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CHARACTERIZATION OF STATIC BIFURCATIONS IN THE FREQUENCY DOMAIN

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In this paper two-dimensional systems with static bifurcations are considered. An analysis of the bifurcation behavior is proposed using a frequency domain approach. The analyzed bifurcations are known as elementary since they are the building blocks to understand other more complex singularities.

1. Introduction

The defining conditions which cause the appearance of static and dynamical bifurcations are essentially important for the proper design of the engineering systems. In almost all cases, it is desirable to avoid the conditions which generate more "complex" dynamics, like the appearance of multiple equilibrium solutions (related to static bifurcations), and oscillatory solutions (related to dynamical bifurcations). In both cases, the bifurcation condition states that one eigenvalue (or a pair of eigenvalues) of the linearized system must cross the imaginary axis, causing (or not) a stability change of the equilibrium point. For the case of a nonlinear system, it is necessary to analyze the "leading" terms which follow the linear one in order to classify the bifurcation.

In this article, the applied methodology comes from the theory of multivariable control systems known as the frequency domain method. The theory for the stability analysis in linear systems was completed by MacFarlane and Postlethwaite [1977], and its applications were pointed out by Edmunds [1979]. In [Mees & Chua, 1979] and [Mees, 1981], the problem of analyzing the dynamical bifurcation (Hopf bifurcation) was dealt with, while in [Moiola & Chen, 1993, 1996], some defining conditions (generic) for the static bifurcations were established. Moreover, a closely related approach was used by Llibre and Ponce [1996] and Llibre and Sotomayor [1996] to classify the global behavior of certain static bifurcations in piecewise-linear control systems.

The present work extends some previous results stated in [Moiola *et al.*, 1997] with the aim of classification of different elementary bifurcations by using the frequency domain formalism. These results should be of interest for bifurcation control of elementary bifurcations, where some of the most representative works have been presented by Abed and Fu [1986, 1987], Kang and Krener [1992], Kang [1998] and Kang *et al.* [1999], although they have used the more traditional time domain formulation.

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2. Preliminaries

Consider a general n-dimensional nonlinear system

$$\begin{aligned} \dot{x} &= f(x;\, \mu)\,, \\ x(0) &= 0\,, \end{aligned} \tag{1}$$

where μ is a real parameter and f satisfies adequate conditions to guarantee the existence and uniqueness of this initial value problem.

If the previous equation is written in a state-variable form, the nonlinear system (1), results in

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)g(y;\, \mu)\,, \\ y &= -C(\mu)x\,, \\ x(0) &= 0\,, \end{aligned} \tag{2}$$

where A is an $n \times n$ matrix, which can be arbitrarily chosen for convenience (invertible and stable for all values of μ), B and C are $n \times p$ and $m \times n$ matrices, respectively, and $g(y;\, \mu)$ is a $p \times 1$ nonlinear vectorial function which belongs to C^4.

Introducing in the system (2) a state-feedback control $u = g(y;\, \mu)$, a linear system is obtained with a nonlinear control variable as follows

$$\begin{aligned} \dot{x} &= A(\mu)x + B(\mu)u\,, \\ y &= -C(\mu)x\,, \\ u &= g(y;\, \mu)\,. \end{aligned} \tag{3}$$

Taking Laplace transform in (3), with zero-initial condition, yields

$$\mathcal{L}(y) = -G(s;\, \mu)\mathcal{L}(g(y;\, \mu))\,, \tag{4}$$

where $G(s;\, \mu) = C(\mu)[sI - A(\mu)]^{-1}B(\mu)$ is the usual transfer matrix of the linear part of (3). From the last equation, the original problem can be solved for the variable y (contained in $\mathcal{L}(.)$) in the so-called frequency domain. Therefore if $\hat{x}(t;\, \mu)$ is an equilibrium solution of (3), then $\hat{y}(t;\, \mu) = -C(\mu)\hat{x}(t;\, \mu)$ can be considered as an equilibrium solution in the frequency domain. Taking inverse Laplace transform in (4), yields

$$\hat{y}(t;\, \mu) = -G(0;\, \mu)g(\hat{y}(t;\, \mu);\, \mu)\,, \tag{5}$$

where $-G(0;\, \mu) = C(\mu)[A(\mu)]^{-1}B(\mu)$.

Linearizing (3) about the equilibrium $\hat{y}(t;\, \mu)$, a system with the following transfer matrix is obtained

$$G(s;\, \mu)J(\mu)\,, \tag{6}$$

where $J(\mu) = (\partial g/\partial y)|_{y=\hat{y}}$.

An application of the generalized Nyquist stability criterion [MacFarlane & Postlethwaite, 1977], where $s = i\omega$ gives the following result

Lemma 1. *If an eigenvalue of the Jacobian of the system (3), in the time domain, takes a purely imaginary value $i\omega_0$ at a particular value $\mu = \mu_0$, then the corresponding eigenvalue of the constant matrix $G(i\omega_0;\, \mu_0)J(\mu_0)$ in the frequency domain must take the value $-1 + i0$ at $\mu = \mu_0$.*

Let $\hat{\lambda} = \hat{\lambda}(i\omega;\, \mu)$ be an eigenvalue of the matrix $G(i\omega;\, \mu)J(\mu)$ which satisfies $\hat{\lambda}(i\omega_0;\, \mu_0) = -1 + i0$. Fixing $\mu = \tilde{\mu}$ and varying ω, the locus of the eigenvalue or "eigenlocus" is obtained. When there is a dynamical bifurcation, this locus is analyzed for $\omega_0 \neq 0$. Thus, a real zero eigenvalue, which is a necessary condition for static bifurcation in the time domain formulation, is associated with an eigenlocus that crosses the point $-1 + i0$ at $\omega_0 = 0$ in the frequency domain counterpart.

In general, the eigenvalues $\lambda(s;\, \mu)$ of (6) are the solutions of the following algebraic equation

$$h(\lambda,\, s;\, \mu) = \det(\lambda I - G(s;\, \mu)J(\mu)) = 0 \tag{7}$$

or

$$h(\lambda,\, s;\, \mu) = \sum_{i=0}^{K} a_i(s;\, \mu)\lambda^i = 0\,, \tag{8}$$

where $K = \min(m,\, p)$ (the matrices G and J have ranks at most K), $a_K(s;\, \mu) \equiv 1$ and the remaining coefficients $a_k(s;\, \mu)$ are rational functions in s as follows:

$$a_k(s;\, \mu) = \frac{\displaystyle\sum_{l_1=0}^{p_k} \beta_{p_k - l_1, k} \cdot s^{p_k - l_1}}{\displaystyle\sum_{l_2=0}^{q_k} \alpha_{q_k - l_2, k} \cdot s^{q_k - l_2}}\,,$$

$$k = 0,\, 1, \ldots,\, K-1\,,$$

where $\beta_{i_1,k}(\mu)$ and $\alpha_{i_2,k}(\mu)$ are real functions of the parameter μ. From now on, suppose that the functions $a_k(s;\, \mu)$ have no poles on the imaginary axis. Moreover, consider that $\lambda = -1$ is a simple root of (7).

