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# CONTROL SCHEMES FOR DISTURBANCE REJECTION: A CASE STUDY

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## Abstract

**Chemical processes are usually designed to work under a specified set of conditions; therefore the first goal of a regulatory control scheme is to keep the controlled variable at or near these conditions in the presence of disturbances and model uncertainty. In this paper, we present some control schemes to deal with some special problems of disturbance rejection. The usefulness of these alternatives is discussed within the context of controlling a simple flowsheet example.**

## 1. Introduction

The main objective of any regulatory feedback control scheme is to keep the controlled outputs "close" to their desired value. What is meant by "close" can be defined in terms of various stability and performance requirements for the closed-loop system. However, some process characteristics make it difficult. One of these factors is the presence of disturbances (often called load changes). The suppression of the impact that disturbances have on the operating behavior of processing units is one of the main reasons for the use of control in the chemical industry (Chang and Yu, 1992). In the last years systematic tools for quantifying the effect of disturbances, either on a simple unit or on a plant-wide level, have been introduced by several authors (Stanley *et al.*, 1985; Luyben, 1988; Skogestad and Morari, 1987; Chang and Yu, 1992, Skogestad and Wolff, 1992; Yi and Luyben, 1995; Belanger and Luyben, 1996).

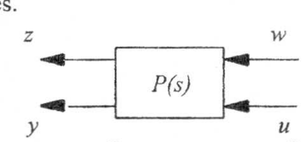
The design of compensators to produce closed-loop systems with certain disturbance attenuation capabilities has also been treated in the literature in terms of optimizing the sensitivity function using a feedback control strategy (for example Bhattacharyya *et al.* (1983), Vidyasagar (1986), etc.). Figueroa *et al.* (1996) propose a loop-shaping methodology to adjust compensators with the same control scheme. Lewin and Scali (1988) introduce the idea of robustness in a feedforward scheme. Later, Lewin (1991) used the concept of disturbance condition number to define a feedforward scheme. But, unfortunately, it is necessary to manipulate the disturbance signal to apply this scheme. Recently, Belanger and Luyben (1997) propose a design for feedback control in order to improve the disturbance rejection of a system with sluggish disturbance using a double integral or a lag compensator.

In this paper, several control schemes are discussed to design controllers with disturbance rejection specifications for diverse level of knowledge of the process. In all cases, the problem is solved as an $H_\infty$ optimal control problem in order to have a common reference of parameter setting for the controllers. All schemes are compared in their application to a simple example where their robustness properties are discussed.

The paper is organized as follows. The next section will highlight some ideas about the disturbance rejection problem and will present some control schemes in a general formulation. The usefulness of these techniques will be discussed within the context of a flowsheet example in section 3. Some general conclusions are included in section 4.

## 2. Problem Assessment

Consider the process model of Fig. 1, where the signal $w$ contains all external inputs (including disturbances, sensor noise, etc.), the output $z$ is an output signal, $y$ is the measured variable, $u$ is the control input and $P(s)$ is the process model. Note that this process could be a simple unit or a plant sector. In general the signal $w$ could be divided into two categories: measurable ($w_m$) and non measurable ($w_u$) disturbances.




$$P(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Fig.1. Process Model

The objective is to explore the use of alternative control schemes to obtain good disturbance rejection properties, depending on the information that can be obtained from the process. All controllers are designed with an $H_\infty$ criterion (a brief description about this criterion can be found in Appendix A) to have a

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common reference for schemes comparison. The goal is to find a stabilizable controller *K(s)* that solves $\min_{K(s)} \|T_{zw}\|_\infty$, where $T_{zw}$ is the closed loop transfer function from $w$ to $z$.

The main control objective in this paper is to reject disturbances. We are not trying to obtain a decentralized scheme. In some cases the response of the system to load changes can be degraded by using decoupling control schemes (Luyben, 1988). In this sense, the $H_\infty$ criterion provides a quasi optimal control in the sense of minimizing the disturbance effect in the process output. In general, the proposed schemes can be applied to a single unit as well as to complete plant sectors.

2.1 Feedback Scheme

Feedback is the most popular scheme for control systems. In feedback control, the information of the output $y$ is used to modify the control action $u$, as is shown in Fig. 2. This scheme is quite insensitive to modeling errors or parameter changes. In this case, the generalized plant *G(s)* in the Linear Fractional Transformation setup is given as

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right],$$

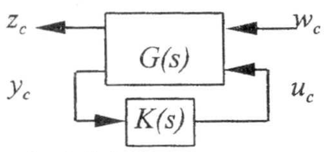



Fig. 2. Feedback Scheme

2.2 Feedforward Scheme

The largest disadvantage of the feedback control is that it takes control action only when the effect of the disturbance is observed at the process output; consequently it can be unsatisfactory for slow processes with large disturbances. When information from the disturbances is available, an alternative scheme to the classic feedback configuration is the use of a feedforward scheme (see for example Stephanopoulos, 1984; Skogestad and Postlethwaite, 1996). In this case the available (measurable) information from the disturbances is used to modify the control action $u$, as is shown in Fig. 3. In this case, the generalized plant *G(s)* in the Linear Fractional Transformation setup is given as

$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ 0 & \hat{E}_m & 0 \end{array}\right],$$

where the $\hat{E}_m$ is a matrix with as many rows as the number of measurable disturbances. This matrix produces the mapping from the set of disturbances $w_c$ into the $w_m$ (i.e., $w_m = \hat{E}_m w_c$). Usually, this scheme presents good disturbance rejection properties and it is inherently stable, but it is sensitive to the presence of non measurable disturbances and error in the models. The use of a feedback loop in combination with the feedforward scheme is a common practice to avoid these problems.

