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Brief Paper

Detecting period-doubling bifurcation: an approximate monodromy matrix approach☆

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Abstract

A quasi-analytical approach is developed for detecting period-doubling bifurcation emerging near a Hopf bifurcation point. The new algorithm employs higher-order Harmonic Balance Approximations (HBAs) to compute the monodromy matrix, useful for the study of limit cycle bifurcations. Prediction of the period-doubling bifurcation is accomplished very accurately by using this algorithm, along with a detailed approximation error analysis, without using numerical integration of the dynamical system. An example is given to illustrate the results. © 2001 Elsevier Science Ltd. All rights reserved.

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1. Introduction

Investigation of complex systems with multiple equilibria, multiple periodic solutions, and particularly chaotic behaviors has posed a real challenge to both systems analysts and control engineers. Recently, one main concern in this study is how to take advantage of such complex behaviors, or how to control the complex dynamics by practical means (Chen & Dong, 1998). In this pursuit, prediction of a period-doubling bifurcation is especially important in the regard of capturing an essential mechanism that generates chaos in a non-linear dynamical system. Several approaches (Phillipson & Schuster, 2000; Seydel, 1994) have proven to be useful in computing bifurcation parameter values for detecting period-doubling bifurcation; among them, the describing function method (a first-order harmonic analysis) has been extensively applied (Basso, Genesio, & Tesi, 1997; Maggio, Kennedy, & Gilli, 1998; Tesi, Abed, Genesio, & Wang, 1996).

In this paper, we propose a more accurate computational scheme for the prediction of a period-doubling bifurcation. This new scheme improves the accuracy of the detection as compared to other closely related approaches (Basso et al., 1997; Tesi et al., 1996). This scheme utilizes the information of higher-order HBAs (Moiola & Chen, 1996) along with an "approximate" evaluation of the associated monodromy matrix. This technique gives approximate characteristic multipliers (Floquet multipliers), which are the simplest tools for determining the stability of limit cycles and for detecting system bifurcations. Furthermore, this technique suggests an accurate detection method without using too many harmonics, in comparison with the method proposed in Bonani and Gilli (1999), which points out that the harmonic balance method can recover a nearly complete picture of limit cycle bifurcations after increasing the number of harmonics to 20. Thus, our approach provides significant savings of computational efforts. Moreover, a detailed analysis of the approximation errors in our approach can be performed, which is also

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provided in this paper. More specifically, a formulation of a general setting for computing limit cycles generated by the Hopf bifurcation mechanism in a nonlinear feedback system is first described. Then, an analysis of the stability of limit cycles is performed based on an associate *approximate* monodromy matrix. Moreover, an evaluation of the approximation error through a Taylor-series expansion is provided for completeness. To this end, a simple algorithm is developed for predicting the appearance of a period-doubling bifurcation in the system, and then applied to a state feedback nonlinear control problem.

2. The graphical Hopf bifurcation method

Consider a parametrized nonlinear system,

$$\dot{x}(t) = f(x;\ \mu) = A(\mu)x(t) + B(\mu)g(C(\mu)x(t)),$$
$$y(t) = C(\mu)x(t), \tag{1}$$

where A, B, and C are $n\times n$, $n\times r$, and $m\times n$ matrices, respectively, $\mu\in R$ is the main bifurcation control parameter, $x\in R^n$ is the state vector, $y\in R^m$ is the system output, $g: R^m \to C^{2q+1}(R^r)$ is a smooth nonlinear feedback function, $f: R^n \to C^{2q+1}(R^n)$ is a smooth nonlinear system function, and n, m, q, and r are positive integers. Define

$$G(s;\ \mu) = C(\mu)[sI \quad - A(\mu)]^{-1}B(\mu) \tag{2}$$

and let $x = -z$ with $g(x;\ \mu) := h(z;\ \mu)$. Then, the equilibrium solution of (1) can be obtained by solving equation

$$G(0;\ \mu)h(\hat{z};\ \mu) = -\hat{z}. \tag{3}$$

Also, define $J_{\hat{z}} = \partial h(z)/\partial z|_{z=\hat{z}}$.

A Hopf bifurcation occurs when one eigenvalue $\hat{\lambda}$ of the linearized system transfer matrix $G(s;\ \mu)J_{\hat{z}}$ satisfies

$$\hat{\lambda}(\mathrm{i}\omega_0;\ \mu_0) = -1 + 0\mathrm{i}, \quad \mathrm{i} = \sqrt{-1} \tag{4}$$

for some values of ω_0 and μ_0. At the moment of bifurcation, a periodic branch arises from criticality, and continues to develop as μ is varied.