Taking into account Lemma 1 and imposing the condition $\lambda = -1$ in (8), a necessary relationship between $s = i\omega$ and μ to find a bifurcation point is obtained. Therefore

$$h(-1,\, i\omega;\, \mu) = (-1)^K + \sum_{k=0}^{K-1} (-1)^k a_k(i\omega;\, \mu) = 0\,. \tag{9}$$

By separating the expression (9) into real ($\Re$) and imaginary ($\Im$) parts, the following is attained

$$F_1(\omega,\,\mu) = \Re\{h(-1,\,i\omega;\,\mu)\} = (-1)^K + \sum_{k=0}^{K-1}(-1)^k\Re\{a_k(i\omega;\,\mu)\} = 0\,, \tag{10}$$

$$F_2(\omega,\,\mu) = \Im\{h(-1,\,i\omega;\,\mu)\} = \sum_{k=0}^{K-1}(-1)^k\Im\{a_k(i\omega;\,\mu)\} = 0\,. \tag{11}$$

It is possible to set up bifurcation conditions upon F_1, F_2, and their partial derivatives to find and classify singular points [Moiola & Chen, 1993].

Definition 1. A static bifurcation condition is obtained satisfying (10) and (11) with $\omega_0 = 0$.

Proposition 1. *To have a static bifurcation condition, it is necessary that* $F_1(0,\,\mu) = 0$ [*Moiola & Chen, 1996*].

3. Two-Dimensional Systems

It is attempted to study two-dimensional systems as (1) (with zero-initial condition, and an equilibrium point $\hat{x} = 0$), whose Jacobian evaluated at $\hat{x}$ has one negative eigenvalue and the other equals to zero. The defining conditions correspond to the named elementary static bifurcations. By calculating the nondegeneracy conditions, the type of bifurcation involved can be determined. This section has the goal of specifying the frequency formulation which allows to calculate the aforementioned conditions.

Consider the general system in a state-variable formulation

$$\begin{aligned}\dot{x}_1 &= f_1(x_1,\,x_2;\,\mu)\,,\\ \dot{x}_2 &= f_2(x_1,\,x_2;\,\mu)\,.\end{aligned} \tag{12}$$

It is known that, changing coordinates and making use of the invariant manifolds, it is possible to rewrite the system (12) locally, as

$$\begin{aligned}\dot{y}_1 &= g_1(y_1;\,\mu)\,,\\ \dot{y}_2 &= -\beta(\mu)y_2\,,\end{aligned} \tag{13}$$

where $\beta(\mu) > 0\ \forall\,\mu$. It is supposed that when μ crosses 0, the number of equilibrium solutions changes. So, there is a bifurcation point at $(\hat{y}_1,\,\mu) = (0,\,0)$. Henceforth, the bifurcation analysis of the system (12) at $(0, 0)$, will be directly connected with the respective analysis of the system (13).

By carrying out a realization of the system (13), just to solve it in the frequency domain, the following is obtained

$$\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix} = A\begin{pmatrix}y_1\\ y_2\end{pmatrix} + Bu(z_1,\,z_2;\,\mu)\,,$$

where

$$A = \begin{pmatrix}-1 & 0\\ 0 & \dfrac{-\beta(\mu)}{2}\end{pmatrix},\quad B = C = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

and as $\begin{pmatrix}z_1\\ z_2\end{pmatrix} = -C\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}-y_1\\ -y_2\end{pmatrix}$ results

$$u(z_1,\,z_2;\,\mu) = \begin{pmatrix}u_1(z_1,\,z_2;\,\mu)\\ u_2(z_1,\,z_2;\,\mu)\end{pmatrix}, = \begin{pmatrix}g_1(-z_1;\,\mu) - z_1\\ \dfrac{\beta(\mu)}{2}z_2\end{pmatrix}.$$

In this case

$$G(0,\,\mu) = \begin{pmatrix}1 & 0\\ 0 & \dfrac{2}{\beta(\mu)}\end{pmatrix}$$

and according to (5), the following nonlinear system must be solved in the frequency domain

$$-\begin{pmatrix}\hat{z}_1\\ \hat{z}_2\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & \dfrac{2}{\beta(\mu)}\end{pmatrix}\begin{pmatrix}g_1(-\hat{z}_1;\,\mu) - \hat{z}_1\\ \dfrac{\beta(\mu)}{2}\hat{z}_2\end{pmatrix}.$$

The equilibrium solutions $\hat{z} = \begin{pmatrix}\hat{z}_1\\ \hat{z}_2\end{pmatrix}$ are qualitatively coincident with those of system (13), specifically

$$\begin{aligned}g_1(-\hat{z}_1;\,\mu) &= 0\,,\\ \hat{z}_2 &= 0\,.\end{aligned}$$

Then, the Jacobian of the system is given by

$$J(\mu) = \left.\frac{\partial u}{\partial z}\right|_{z=\hat{z}} = \left.\begin{pmatrix}\dfrac{\partial u_1}{\partial z_1} & \dfrac{\partial u_1}{\partial z_2}\\ \dfrac{\partial u_2}{\partial z_1} & \dfrac{\partial u_2}{\partial z_2}\end{pmatrix}\right|_{z=\hat{z}} = \begin{pmatrix}\left.\dfrac{\partial g_1(-z_1;\,\mu)}{\partial z_1}\right|_{z_1=\hat{z}_1} - 1 & 0\\ 0 & \dfrac{\beta(\mu)}{2}\end{pmatrix}.$$

By using Lemma 1, the Jacobian of the system (13) evaluated at the equilibrium has a zero eigenvalue, it is known that, in the frequency domain, an eigenlocus of the matrix $G(0;\,\mu)J(\mu)$ crosses $-1+i0$.

Carrying out the aforementioned realization, the type of static bifurcation found in the frequency domain will be analyzed, through $h(-1,\,0;\,\mu) = F_1(0,\,\mu) = 0$. In this case and in agreement with the general definition of $h(\lambda,\,s;\,\mu)$ given in (7),

$$\begin{aligned} h(-1,\,0;\,\mu) &= \det(-I - G(0;\,\mu)J(\mu)) \\ &= 2\frac{\partial g_1(-z_1;\,\mu)}{\partial z_1}\bigg|_{z_1=\hat{z}_1} . \end{aligned} \tag{14}$$

Henceforth, for convenience, it will be considered $F_1(0,\,\mu) = h(-1,\,0;\,\mu) = F_1(z_1;\,0,\,\mu)$ as a two-variable function of z_1 and μ, which must be evaluated at $(\hat{z}_1,\,\mu) = (0,\,0)$. Moreover, it is known that, depending on the type of static bifurcation which appears in (13), the function g_1 with its partial derivatives must satisfy certain precise conditions [Glendinning, 1994] at the bifurcation point $(0, 0)$.