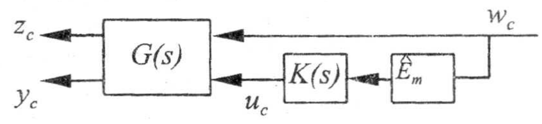
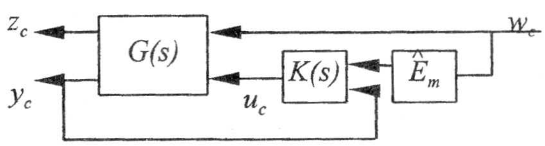



Fig. 3. Feedforward Scheme

2.3 Feedback - Feedforward Scheme

The combination of feedback and feedforward loops is shown in Fig. 4. This scheme combines the advantages of previous schemes. The idea is to use a feedforward compensator to reduce the influence of the disturbances on the outputs and a feedback controller to reduce the effect of non measurable disturbances or model errors. It requires the measurement of process outputs and disturbances. The control quality will be better as more information from the process is available. In general, the controller is not square because it has as many inputs as the number of the measurable variables ($y_c$) plus the number of measurable disturbances ($w_m$). In this case the control variable is $u_c = K\begin{bmatrix} w_m \\ y_c \end{bmatrix}$.

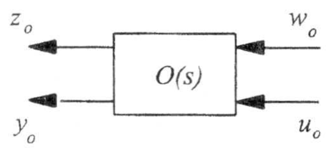



$$G(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ \begin{bmatrix} 0 \\ C_y \end{bmatrix} & \begin{bmatrix} \hat{E}_m \\ D_{yw} \end{bmatrix} & \begin{bmatrix} 0 \\ D_{yu} \end{bmatrix} \end{array}\right]$$

Fig. 4. Feedback-FeedForward Scheme

2.4 Disturbance Estimation Scheme

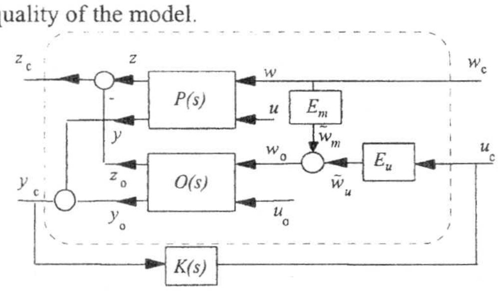
The quality of the feedforward schemes (Sections 2.2 and 2.3) will be as good as the knowledge of the disturbances is complete. Then, to improve the performance of these controllers, it is possible to use an estimator to complete the information of the non measurable disturbances (Colantonio *et al.* 1995; Muske and Edgar, 1997). Consider the model for the estimator as it is shown in Fig. 5 and let us assume that the disturbance vector is partitioned as $w = \begin{bmatrix} \tilde{w}_m \\ \tilde{w}_u \end{bmatrix}$.




$$O(s) = \left[\begin{array}{c|cc} A & B_w & B_u \\ \hline C_z & D_{zw} & D_{zu} \\ C_y & D_{yw} & D_{yu} \end{array}\right]$$

Fig. 5. Process Model

We can design an estimator scheme using the configuration of Fig. 6. The control objective of the estimator is to compute the non measurable disturbances (i.e., $u_c$ will follow $w_u$) to reduce the difference between the outputs of the process and its model ($z_c = z - z_o$) using the information of the difference in measured variables $y_c = y - y_o$. In this case, the matrices $E_m$ and $E_u$ are defined to equate $w_o = E_m w_c + E_u u_c$. The disturbance vector for the observer comprises information from measurable disturbances ($\tilde{w}_m = E_m w_c$) and estimation of the unmeasurable ones ($\tilde{w}_u = E_u u_c$). In this way, the Linear Fractional Transformation setup is given as: $\begin{bmatrix} z_c \\ y_c \end{bmatrix} = G(s) \begin{bmatrix} w_c \\ u_c \end{bmatrix}$. The advantage of this structure is that it allows us to complete the disturbance information using the output measurements. The quality of this estimation strongly depends on the quality of the model.




$$G(s) = \left[\begin{array}{c|c|c} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} & \begin{bmatrix} B_w \\ B_w E_m \end{bmatrix} & \begin{bmatrix} 0 \\ B_w E_u \end{bmatrix} \\ \hline \begin{bmatrix} C_z & -C_z \end{bmatrix} & D_{zw} - D_{zw} E_m & -D_{zw} E_u \\ \begin{bmatrix} C_y & -C_y \end{bmatrix} & D_{yw} - \varepsilon D_{yw} E_m & -\varepsilon D_{yw} E_u \end{array}\right]$$

Fig. 6. Estimator Scheme

To verify the conditions of existence of the $H_\infty$ controllers (A2 in Appendix A) it is necessary to consider an arbitrarily small variable $0<\varepsilon<<1$ that represents the model error. Now, we can use this scheme to estimate the complete set of disturbances to introduce a complete feedforward scheme.