A periodic solution of Eq. (1) can be written as (Moiola & Chen, 1996)

$$x_H(t) = -z(t) = -\left(\hat{z} + \Re\left\{\sum_{k=0}^{\infty} Z^k \mathrm{e}^{\mathrm{i}k\omega_H t}\right\}\right), \tag{5}$$

where $\Re\{.\}$ is the real part, ω_H is the fundamental frequency of the nonlinear oscillations, and Z^k are the k-harmonic complex amplitudes satisfying the harmonic balance equations

$$Z^k = -G(\mathrm{i}k\omega_H, \mu)H^k, \quad k\in[0,1,2,\dots), \tag{6}$$

where $\{H^k\}$ are the Fourier coefficients of the output signal of the nonlinear feedback

$$h(z(t)) = \Re\left\{\sum_{k=0}^{\infty} H^k \mathrm{e}^{\mathrm{i}k\omega_H t}\right\},$$

written as polynomial functions of $\{Z^k\}$. In other words, the harmonic balance technique assumes that a periodic solution can be expanded as a superposition of different harmonics.

The graphical Hopf bifurcation method (GHBM) provides the following qth-order prediction of the true limit cycle

$$z(t) \approx z_q(t) = \hat{z} + \Re\left\{\sum_{k=0}^{2q} Z_q^k \mathrm{e}^{\mathrm{i}k\omega_q t}\right\}, \tag{7}$$

where ω_q and Z_q^k are the frequency and the k-harmonic complex amplitudes predicted by a qth-order HBA.

It amounts to solving a finite set of equations:

$$Z_q^k = -G(\mathrm{i}k\omega_q, \mu)H_q^k, \quad k\in[0,1,\dots,2q]. \tag{8}$$

Here, only the first $2q+1$ Fourier coefficients $\{H_q^k\}$ of $h(z(t))$ are considered:

$$h(z(t)) = \Re\left\{\sum_{k=0}^{2q} H_q^k \mathrm{e}^{\mathrm{i}k\omega_q t}\right\} + \Re\left\{\sum_{k=2q+1}^{\infty} H_q^k \mathrm{e}^{\mathrm{i}k\omega_q t}\right\},$$
$$q = 1,2,3,4.$$

These equations are solved in terms of $Z_q^1 = Z_q^1(v, \theta_q)$, where v is the right eigenvector of $G(\mathrm{i}\omega, \mu)J_{\hat{z}}$ associated with the eigenvalue $\hat{\lambda}$, and θ_q is a measure of the amplitude of the periodic solution. In simple terms, H_q^k is a function of Z_q^k, which involves partial derivatives of the generic function $g(x;\ \mu)$ with respect to x. The explicit formulas up to the eighth-order (involving the ninth-order partial derivatives) for a multivariable nonlinear system were given in Moiola and Chen (1996).

The amplitude and frequency of the periodic solution are evaluated by computing the complex functions $\xi_q(v,\omega)$, $q = 1,2,3,4$ (see Moiola & Chen, 1996 for more details), and the corresponding complex functions

$$L_q(v,\ \omega,\ \theta) = -1 + \sum_{k=1}^{q} \xi_k(v,\ \omega)\theta^{2k}.$$

The following equation has to be solved, in a way reminiscent to the describing function method, to obtain the frequency $\hat{\omega}$ and the amplitude $\hat{\theta}$ of the periodic solutions:

$$\hat{\lambda}(\mathrm{i}\omega;\ \mu) = L_q. \tag{9}$$

We refer to L_q as the *amplitude locus* in contrast to the *characteristic locus* $\hat{\lambda}(\mathrm{i}\omega;\ \mu)$. The predicted periodic solution can be obtained by placing the solution pair $(\hat{\omega}_q, \hat{\theta}_q)$, known as the L_q-approximation solution, in Eq. (7).

3. Stability analysis of the limit cycles

The GHBM provides a series of approximations, x_q ($q = 1,2,3,4$), to a periodic solution x_H of a nonlinear differential equation $\dot{x} = f(x)$. Suppose first that a limit cycle, $x_H(t)$, has been generated by the Hopf bifurcation mechanism. Define a perturbed trajectory by

$$x_p(t) = x_H(t) + x_D(t), \tag{10}$$

where $x_D(t)$ is a perturbation of $x_H(t)$. Taking a time derivative gives

$$\dot{x}_p(t) = \dot{x}_H(t) + \dot{x}_D(t). \tag{11}$$

Also, we have

$$\dot{x}_p(t) = A(x_H(t) + x_D(t)) + Bg(C(x_H(t) + x_D(t))). \tag{12}$$

Some algebraic manipulations give

$$\dot{x}_D(t) = Ax_D(t) + B[g(C(x_H(t) + x_D(t))) - g(Cx_H(t))]. \tag{13}$$

It is then possible to demonstrate the growth or decay of $x_D(t)$ and, thereafter, the instability or stability of $x_H(t)$, by using an appropriate Poincaré section and looking for an equilibrium solution of the return map (Seydel, 1994). The stability of this equilibrium depends on the eigenvalues of the associated monodromy matrix with respect to the unit circle. The monodromy matrix M is defined by

