*

(b) *If* $M(\omega_0,\mu_0) = 0$ *and* $\partial F_2/\partial\omega|_{(\omega_0,\,\mu_0)} \neq 0$, *then* $(\omega_0,\,\mu_0)$ *is a critical point of the function* $G_1 = G_1(\omega,\,\mu)$ *under the condition* $G_2 = G_2(\omega,\,\mu) = 0$.

Observation. The previous proposition can also be formulated under the assumption that $\partial F_2/\partial\mu|_{(\omega_0,\,\mu_0)} \neq 0$.

Under the general hypothesis established in Proposition 3.2, given that for certain $(\omega_0,\,\mu_0)$, $\omega_0 \neq 0$, is satisfied

$$\left.\frac{\partial G_2}{\partial\omega}\right|_{(\omega_0,\,\mu_0)} = \left.\frac{\partial F_2}{\partial\omega}\right|_{(\omega_0,\,\mu_0)} \neq 0\,,$$

it is possible to define a function

$$\omega = \omega(\mu)\,, \tag{11}$$

in a neighborhood $\mathcal{E}(\mu_0)$, which solves the equation $G_2(\omega,\,\mu) = 0$. Moreover, according with (10), we see the following results particularly

$$\left.\frac{d\omega}{d\mu}\right|_{(\omega_0,\,\mu_0)} = -\left.\frac{\dfrac{\partial G_2}{\partial\mu}}{\dfrac{\partial G_2}{\partial\omega}}\right|_{(\omega_0,\,\mu_0)} = -\left.\frac{\dfrac{\partial F_2}{\partial\mu}}{\dfrac{\partial F_2}{\partial\omega}}\right|_{(\omega_0,\,\mu_0)}. \tag{12}$$

If $G_1 = G_1(\omega,\,\mu)$ has an extreme value at $(\omega_0,\,\mu_0)$, the function

$$H(\mu) = G_1(\omega(\mu),\,\mu) \tag{13}$$

can be considered for any $\mu \in \mathcal{E}(\mu_0)$ and it can be deduced, under the general assumptions and (12), that

$$\left.\frac{dH}{d\mu}\right|_{\mu=\mu_0} = 0\,.$$

Connected with degenerate Hopf bifurcations where the transversality condition of the classical Hopf theorem fails, the function H defined in (13) can be used in order to generalize the result already stated in [Moiola *et al.*, 1990], as follows:

Proposition 3.3. *Suppose that*

$$F_1(\omega_0,\mu_0) = F_2(\omega_0,\mu_0) = 0\,,$$

$$M(\omega_0,\,\mu_0) = 0 \quad \textit{and}$$

$$\left.\frac{\partial F_2}{\partial\omega}\right|_{(\omega_0,\,\mu_0)} \neq 0$$

where $\omega_0 \neq 0$. *Let be*

$$\Delta_n = \left.\frac{d^n H}{d\mu^n}\right|_{\mu=\mu_0} \neq 0\,, \quad n \geq 2$$

the first nonzero derivative of H *at* $\mu = \mu_0$, *then*:

(a) *If* n *is even, the transversality condition of the classical Hopf theorem fails.*
(b) *If* n *is odd, the transversality condition of the classical Hopf theorem degenerates.*

Observation. Henceforth, according to the previous result, the following notation will be used

$$\Delta_i = \left.\frac{d^i H}{d\mu^i}\right|_{\mu=\mu_0}\,, \quad \forall i \geq 2\,.$$

By supposing that the sign of the curvature coefficient "a" is always defined, the degenerate and nondegenerate conditions established in the preceding proposition characterize the singularity family $H_{0,n-1}$, which involves up to n Hopf bifurcation points in its unfoldings [Alhumaizi & Aris, 1995; Golubitsky & Langford, 1981].

4. Characterization of Some Bifurcations of Types H_{01} and H_{02}

Under the assumptions of Proposition 3.3 (a) at the point (ω_0, μ_0), the following equation is considered

$$H(\mu, \rho) = G_1(\omega(\mu), \mu, \rho) = -1\,, \tag{14}$$

where $\omega(\mu)$ is the function defined in (11) and ρ is a vector of auxiliary parameters introduced in order to analyze the H_{01} and H_{02} degeneracies. For simplicity, let us suppose at first that $\rho \in R^1$. Moreover, the next equality holds

$$H(\mu_0, 0) = G_1(\omega_0, \mu_0, 0) = -1\,. \tag{15}$$

Under the stated conditions, it is known that the hypothesis of transversality of the Hopf bifurcation theorem fails. By analyzing a system like (3) with an auxiliary parameter ρ, sufficient conditions can be specified to guarantee the existence of a fold singularity at the point $(\mu, \rho) = (\mu_0, 0)$ in the parameter space.

Observation. This notation will be used in the subsequent theorems

$$\left.\frac{\partial^i H}{\partial\mu^i}\right|_{(\mu_0,\,0)} = \left.\frac{d^i H}{d\mu^i}\right|_{\mu=\mu_0}\,, \quad \forall i \geq 2\,.$$

Theorem 4.1. *Under the assumptions of Proposition 3.3* (*a*) *at the point* (ω_0, μ_0), (*15*) *and if* $\partial H/\partial\rho|_{(\mu_0,\,0)} \neq 0$, *there is a fold singularity that results at* $(\mu, \rho) = (\mu_0, 0)$ *in the parameter space.*

Proof. First, the case with $n = 2$ is considered, this means that $\Delta_2 = d^2H/d\mu^2|_{\mu=\mu_0} \neq 0$. Given that $\partial H/\partial\rho|_{(\mu_0,\,0)} \neq 0$ and due to (15), applying the implicit function theorem, it is possible to find a function $\rho = \rho(\mu)$ which solves Eq. (14) and satisfies the following properties:

(i) $d\rho/d\mu|_{\mu=\mu_0} = 0$, due to

$$\frac{\partial H}{\partial\mu} + \frac{\partial H}{\partial\rho}\frac{d\rho}{d\mu} = 0\,, \tag{16}$$

and evaluating at $\mu = \mu_0$, results in

$$\left.\frac{d\rho}{d\mu}\right|_{\mu=\mu_0} = -\frac{\left.\dfrac{\partial H}{\partial\mu}\right|_{(\mu_0,\,0)}}{\left.\dfrac{\partial H}{\partial\rho}\right|_{(\mu_0,\,0)}} = 0\,.$$