In accordance with (14), as the function F_1 is known, the type of static bifurcation which appears in (12) will be determined. Thus, the following theorems can be stated:

Theorem 1 (Saddle-Node Bifurcation). *Suppose that* $F_1(\hat{z}_1,\,0,\,0) = 0$, $U = (\partial g_1(-z_1;\,\mu)/\partial\mu)|_{(\hat{z}_1,0)} \neq 0$, *and* $V = (\partial/\partial z_1)F_1(z_1;\,0,\,\mu)|_{(\hat{z}_1,0)} \neq 0$. *The system (13) has a continuous curve of equilibrium points in a neighborhood of* $(y_1,\,\mu) = (0,\,0)$ *which is tangent to* $\mu = 0$ *at* $(0,\,0)$. *If* $UV < 0$ (*respectively* $UV > 0$) *there are no stationary points near* $(0,\,0)$ *if* $\mu < 0$ (*respectively* $\mu > 0$) *while for each value of* $\mu > 0$ (*respectively* $\mu < 0$), *in some sufficiently small neighborhood of* $\mu = 0$, *there are two equilibrium points near* $y_1 = 0$. *For* $\mu \neq 0$, *both equilibrium points are hyperbolic, the upper one is unstable and the lower is stable if* $V > 0$. *The stability properties are reversed if* $V < 0$.

Proof. This statement is the frequency domain counterpart of the result proved in [Glendinning, 1994]. ■

Example 1. Consider the product-system [Hale & Koçak, 1991]

$$\begin{aligned} \dot{y}_1 &= \mu + y_1^2\,, \\ \dot{y}_2 &= -y_2\,. \end{aligned}$$

A realization of the given system is proposed, as follows,

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix}, \quad B = C = I\,,$$

$$u(z_1,\,z_2;\,\mu) = \begin{pmatrix} -z_1 + \mu + z_1^2 \\ \dfrac{z_2}{2} \end{pmatrix} = \begin{pmatrix} -z_1 + g_1(-z_1;\,\mu) \\ \dfrac{z_2}{2} \end{pmatrix}.$$

Due to (14),

$$F_1(z_1;\,0,\,\mu) = 4z_1\,.$$

Provided that $\hat{z}_1 = 0$, $F_1(\hat{z}_1,\,0,\,0) = 0$, $U = 1$ and $V = 4$, in accordance with Theorem 1, there is a saddle-node bifurcation at $(\hat{y}_1,\,\mu) = (0,\,0)$. The equilibrium points, which exist if $\mu < 0$, are hyperbolic, the upper one is unstable and the lower one is stable because $V > 0$.

Theorem 2 (Transcritical Bifurcation). *Suppose that* $F_1(\hat{z}_1,\,0,\,0) = 0$, $U = (\partial g_1(-z_1;\,\mu)/\partial\mu)|_{(\hat{z}_1,0)} = 0$, $V = (\partial/\partial z_1)F_1(z_1;\,0,\,\mu)|_{(\hat{z}_1,0)} \neq 0$ *and if* $W = (\partial/\partial\mu)F_1(z_1;\,0,\,\mu)|_{(\hat{z}_1,0)}$, $W^2 - V\cdot(\partial^2 g_1(-z_1;\,\mu)/\partial\mu^2)|_{(\hat{z}_1,0)} > 0$. *The system (13) has two curves of equilibrium points in a neighborhood of* $(y_1,\,\mu) = (0,\,0)$. *These curves intersect transversely at* $(0,\,0)$ *and for each* $\mu \neq 0$ *sufficiently small, there are two hyperbolic equilibrium points near* $y_1 = 0$. *The upper equilibrium point is stable* (*respectively unstable*) *and the lower equilibrium point is unstable* (*respectively stable*) *if* $V < 0$ (*respectively* $V > 0$).

Proof. This statement is the frequency domain counterpart of the result proved in [Glendinning, 1994]. ■

Example 2. Consider the product-system [Hale & Koçak, 1991]

$$\begin{aligned} \dot{y}_1 &= \mu y_1 + y_1^2\,, \\ \dot{y}_2 &= -y_2\,. \end{aligned}$$

A realization of the given system is proposed, as follows,

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix}, \quad B = C = I\,,$$

$$u(z_1,\, z_2;\, \mu) = \begin{pmatrix} -(\mu+1)z_1 + z_1^2 \\ \dfrac{z_2}{2} \end{pmatrix},$$

$$= \begin{pmatrix} -z_1 + g_1(-z_1;\, \mu) \\ \dfrac{z_2}{2} \end{pmatrix}.$$

Due to (14),

$$F_1(z_1;\, 0,\, \mu) = -2\mu + 4z_1\,.$$

Provided that $\hat{z}_1 = 0$, $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $U = 0$, $V = 4$, $W = -2$, $W^2 - V \cdot (\partial^2 g_1(-z_1;\, \mu)/\partial\mu^2)|_{(\hat{z}_1,0)} = 4 > 0$, in accordance with Theorem 2, there is a transcritical bifurcation at $(\hat{y}_1,\, \mu) = (0,\, 0)$. There are two equilibrium points for each $\mu \neq 0$ sufficiently small. These points are hyperbolic, the upper one is unstable and the lower is stable because $V > 0$.

Theorem 3 (Pitchfork Bifurcation). *Suppose that* $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $U = (\partial g_1(-z_1;\, \mu)/\partial\mu)|_{(\hat{z}_1,0)} = 0$, $V = (\partial/\partial z_1)F_1(z_1;\, 0,\, \mu)|_{(\hat{z}_1,0)} = 0$, $W = (\partial/\partial\mu)F_1(z_1;\, 0,\, \mu)|_{(\hat{z}_1,0)} \neq 0$ *and* $X = (\partial^2/\partial z_1^2)F_1(z_1;\, 0,\, \mu)|_{(\hat{z}_1,0)} \neq 0$. *The system (13) has two curves of equilibrium points in a neighborhood of* $(y_1,\, \mu) = (0,\, 0)$. *One of these passes through* $(0,\, 0)$ *transverse to the axis* $\mu = 0$ *while the other is tangential to* $\mu = 0$ *at* $(0,\, 0)$. *If* $WX < 0$, *then for each* μ *with* $|\mu|$ *sufficiently small, there exist three equilibrium points near* $y_1 = 0$ *if* $\mu > 0$ (*the outer pair are stable and the inner point is unstable if* $X > 0$) *and an equilibrium point near* $y_1 = 0$ *if* $\mu < 0$ (*stable if* $X > 0$). *The stability properties are reversed if* $X < 0$. *On the other hand, if* $WX > 0$, *then there exist three equilibrium points near* $y_1 = 0$ *if* $\mu < 0$ (*the outer pair is unstable and the inner point is stable if* $X < 0$) *and an equilibrium point near* $y_1 = 0$ *if* $\mu > 0$ (*unstable if* $X < 0$). *The stability properties are reversed if* $X > 0$.