$$\begin{aligned} &\dot{X}(t) = J_D(t)X(t), \\ &X(0) = I, \\ &M = X\left(\frac{2\pi}{\omega_H}\right), \end{aligned} \tag{14}$$

where $J_D(t;\,\mu)$ is a periodic matrix defined by

$$\begin{aligned} &J_D(t) = \left.\frac{\partial \dot{x}_D(t)}{\partial x_D}\right|_{x_D=0} = A + BJ_{x_H}(t), \\ &J_{x_H}(t) = \left.\frac{\partial g(x)}{\partial x}\right|_{x=x_H(t)}. \end{aligned} \tag{15}$$

The eigenvalues of M are known as characteristic (or Floquet) multipliers, denoted λ_i, $i = 1,\ldots,n$, where one of them, λ_1, is always equal to $+1$. In a planar system ($n = 2$), there is only one possible crossing of the unit circle by another real eigenvalue, at the point $+1 + 0\mathrm{i}$, leading to, for example, a cyclic-fold, transcritical or pitchfork bifurcation. When $n = 3$, there are two additional possibilities: Neimark–Sacker bifurcation (two complex conjugate characteristic multipliers cross the unit circle simultaneously), and period-doubling bifurcation (one real characteristic multiplier cross the unit circle at the point $-1 + 0\mathrm{i}$). As to the stability, a periodic orbit becomes stable when all the characteristic multipliers (except the one at $+1$) stay within the unit circle, and it becomes unstable when they move out. Moreover, the corresponding period-doubling bifurcation can be either supercritical or subcritical.

One problem in using matrix M is the need to integrate two dynamical systems simultaneously: the original nonlinear system and the variational equation (14). Moreover, transients of the nonlinear system may be very large, leading to a long simulation time before the actual calculation of M can be completed. In order to drastically reduce the computational burden, we propose to calculate the following approximate matrices M_q (when $2q$, $q = 1$, 2 and 3, are the numbers of harmonics of the periodic solution), instead of the original M, so that we only need to deal with the integration of the (approximate) variational equation:

$$\begin{aligned} &\dot{Y}(t) = J_{D_q}(t)Y(t), \\ &Y(0) = I, \\ &M_q = Y\left(\frac{2\pi}{\omega_q}\right), \end{aligned} \tag{16}$$

where $J_{D_q}(t)$ is a periodic matrix (obtained by using the information of L_q-approximation for the limit cycle), defined by

$$J_{Dq}(t) = A + BJ_{x_q}(t), \quad J_{x_q}(t) = \left.\frac{\partial g(x)}{\partial x}\right|_{x=x_q(t)}. \tag{17}$$

4. Analysis on approximation errors

Here, we provide some formulas to deal with the approximation errors in a semi-analytical manner. There are some related works using higher-order methods for the continuation of periodic solutions (Guckenheimer & Meloon, 2000; Phillipson & Schuster, 2000), or using Taylor series, which are believed to be better than direct numerical integration regarding efficiency and accuracy (specially, Guckenheimer & Meloon, 2000). The present analysis of the approximation error over the predicted higher-order HBAs distinguishes our approach from many others in the current literature.

4.1. A Taylor-series expansion

Consider the parametrized nonlinear system

$$\dot{x}(t) = f(x,t;\,\mu), \quad x(t_0) = x_0,$$

which, for notational convenience, is denoted as

$$g_1(u,t;\,\mu) := f(u,t;\,\mu).$$

Note that

$$\left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=t_0} = g_1(x(t),t;\ \mu)|_{t=t_0} = f(x(t_0),t_0;\ \mu)$$

$$= f(x_0,t_0;\ \mu).$$

Then, using the chain rule, we can write

$$\left.\frac{\mathrm{d}^2x(t)}{\mathrm{d}t^2}\right|_{t=t_0} = g_2(x(t),t;\ \mu)|_{t=t_0}$$

$$= \left.\frac{\partial g_1(u,t_0)}{\partial u}\right|_{u=x_0}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \left.\frac{\partial g_1(x_0,t)}{\partial t}\right|_{t=t_0}.$$

In general, it is easy to deduce that

$$\left.\frac{\mathrm{d}^kx(t)}{\mathrm{d}t^k}\right|_{t=t_0} = g_k(x(t),t)|_{t=t_0}$$

$$= \left.\frac{\partial g_{k-1}(u,t_0)}{\partial u}\right|_{u=x_0}\frac{\mathrm{d}x(t)}{\mathrm{d}t} + \left.\frac{\partial g_{k-1}(x_0,t)}{\partial t}\right|_{t=t_0}.$$