2.5 Complete Feedforward Scheme

When a slow process is subject to large non measurable disturbances, we can use the disturbance estimator previously developed (section 2.4) to introduce a complete feedforward scheme as is shown in Fig. 7. In this case, the control variable affects both the disturbance input of the observer (i.e., $w_o$) to estimate the unmeasurable ones and the control inputs ($u$ and $u_o$) to minimize the effect of the disturbances in the process output variable ($z$). Note that the control objective will be to minimize the vector $z_c = \left[(z - z_o)^T \quad z^T\right]^T$ to both perform the estimate and reject the disturbances simultaneously. The measured variables in this scheme are a combination of the difference of the measured variable of the process and the measured disturbances (i.e., $y_c = \left[(y - y_o)^T \quad w_m{}^T\right]^T$, where $w_m = \hat{E}_m w$). The quality of this scheme depends on the quality of the disturbance estimation; hence, a good model for the process is necessary.

$$G(s) = \left[\begin{array}{c|cc} \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} & \begin{bmatrix} B_w \\ B_w E_m \end{bmatrix} & \begin{bmatrix} 0 & B_u \\ B_w E_u & B_u \end{bmatrix} \\ \hline \begin{bmatrix} C_z & -C_z \\ C_z & 0 \end{bmatrix} & D_{11} & D_{12} \\ \begin{bmatrix} C_y & -C_y \\ 0 & 0 \end{bmatrix} & D_{21} & D_{22} \end{array}\right]$$

$$D_{11}(s) = \begin{bmatrix} D_{zw} - D_{zw} E_m \\ D_{zw} \end{bmatrix} \quad D_{12} = \begin{bmatrix} -D_{zw} E_u & D_{zu} - \varepsilon D_{zu} \\ 0 & D_{zu} \end{bmatrix}$$

$$D_{21} = \begin{bmatrix} D_{yw} - \varepsilon D_{yw} E_m \\ \hat{E}_m \end{bmatrix} \quad D_{22} = \begin{bmatrix} -\varepsilon D_{yw} E_u & D_{yu} - \varepsilon D_{yu} \\ 0 & 0 \end{bmatrix}$$

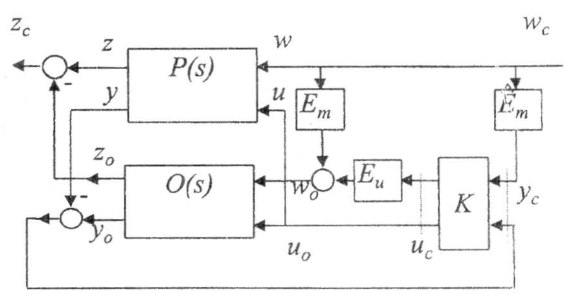



Fig. 7. Complete Feedforward Scheme

## 3. Example

The control structures discussed previously will now be tested with a flowsheet example. The case study consists of two stirred tank reactors (Fig. 8) in series, with an intermediate mixer introducing a second feed (de Hennin and Perkins, 1993; de Hennin *et al.*, 1994). A linear model for this process is defined in Appendix B, where $u=\left[F_{cw}^1 \;\; F_{cw}^2\right]^T$, $w=\left[C_F^1 \;\; T_F^1 \;\; C_2 \;\; T_2\right]^T$, $z\cong\left[C^1 \;\; T^1 \;\; C^2 \;\; T^2\right]^T$ and $y\cong\left[T^1 \;\; T^2\right]^T$. In this process the critical control objective is the disturbance rejection. The interaction between both reactors is not a control problem.

In this example, only the disturbances associated with the temperature are measurable (i.e., $w_m=\left[T_F^1 \;\; T_2\right]^T$). The expressions for $z$ and $y$ are very good approximations of those given in Appendix B.

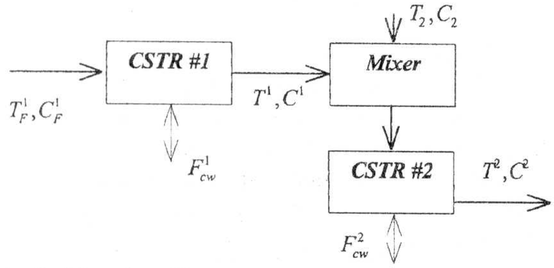



Fig. 8. Flowsheet Example

In order to normalize the variables (Rossi and Figueroa, 1997), we include two weighting matrices. The following variable transformations are considered for design purposes; $W_w w \mapsto w$ $\quad W_z z \mapsto z$,

where
$$W_w=\frac{1}{s+0.001}\begin{bmatrix} 4/20 & 0 & 0 & 0 \\ 0 & 4/30 & 0 & 0 \\ 0 & 0 & 4/20 & 0 \\ 0 & 0 & 0 & 4/30 \end{bmatrix}$$
and
$$W_z=\begin{bmatrix} 1/10 & 0 & 0 & 0 \\ 0 & 1/100 & 0 & 0 \\ 0 & 0 & 1/10 & 0 \\ 0 & 0 & 0 & 1/100 \end{bmatrix}.$$

Note that the quasi integrator term in $W_w$ is included to allow the $H_\infty$ optimal controllers to manage step disturbance inputs. In a similar way, we can include a "double" integrator to reject "ramp" disturbances. The performance of the following control schemes are analyzed: Feedback, Feedforward, Feedback-Feedforward, Disturbance Estimation and Complete Feedforward.