(ii) $d^2\rho/d\mu^2|_{\mu=\mu_0} \neq 0$, since taking derivatives in (16) with respect to the main parameter μ, it is obtained

$$\frac{\partial^2 H}{\partial\mu^2} + 2\frac{\partial^2 H}{\partial\mu\partial\rho}\frac{d\rho}{d\mu} + \frac{\partial^2 H}{\partial\rho^2}\left(\frac{d\rho}{d\mu}\right)^2 + \frac{\partial H}{\partial\rho}\frac{d^2\rho}{d\mu^2} = 0\,, \tag{17}$$

and evaluating at $\mu = \mu_0$ follows,

$$\left.\frac{d^2\rho}{d\mu^2}\right|_{\mu=\mu_0} = -\frac{\left.\dfrac{\partial^2 H}{\partial\mu^2}\right|_{(\mu_0,\,0)}}{\left.\dfrac{\partial H}{\partial\rho}\right|_{(\mu_0,0)}} \neq 0\,,$$

taking into account the result of (i) and the hypothesis about the second derivative of function H. Then, there is a fold singularity at the point $(\mu, \rho) = (\mu_0, 0)$ in the parameter space. In the general case, with an even n greater than two, following the same argument, $d^2\rho/d\mu^2|_{\mu=\mu_0} = 0$. Nevertheless, starting from (17), it can be proved that if

$$\Delta_i = \left.\frac{\partial^i H}{\partial \mu^i}\right|_{\mu=\mu_0} = 0, \quad \text{for} \quad 2 \le i \le n-1\,,$$

$$\Delta_n = \left.\frac{\partial^n H}{\partial \mu^n}\right|_{\mu=\mu_0} \ne 0\,, \tag{18}$$

then

$$\left.\frac{d^i \rho}{d\mu^i}\right|_{\mu=\mu_0} = 0, \quad \text{for} \quad 2 \le i \le n-1 \quad \text{and}$$

$$\left.\frac{d^n \rho}{d\mu^n}\right|_{\mu=\mu_0} \ne 0\,. \tag{19}$$

Thus, as (18) is satisfied with an even n greater than two, due to (19) and Taylor's formula, there results again a fold singularity at the point $(\mu, \rho) = (\mu_0, 0)$ in the parameter space. ■

Under the assumptions of Proposition 3.3 (b) with $n = 3$ at the point (ω_0, μ_0), the following equation is considered:

$$H(\mu, (\rho_1, \rho_2)) = G_1(\omega(\mu), \mu, (\rho_1, \rho_2)) = -1\,, \tag{20}$$

where $\omega(\mu)$ is the function defined in (11) and (ρ_1, ρ_2) is a pair of auxiliary parameters that appear in the function G_1. Furthermore, the next equality is valid

$$H(\mu_0, (0, 0)) = G_1(\omega_0, \mu_0, (0, 0)) = -1\,. \tag{21}$$

Thereby, the hypothesis about the transversality of the Hopf bifurcation theorem degenerates.

Observation. From now on, it will be noted $\rho = (\rho_1, \rho_2)$ and $(\mu, \rho) = (\mu, (\rho_1, \rho_2))$. Thus, it can be considered $H = H(\mu, \rho) = H(\mu, (\rho_1, \rho_2))$. Moreover, it will be supposed that $\mu_0 = 0$.

Forthwith, analyzing a system like (3) with a pair of auxiliary parameters $\rho = (\rho_1, \rho_2)$, sufficient conditions to guarantee the existence of a cusp singularity at the point $(\mu, \rho) = (\mu_0, 0) = (0, 0)$ in the parameter space are established.

Theorem 4.2. *Under the assumptions of Proposition 3.3 (b) with $n = 3$ at the point $(\omega_0, 0)$, (21) and if the Jacobian*

$$J\left(H, \frac{\partial H}{\partial \mu}\right) = \left|\begin{matrix} \dfrac{\partial H}{\partial \rho_1} & \dfrac{\partial H}{\partial \rho_2} \\ \dfrac{\partial\left(\dfrac{\partial H}{\partial \mu}\right)}{\partial \rho_1} & \dfrac{\partial\left(\dfrac{\partial H}{\partial \mu}\right)}{\partial \rho_2} \end{matrix}\right|_{(0,0)} \ne 0\,,$$

where $H = H(\mu, \rho)$, there is a cusp singularity that results at the point $(\mu_0, 0) = (0, 0)$ in the parameter space.

Proof. From the assumed hypotheses it follows that

$$H(0, 0) = -1, \quad \left.\frac{\partial H}{\partial \mu}\right|_{(0,0)} = \left.\frac{\partial^2 H}{\partial \mu^2}\right|_{(0,0)} = 0\,,$$

$$\left.\frac{\partial^3 H}{\partial \mu^3}\right|_{(0,0)} \ne 0\,.$$

Therefore, the function $\hat{H}(\mu, \rho) = H(\mu, \rho) + 1$ has a Taylor expansion about the origin like

$$\hat{H}(\mu, \rho) = a(\rho) + b(\rho)\mu + \frac{c(\rho)}{2!}\mu^2 + \frac{d(\rho)}{3!}\mu^3 + R(\mu, \rho)\,,$$

where $a(0) = b(0) = c(0) = 0$, $d(0) \ne 0$ and $\forall \varepsilon > 0$ there exist $\delta_\mu > 0$ and $\delta_\rho > 0$ such that $|R(\mu, \rho)| < \varepsilon|\mu|^3$ when $|\mu| < \delta_\mu$, $\|\rho\| < \delta_\rho$.

Provided that the Jacobian

$$J\left(\hat{H}, \frac{\partial \hat{H}}{\partial \mu}\right) = J\left(H, \frac{\partial H}{\partial \mu}\right)$$

$$= \left|\begin{matrix} \dfrac{\partial H}{\partial \rho_1} & \dfrac{\partial H}{\partial \rho_2} \\ \dfrac{\partial\left(\dfrac{\partial H}{\partial \mu}\right)}{\partial \rho_1} & \dfrac{\partial\left(\dfrac{\partial H}{\partial \mu}\right)}{\partial \rho_2} \end{matrix}\right|_{(0,0)} \ne 0\,,$$

and according to the results established in [Hale & Koçak, 1991], the solution set of Hopf points of (20), in the neighborhood of $(\mu_0, 0) = (0, 0)$, is defined through a cusp singularity in the (ρ_1, ρ_2) parameter space. ■

Hereafter, some application examples about the aforementioned theorems are analyzed in detail.

5. Applications

Example 1. The nonlinear system analyzed in [Moiola *et al.*, 1999] is considered

$$\begin{aligned}\dot{x}_1 &= -x_1x_2 + x_2^2 + \mu + u(x;\, \mu,\, \alpha)\,,\\ \dot{x}_2 &= -x_2 + x_1^3 + \mu x_2\,,\end{aligned} \tag{22}$$

where $x = (x_1,\, x_2)$, μ is the main bifurcation parameter, u is a nonlinear control law and α is an auxiliary control parameter.