Thus, the Taylor-series expansion of a smooth function $x(t)$, if it converges, takes the form

$$x(t) = \sum_{k=0}^{\infty} a_k\frac{(t-t_0)^k}{k!},$$

where the coefficients a_k satisfy

$$x(t)|_{t=t_0} = x(t_0) = a_0, \quad \left.\frac{\mathrm{d}x(t)}{\mathrm{d}t}\right|_{t=t_0} = a_1$$

and, in general, $\mathrm{d}^kx(t)/\mathrm{d}t^k|_{t=t_0} = a_k$. To this end, it is easy to derive the following expressions:

$$a_0 = x_0, \tag{18}$$

$$a_k = \frac{\partial a_{k-1}}{\partial x_0}f(x_0,t_0) + \frac{\partial a_{k-1}}{\partial t_0}, \quad k>0. \tag{19}$$

The corresponding Taylor series can be written as

$$x(t) = x_0 + \left(\sum_{k=1}^{\infty}\frac{\partial a_{k-1}}{\partial x_0}\frac{(t-t_0)^k}{k!}\right)f(x_0,t_0) + \left(\sum_{k=1}^{\infty}\frac{\partial a_{k-1}}{\partial t_0}\frac{(t-t_0)^k}{k!}\right).$$

Note that a_k depends only on $a_0,\ldots,a_{k-1}$, so evaluations of formulas (18) and (19) can be computed either analytically or numerically.

4.2. *Evaluation of the approximation error*

Suppose that a solution

$$x(t) = \sum_{k=0}^{\infty} a_k\frac{(t-t_0)^k}{k!},$$

of an ODE $\dot{x}(t) = f(x(t),t;\ \mu)$, is approximated by its rth-order polynomial truncation

$$x_r(t) = \sum_{k=0}^{r} a_k\frac{(t-t_0)^k}{k!}.$$

Note that the polynomial has the same coefficients $a_0,a_1,\ldots,a_r$ as the original ODE solution, and that $\dot{x}_r \approx f(x_r(t),t;\ \mu)$, which becomes equal if and only if $r=\infty$. Thus, it is reasonable to introduce an error measure (or error index) defined by

$$\Delta_r = \|f(x_r(t),t;\ \mu) - \dot{x}_r(t)\|.$$

Obviously, $\Delta_r \to 0$ as $r\to\infty$. To have more insights about this measure, rewrite

$$\Delta_r = \|\dot{x}(t) - \dot{x}_r(t) + f(x_r(t),t;\ \mu) - f(x(t),t;\ \mu)\|.$$

Triangular inequality then yields

$$\Delta_r \leqslant \|\dot{x}(t) - \dot{x}_r(t)\| + \|f(x_r(t),t;\ \mu) - f(x(t),t;\ \mu)\|.$$

Then, the Mean Value Theorem gives

$$\|f(x_r(t),t;\ \mu) - f(x(t),t;\ \mu)\| = \left\|\left.\frac{\partial f(u,t;\ \mu)}{\partial u}\right|_{u=u_\alpha}\right\|\|x_r(t)-x(t)\|,$$

where, for each t,

$$u_\alpha = \alpha x_r(t) - (1-\alpha)x(t) \quad \text{for some } \alpha\in[0,1].$$

Thus,

$$\Delta_r \leqslant \|\dot{x}(t) - \dot{x}_r(t)\| + \left\|\left.\frac{\partial f(u,t;\ \mu)}{\partial u}\right|_{u=u_\alpha}\right\|\|x_r(t)-x(t)\|,$$

which can be further elaborated to yield

$$\Delta_r \leqslant \hat{\Delta}_r := \left\|\sum_{k=r+1}^{\infty} a_k\frac{(t-t_0)^{k-1}}{(k-1)!}\right\| + \left\|\left.\frac{\partial f(u,t;\ \mu)}{\partial u}\right|_{u=u_\alpha}\right\|\left\|\sum_{k=r+1}^{\infty} a_k\frac{(t-t_0)^k}{k!}\right\|.$$

Now, notice that $\hat{\Delta}_r$ reduces to zero if either one of the following conditions is satisfied:

- $t=t_0$,
- $a_k = 0, \quad k>r$.

Also, note that because the original Taylor series converges, we have $\lim_{r\to\infty} a_r = 0$, so that $\lim_{r\to\infty}\hat{\Delta}_r = 0$.

The above formulation prepares the following evaluation and analysis on the polynomial approximation errors.