Proof. This statement is the frequency domain counterpart of the result proved in [Glendinning, 1994]. ■

Example 3. Consider the product-system [Hale & Koçak, 1991]

$$\dot{y}_1 = \mu y_1 - y_1^3\,,$$
$$\dot{y}_2 = -y_2\,.$$

A realization of the given system is proposed, as follows,

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix}, \quad B = C = I\,,$$

$$u(z_1,\, z_2;\, \mu) = \begin{pmatrix} -(\mu+1)z_1 + z_1^3 \\ \dfrac{z_2}{2} \end{pmatrix}$$

$$= \begin{pmatrix} -z_1 + g_1(-z_1;\, \mu) \\ \dfrac{z_2}{2} \end{pmatrix}.$$

Due to (14),

$$F_1(z_1;\, 0,\, \mu) = -2\mu + 6z_1^2\,.$$

Provided that $\hat{z}_1 = 0$, $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $U = 0$, $V = 0$, $W = -2$ and $X = 12 > 0$, in accordance with Theorem 3, there is a pitchfork bifurcation at $(\hat{y}_1,\, \mu) = (0,\, 0)$. Given that $WX < 0$, then there exist three equilibrium points near $\hat{y}_1 = 0$ if $\mu > 0$ (the outer pair is stable and the inner point is unstable) and an equilibrium point near $\hat{y}_1 = 0$ if $\mu < 0$ (which is stable). The stability conditions arise from the condition $X > 0$.

It must be emphasize that the described static bifurcation analysis can be applied directly to product-systems, specifically

$$\dot{x}_1 = f_1(x_1;\, \mu)\,,$$
$$\dot{x}_2 = f_2(x_2;\, \mu)\,,$$

when the linearization about the equilibrium point has one negative eigenvalue and the other equals to zero. However, notice that it is possible to apply the previous results to more general systems, through an "appropriate" first reduction. Connected to this, some more complex examples are developed.

Example 4. Consider the system [Strogatz, 1994]

$$\dot{x}_1 = x_2 - 2x_1\,,$$
$$\dot{x}_2 = \mu + 1 + x_1^2 - x_2\,.$$

The bifurcation analysis of this example can be developed via the bifurcation equation of the system [Hale & Koçak, 1991]. In this case, it yields

$$\dot{y}_1 = \mu + 1 + y_1^2 - 2y_1 = g_1(y_1;\, \mu)\,.$$

Due to (14),

$$F_1(z_1;\, 0,\, \mu) = 4z_1 + 4\,.$$

Provided that $\hat{z}_1 = -1$ and $\mu = 0$, $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $U = 1$ and $V = 4$, in accordance with Theorem 1, there is a saddle-node bifurcation at $(\hat{x}_1,\, \mu) = (\hat{y}_1,\, \mu) = (1,\, 0)$. The equilibrium points, which exist if $\mu < 0$, are hyperbolic, the upper one is unstable and the lower one is stable. These results agree with those which can be obtained in the time domain counterpart.

Example 5. Consider the system [Strogatz, 1994]

$$\begin{aligned}\dot{x}_1 &= x_2 + (\mu - 2)x_1 + \sin(x_1)\,,\\ \dot{x}_2 &= x_1 - x_2\,.\end{aligned}$$

The bifurcation equation of the system is

$$\dot{y}_1 = y_1 + (\mu - 2)y_1 + \sin(y_1) = g_1(y_1;\, \mu)\,.$$

Due to (14),

$$F_1(z_1;\, 0,\, \mu) = 2(1 - (\mu + \cos(z_1)))\,.$$

Provided that $\hat{z}_1 = 0$ and $\mu = 0$, $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $U = 0$, $V = 0$, $W = -2$, $X = 2$ and $WX < 0$, in accordance with Theorem 3, there is a pitchfork bifurcation at $(\hat{x}_1,\, \mu) = (\hat{y}_1,\, 0) = (0,\, 0)$. Then, there exist three equilibrium points near $\hat{x}_1 = 0$ if $\mu > 0$ and an equilibrium point near $\hat{x}_1 = 0$ if $\mu < 0$. Provided that $X > 0$, when $\mu > 0$, the outer pair is stable and the inner point is unstable and when $\mu < 0$, the equilibrium point is stable. Once again, these results agree with those that can be obtained with the usual analysis of the given system in the time domain.

Henceforth, the appearance of cusp points will be analyzed for two dimensional systems in the frequency domain formulation. Consider the case of a two-dimensional system as Eq. (12) but now $\mu = (\mu_1,\, \mu_2) \in \mathbf{R}^2$, $\hat{x} = (\hat{x}_1,\, \hat{x}_2) = (0,\, 0)$ is an equilibrium point, and the Jacobian evaluated at this point has one negative eigenvalue and the other equals to zero. Moreover, when μ varies in a neighborhood of 0, the number of equilibrium solutions changes. Thus, the given system has a bifurcation point at $(\hat{x},\, \mu) = (0,\, 0)$.

Again, it is possible to rewrite the system (12) locally, as in Eq. (13), where now $\beta(\mu) > 0\ \forall\, \mu \in \mathbf{R}^2$. Thus, the bifurcation point can be completely analyzed in the $(y_1,\, \mu)$-space.

Consider that the function g_1 has a Taylor expansion about the origin like

$$g_1(y_1;\, \mu) = a(\mu) + b(\mu)y_1 + c(\mu)\frac{y_1^2}{2!} + d(\mu)\frac{y_1^3}{3!} + G(y_1;\, \mu)\,, \qquad (15)$$

where $a(0) = b(0) = c(0) = 0$, $d(0) \neq 0$ and $\forall\, \varepsilon > 0$ there exist δ and η such that $|G(y_1;\, \mu)| < \varepsilon|y_1|^3$ when $|y_1| < \delta$, $\|\mu\| < \eta$.

Carrying out a realization similar to the one done before to (13), follows that

$$h(-1,\, 0;\, \mu) = 2\frac{\partial g_1(-z_1;\, \mu)}{\partial z_1}\bigg|_{z_1 = \hat{z}_1}$$

where $\hat{z} = (\hat{z}_1,\, 0)$ is an equilibrium solution in the frequency domain.

Now, the functional $F_1(0,\, \mu)$ depends on z_1 and $\mu = (\mu_1,\, \mu_2)$, and must be evaluated at $(\hat{z}_1,\, \mu) = (0,\, 0)$, which is the bifurcation point of the system (12). According to the results stated in [Hale & Koçak, 1991], the following theorem can be established, which allows to set up sufficient conditions to find a cusp point bifurcation via frequency analysis:

Theorem 4 (Cusp Singularity). *Suppose that* $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $T = g_1(-z_1;\, \mu)|_{(\hat{z}_1, 0)} = 0$, $V = (\partial/\partial z_1)F_1(z_1;\, 0,\, \mu)|_{(\hat{z}_1, 0)} = 0$, $X = (\partial^2 F_1(z_1;\, 0,\, \mu)/\partial z_1^2)|_{(\hat{z}_1, 0)} \neq 0$ *and the Jacobian*

$$J(g_1,\, F_1)_{(\mu_1, \mu_2)} = \begin{vmatrix} \dfrac{\partial g_1}{\partial \mu_1} & \dfrac{\partial g_1}{\partial \mu_2} \\[2mm] \dfrac{\partial F_1}{\partial \mu_1} & \dfrac{\partial F_1}{\partial \mu_2} \end{vmatrix}_{(\hat{z}_1, 0)} \neq 0\,.$$