### 3.1 Feedback Scheme

In this case the manipulated variables are the cooling water flow in both reactors (i.e. $u_c = u$), the measured output is $y$ and the control objective is $z$. In this scheme three settings are considered:

*3.1.a. Ziegler – Nichols:* Two simple PID controllers are tuned using the classical Ziegler - Nichols settings. The performance of the closed-loop system is shown in Fig. 9, where simulation response for steps of amplitude 2 and 20 applied to $C_F^1$ and $T_F^1$, respectively, are presented. From this plot, it is clear that the behavior of the controllers is poor when load changes are present. Other alternatives, such as those presented by Belanger and Luyben (1997), are not useful because the disturbances do not have large low frequency components.

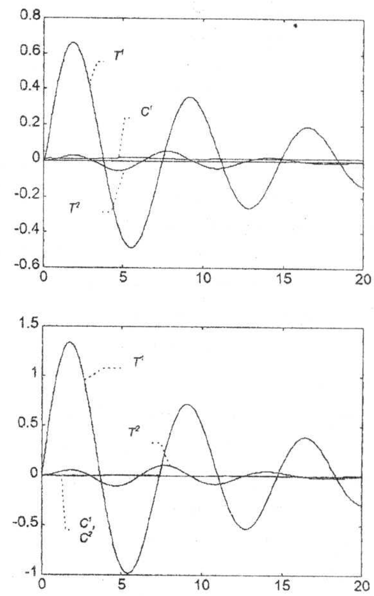


Fig. 9: Time simulation responses for Feedback Ziegler - Nichols Controller

*3.1.b. Decentralized $\mathcal{H}_\infty$ controller:* Two individual decentralized $\mathcal{H}_\infty$ controllers are tuned. For these controllers, we obtain a closed loop norm of $\|T_{zw}\|_\infty = 0.56$. Fig. 10 shows a simulation response for steps applied to $C_F^1$ and $T_F^1$, respectively. From this plot, it is clear that the behavior of these controllers is significantly better than the ones presented in Section 3.1.a.

*3.1.c. Multivariable $\mathcal{H}_\infty$ controller:* A complete $\mathcal{H}_\infty$ controller is tuned. In this case a closed loop norm of $\|T_{zw}\|_\infty = 0.2340$ is obtained. This value is smaller than the one obtained with the decentralized controller. Fig. 11 shows a simulation response for steps applied to $C_F^1$ and $T_F^1$, respectively. Note that simulation results are close to the ones obtained for the decentralized controller. However, the lower value of the norm implies that, for a general disturbance, the behavior of this controller will be preferred.

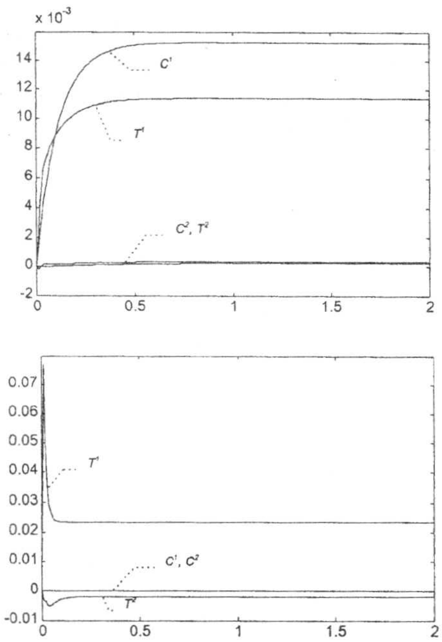



Fig. 10: Time simulation responses for decentralized $\mathcal{H}_\infty$ Controller

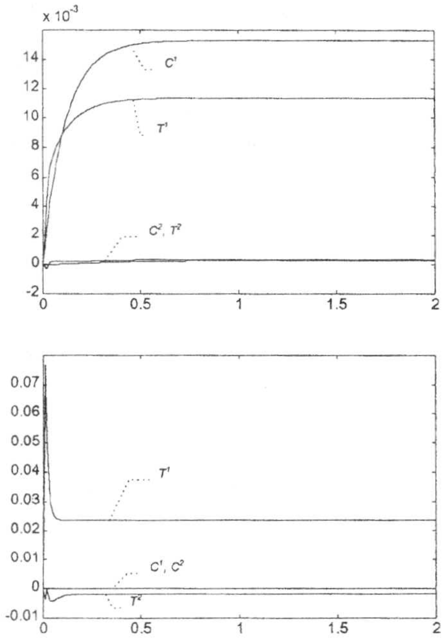



Fig. 11: Time simulation responses for Complete $\mathcal{H}_\infty$ Controller

3.2 Feedforward

In the Feedforward scheme the manipulated variables $u$ are modified using information of the measurable disturbances ($w_m$). For this compensator we obtain a closed loop norm of $\|T_{zw}\|_\infty = 7.3074$ while a time simulation for steps applied in $C_F^1$ and $T_F^1$ is shown in Fig. 12. The value of $\|T_{zw}\|_\infty$ is larger than the one obtained for the feedback controller. Also, the performance of the simulation for a $C_F^1$ input is worse than the one obtained in the feedback scheme. This is due to the fact that $C_F^1$ is not a measurable disturbance; then the controller does not record the presence of a disturbance (this is an open loop scheme). However, when the disturbance is within the set of measurable ones ($T_F^1$), the performance of this loop is excellent.