It is established as the control law: $u(x;\, \mu,\, \alpha) = \alpha\dot{x}_2$ and thus, the following system is attained

$$\begin{aligned}\dot{x}_1 &= -x_1x_2 + x_2^2 + \mu + \alpha(-x_2 + x_1^3 + \mu x_2)\,,\\ \dot{x}_2 &= -x_2 + x_1^3 + \mu x_2\,.\end{aligned} \tag{23}$$

Through a convenient realization in the frequency domain, (23) can be written as

$$A = \begin{bmatrix} -\beta & 0 \\ 0 & -1 \end{bmatrix}, \quad B = C = I_2\,,$$

where β is a positive constant, introduced to guarantee that the open-loop poles are located in the left half-plane. In agreement with the notation of Sec. 2,

$$G(s) = C(sI - A)^{-1}B = \begin{bmatrix} \dfrac{1}{s+\beta} & 0 \\ 0 & \dfrac{1}{s+1} \end{bmatrix}$$

and

$$\begin{aligned} g(y;\, \mu,\, \alpha,\, \beta) &= \begin{bmatrix} g_1((y_1,\, y_2);\, \mu,\, \alpha,\, \beta) \\ g_2((y_1,\, y_2);\, \mu,\, \alpha,\, \beta) \end{bmatrix} \\ &= \begin{bmatrix} -y_1y_2 + y_2^2 + \mu - \beta y_1 + \alpha(y_2 - y_1^3 - \mu y_2) \\ -y_1^3 - \mu y_2 \end{bmatrix}, \end{aligned}$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = -C \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

We attempt to determine the solutions $\hat{y}^t = (\hat{y}_1,\, \hat{y}_2)$ such that

$$\hat{y} = -G(0) \cdot g(\hat{y};\, \mu,\, \alpha,\, \beta)\,. \tag{24}$$

From now on, let $\beta = 1$ for simplicity. Thereby, the resulting algebraic system is

$$\begin{aligned}\hat{y}_1\hat{y}_2 - \hat{y}_2^2 - \mu &= 0\,,\\ \hat{y}_1^3 + (\mu - 1)\hat{y}_2 &= 0\,,\end{aligned} \tag{25}$$

which coincides with the equilibrium points of the system (22). Apropos of analyzing the Hopf bifurcations of the system, the eigenvalue of the system $G(s)J$ which crosses through $P_c = -1 + i0$ is considered for some frequency $s = i\omega$, $\omega \neq 0$. In other words, it is required to find the solution of the following equation:

$$h(-1,\, i\omega,\, \mu,\, \alpha) = \det(-1 - G(i\omega)J(\mu,\, \alpha)) = 0\,,$$

where

$$J(\mu,\, \alpha) = \begin{bmatrix} \dfrac{\partial g_1}{\partial y_1} & \dfrac{\partial g_1}{\partial y_2} \\ \dfrac{\partial g_2}{\partial y_1} & \dfrac{\partial g_2}{\partial y_2} \end{bmatrix}_{y=\hat{y}} = \begin{bmatrix} -J_{11} & -J_{12} \\ -3\hat{y}_1^2 & -\mu \end{bmatrix},$$

and $J_{11} = 1 + \hat{y}_2 + 3\alpha\hat{y}_1^2$, $J_{12} = \hat{y}_1 - 2\hat{y}_2 - \alpha(1-\mu)$. Thus,

$$\begin{aligned} h(-1,\, i\omega,\, \mu,\, \alpha) &= 1 - \frac{J_{11} + \mu}{(1+i\omega)} + \frac{1}{(1+i\omega)^2}[\mu J_{11} - 3\hat{y}_1^2 J_{12}] \\ &= 1 - \frac{1}{(1+i\omega)}D + \frac{1}{(1+i\omega)^2}E = 0\,, \end{aligned}$$

where $D = J_{11} + \mu$, $E = \mu J_{11} - 3\hat{y}_1^2 J_{12}$.

In order to find the dynamic bifurcation points of (23) in the frequency domain, it is necessary to solve the system (8) for $\omega \neq 0$, which is particularly

$$\begin{aligned} F_1(\omega,\, \mu,\, \alpha) &= 1 - \frac{D}{1+\omega^2} + \frac{E(1-\omega^2)}{(1+\omega^2)^2} = 0\,,\\ F_2(\omega,\, \mu,\, \alpha) &= \frac{D\omega}{1+\omega^2} - \frac{2E\omega}{(1+\omega^2)^2} = 0\,. \end{aligned} \tag{26}$$

The last equation of (26) yields

$$D(1+\omega^2) - 2E = 0\,, \tag{27}$$

and finally, substituting D into $F_1 = 0$, results in $E = 1 + \omega^2$ and $D = 2$. Furthermore, considering the following equilibrium values of the system (25) $(\mu_0,\, \hat{y}_{1_0},\, \hat{y}_{2_0}) = (-0.3237359,\, 1.248025,\, 1.468481)$ with $\alpha_0 = -0.0309768$, it can be determined that

$E = 6.948013$ and $\omega_0 = 2.438854$. This starting point has been obtained by using LOCBIF [Khibnik *et al.*, 1993] when continuing a Hopf bifurcation curve and after arriving at a degenerate Hopf point. It can be observed that in the neighborhood of the aforementioned equilibrium point, the implicit function theorem can be applied. In other words, there are two functions $\hat{y}_1(\mu)$ and $\hat{y}_2(\mu)$ defined in a neighborhood of $(\mu_0, \hat{y}_{1_0}, \hat{y}_{2_0})$, whose first and second derivatives evaluated at $\mu = \mu_0$ are

$$\begin{aligned} \left.\frac{d\hat{y}_1}{d\mu}\right|_{\mu=\mu_0} &= -0.6395257\,, \\ \left.\frac{d^2\hat{y}_1}{d\mu^2}\right|_{\mu=\mu_0} &= -0.4775182\,, \\ \left.\frac{d\hat{y}_2}{d\mu}\right|_{\mu=\mu_0} &= -1.1481372\,, \\ \left.\frac{d^2\hat{y}_2}{d\mu^2}\right|_{\mu=\mu_0} &= -1.1066949\,. \end{aligned} \tag{28}$$

Moreover, it can be obtained that

$$\begin{aligned} F_1(\omega_0, \mu_0) &= -4.2707122 * 10^{-7}\,, \\ F_2(\omega_0, \mu_0) &= -4.2100343 * 10^{-7}\,. \end{aligned}$$