The dynamics of the system (12) *is defined through a cusp in the* $(\mu_1,\, \mu_2)$ *plane. The equation of the bifurcation curve is approximately given by*

$$2[F_1(\hat{z}_1;\, 0,\, \mu)]^3 = -9X[g_1(-\hat{z}_1;\, \mu)]^2\,.$$

Proof. This statement is the frequency domain counterpart of the result proved in [Hale & Koçak, 1991]. ■

Example 6. Consider the following product-system

$$\begin{aligned}\dot{y}_1 &= \mu_1 + \mu_2 y_1 + y_1^3\,,\\ \dot{y}_2 &= -y_2\,.\end{aligned}$$

It must be observed that, in this case,

$$g_1(-z_1;\, \mu) = \mu_1 - \mu_2 z_1 - z_1^3\,,$$

and then

$$F_1(z_1;\, 0,\, \mu) = -2(\mu_2 + 3z_1^2)\,.$$

Provided that $\hat{z}_1 = 0$ and $\mu = 0$, $F_1(\hat{z}_1,\, 0,\, 0) = 0$, $T = 0$, $V = 0$, $X = -12$, and as

$$J(g_1,\, F_1)_{(\mu_1,\mu_2)} = \left|\begin{matrix} \dfrac{\partial g_1}{\partial \mu_1} & \dfrac{\partial g_1}{\partial \mu_2} \\ \dfrac{\partial F_1}{\partial \mu_1} & \dfrac{\partial F_1}{\partial \mu_2} \end{matrix}\right|_{(\hat{z}_1,0)}$$

$$= \left|\begin{matrix} 1 & 0 \\ 0 & -2 \end{matrix}\right| = -2 \neq 0\,,$$

all the conditions established in Theorem 4 are satisfied, so the considered system has a cusp point bifurcation at $(\hat{y}_1,\, \mu) = (0,\, 0)$ and the defining bifurcation curve is approximately given by

$$2[-2\mu_2)]^3 = -9(-12)[\mu_1]^2\,,$$
$$4\mu_2^3 = -27\mu_1^2\,.$$

Example 7. Consider the system analyzed in [Moiola & Chen, 1996], the mathematical model of a continuously stirred tank reactor (CSTR) in which a first-order irreversible, exothermic reaction $A \to B$ takes place. Under certain convenient conditions [Uppal *et al.*, 1974], the system can be written in its dimensionless form as

$$\dot{x}_1 = -x_1 + D(1 - x_1)\exp(x_2)\,,$$
$$\dot{x}_2 = -(1+\beta)x_2 + \tilde{B}D(1 - x_1)\exp(x_2)\,.$$

It is known that this system has an equilibrium point at $\hat{x} = (\hat{x}_1,\, \hat{x}_2) = (1/2,\, 2)$ and that there is a bifurcation point at $(\hat{x};\, \tilde{B},\, D,\, \beta) = (1/2,\, 2,\, 4,\, \exp(-2),\, 0)$ since the eigenvalues of the Jacobian evaluated at this point are 0 and -1. The bifurcation analysis can be developed via the bifurcation equation of the given system. In this case, solving the first equation for x_1, results

$$x_1 = \frac{D\,\exp(x_2)}{1 + D\,\exp(x_2)}\,,$$

dividing the second equation by $(1+\beta)$, which is always nonzero, and substituting the last expression of x_1 into it, yields

$$-x_2 + \frac{\tilde{B}D}{(1+\beta)}\frac{\exp(x_2)}{(1 + D\exp(x_2))} = 0\,.$$

Calling $B_1 = \tilde{B}/(1+\beta)$, follows

$$g_2(x_2;\, B_1,\, D) = B_1 D\,\exp(x_2) - x_2(1 + D\,\exp(x_2)) = 0$$

which is the bifurcation equation of the CSTR system.

Thus, the study of the equilibrium points and their stability types can be obtained from the following scalar differential equation

$$\dot{y}_2 = g_2(y_2;\, B_1,\, D) = B_1 D\,\exp(y_2) - y_2(1 + D\,\exp(y_2)) = 0$$

which are in one-to-one correspondence with those of the original system.

Provided that

$$F_1(z_2;\, 0,\, B_1,\, D) = 2\frac{\partial g_2(-z_2;\, B_1,\, D)}{\partial z_2} = 2[1 + D(1 - z_2 - B_1)\exp(-z_2)]\,,$$

where $\hat{z}_2 = -2$, $B_1 = 4$ and $D = \exp(-2)$ follows

$$F_1(z_2;\, 0,\, B_1,\, D)|_{(\hat{z}_2,4,\exp(-2))} = 0\,,$$
$$T = g_2(-z_2;\, B_1,\, D)|_{(\hat{z}_2,4,\exp(-2))} = 0\,,$$

$$V = \frac{\partial}{\partial z_2}F_1(z_2;\, 0,\, B_1,\, D)\Big|_{(\hat{z}_2,4,\exp(-2))} = 2D\,\exp(-z_2)(-2 + z_2 + B_1)|_{(\hat{z}_2,4,\exp(-2))} = 0\,,$$

$$X = \frac{\partial^2 F_1(z_2;\, 0,\, B_1,\, D)}{\partial z_2^2}\Big|_{(\hat{z}_2,4,\exp(-2))} = 2D\,\exp(-z_2)(3 - z_2 - B_1)|_{(\hat{z}_2,4,\exp(-2))} = 2 \neq 0$$

and the Jacobian

$$J(g_2,\, F_1)_{(B_1,D)} = \left|\begin{matrix} \dfrac{\partial g_2}{\partial B_1} & \dfrac{\partial g_2}{\partial D} \\ \dfrac{\partial F_1}{\partial B_1} & \dfrac{\partial F_1}{\partial D} \end{matrix}\right|_{(\hat{z}_2,4,\exp(-2))}$$

$$= \left|\begin{matrix} 1 & 2\,\exp\,2 \\ -2 & -2\,\exp\,2 \end{matrix}\right| = 2\,\exp\,2 \neq 0\,.$$

According to the hypothesis stated in Theorem 4 in the frequency formulation, it can be asserted that the dynamics of the considered CSTR system is defined through a cusp point in the $(D,\, B_1)$ plane, as is shown in Fig. 1.

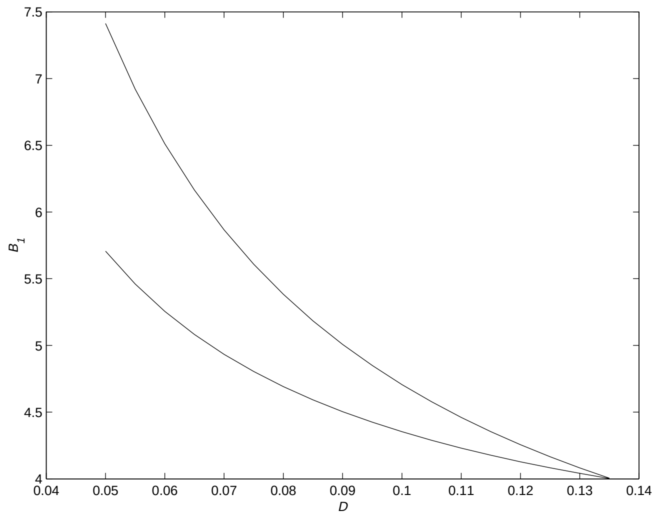

Fig. 1. Continuation of fold points ending in a cusp point in the CSTR.