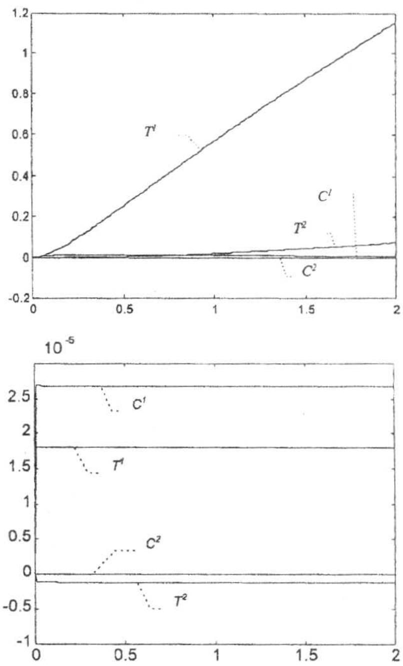



Fig. 12: Time simulation responses for Feedforward-Controller

3.3 Feedback - Feedforward

In this case we try to obtain the benefits of the Feedback and Feedforward schemes working together. The manipulated variables $u$ are modified using information of the measurable disturbances ($w_m$) and the measured variables ($y$). For this controller we obtain a closed loop norm of $\|T_{zw}\|_\infty = 0.2278$ while a time simulation for steps applied in $C_F^1$ and $T_F^1$ is shown in Fig. 13. The value of $\|T_{zw}\|_\infty$ is smaller than the one obtained for the feedback controller. This is due to the fact that we are using information from measurable disturbances ($w_m = \left[T_F^1 \quad T_2\right]^T$) to reject them. In this case the system's response to a disturbance in $C_F^1$ is similar to that obtained for a feedback scheme, while the response to a step in $T_F^1$ is close to the behavior of the feedforward scheme. We obtain the benefits of both control schemes.

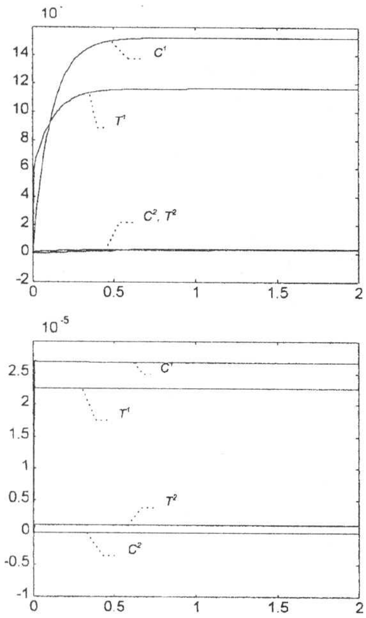



Fig. 13 Time simulation responses for Feedback-Feedforward Controller

3.4 Disturbance Estimation

In order to make good use of the advantages of the feedforward control in response to non measured disturbances, we will try to reproduce the values of these variables making $u_c = [C_F^1 \;\; C_2]^T$ and using as measurable inputs the difference between the measured variables of the process model and of the observer. For this estimator, we obtain a closed loop norm of $\|T_{zw}\|_\infty = 7.8516$ while a time simulation of the states $C^I$ and $T^I$ and their estimation for steps applied in $C_F^1$ and $T_F^1$ are shown in Fig. 14. There is no appreciable difference of the process states and their estimation.

3.5 Complete Feedforward

In this case we will use the results of the state estimator studied in the last section to perform a complete feedforward. To do this, we will try to reproduce the values of the non measurable disturbances, and to use this result to reject disturbances. We will use the difference between the measured variables of the process model and the observer and the measurable disturbances as measurable output ($y_c$). Then, we will use this information to modify the manipulated variables, defined as $u_c = [C_F^1 \;\; C_2 \;\; F_{cw}^1 \;\; F_{cw}^2]^T$.

This controller is shown in Appendix C. We obtain a closed loop norm of $\|T_{zw}\|_\infty = 0.2354$. This value is larger than in the case of Feedback-Feedforward, because it also includes information about the estimation. Time simulations of the states $C^I$ and $T^I$ for steps applied in $C_F^1$ and $T_F^1$ are shown in Fig. 15. Note the excellent disturbance rejection properties of this control scheme.