Remarks: Define

$$a_{0,0} = x_0,$$

$$a_{0,k} = \frac{\partial a_{0,k-1}}{\partial a_{0,0}}f(a_{0,0},t_0) + \frac{\partial a_{0,k-1}}{\partial t_0}.$$

Suppose we use the approximation

$$x_{r,0}(t_1) = \hat{x}_1(t_1) = \sum_{k=0}^{r} a_{0,k}\frac{(t_1 - t_0)^k}{k!}. \tag{20}$$

To get a point, $\hat{x}_1(t_1)$, close enough to the "true" point $x(t_1) = \sum_{k=0}^{\infty} a_{0,k}(t_1 - t_0)^k/k!$. The question now is: Can we actually obtain an approximation,

$$x_{r,1}(t) = \sum_{k=0}^{r} a_{1,k}\frac{(t - t_1)^k}{k!},$$

$$a_{1,0} = \hat{x}_1(t_1),$$

$$a_{1,k} = \frac{\partial a_{0,k-1}}{\partial a_{1,0}} f(a_{1,0}, t_1) + \frac{\partial a_{1,k-1}}{\partial t_1}, \tag{21}$$

such that it is close enough to the true solution $x(t)$? Fortunately, the answer is *yes*. The Shadowing Lemma (Guckenheimer & Holmes, 1993) implies that while we may not be able to calculate the real solution, we can find a close enough approximation to it. The condition for applying this lemma is that the points $x_0,\ \hat{x}_1(t_1), \ldots,$ belong to an hyperbolic invariant set. This means that the trajectories initiated inside a given neighborhood will never leave it, and that the discrete map (21) inside this neighborhood contracts at an uniformly exponential rate, λ_h.

So, we are entitled to define a special discrete map and expect to iteratively calculate a sequence of points that are close enough to the true solution of the concerned equation. Here, the key is that the discrete map defined by

$$t_m = t_0 + m t_Q, \quad m \geqslant 0,$$

$$a_0(x_m, t_m) = x_m,$$

$$a_k(x_m, t_m) = \frac{\partial a_{k-1}}{\partial x_m} f(x_m, t_m;\ \mu) + \frac{\partial a_{k-1}}{\partial t_m},$$

$$x_{m+1} = \sum_{k=0}^{Q} a_k(x_m, t_m)\frac{t_Q^k}{k!},$$

can be evaluated both analytically and numerically, as a function of $x_0,\ t_0$, and t_Q, where x_0 and t_0 are the initial conditions and t_Q is a time step in integration.

Generally speaking, performing a numerical evaluation is usually much faster than carrying out an analytical approximation, and these two types of results should provide the same error for a given value of Q if they are both correctly completed. Then, why should one bother to consider analytical or semi-analytical calculations? Our answer is that an analytic solution, even an approximate one, is a function that has a useful and precise structure. If the approximation is accurate enough, this structure describes the true structure at least qualitatively, from which one can obtain some hidden relationships or conditions on system parameters. Among a few remarkable advantages that can be mentioned, one can easily imagine of applying it to feedback controllers design, where a precise structure is not only important but actually is necessary. Another important note in point is that for small values of Q, analytical expressions for the coefficients a_k are quite compact, but the interesting values of Q are rather large. However, the Mathematica program used to generate them is still quite simple (less than 15 lines in the code).

4.2.1. *An example: a linear time-invariant system*

First, note that for a linear time-invariant system,

$$\dot{x} = Ax,$$

the coefficients of the corresponding Taylor series are

$$a_0 = x_0, \quad a_1 = Ax_0, \quad a_2 = \frac{\partial a_1}{\partial x_0} a_1 = AAx_0 = A^2 x_0, \cdots$$

$$a_k = \frac{\partial a_{k-1}}{\partial x_0} a_1 = A^{k-1}Ax_0 = A^k x_0.$$

4.3. *A nonlinear system with a limit cycle*

For a given nonlinear system,

$$\dot{x} = f(x;\ \mu),$$

$$x(t_0) = x_0,$$

with a limit cycle $x_T(t)$ of period T, we are interested in the evaluation of its monodromy matrix M:

$$\dot{X} = \left.\frac{\partial f(x;\ \mu)}{\partial x}\right|_{x = x_T(t)} X,$$

$$X(0) = I,$$

$$X(T) = M.$$

Here, x is an $n \times 1$ vector and X an $n \times n$ matrix. Their respective coefficients of the Taylor series, a_k and A_k, can be evaluated as follows:

$$a_0 = x_0, \quad A_0 = I,$$

$$a_k = \frac{\partial a_{k-1}}{\partial x_0} f(x_0;\ \mu), \quad k > 0,$$

$$A_k = A_{k-1}\frac{\partial f(x_0;\ \mu)}{\partial x_0} + \frac{\partial A_{k-1}}{\partial x_0} f(x_0;\ \mu), \quad k > 0.$$

It is easy to prove, by induction, that $A_k = \partial a_k/\partial x_0$. So, the monodromy matrix can be evaluated as $M = \partial x_T(T)/\partial x_0$ (Seydel, 1994). However, a rather large value of r is needed in order to obtain a fairly accurate approximation:

$$\hat{M}_r = \sum_{k=0}^{r} A_k \frac{T^k}{k!}. \tag{22}$$

Actually, there are two alternative methods to carry out the evaluation of $\hat{M}$, as further discussed below.