The results stated in [Hale & Koçak, 1991] for (15) can be extended from the time domain, in order to determine the existence of the swallowtail singularity that appears in the study of the bifurcation set of a three-parameter perturbation of the quartic degeneracy as

$$g_1(y_1;\,\mu) = a(\mu) + b(\mu)y_1 + c(\mu)\frac{y_1^2}{2!} + d(\mu)\frac{y_1^3}{3!} + e(\mu)\frac{y_1^4}{4!} + G(y_1;\,\mu)\,,$$

under certain conditions, similar to those considered before. This is attained through the following two lemmas.

Lemma 2. *Let be* $\dot{y}_1 = g_1(y_1;\,\lambda)$, $\lambda = (\lambda_1,\,\lambda_2,\,\lambda_3)$ *and*

$$g_1(y_1;\,\lambda) = \lambda_1 + \lambda_2 y_1 + \lambda_3\frac{y_1^2}{2!} + d(\lambda)\frac{y_1^3}{3!} + e(\lambda)\frac{y_1^4}{4!} + G(y_1;\,\lambda)\,, \qquad (16)$$

where $d(0) = 0$, $e(0) \neq 0$ *and* $\forall\,\varepsilon > 0$ *there exist* δ *and* η *such that* $|G(y_1;\,\lambda)| < \varepsilon|y_1|^4$ *when* $|y_1| < \delta$, $\|\lambda\| < \eta$.

Then the bifurcation set of the given differential equation is a swallowtail surface in the $(\lambda_1,\,\lambda_2,\,\lambda_3)$ *space, described by the following equations*:

$$\lambda_1 = \lambda_1(r,\,s) = \frac{r^2 s}{2} + \frac{3e(0)}{4!}r^4 + \cdots\,,$$

$$\lambda_2 = \lambda_2(r,\,s) = -rs - \frac{e(0)}{3!}r^3 + \cdots\,,$$

$$\lambda_3 = s\,.$$

Proof. It is based on a laborious and reiterative application of the Implicit Function Theorem. The system considered is given by

$$g_1(y_1;\,\lambda) = 0\,,$$

$$\frac{\partial g_1(y_1;\,\lambda)}{\partial y_1} = 0\,.$$

Under the stated assumptions, it is possible to determine the existence of two functions

$$\lambda_1 = \lambda_1(y_1;\,\lambda_3)\,,$$

$$\lambda_2 = \lambda_2(y_1;\,\lambda_3)\,,$$

in a neighborhood of the origin which solve the previous system. Moreover, the first terms of their Taylor expansions about $(0; 0)$ can be calculated, by attaining the following expressions:

$$\lambda_1 = \lambda_1(y_1;\,\lambda_3) = \frac{y_1^2\lambda_3}{2} + \frac{3e(0)}{4!}y_1^4 + \cdots\,,$$

$$\lambda_2 = \lambda_2(y_1;\,\lambda_3) = -y_1\lambda_3 - \frac{e(0)}{3!}y_1^3 + \cdots\,,$$

which agree with those given above, after making a change of variables. ■

Lemma 3. *Let* $\dot{y}_1 = g_1(y_1;\,\mu)$, $\mu = (\mu_1,\,\mu_2,\mu_3)$ *and*

$$g_1(y_1;\,\mu) = a(\mu) + b(\mu)y_1 + c(\mu)\frac{y_1^2}{2!} + \hat{d}(\mu)\frac{y_1^3}{3!} + \hat{e}(\mu)\frac{y_1^4}{4!} + \hat{G}(y_1;\,\mu)\,, \qquad (17)$$

where $a(0) = b(0) = c(0) = \hat{d}(0) = 0$, $\hat{e}(0) \neq 0$ *and* $\forall\,\varepsilon > 0$, *there exist* δ *and* η *such that* $|\hat{G}(y_1;\,\mu)| < \varepsilon|y_1|^4$ *when* $|y_1| < \delta$, $\|\mu\| < \eta$. *It is assumed that*

$$J(a(\mu),\,b(\mu),\,c(\mu))_{(\mu_1,\mu_2,\mu_3)} = \left|\begin{matrix} \dfrac{\partial a}{\partial \mu_1} & \dfrac{\partial a}{\partial \mu_2} & \dfrac{\partial a}{\partial \mu_3} \\ \dfrac{\partial b}{\partial \mu_1} & \dfrac{\partial b}{\partial \mu_2} & \dfrac{\partial b}{\partial \mu_3} \\ \dfrac{\partial c}{\partial \mu_1} & \dfrac{\partial c}{\partial \mu_2} & \dfrac{\partial c}{\partial \mu_3} \end{matrix}\right|_{(\mu_1,\mu_2,\mu_3)=(0,0,0)} \neq 0\,.$$

Then, the bifurcation set of the given differential equation is approximately a swallowtail surface in the (μ_1, μ_2, μ_3)*-space.*

Proof. Due to the condition about the Jacobian J, the transformation of parameters given by

$$\lambda_1 = a(\mu_1, \mu_2, \mu_3),$$
$$\lambda_2 = b(\mu_1, \mu_2, \mu_3),$$
$$\lambda_3 = c(\mu_1, \mu_2, \mu_3),$$

sends a neighborhood of the origin in the μ-space onto another neighborhood of the origin in the λ-space, in a 1–1 correspondence. Writing $d(\lambda) = \hat{d}(\mu(\lambda))$, $e(\lambda) = \hat{e}(\mu(\lambda))$ and $G(x; \lambda) = \hat{G}(x; \mu(\lambda))$ in the general perturbation (17), the original perturbation (16) is attained. Then, the results follow directly from the Lemma 2. ■

Thus, the next theorem is established, which allows to determine the existence of a swallowtail singularity in two-dimensional systems like (13), via frequency analysis. This can be carried out, taking into account the previous lemma and setting sufficient conditions over g_1 and F_1 and its derivatives.