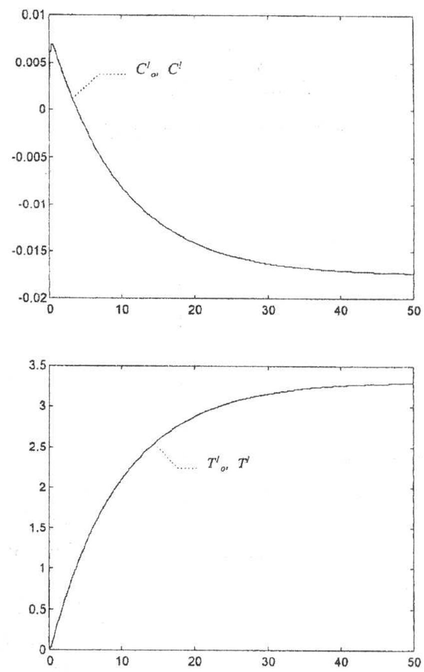



Fig. 14 Time simulation responses for Disturbance Estimator

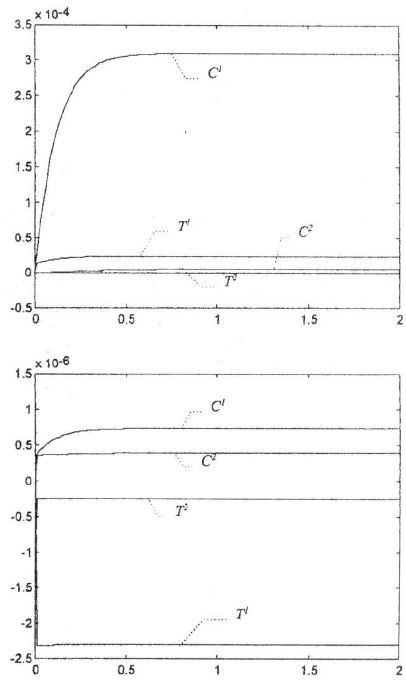



Fig. 15 Time simulation responses for Complete Feedforward

3.6 Robustness Analysis

At this point, it is important to perform a robustness analysis to determine the behavior of these control schemes when model uncertainties are considered. An input multiplicative uncertainty ($\Delta_u$) is introduced to the model as is shown in Fig. 16, where $\gamma_u = \left|\frac{s+1.2}{20s+6}\right|$ and $\|\Delta_u\| \le 1$. In order to normalize the performance measure a weighting constant $\gamma$ is included, where $\gamma = 1/\|T_{zw}\|_\infty$. As a measure of robust performance the $\mu$-test is analyzed (Zhou *et al.*, 1996) for the control schemes studied above. Numerical results are shown in Fig. 17. From the plot (A) it is clear that the feedback controller, which has a good nominal performance, presents a high value at high frequencies. This peak is due to the incapability of the controller to deal with uncertainties at this frequency range. The same problem is present in the combination of feedforward-feedback. This peak is not present in plots (B) and (D). In plot B the value of the $\mu$ function is small because in this case $\gamma$=7.3054. However, plot (D) shows a value larger than one at low frequencies. This is specifically due to the robust performance analysis (i.e., it is not a stability problem). Moreover, this value is larger than one only because the normalization constant $\gamma$=0.2354 used in this analysis. In conclusion, from this analysis we can appreciate the good performance of the complete feedforward scheme.

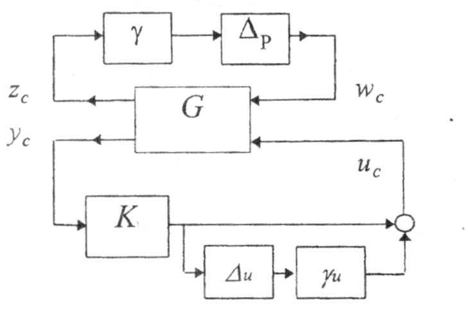



Fig. 16. Block Diagram for Robustness Analysis

## 4. Conclusions

In this paper, we studied the problem of disturbance rejection in the frequency domain. Several control schemes to design controllers with disturbance rejection specifications are discussed and their robust stability properties are analyzed. In all cases, the design problem is solved as an $H_\infty$ optimal control problem. The so called *complete feedforward* scheme presented excellent properties as regards time and frequency domains analysis. The usefulness of this technique was discussed in terms of controlling a flowsheet example where some robustness aspects were also discussed.

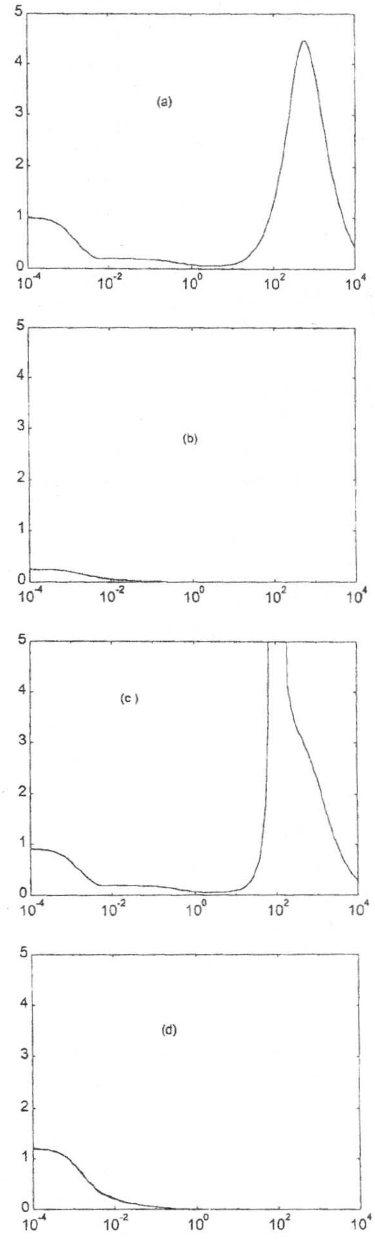



Fig. 17. $\mu$ as function of the Frequen for (a) Feedback, (B) Feedforward, (c) dback-Feedforward and (d) Complete Feedforw

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**Appendix A: Optimal Control Theory**

Consider now the linear system described by the block diagram of Fig. A1,

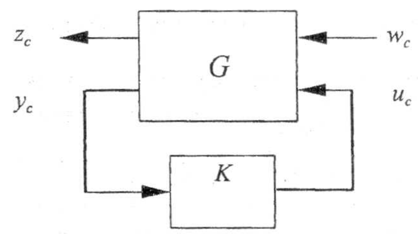


Fig. A1: Linear Fractional Representation

where $G(s)=\left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]$ is the generalized plant, $K$ is the controller and the diagram in Figure A1 is referred to as a linear fractional transformation (LFT). The resulting closed loop transfer function from $w_c$ to $z_c$ is denoted by $T_{zw}=\mathcal{F}_\ell(G,K)$. $G$ and $K$ are assumed to be real rational and proper with $K$ constrained to provide internal stability, such a controller is called admissible.