Furthermore, taking into account the values of all the variables and the results in (28),

$$\left.\frac{\partial F_1}{\partial \mu}\right|_{(\omega_0,\mu_0)} = 1.3161361, \quad \left.\frac{\partial F_2}{\partial \mu}\right|_{(\omega_0,\mu_0)} = 1.2975384,$$

and

$$\left.\frac{\partial F_1}{\partial \omega}\right|_{(\omega_0,\mu_0)} = 0.4999489, \quad \left.\frac{\partial F_2}{\partial \omega}\right|_{(\omega_0,\mu_0)} = 0.4928460\,.$$

Therefore, the value of the determinant of degeneracies is obtained

$$M(\omega_0, \mu_0) = \left|\begin{matrix} \dfrac{\partial F_1}{\partial \mu} & \dfrac{\partial F_1}{\partial \omega} \\ \dfrac{\partial F_2}{\partial \mu} & \dfrac{\partial F_2}{\partial \omega} \end{matrix}\right|_{(\omega_0,\mu_0)} = -5.048344 * 10^{-5}.$$

This value is in correspondence with the precision of LOCBIF in order to detect the failure of the transversality condition. Then, according to Definition 3.1, the system (23) has a degenerate Hopf bifurcation at (ω_0, μ_0).

To see that all the assumptions of Proposition 3.3 (a) with $n = 2$ are satisfied, it remains be seen that $\Delta_2 = d^2H/d\mu^2|_{\mu=\mu_0} \neq 0$, and in that case, then it could be asserted that the transversality condition of the Hopf theorem fails. In the considered example, the function $\omega = \omega(\mu)$ is the solution of (27) in a neighborhood of (ω_0, μ_0). Taking derivatives implicitly in that equation, the following can be deduced

$$\left.\frac{d\omega}{d\mu}\right|_{\mu=\mu_0} = -2.6327484, \quad \left.\frac{d^2\omega}{d\mu^2}\right|_{\mu=\mu_0} = -1.2664337.$$

Hence, we attempt to analyze the function

$$H(\mu) = G_1(\omega(\mu), \mu) = F_1(\omega(\mu), \mu, \alpha) - 1\,,$$

and its derivatives in the neighborhood of $\mu_0 = -0.3237359$ when $\alpha = \alpha_0$. Thus, given that

$$\frac{dH}{d\mu} = \frac{\partial F_1}{\partial \omega}\frac{d\omega}{d\mu} + \frac{\partial F_1}{\partial \mu}\,,$$

and

$$\frac{d^2H}{d\mu^2} = \frac{\partial^2 F_1}{\partial \omega^2}\left(\frac{d\omega}{d\mu}\right)^2 + 2\frac{\partial^2 F_1}{\partial\omega\partial\mu}\frac{d\omega}{d\mu} + \frac{\partial F_1}{\partial \omega}\frac{d^2\omega}{d\mu^2} + \frac{\partial^2 F_1}{\partial \mu^2},$$

we obtain results

$$\begin{aligned} \left.\frac{dH}{d\mu}\right|_{\mu=\mu_0} &= -1.0355665 * 10^{-4}\,, \\ \left.\frac{d^2H}{d\mu^2}\right|_{\mu=\mu_0} &= 0.5360830 \neq 0\,. \end{aligned}$$

It can be then concluded that the transversality condition of the Hopf bifurcation theorem fails at (ω_0, μ_0) giving a type of H_{01} degeneracy.

With the purpose of determining the set of Hopf points that exists in the neighborhood of (μ_0, α_0), the solution set is considered of the following two-variable equation

$$H(\mu, \alpha) = G_1(\omega(\mu), \mu, \alpha) = -1\,.$$

Furthermore, it is known that the next equality holds

$$\begin{aligned} H(\mu_0, \alpha_0) &= G_1(\omega_0, \mu_0, \alpha_0) \\ &= F_1(\omega_0, \mu_0, \alpha_0) - 1 = -1\,. \end{aligned}$$

We attempt to verify Theorem 4.1, so it remains to prove that

$$\left.\frac{\partial H}{\partial \alpha}\right|_{(\mu_0,\,\alpha_0)} \neq 0\,.$$

Given

$$H(\mu,\,\alpha) = G_1(\omega(\mu),\,\mu,\,\alpha)$$
$$= -\frac{D}{1+\omega^2(\mu)} + \frac{E(1-\omega^2(\mu))}{(1+\omega^2(\mu))^2}\,,$$

and in agreement with (27),

$$H(\mu,\,\alpha) = G_1(\omega(\mu),\,\mu,\,\alpha) = -\frac{D}{2}\,.$$

Thereby,

$$\left.\frac{\partial H}{\partial \alpha}\right|_{(\mu_0,\alpha_0)} = \left.\frac{\partial G_1}{\partial \alpha}\right|_{(\omega_0,\mu_0,\alpha_0)}$$
$$= -\frac{1}{2}\left.\frac{dD}{d\alpha}\right|_{(\mu_0,\alpha_0)} = -2.3363496 \neq 0\,.$$

Then, it can be asserted that the system (23) has a fold singularity at $(\mu_0,\,\alpha_0)$ in the parameter space.

Example 2. Consider the system investigated by Byeon and Chung [1989], Farr [1986], and Moiola and Chen [1993, 1996], the mathematical model for a perfectly mixed reactor with a coiling coil, in which two consecutive, irreversible, exothermic and first-order reactions $A \to B \to C$ occur. The described system can be written in its dimensionless form as

$$\begin{aligned}
\dot{x}_1 &= -x_1 + D(1-x_1)\exp(x_3)\,,\\
\dot{x}_2 &= -x_2 + D(1-x_1)\exp(x_3) - DSx_2\exp(x_3)\,,\\
\dot{x}_3 &= -(1+\beta)x_3 + \tilde{B}D(1-x_1)\exp(x_3)\\
&\quad + \tilde{B}DS\alpha x_2\exp(x_3)\,,
\end{aligned} \tag{29}$$

where D is the main bifurcation parameter and $\tilde{B}$, S, α and β are the auxiliary system parameters. In the frequency domain, the following realization is proposed, considering the matrices A, B, C and the vectorial function g