Theorem 5 (Swallowtail Singularity). *Suppose that* $F_1(\hat{z}_1, 0, 0) = 0$, $T = g_1(-z_1; \mu)|_{(\hat{z}_1,0)} = 0$, $V = (\partial/\partial z_1)F_1(z_1; 0, \mu)|_{(\hat{z}_1,0)} = 0$, $X = (\partial^2 F_1(z_1; 0, \mu)/\partial z_1^2)|_{(\hat{z}_1,0)} = 0$, $Y = (\partial^3 F_1(z_1; 0, \mu)/\partial z_1^3)|_{(\hat{z}_1,0)} \neq 0$ *and the Jacobian*

$$J\left(g_1, F_1, \frac{\partial F_1}{\partial z_1}\right)_{(\mu_1,\mu_2,\mu_3)} = \left|\begin{matrix} \frac{\partial g_1}{\partial \mu_1} & \frac{\partial g_1}{\partial \mu_2} & \frac{\partial g_1}{\partial \mu_3} \\ \frac{\partial F_1}{\partial \mu_1} & \frac{\partial F_1}{\partial \mu_2} & \frac{\partial F_1}{\partial \mu_3} \\ \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_1} & \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_2} & \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_3} \end{matrix}\right|_{(\hat{z}_1,0)} \neq 0.$$

The dynamics of the system (12) is defined through a swallowtail in the (μ_1, μ_2, μ_3) *space. The parametric representation of the bifurcation surface is approximately given by*

$$\mu_1 = \mu_1(r, s) = \frac{r^2 s}{2} + \frac{1}{16} Y r^4 + \cdots,$$
$$\mu_2 = \mu_2(r, s) = -rs - \frac{1}{12} Y r^3 + \cdots,$$
$$\mu_3 = s.$$

Proof. This statement is the frequency domain counterpart of the result proved in Lemma 3. ■

Finally two examples are developed to show how the last theorem can be applied.

Example 8. Consider the following product-system inspired in [Seydel, 1994]

$$\dot{y}_1 = \mu_1 + \mu_2 y_1 + \mu_3 y_1^2 + y_1^4,$$
$$\dot{y}_2 = -y_2.$$

It must be observed that, in this case,

$$g_1(-z_1; \mu) = \mu_1 - \mu_2 z_1 + \mu_3 z_1^2 + z_1^4,$$

where $\mu = (\mu_1, \mu_2, \mu_3)$ and

$$F_1(z_1; 0, \mu) = -2(\mu_2 - 2\mu_3 z_1 - 4z_1^3).$$

Provided that $\hat{z}_1 = 0$ and $\mu = 0$, $F_1(\hat{z}_1, 0, 0) = 0$, $T = 0$, $V = 0$, $X = 0$, $Y = 48 \neq 0$ and as

$$J\left(g_1, F_1, \frac{\partial F_1}{\partial z_1}\right)_{(\mu_1,\mu_2,\mu_3)} = \left|\begin{matrix} \frac{\partial g_1}{\partial \mu_1} & \frac{\partial g_1}{\partial \mu_2} & \frac{\partial g_1}{\partial \mu_3} \\ \frac{\partial F_1}{\partial \mu_1} & \frac{\partial F_1}{\partial \mu_2} & \frac{\partial F_1}{\partial \mu_3} \\ \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_1} & \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_2} & \frac{\partial\left(\frac{\partial F_1}{\partial z_1}\right)}{\partial \mu_3} \end{matrix}\right|_{(\hat{z}_1,0)}$$

$$= \left|\begin{matrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{matrix}\right| = -8 \neq 0,$$

all the conditions established in Theorem 5 are satisfied, so the considered system has a swallowtail bifurcation at $(\hat{y}_1, \mu) = (0, 0)$ and the defining bifurcation surface in the (μ_1, μ_2, μ_3)-space is given by

$$\mu_1 = \mu_1(r, s) = \frac{r^2 s}{2} + 3r^4,$$
$$\mu_2 = \mu_2(r, s) = -rs - 4r^3,$$
$$\mu_3 = \frac{s}{2}.$$

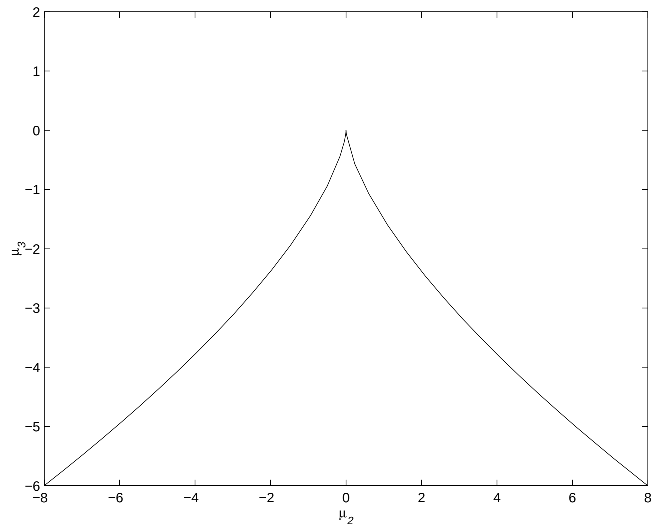

Fig. 2. Continuation of cusp points ending in a swallowtail singularity.

This situation can be checked through in Fig. 2, which shows a cusp continuation in the (μ_2, μ_3) plane. The distinguished singularity, which looks like another cusp corresponds to the aforementioned swallowtail.

Example 9. Consider the system analyzed in [Moiola & Chen, 1996], the mathematical model for a perfectly mixed reactor with a coiling coil, in which two consecutive, irreversible, exothermic and first-order reactions $A \to B \to C$ occur, as investigated by Halbe and Poore [1981]. The described system can be written in its dimensionless form as:

$$\begin{aligned}\dot{x}_1 &= -x_1 + D(1-x_1)\exp(x_3)\,,\\ \dot{x}_2 &= -x_2 + D(1-x_1)\exp(x_3) - DSx_2\exp(x_3)\,,\\ \dot{x}_3 &= -(1+\beta)x_3 + \tilde{B}D(1-x_1)\exp(x_3)\\ &\quad + \tilde{B}DS\alpha x_2\exp(x_3)\,,\end{aligned}$$

where D is the main bifurcation parameter and $\tilde{B}$, S, α and β are the auxiliary system parameters.

This system has an equilibrium point at $(\hat{x};\, \tilde{B},\, S) = ((\hat{x}_1,\, \hat{x}_2,\, \hat{x}_3);\, \tilde{B},\, S)$ where

$$\begin{aligned}\hat{x}_1 &= 0.7142155\,,\\ \hat{x}_2 &= 0.6348825\,,\\ \hat{x}_3 &= 2.6559060\,,\\ \tilde{B} &= 8\,,\\ S &= 0.05\,,\end{aligned}$$

for

$$(\hat{D},\, \hat{\alpha},\, \hat{\beta}) = (0.1755276,\, 1.023397,\, 1.395882)\,.$$

Analyzing the Jacobian at $\hat{x}$ with the distinguished values of the parameters, it can be observed that one of the eigenvalues is equal to zero.