4.3.1. First alternative method

To approximate a limit cycle of period T, we write the equations as

$$\hat{x}_1 = \sum_{k=0}^{r} \frac{a_{k,1}(x_0, t_0)}{k!}\left(\frac{T}{s}\right)^k,$$

$$\ldots$$

$$\hat{x}_s = \sum_{k=0}^{r} \frac{a_{k,N}(x_{N-1}, t_{N-1})}{k!}\left(\frac{T}{s}\right)^k, \tag{23}$$

where the constants r and s must be chosen such that Δ_r, at the points $\hat{x}_k$, $s+1>k>0$, are small enough. Ideally, we want $x_0 = x_s$ to hold, but actually we get $x_0 = x_s + e$, where the magnitude of e is a measure of the prediction error.

What is interesting in the set of equations (23) is the possibility of evaluating $\partial x_s/\partial x_0$ in order to approximate $\partial x(t)/\partial x_0|_{t=T}$: applying the chain rule we get

$$\frac{\partial x_s}{\partial x_0} = \frac{\partial x_s}{\partial x_{s-1}} \times \frac{\partial x_{s-1}}{\partial x_{s-2}} \times \cdots \times \frac{\partial x_1}{\partial x_0}. \tag{24}$$

The only troublesome situation inherent to this method is that if we want to obtain some numbers, a value for x_0 is needed. For this, however, the GHBM can be used to provide expressions for the starting point x_0.

4.3.2. Second alternative method

Suppose now we have a linear time-variant system,

$$\dot{X} = B(t)X, \tag{25}$$

where

$$B(t) = \sum_{n=0}^{\infty} B_n \frac{(t-t_0)^n}{n!}. \tag{26}$$

The coefficients of the corresponding Taylor series are

$$A_0 = X_0, \quad A_k = \frac{\partial A_{k-1}}{\partial X_0} B(t_0) X_0 + \frac{\partial A_{k-1}}{\partial t_0}. \tag{27}$$

It is easy to derive the following equalities (dependence on t_0 has been dropped for notational convenience):

$$\frac{A_1}{0!} = B_0 A_0, \tag{28}$$

$$\frac{A_2}{1!} = B_1 A_0 + B_0 A_1, \tag{29}$$

$$\frac{A_3}{2!} = \frac{B_2}{2!} A_0 + B_1 A_1 + B_0 \frac{A_2}{2!}, \tag{30}$$

$$\frac{A_4}{3!} = \frac{B_3}{3!} A_0 + \frac{B_2}{2!} A_1 + B_1 \frac{A_2}{2!} + B_0 \frac{A_3}{3!} \tag{31}$$

and, in general,

$$\frac{A_k}{(k-1)!} = \sum_{n=0}^{k-1} \frac{B_n}{n!} \frac{A_{k-n-1}}{(k-n-1)!}, \quad k>0. \tag{32}$$

In order to have a quick check on the equations given above, consider the following "temporal" balance:

$$\left(\sum_{k=1}^{\infty} A_k \frac{t^{k-1}}{(k-1)!}\right) = \left(\sum_{n=0}^{\infty} B_n \frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty} A_k \frac{t^k}{k!}\right). \tag{33}$$

Here, to simplify the evaluation of formulas (28)–(32), we divide them by k, and define $C_k = B_k/k!$, and

$$D_0 = A_0,$$

$$D_1 = \frac{A_1}{1!} = B_0 A_0 = C_0 D_0,$$

$$D_2 = \frac{A_2}{2!} = \frac{B_1 A_0 + B_0 A_1}{2} = \frac{C_1 D_0 + C_0 D_1}{2},$$

$$D_3 = \frac{A_3}{3!} = \frac{(B_2/2!)A_0 + B_1 A_1 + B_0 A_2/2!}{3}$$

$$= \frac{C_2 D_0 + C_1 D_1 + C_0 D_2}{3},$$

$$D_4 = \frac{A_4}{4!} = \frac{B_3/3! A_0 + B_2/2! A_1 + B_1 A_2/2! + B_0 A_3/3!}{4}$$

$$= \frac{C_3 D_0 + C_2 D_1 + C_1 D_2 + C_0 D_3}{4}.$$

In general,

$$D_k = \frac{A_k}{k!} = \frac{\sum_{n=0}^{k-1} B_n/n! A_{k-n-1}/(k-n-1)!}{k}$$

$$= \frac{\sum_{n=0}^{k-1} C_n D_{k-n-1}}{k}, \quad k>0. \tag{34}$$

Here, note that D_k depends on $D_0, \ldots, D_{k-1}$ (previously calculated), so an evaluation of formula (34) is straightforward, which can be done either analytically or numerically.