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1-\beta \end{bmatrix},$$

$$B = \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ \tilde{B} & \tilde{B}\alpha \end{bmatrix}, \quad C = I_3$$

and

$$g(y;\,D,\,S) = \begin{bmatrix} g_1(y;\,D,\,S) \\ g_2(y;\,D,\,S) \end{bmatrix}$$
$$= \begin{bmatrix} D(1+y_1)\exp(-y_3) \\ -DSy_2\exp(-y_3) \end{bmatrix},$$

where

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = -C\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Thereby, the transfer matrix $G(s)$:

$$G(s) = C(sI-A)^{-1}B = \begin{bmatrix} \dfrac{1}{s+1} & 0 \\ \dfrac{1}{s+1} & -\dfrac{1}{s+1} \\ \dfrac{\tilde{B}}{s+1+\beta} & \dfrac{\tilde{B}\alpha}{s+1+\beta} \end{bmatrix}.$$

Henceforth, we attempt to find the equilibrium solutions $\hat{y}$ of (29), $\hat{y}^t = (\hat{y}_1,\,\hat{y}_2,\,\hat{y}_3)$ that satisfy the following equation

$$\hat{y} = -G(0)\cdot g(\hat{y};\,D,\,S)\,.$$

Solving this system, we obtain

$$\begin{aligned}
D &= -\frac{\hat{y}_1}{1+\hat{y}_1}\exp(\hat{y}_3)\,,\\
\hat{y}_2 &= \frac{\hat{y}_1(1+\hat{y}_1)}{1+(1-S)\hat{y}_1}\,,\\
\hat{y}_3 &= \frac{\tilde{B}\hat{y}_1}{(1+\beta)} - \frac{\tilde{B}\alpha S\hat{y}_1^2}{(1+\beta)[1+(1-S)\hat{y}_1]}\,,
\end{aligned} \tag{30}$$

where it can be observed that the values of the bifurcation parameter D are in one-to-one correspondence with the values of $\hat{y}_1$. The Jacobian matrix $J(D)$ has the following form:

$$J(D) = \begin{bmatrix} J_{11} & 0 & J_{13} \\ 0 & J_{22} & J_{23} \end{bmatrix}$$
$$= \begin{bmatrix} J_{11} & 0 & -(1+\hat{y}_1)J_{11} \\ 0 & -SJ_{11} & S\hat{y}_2J_{11} \end{bmatrix},$$

where $J_{11} = D\exp(-\hat{y}_3) = -\hat{y}_1/(1+\hat{y}_1)$.

With the purpose of determining the bifurcations of the system, the eigenvalues λ_i; $i = 1,\,2,\,3,$

of the matrix $G(s)J(D)$ are considered. They are: $\lambda_3 = 0$ and the roots of the equation

$$\lambda^2 + a_1(s;\, \hat{y}_1)\lambda + a_0(s;\, \hat{y}_1) = 0\,, \qquad (31)$$

where

$$a_1(s;\, \hat{y}_1) = \frac{J_{22} - J_{11}}{s+1} - \frac{\tilde{B}(J_{13} + \alpha J_{23})}{s+1+\beta}\,,$$

$$a_0(s;\, \hat{y}_1) = \frac{\tilde{B}[\alpha J_{23} J_{11} - J_{22} J_{13}(1+\alpha)]}{(s+1)(s+1+\beta)} - \frac{J_{11} J_{22}}{(s+1)^2}\,.$$

It is important to note that an *artificial* bifurcation parameter $\hat{y}_1$ has been introduced, from which the main bifurcation parameter D can be obtained immediately, as it has been pointed out before.

Substituting $\lambda = -1$ and $s = i\omega$ in (31), applying (30) and the expression of $J(D)$, the functions F_1 and F_2 can be obtained

$$F_1(\omega,\, \hat{y}_1) = 1 - \frac{a}{p_1} + \frac{b(1+\beta)}{p_2} + \frac{c(1+\beta-\omega^2)}{p_3} + \frac{d(1-\omega^2)}{p_4}\,,$$

$$F_2(\omega,\, \hat{y}_1) = \frac{a\omega}{p_1} - \frac{b\omega}{p_2} - \frac{c(2+\beta)\omega}{p_3} - \frac{2d\omega}{p_4}\,, \qquad (32)$$

where

$$a(\hat{y}_1) = \frac{\hat{y}_1(S+1)}{1+\hat{y}_1}\,,$$

$$b(\hat{y}_1) = \frac{\tilde{B}\hat{y}_1[1 + (1 - S - S\alpha)\hat{y}_1]}{1 + (1-S)\hat{y}_1}\,,$$

$$c(\hat{y}_1) = \frac{-\tilde{B}S\hat{y}_1^2[1 + \alpha + (1 - S - S\alpha)\hat{y}_1]}{(1 + (1-S)\hat{y}_1)(1+\hat{y}_1)}\,,$$

$$d(\hat{y}_1) = \frac{S\hat{y}_1^2}{(1+\hat{y}_1)^2}\,,$$

$$p_1 = 1 + \omega^2, \quad p_2 = (1+\beta)^2 + \omega^2\,,$$

$$p_3 = (1+\omega^2)((1+\beta)^2 + \omega^2), \quad p_4 = (1+\omega^2)^2\,.$$

Fixing the values of the parameters $\tilde{B} = 8$ and $S = 0.06$, it can be proved that a dynamic degenerate bifurcation of the system (32) appears at the point $(\hat{y}_{1_1};\, \omega_1,\, (\alpha_1,\, \beta_1)) = (-0.7973357;\, 1.4509548,\, (1.1365924,\, 2.2734795))$ where ω_1 is the frequency at criticality. This situation repeats at $(\hat{y}_{1_2};\, \omega_2,\, (\alpha_2,\, \beta_2)) = (-0.9235686;\, 0.8618195,\, (0.7241262,\, 1.2856351))$. Both results have been obtained by using LOCBIF [Khibnik *et al.*, 1993], but in these cases we have looked for the failure of higher-order derivatives in the transversality condition. Next, it will be showed that the (higher-order) transversality condition fails at the aforementioned equilibrium points and cusp singularities can be found next to $(\alpha_1,\, \beta_1)$ and $(\alpha_2,\, \beta_2)$.