Moreover, the following assumptions (Zhou *et al.*, 1995) are made about the system $G$:

**(A1)** $(A,B_2)$ is stabilizable and $(C_2,A)$ is detectable.

**(A2)** $D_{12}$ is full column rank and $D_{21}$ is full row rank.

**(A3)** $\begin{bmatrix} A-j\omega I & B_2 \\ C_1 & D_{12} \end{bmatrix}$ has full column rank for all $\omega$.

**(A4)** $\begin{bmatrix} A-j\omega I & B_1 \\ C_2 & D_{21} \end{bmatrix}$ has full row rank for all $\omega$.

and we will also suppose that $D_{11}=D_{22}=0$.

**Suboptimal $H_\infty$ Control.** *Given $\gamma>0$, find all admissible controllers K(s), if there are any, such that* $\|T_{zw}\|_\infty<\gamma$.

The $H_\infty$ solution involves the following Hamiltonian matrices,

$$H_\infty \triangleq \begin{bmatrix} A & \gamma^{-2}B_1B_1' - B_2B_2' \\ -C_1'C_1 & -A' \end{bmatrix}$$

$$J_\infty \triangleq \begin{bmatrix} A' & \gamma^{-2}C_1'C_1 - C_2'C_2 \\ -B_1B_1' & -A \end{bmatrix}$$

Then, for a given γ value it is possible to determine whether or not a solution exists. The conditions for existence of a solution are (Zhou *et al.*, 1996),

(i) $H_\infty$ and $J_\infty$ must have no imaginary-axis eigenvalues.

(ii) Eigenvalues of $X_\infty = Ric(H_\infty)$ and $Y_\infty = Ric(J_\infty)$ must be greater than zero, where $X_\infty$ and $Y_\infty$ are the solutions to the Riccati equations related to the matrices $H_\infty$ and $J_\infty$, respectively.

(iii) $\rho(X_\infty Y_\infty) < \gamma^2$, where $\rho(.)$ is the spectral radius.

Moreover, when these conditions hold, one such controller is

$$K_{sub}(s) = \left[\begin{array}{c|c} \hat{A} & \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1} Y_\infty C_2' \\ \hline -B_2' X_\infty & 0 \end{array}\right]$$

with $\hat{A} = A + \gamma^{-2} B_1 B_1' X_\infty - B_2 B_2' X_\infty - \left(I - \gamma^{-2} Y_\infty X_\infty\right)^{-1} Y_\infty C_2' C_2$

Conditions (i)-(iii) are necessary and sufficient for the existence of an admissible controller *K(s)* such that $\|T_{zw}\|_\infty < \gamma$ for a given γ. If $\gamma_{opt} = \min\{\|T_{zw}\|_\infty : K \ adm.$ is the optimum value for γ, then it is obvious that γ must be larger than $\gamma_{opt}$ for the existence of $H_\infty$ suboptimal controllers. Moreover, conditions (i)-(iii) allow us to formulate a bisection method to compute a value of γ close to its optimal value $\gamma_{opt}$.

## Appendix B: Linear Model for the CSTR System

In this example, we consider four states ($C^1$, $T^1$, $C^2$ and $T^2$), two control inputs ($F_{cw}^1$ and $F_{cw}^2$) and four disturbances ($C_F^1, T_F^1$, $C_2$ and $T_2$). The model is as follows,

$$\begin{bmatrix} \dot{C}^1 \\ \dot{T}^1 \\ \dot{C}^2 \\ \dot{T}^2 \end{bmatrix} = \begin{bmatrix} -8.5297 & -0.0647 & 0 & 0 \\ 42.3122 & 0.2229 & 0 & 0 \\ 0.0655 & 0 & -3.9510 & -0.0485 \\ 0 & 0.0655 & 19.2008 & 0.0829 \end{bmatrix} \begin{bmatrix} C^1 \\ T^1 \\ C^2 \\ T^2 \end{bmatrix} + \ldots$$

$$\begin{bmatrix} 0.0655 & 0 & 0 & 0 \\ 0 & 0.0655 & 0 & 0 \\ 0 & 0 & 0.0453 & 0 \\ 0 & 0 & 0 & 0.0453 \end{bmatrix} \begin{bmatrix} C_F^1 \\ T_F^1 \\ C_2 \\ T_2 \end{bmatrix} + \ldots$$

$$\begin{bmatrix} 0 & 0 \\ -4.8630 & 0 \\ 0 & 0 \\ 0 & -1.5204 \end{bmatrix} \begin{bmatrix} F_{cw}^1 \\ F_{cw}^2 \end{bmatrix}$$

$$z_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C^1 \\ T^1 \\ C^2 \\ T^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_F^1 \\ T_F^1 \\ C_2 \\ T_2 \end{bmatrix} + \ldots$$

$$\begin{bmatrix} 10^{-4} & 0 \\ 0 & 10^{-4} \\ 10^{-4} & 0 \\ 0 & 10^{-4} \end{bmatrix} \begin{bmatrix} F_{cw}^1 \\ F_{cw}^2 \end{bmatrix}$$

$$y_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C^1 \\ T^1 \\ C^2 \\ T^2 \end{bmatrix} + \begin{bmatrix} 10^{-4} & 0 & 10^{-4} & 0 \\ 0 & 10^{-4} & 0 & 10^{-4} \end{bmatrix} \cdot$$

$$\begin{bmatrix} C_F^1 \\ T_F^1 \\ C_2 \\ T_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} F_{cw}^1 \\ F_{cw}^2 \end{bmatrix}$$