The bifurcation analysis can be developed via the bifurcation equation of the given system. In this case, solving the first equation for x_1,

$$x_1 = \frac{D\,\exp(x_3)}{1 + D\,\exp(x_3)}\,,$$

putting this expression in the second one, and solving this for x_2 yields

$$x_2 = \frac{D\,\exp(x_3)}{(1 + D\,\exp(x_3))(1 + DS\,\exp(x_3))}\,.$$

Finally, operating with the third equation, one can obtain the bifurcation equation of the given system, which is:

$$\begin{aligned}&g_3(x_3;\, D,\, \tilde{B},\, S,\, \alpha,\, \beta)\\ &\quad = -(1+\beta)x_3 + \tilde{B}\frac{D\,\exp(x_3)}{1 + D\,\exp(x_3)}\\ &\qquad + \tilde{B}D^2S\alpha\frac{\exp^2(x_3)}{(1 + D\,\exp(x_3))(1 + DS\,\exp(x_3))}\\ &\quad = 0\end{aligned}$$

or

$$\begin{aligned}&g_3(x_3;\, D,\, \tilde{B},\, S,\, \alpha,\, \beta)\\ &\quad = -(1+\beta)x_3(1 + D\,\exp(x_3))(1 + DS\,\exp(x_3))\\ &\qquad + \tilde{B}D\,\exp(x_3)(1 + DS\,\exp(x_3))\\ &\qquad + \tilde{B}D^2S\alpha\,\exp^2(x_3) = 0\,.\end{aligned}$$

Taking into account the expression of the function $g_3(\cdot)$, with the fixed values of $\tilde{B}$ and S, and

$$F_1(z_3;\, 0,\, D,\, \alpha,\, \beta) = 2\frac{\partial g_3(-z_3;\, D,\, \alpha,\, \beta)}{\partial z_3}\,,$$

it is possible to prove via Theorem 5 that there is a swallowtail singularity at

$$(\hat{D},\, \hat{\alpha},\, \hat{\beta}) = (0.1755276,\, 1.023397,\, 1.395882)\,.$$

for $\hat{x}_3 = 2.6559060$.

Provided that $\hat{z}_3 = -\hat{x}_3$ results

$$F_1(\hat{z}_3,\, 0,\, \hat{D},\, \hat{\alpha},\, \hat{\beta}) = -3.52575 * 10^{-6}\,,$$

$$T = g_3(-z_3;\, D,\, \alpha,\, \beta)|_{(\hat{z}_3,\hat{D},\hat{\alpha},\hat{\beta})} = 2.78230 * 10^{-6}\,,$$

$$V = \frac{\partial}{\partial z_3} F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)\Big|_{(\hat{z}_3,\hat{D},\hat{\alpha},\hat{\beta})} = 2.56511 * 10^{-6}\,,$$

$$X = \frac{\partial^2 F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)}{\partial z_3^2}\Big|_{(\hat{z}_3,\hat{D},\hat{\alpha},\hat{\beta})} = 2.26394 * 10^{-3}\,,$$

$$Y = \frac{\partial^3 F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)}{\partial z_3^3}\Big|_{(\hat{z}_3,\hat{D},\hat{\alpha},\hat{\beta})} = 6.58167 \neq 0$$

and the Jacobian

$$J\left(g_3,\, F_1,\, \frac{\partial F_1}{\partial z_3}\right)_{(D,\alpha,\beta)} = \begin{vmatrix} \dfrac{\partial g_3}{\partial D} & \dfrac{\partial g_3}{\partial \alpha} & \dfrac{\partial g_3}{\partial \beta} \\ \dfrac{\partial F_1}{\partial D} & \dfrac{\partial F_1}{\partial \alpha} & \dfrac{\partial F_1}{\partial \beta} \\ \dfrac{\partial\left(\dfrac{\partial F_1}{\partial z_3}\right)}{\partial D} & \dfrac{\partial\left(\dfrac{\partial F_1}{\partial z_3}\right)}{\partial \alpha} & \dfrac{\partial\left(\dfrac{\partial F_1}{\partial z_3}\right)}{\partial \beta} \end{vmatrix}_{(\hat{z}_3,\hat{D},\hat{\alpha},\hat{\beta})} = -2.27595 * 10^3 \neq 0\,,$$

so the last assertion follows.

Similar results can be obtained with

$$(\breve{D},\, \breve{\alpha},\, \breve{\beta}) = (0.1325816,\, 0.7992681,\, 1.1304940)\,,$$

for $\breve{x}_3 = 4.0999860$. In this case, with $\breve{z}_3 = -\breve{x}_3$, the calculations give:

$$F_1(\breve{z}_3,\, 0,\, \breve{D},\, \breve{\alpha},\, \breve{\beta}) = 5.35974 * 10^{-5}\,,$$

$$T = g_3(-z_3;\, D,\, \alpha,\, \beta)|_{(\breve{z}_3,\breve{D},\breve{\alpha},\breve{\beta})} = -1.68357 * 10^{-5}\,,$$

$$V = \frac{\partial}{\partial z_3} F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)\Big|_{(\breve{z}_3,\breve{D},\breve{\alpha},\breve{\beta})} = -8.08018 * 10^{-5}\,,$$

$$X = \frac{\partial^2 F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)}{\partial z_3^2}\Big|_{(\breve{z}_3,\breve{D},\breve{\alpha},\breve{\beta})} = -9.94328 * 10^{-5}\,,$$

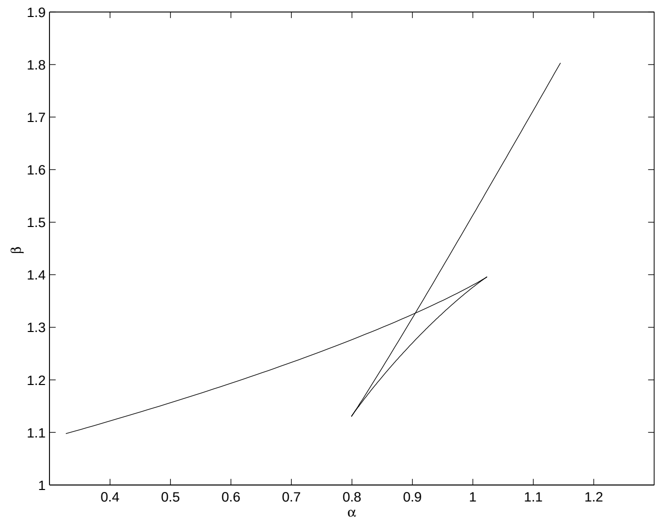

Fig. 3. Continuation of cusp points ending in two swallowtail singularities in the CSTR.

$$Y = \frac{\partial^3 F_1(z_3;\, 0,\, D,\, \alpha,\, \beta)}{\partial z_3^3}\Big|_{(\breve{z}_3,\breve{D},\breve{\alpha},\breve{\beta})} = -1.87469 * 10 \neq 0$$

and the Jacobian

$$J\left(g_3,\, F_1,\, \frac{\partial F_1}{\partial z_3}\right)_{(D,\alpha,\beta)}\Big|_{(\breve{z}_3,\breve{D},\breve{\alpha},\breve{\beta})} = -9.88604 * 10^4 \neq 0\,.$$

These two swallowtails singularities were computed by using LOCBIF [Khibnik *et al.*, 1993] as well as the continuation of cusp singularities as depicted in Fig. 3. Starting the continuation from these two swallowtails, it is possible to find a butterfly singularity after a suitable (extra) parameter variation.

4. Conclusions

In this work, the analysis of determining saddle-node, transcritical and pitchfork bifurcations for two-dimensional product systems with the frequency domain formulation has been completed. The analysis has been made by using the generalized Nyquist stability criterion and the nondegeneracy conditions of the aforementioned singularities. The precise conditions have been established over the functional $F_1(z_1;\, 0,\, \mu)$ (characteristic in the frequency domain formulation) to distinguish from the elementary static bifurcations up to swallowtail singularities.

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