First of all, some calculations are given

	$(\omega_1,\, \hat{y}_{1_1})$	$(\omega_2,\, \hat{y}_{1_2})$
F_1	$1.59511271 * 10^{-4}$	$-1.79688476 * 10^{-3}$
F_2	$-8.41590000 * 10^{-5}$	$-1.60613163 * 10^{-5}$

(33)

Apropos of evaluating the determinants of degeneracies M

$$M(\omega,\, \hat{y}_1) = \begin{vmatrix} \dfrac{\partial F_1}{\partial \hat{y}_1} & \dfrac{\partial F_1}{\partial \omega} \\ \dfrac{\partial F_2}{\partial \hat{y}_1} & \dfrac{\partial F_2}{\partial \omega} \end{vmatrix}\,,$$

at each one of the points $(\omega_i,\, \hat{y}_{1_i})$; $i = 1, 2$, the expressions of the partial derivatives of F_1 and F_2 with respect to $\hat{y}_1$ and ω are given below

$$\frac{\partial F_1}{\partial \hat{y}_1} = -\frac{1}{p_1}\frac{\partial a}{\partial \hat{y}_1} + \frac{(1+\beta)}{p_2}\frac{\partial b}{\partial \hat{y}_1} + \frac{(1+\beta-\omega^2)}{p_3}\frac{\partial c}{\partial \hat{y}_1} + \frac{(1-\omega^2)}{p_4}\frac{\partial d}{\partial \hat{y}_1}\,,$$

$$\frac{\partial F_2}{\partial \hat{y}_1} = \omega\left(\frac{1}{p_1}\frac{\partial a}{\partial \hat{y}_1} - \frac{1}{p_2}\frac{\partial b}{\partial \hat{y}_1} - \frac{(2+\beta)}{p_3}\frac{\partial c}{\partial \hat{y}_1} - \frac{2}{p_4}\frac{\partial d}{\partial \hat{y}_1}\right),$$

where

$$\frac{\partial a}{\partial \hat{y}_1} = \frac{S+1}{(1+\hat{y}_1)^2}\,,$$

$$\frac{\partial b}{\partial \hat{y}_1} = \frac{\tilde{B}[(1-S)(1-S-S\alpha)\hat{y}_1^2 + 2(1-S-S\alpha)\hat{y}_1 + 1]}{[1+(1-S)\hat{y}_1]^2}\,,$$

$$\frac{\partial c}{\partial \hat{y}_1} = -\tilde{B}S\left\{\frac{(1-S)(1-S-S\alpha)\hat{y}_1^4 + 2(2-S)(1-S-S\alpha)\hat{y}_1^3}{[(1-S)\hat{y}_1^2 + (2-S)\hat{y}_1 + 1]^2}\right.$$
$$\left. + \frac{[5+2\alpha-4S(1+\alpha)]\hat{y}_1^2 + 2(1+\alpha)\hat{y}_1}{[(1-S)\hat{y}_1^2 + (2-S)\hat{y}_1 + 1]^2}\right\},$$
$$\frac{\partial d}{\partial \hat{y}_1} = \frac{2S\hat{y}_1}{(1+\hat{y}_1)^3},$$

and with respect to ω are

$$\frac{\partial F_1}{\partial \omega} = -a\frac{\partial}{\partial \omega}\left(\frac{1}{p_1}\right) + b(1+\beta)\frac{\partial}{\partial \omega}\left(\frac{1}{p_2}\right) - \frac{2\omega c}{p_3} + c(1+\beta-\omega^2)\frac{\partial}{\partial \omega}\left(\frac{1}{p_3}\right)$$
$$- \frac{2\omega d}{p_4} + d(1-\omega^2)\frac{\partial}{\partial \omega}\left(\frac{1}{p_4}\right),$$
$$\frac{\partial F_2}{\partial \omega} = \frac{a}{p_1} - \frac{b}{p_2} - \frac{c(2+\beta)}{p_3} - \frac{2d}{p_4} + \omega\left(a\frac{\partial}{\partial \omega}\left(\frac{1}{p_1}\right) - b\frac{\partial}{\partial \omega}\left(\frac{1}{p_2}\right)\right.$$
$$\left. - c(2+\beta)\frac{\partial}{\partial \omega}\left(\frac{1}{p_3}\right) - 2d\frac{\partial}{\partial \omega}\left(\frac{1}{p_4}\right)\right),$$

where

$$\frac{\partial}{\partial \omega}\left(\frac{1}{p_1}\right) = \frac{-2\omega}{p_1^2} = \frac{-2\omega}{(1+\omega^2)^2},$$
$$\frac{\partial}{\partial \omega}\left(\frac{1}{p_2}\right) = \frac{-2\omega}{p_2^2} = \frac{-2\omega}{((1+\beta)^2+\omega^2)^2},$$
$$\frac{\partial}{\partial \omega}\left(\frac{1}{p_3}\right) = \frac{-2\omega(2+2\beta+2\omega^2+\beta^2)}{p_3^2} = \frac{-2\omega(2+2\beta+2\omega^2+\beta^2)}{(1+\omega^2)^2((1+\beta)^2+\omega^2)^2},$$
$$\frac{\partial}{\partial \omega}\left(\frac{1}{p_4}\right) = \frac{-4\omega}{(p_4)^{\frac{3}{2}}} = \frac{-4\omega}{(1+\omega^2)^3}.$$

Thus, it is obtained

	$(\omega_1, \hat{y}_{1_1})$	$(\omega_2, \hat{y}_{1_2})$
M	$9.5251799 * 10^{-5}$	$-2.6016338 * 10^{-3}$
$\frac{\partial F_2}{\partial \omega}$	$0.57795049\ (\neq 0)$	$3.0282261\ (\neq 0)$

(34)

Due to (33) and (34) and in agreement with Definition 3.1, it has been shown that degenerate Hopf bifurcations occur at $(\omega_i, \hat{y}_{1_i})$; $i = 1, 2$. Moreover, it can be verified that Proposition 3.3 (b) with $n = 3$ is satisfied at the analyzed points. Related to this, the following results have been attained

	$\hat{y}_{1_1}$	$\hat{y}_{1_2}$
$\frac{dH}{d\hat{y}_1}$	$1.6025978 * 10^{-4}$	$-8.5043342 * 10^{-4}$
$\frac{d^2H}{d\hat{y}_1^2}$	$-1.3805291 * 10^{-3}$	$2.1660876 * 10^{-2}$
$\frac{d^3H}{d\hat{y}_1^3}$	$4.8048344 * 10^{2}$	$-1.1006330 * 10^{4}$

(35)

Taking into account the functional dependence $D = D(\hat{y}_1)$, it is possible to transform all the results in the previous table, in terms of the main bifurcation parameter D, as stated in Proposition 3.3. Thereby,

	$(\omega_1,\, D_1)$	$(\omega_2,\, D_2)$
M	$-2.3657267 * 10^{-4}$	$9.5931639 * 10^{-3}$
Δ_1	$-3.9803013 * 10^{-4}$	$3.1358552 * 10^{-3}$
Δ_2	$-6.0164822 * 10^{-3}$	$-6.5966378 * 10^{-3}$
Δ_3	$-7.3619480 * 10^{3}$	$5.5182339 * 10^{5}$

where $\Delta_j = d^jH/dD^j$; $j = 1, 2, 3$. The last table allows to assert that the transversality condition of the Hopf theorem degenerates at the points (ω_i, D_i); $i = 1, 2$.