Now, suppose that $B(t)$ is a period T-function:

$$B(t) = \sum_{m=-\infty}^{\infty} H_m \mathrm{e}^{\mathrm{i}m(2\pi/T)(t-t_0)}. \tag{35}$$

We are interested in the special case

$$C_k = \frac{\sum_{m=-\infty}^{\infty} H_m (\mathrm{i}m2\pi/T)^k}{k!}, \quad k \geqslant 0, \tag{36}$$

$$D_0 = I, \tag{37}$$

$$D_k = \frac{\sum_{n=0}^{k-1} C_n D_{k-n-1}}{k}, \quad k>0 \tag{38}$$

and the evaluation of

$$X(T) = \sum_{k=0}^{\infty} D_k T^k = I + \left[\sum_{k=1}^{\infty} \left(\frac{\sum_{n=0}^{k-1} C_n D_{k-n-1}}{k} \right) T^k \right]. \quad (39)$$

Note that, in order to evaluate an approximation to $X(T)$,

$$\hat{X}_r(T) = \sum_{k=0}^{r} D_k T^k = I + \left[\sum_{k=1}^{r} \left(\frac{\sum_{n=0}^{k-1} C_n D_{k-n-1}}{k} \right) T^k \right], \quad (40)$$

we can use the following algorithm:

Algorithm A

(1) Get an approximate $\hat{B}(t) = \sum_{m=-q}^{q} \hat{H}_m e^{im(2\pi/\hat{T})(t-t_0)}$.

(2) Evaluate, for $p = 0, \ldots, r-1$, $C_p =$

$$\left. \frac{d^p \hat{B}(t)}{dt^p} \right|_{t=T} \frac{1}{p!} = \frac{\sum_{m=-q}^{q} \hat{H}_m (im2\pi/\hat{T})^P}{k!}.$$

(3) Evaluate, for $p = 0, \ldots, r$,

$$\begin{cases} p = 0 \to D_0 = I, \\ p > 0 \to D_p = \dfrac{\sum_{n=0}^{p-1} C_n D_{p-n-1}}{p}. \end{cases}$$

(4) Evaluate, for $p = 0, \ldots, r$,

$$\begin{cases} p = 0 \to F_0 = D_r, \\ p > 0 \to F_p = F_{p-1} T + D_{r-p}. \end{cases}$$

(5) Evaluate $\hat{X}_r(T) = M_q = F_r$, where M_q is the q-order HBA of the monodromy matrix.

5. The new scheme for detecting limit cycle bifurcations

To detect the stability changes of periodic orbits and, hence, limit cycle bifurcations, the following computational algorithm is proposed:

(1) Find all the equilibria of the given system.

(2) Find all the equilibria that undergo a Hopf bifurcation for a given value of the main bifurcation parameter μ.

(3) Choose an equilibrium, and find the corresponding L_1 prediction as well as its approximate monodromy matrix M_1.

(4) If a value of μ can be found such that a characteristic multiplier $\hat{\lambda}$ crosses the unit circle, then evaluate L_2 and L_3 predictions with their corresponding approximate monodromy matrices M_2 and M_3, in a neighborhood of μ. This can be done with *Algorithm A* indicated in the previous section. Otherwise, go to Step 3.

6. An example of the new scheme

Consider the dynamical system introduced in Tesi et al. (1996) and analyzed in Berns, Moiola, and Chen (1999):

$$\dot{x} = A(\mu)x + Bg(x),$$
$$y = Cx, \quad (41)$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1-\alpha & -1.2 & \mu \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$C = [1 \quad 0 \quad 0],$$

$g(x) = x_1^2 + \alpha x_1$, and $\mu \in [-5/6, 0)$ is the bifurcation parameter. The equilibria of the system are $x_a(0,0,0)$ and $x_b(1,0,0)$.

A branch of periodic solutions emerges from the first equilibrium when a supercritical Hopf bifurcation takes place at the value $\mu_{HB} = -5/6$. In Fig. 1, the continuation of limit cycles is depicted from the Hopf bifurcation (μ_{HB}) until the first period-doubling bifurcation takes place (μ_{PD}). The computation is done by using the software LOCBIF (Khibnik, Kuznetsov, Levitin, & Nikolaev, 1993), which gives the prediction for the first period-doubling bifurcation at $\mu_{PD} \approx -0.481045$.

Evaluating the approximations of the monodromy matrices M_1, M_2, and M_3 by using the periodic predictions L_1 (two harmonic approximations), L_2 (four harmonics) and L_3 (six harmonics), respectively, we obtained the characteristic multipliers shown in Tables 1–3.