## Appendix C: Complete Feedforward Controller

Matrix A (Columns 1-7)

| | | | | | | |
|---|---|---|---|---|---|---|
| -331.21 | -351.29 | 7.54665 | 327.901 | 321.844 | 353.403 | -7.7985 |
| -10377. | -16038. | -11252.8 | -1796.8 | 11656.9 | 12812.3 | 11630.3 |
| 4.9031 | 7.21990 | -226.394 | -249.56 | -5.4178 | -5.7167 | 2.22269 |
| -683.24 | 1831.29 | 6408.719 | -10539. | -2.7692 | 0.16340 | -1.2786 |
| 322.687 | -303.76 | -7.5466 | 327.098 | -3.3037 | 3.01531 | 0.07798 |
| -10420. | -16038. | -11252.89 | -0.1796 | 11699.2 | 12812.5 | 11630.3 |
| -4.8376 | -7.2199 | 222.444 | -203.88 | 5.4833 | 5.7167 | -226.22 |
| -683.24 | 1831.2 | 6389.5 | -10540. | -2.7692 | 0.22890 | 17.9221 |
| -492652 | -536234 | 115216.1 | 500612 | 491366 | 539547 | -119061 |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |
| 106696 | 159239 | -4906131 | -550315 | -119493 | -126085 | 490227 |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 0.00 |

Matrix A (Columns 8-14)

| | | | | | | |
|---|---|---|---|---|---|---|
| -327.52 | -0.0051 | 0.000405 | 0.00157 | 0.00005 | 4491.34 | -4600.8 |
| 1236.42 | 7.4232 | -0.673942 | -2.3115 | -0.0504 | 162662. | 0.00 |
| 2.49774 | -0.0035 | 0.000282 | 0.00109 | 0.00003 | -75.828 | 3184.79 |
| 0.08632 | -5.0774 | -0.50812 | 77.1777 | -1.7446 | 0.00 | 0.00 |
| -3.2747 | 0.00005 | 0.00 | 0.00 | 0.00 | 4.7104 | -4.6008 |
| 0.12364 | 7.42327 | -0.673942 | -2.3115 | -0.0504 | 162662. | 0.00 |
| 203.576 | 0.00358 | -0.00028 | -0.0010 | -0.0000 | 75.8284 | 3.18479 |
| 0.16923 | -5.0774 | -0.50812 | 77.1777 | -1.7446 | 0.0000 | 0.0000 |
| -500036 | -1078.4 | 6.19290 | 24.1095 | 0.77475 | $6.8x10^7$ | $-7.x10^7$ |
| 0.00 | 0.00 | 1000.000 | 0.00 | 0.00 | 0.00 | 0.00 |
| 550892 | -79.035 | 6.23066 | -975.80 | 0.66479 | $-.1x10^7$ | $7.0x10^7$ |
| 0.00 | 0.00 | 0.00 | 0.00 | 1000.00 | 0.00 | 0.00 |

Matrix B

| | |
|---|---|
| -0.04600 | -0.00109 |
| 0.920177 | 1.62662 |
| 0.03184 | 0.03108 |
| 10.57400 | 11.21096 |
| -0.00005 | 0.000001 |
| 0.920177 | 1.62662 |
| 0.03184 | 0.032606 |
| 10.5740 | 11.21096 |
| -702.42 | -16.724 |
| 14048.506 | 0.00 |
| 702.425 | 685.700 |
| 0.00 | 14048.5 |

Matrix C (Columns 1-7)

| | | | | | | |
|---|---|---|---|---|---|---|
| 350.679 | -330.11 | -8.2013 | 355.47 | -349.76 | 327.759 | 8.4750 |
| -7.594 | -11.334 | 349.22 | -320.09 | 8.5058 | 8.97503 | -348.95 |
| 152.525 | 234.76 | 164.71 | 26.301 | -170.62 | -187.53 | -170.23 |
| 31.9877 | -85.733 | -299.14 | 493.457 | 0.1296 | -0.0076 | 0.05986 |

Matrix C (Columns 8-14)

| | | | | | | |
|---|---|---|---|---|---|---|
| -355.88 | 0.00558 | -0.0004 | -0.0017 | $-5x10^{-5}$ | 5119.04 | -5000.0 |
| 319.68 | 0.00562 | -0.0004 | -0.0017 | $-4x10^{-5}$ | 119.047 | 5000.0 |
| -18.097 | -0.1086 | 0.00986 | 0.03383 | 0.00073 | -2380.9 | 0.000 |
| -0.0040 | 0.23771 | 0.02378 | -3.6132 | 0.08168 | 0.00 | 0.00 |

Matrix D

| | |
|---|---|
| -0.05000 | 0.00119 |
| 0.0500 | 0.05119 |
| 0.00 | -0.023809 |
| -0.495049 | -0.495049 |

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