Henceforth, it remains to verify if all the assumptions of Theorem 4.3 satisfy at the points $(D_i, (\alpha_i, \beta_i))$; $i = 1, 2$. Therefore, we will analyze the solution set of the following equation with three variables

$$H(\hat{y}_1, (\alpha, \beta)) = G_1(\omega(\hat{y}_1), \hat{y}_1, (\alpha, \beta)) = -1\,.$$

It is known that

$$G_1(\omega_i, \hat{y}_{1_i}, (\alpha_i, \beta_i)) = F_1(\omega_i, \hat{y}_{1_i}, (\alpha_i, \beta_i)) - 1\,, \quad \text{where} \quad i = 1, 2\,.$$

At first, it will be proved that

$$J\left(H, \frac{\partial H}{\partial \hat{y}_1}\right) = \left|\begin{matrix} \dfrac{\partial H}{\partial \alpha} & \dfrac{\partial H}{\partial \beta} \\ \dfrac{\partial\left(\dfrac{\partial H}{\partial \hat{y}_1}\right)}{\partial \alpha} & \dfrac{\partial\left(\dfrac{\partial H}{\partial \hat{y}_1}\right)}{\partial \beta} \end{matrix}\right|_{(\alpha_i,\beta_i)} \neq 0, \quad \text{where} \quad i = 1, 2\,.$$

The required expressions for the calculation of the previous determinant are

$$\frac{\partial H}{\partial \alpha} = \frac{(1+\beta)}{p_2}\frac{\partial b}{\partial \alpha} + \frac{(1+\beta-\omega^2)}{p_3}\frac{\partial c}{\partial \alpha}\,,$$

$$\frac{\partial H}{\partial \beta} = \left[\frac{1}{p_2} + (1+\beta)\frac{\partial}{\partial \beta}\left(\frac{1}{p_2}\right)\right] b + \left[\frac{1}{p_3} + (1+\beta-\omega^2)\frac{\partial}{\partial \beta}\left(\frac{1}{p_3}\right)\right] c\,,$$

$$\frac{\partial\left(\dfrac{\partial H}{\partial \hat{y}_1}\right)}{\partial \alpha} = \frac{(1+\beta)}{p_2}\frac{\partial^2 b}{\partial \hat{y}_1 \partial \alpha} + \frac{(1+\beta-\omega^2)}{p_3}\frac{\partial^2 c}{\partial \hat{y}_1 \partial \alpha}\,,$$

$$\frac{\partial\left(\dfrac{\partial H}{\partial \hat{y}_1}\right)}{\partial \beta} = \left[\frac{1}{p_2} + (1+\beta)\frac{\partial}{\partial \beta}\left(\frac{1}{p_2}\right)\right]\frac{\partial b}{\partial \hat{y}_1} + \left[\frac{1}{p_3} + (1+\beta-\omega^2)\frac{\partial}{\partial \beta}\left(\frac{1}{p_3}\right)\right]\frac{\partial c}{\partial \hat{y}_1}\,,$$

where

$$\frac{\partial b}{\partial \alpha} = \frac{-\tilde{B}S\hat{y}_1^2}{1+(1-S)\hat{y}_1}, \quad \frac{\partial c}{\partial \alpha} = \frac{-\tilde{B}S\hat{y}_1^2(1-S\hat{y}_1)}{(1+(1-S)\hat{y}_1)\,(1+\hat{y}_1)}\,,$$

$$\frac{\partial}{\partial \beta}\left(\frac{1}{p_2}\right) = \frac{-2\,(1+\beta)}{((1+\beta)^2+\omega^2)^2}, \quad \frac{\partial}{\partial \beta}\left(\frac{1}{p_3}\right) = \frac{-2\,(1+\beta)}{(1+\omega^2)((1+\beta)^2+\omega^2)^2}\,,$$

$$\frac{\partial^2 b}{\partial \hat{y}_1 \partial \alpha} = \frac{-\tilde{B}S\hat{y}_1(2+y_1(1-S))}{(1+(1-S)\hat{y}_1)^2}\,,$$

$$\frac{\partial^2 c}{\partial \hat{y}_1 \partial \alpha} = \frac{-\tilde{B}S[-(1-S)S\hat{y}_1^4 - 2(2-S)S\hat{y}_1^3 + (2-4S)\hat{y}_1^2 + 2\hat{y}_1]}{[(1-S)\hat{y}_1^2+(2-S)\hat{y}_1+1]^2}\,.$$

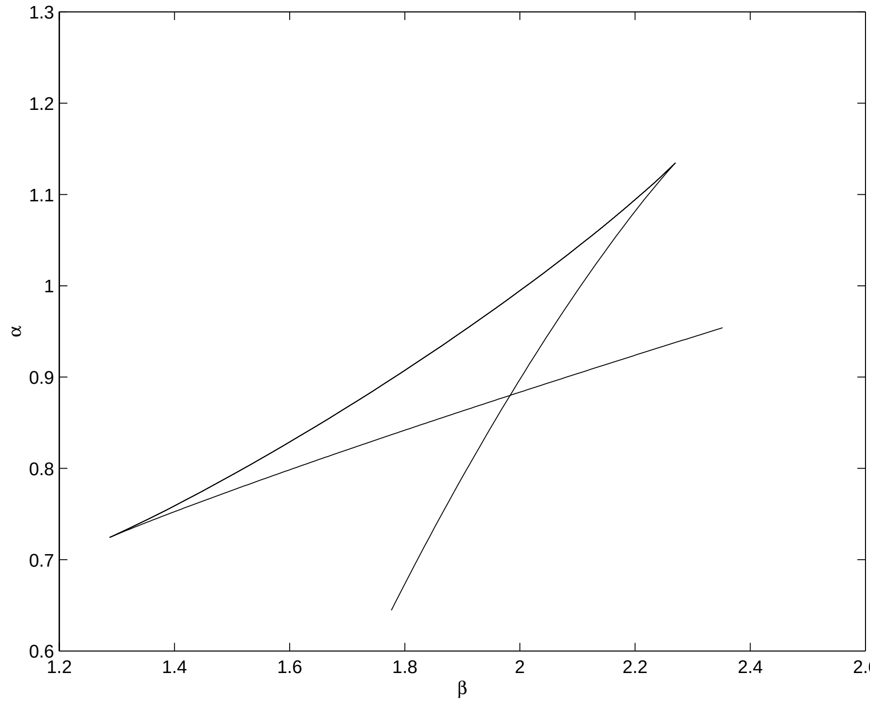

Fig. 1. Continuation of H_{01} singularities in the parameter space. The cusp points are H_{02} degeneracies ($\tilde{B} = 8$ and $S = 0.06$).