As can be easily seen from Table 3, the approximate monodromy matrix M_3 gives one multiplier, $\hat{\lambda}_2 \approx -1 + i0$ (the defining condition for the period-doubling

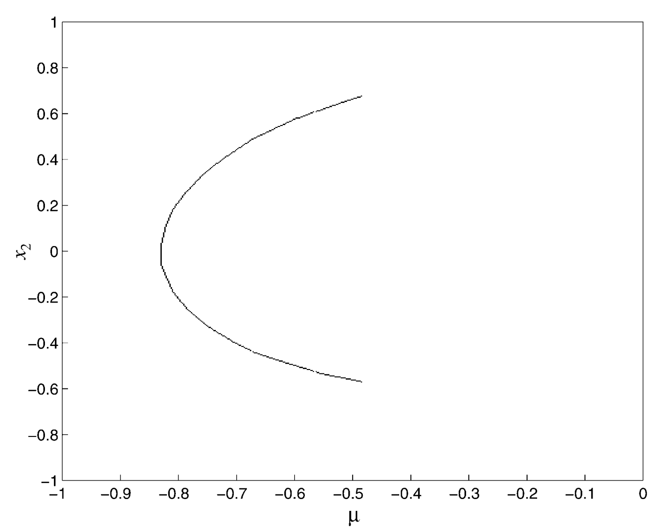

Fig. 1. Continuation of limit cycles from Hopf bifurcation to the first period-doubling bifurcation.

Table 1
Characteristic multipliers of M_1 for detecting a period-doubling bifurcation

μ	λ_1	$\hat{\lambda}_2$	λ_3
− 0.53	1.00558	− 0.999267	− 0.046272
− 0.52	1.00601	− 1.11676	− 0.043808
− 0.51	1.00646	− 1.23802	− 0.041813

Table 2
Characteristic multipliers of M_2 for detecting a period-doubling bifurcation

μ	λ_1	$\hat{\lambda}_2$	λ_3
− 0.51	1.00065	− 0.87955	− 0.05950
− 0.50	1.00072	− 0.98074	− 0.05649
− 0.49	1.00078	− 1.08462	− 0.05408

Table 3
Characteristic multipliers of M_3 for detecting a period-doubling bifurcation

μ	λ_1	$\hat{\lambda}_2$	λ_3
− 0.4900	1.00022	− 0.98558	− 0.05962
− 0.4885	1.00023	− 1.00027	− 0.05926
− 0.4875	1.00023	− 1.01009	− 0.05902

bifurcation) at $\mu_{\mathrm{PD}M_3} \approx -0.4885$, which is very close to the value obtained by LOCBIF: $\mu_{\mathrm{PD}} \approx -0.481045$.

Notice also that one of the multipliers must be $+1.000$, since it is a requirement for the technical construction of the Poincaré section (see also the discussion in [Guckenheimer & Meloon, 2000]). This is a measure of the improvement in the convergence of the increasing order approximations when compared to the values of λ_1 given by M_1 and M_2. In this regard, it is easily seen that the error of the approximate M_1 is of order 6.0×10^{-3}, the error of the approximate M_2 is of order 7.2×10^{-4}, and the one for M_3 is of order 2.3×10^{-4}.

The GHBM results are checked using a Taylor partial sum involving up to ten terms:

$$x_{k+1} = \sum_{k=0}^{10} \frac{a_p}{p!}\left(\frac{T}{8}\right)^p, \tag{42}$$

where T is the predicted period and the coefficients a_p are defined as

$$a_0 = x_0 = -\left(\hat{z} + \Re\left\{\sum_{k=0}^{2q} Z_q^k\right\}\right), \tag{43}$$

$$a_{p+1} = \frac{\partial a_{p-1}}{\partial x_0} f(x_0;\, \mu), \quad p > 0. \tag{44}$$

The test points are x_0 up to x_8, and the error index is

$$e = \sqrt{(x_0 - x_8)^T (x_0 - x_8)}. \tag{45}$$

As a particular case, calculations are made with the following data:

- $\mu = -0.4809$, $T = 5.78301$;
- The initial point $x_0 = (0.797803, -0.0461253, -0.66704)$.

Thus, we obtained $e = 0.00684694$. Furthermore, the evaluation of the monodromy matrix is carried out by using the two proposed alternative algorithms. Using the first method, we evaluated both $\hat{X}_{300}(t)$ and $\hat{X}_{400}(t)$ to get virtually unchanged characteristic multipliers $(0.99989, -1.0009, -0.0618716)$. By the second method, with eight test points, we got slightly different numerical results. Increasing significantly the number of test points (up to 256 and beyond), it converges to the result of the first method.

7. Conclusions

A semi-analytical approach has been developed in this paper for detecting period-doubling bifurcation in a nonlinear dynamical system. The technique has the advantage of utilizing the analytic structure of the system for controller design and other purposes. This, and several other special features, distinguishes the proposed approach from other existing ones in the literature.

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