Thus,

$$J\left(H, \frac{\partial H}{\partial \hat{y}_1}\right)\Big|_{(\alpha_1,\beta_1)} = -1.3120942\,,$$

$$J\left(H, \frac{\partial H}{\partial \hat{y}_1}\right)\Big|_{(\alpha_2,\beta_2)} = -1.5422259 * 10^2\,.$$

Finally, using (35), the recently obtained values and keeping in mind $\hat{y}_1 = \hat{y}_1(D, \alpha, \beta)$ and some of its derivatives up to second order,

$$J\left(H, \frac{\partial H}{\partial D}\right)\Big|_{(\alpha_1,\beta_1)} = 3.2596971 \neq 0\,,$$

$$J\left(H, \frac{\partial H}{\partial D}\right)\Big|_{(\alpha_2,\beta_2)} = 5.6793746 * 10^2 \neq 0\,,$$

which imply, according with Theorem 4.3, that the Hopf points next to (α_1, β_1) and (α_2, β_2) define cusp singularities in the parameter space (α, β). A continuation algorithm is developed using the defining and nondegeneracy conditions in the frequency domain to compute the H_{01} curves in the parameter space with great accuracy, and then finding by continuation the two higher-order singularities (cusps), i.e. H_{02} degeneracies. These results are depicted in Fig. 1.

We have also checked here that carrying out a continuation in the parameter S from both H_{02} singularities, these H_{02} curves coalesce in a higher-order H_{03} singularity (reported before by Moiola and Chen [1996], but without giving the exact location in the parameter space). This computation result is depicted in Fig. 2. Thus, the analysis of the existing dynamic swallowtail, in the examined region of the space (α, β, S), is now completed and the calculations using the results of this paper follow.

For $S = S^* = 0.1034$, the described continuation process allows to find the following H_{02} points:

$$(\hat{y}^*_{1_1};\, \omega^*_1,\, (\alpha^*_1,\, \beta^*_1)) = (-0.8507932;\, 1.8393676,\, (0.8273686,\, 2.5895093))\,,$$

$$(\hat{y}^*_{1_2};\, \omega^*_2,\, (\alpha^*_2,\, \beta^*_2)) = (-0.8720325;\, 1.8760678,\, (0.8258880,\, 2.5843110))\,,$$

where these values are obtained

	$(\omega^*_1,\, D^*_1)$	$(\omega^*_2,\, D^*_2)$
M	$1.8137826 * 10^{-10}$	$8.2322366 * 10^{-10}$
Δ_1	$2.8416388 * 10^{-10}$	$1.6954332 * 10^{-9}$
Δ_2	$-2.5600696 * 10^{-9}$	$-2.9460126 * 10^{-8}$
Δ_3	$-2.0515642 * 10^2$	$1.8599129 * 10^2$.

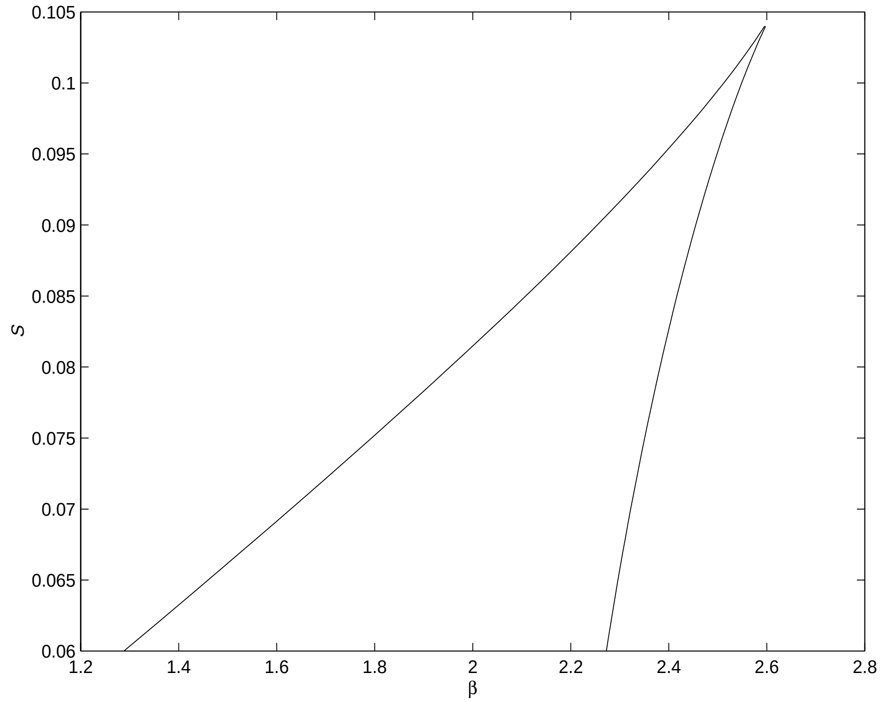

Fig. 2. Continuation of H_{02} degeneracies in the parameter space. In this case, the cusp point is a H_{03} singularity ($\tilde{B} = 8$).

Now, notice that the accuracy of the computations has been improved in relation with the results employing the previous seeds (which were obtained by using LOCBIF), since we have computed the zeroes of the defining conditions using the frequency domain formulas. However, the use of LOCBIF helped us to quite easily find the starting points.

For $S = S^{**} = 0.1046688$, the H_{02} points coalesce at

$$(\hat{y}_1^{**}; \omega^{**}, \alpha^{**}, \beta^{**}) = (-0.8615206; 1.8725690, \\ 0.8255016, 2.6074275),$$

where the following results appear

	(ω^{**}, D^{**})
M	$-2.9877608 * 10^{-12}$
Δ_1	$-2.9035251 * 10^{-12}$
Δ_2	$5.0495017 * 10^{-11}$
Δ_3	$-1.9039496 * 10^{-9}$

The last table points out the degenerate condition of the cusp singularity, at the found H_{03} point. Near this singularity under small perturbation in the auxiliary parameters up to four Hopf bifurcation points are organized.

6. Conclusions

In this work, certain types of degenerate Hopf bifurcations have been characterized in the frequency domain. Specifically, the cases where the transversality condition of the classical Hopf theorem fails and give place to the appearance of fold and cusp singularities, respectively, are analyzed in full detail. Furthermore, some nontrivial engineering examples of the application of stated results have been shown.

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