CHAOS PREDICTION AND BIFURCATION ANALYSIS IN CONTROL ENGINEERING

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Abstract— **Two different methods to compute the period-doubling route to chaos (or Feigenbaum chaos) in nonlinear systems are presented. The first one is a semi-analytical procedure, based on a symbolic calculation of an approximate monodromy matrix. The second one takes advantage of software packages for continuation of periodic solutions. Both procedures are used to analyze Chua's circuit. The second method is also applied to the Rössler system and one of the chaotic systems of Sprott. In all three cases, several period-doubling bifurcation points in the parameter space are detected using standard continuation software packages, allowing to compute a sequence of values supposedly converging to Feigenbaum's constant. This "experimental" computer verification agrees with experiments performed by other researchers in real systems. This material has been used in final projects in a graduate course on dynamical systems.**

Keywords— **period-doubling bifurcations, Feigenbaum's constant, chaotic systems.**

I. INTRODUCTION

The aim of this paper is to present two different methods for prediction of the period-doubling bifurcation in nonlinear systems, mainly for analysis, but with the perspective to alert (control) the birth of chaos using the information provided by the Feigenbaum constant. As it is well known, the cascade route of period-doubling bifurcations is a very important scenario to understand chaotic motions, and several notable recent contributions have dealt with this detection using different orders of accuracy (Basso *et al.*, 1997; Bonani and Gilli, 1999; Collantes and Suárez, 2000; Maggio *et al.*, 1998; Phillipson and Schuster, 2000) as well as its control using different methodologies (Chen and Dong, 1998; Chen *et al.*, 2000, and Tesi *et al.*, 1996). Some of them proposed to use the describing function method (generally a first-order harmonic analysis), while some others suggested to use the harmonic balance approach with several harmonics, and more recently quasi-analytical methods.

In this paper, we propose a more accurate computational scheme for predicting the *first* period-doubling bifurcation, by taking advantage of some explicit formulas of periodic solutions (Moiola and Chen, 1996) in terms of higher-order harmonic balance approximations (HBAs, in short). This new scheme improves the accuracy of the detection when compared to other closely related approaches (Basso *et al.*, 1997; Maggio *et al.*, 1998; Tesi *et al.*, 1996). It also reveals some insights about the hidden relationships of period-doubling mechanisms and the coefficients of the Fourier series in quasi-analytical computations. The proposed algorithm uses the information of higher-order HBAs, in conjunction with an "approximate" evaluation of the associated monodromy matrix (Berns *et al.*, 1999; 2001). This algorithm provides approximate characteristic multipliers (Floquet multipliers), without using too many harmonics in comparison with the method recently proposed in (Bonani and Gilli, 1999) and, at the same time, emphasizes some advantages of series methods in the computation of periodic solutions. These advantages have been very recently rediscovered and carefully analyzed (Guckenheimer and Meloon, 2000). Some preliminary results show that this technique could also be used to compute the second period-doubling bifurcation, although a higher number of harmonics may be needed to obtain accurate detection; this issue is not pursued in the present paper.

An alternative to detecting period-doubling bifurcations in systems of Ordinary Differential Equations (ODEs) is to use a software package for continuation of periodic solutions based on standard numerical techniques, such as, for example, the LBLC program from LOCBIF library (Khibnik *et al.*, 1993b). These powerful tools allow the detection of a higher number of period-doubling bifurcations in the parameter space than the semi-analytical method described above. This enables the "prediction" of Feigenbaum's constant from the first period-doubling bifurcations (up to period 16 or 32). From the point of view of engineering applications, it is also possible to approximate the birth of the attractor with a reasonable good accuracy from the values of the parameter for solutions of period 4 and 8. In this regard, several values approaching the Feigenbaum constant are obtained for very well known chaotic attractors, such as Chua's circuit, Rössler's system and one of the chaotic systems recently discovered by Sprott (1994). In all of these systems, the dynamics of the so-called Lorenz map is "unimodal" and thus the requirements of Feigenbaum's theory is satisfied (Cvitanovic, 1989). These results help us to interpret the "order" that appears in chaotic systems to predict approximately serial values of the cascade of period-doubling bifurcations.

II. BACKGROUND MATERIAL

In this section two different methods to detect period-doubling bifurcations are presented. The first one is based on the computation of the so-called monodromy matrix of the system. The second one relies on numerical procedures for computing the continuation curve of periodic solutions in the parameter space.

A. Quasi-analytical method to predict period-doubling bifurcations

A brief review of a frequency-domain method to detect Hopf bifurcations, as well as a semi-analytical procedure to compute the monodromy matrix are described here. More details can be found in Berns *et al.* (1999, 2001).

A.1. The Graphical Method for Hopf Bifurcation Analysis

Consider the following feedback connection of a parametrized linear system and a memoryless nonlinearity

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(\mu)\,\mathbf{x}(t) + \mathbf{B}(\mu)\,\mathbf{u}, \\ \mathbf{y}(t) = \mathbf{C}(\mu)\,\mathbf{x}(t), \end{cases} \quad (1)$$
$$\mathbf{u} = \mathbf{g}(\mathbf{y};\mu).$$

This system can be thought as an autonomous (parametrized) nonlinear system

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x};\mu) \doteq \mathbf{A}(\mu)\,\mathbf{x}(t) + \mathbf{B}(\mu)\,\mathbf{g}\left[\mathbf{C}(\mu)\mathbf{x}(t);\mu\right], \quad (2)$$

where $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ are $n\times n$, $n\times r$ and $m\times n$ matrices, respectively, $\mu\in\mathbb{R}$ is the main bifurcation control parameter, $\mathbf{x}\in\mathbb{R}^n$ is the state vector, $\mathbf{y}\in\mathbb{R}^m$ is the system output, $\mathbf{g}:\mathbb{R}^m\times\mathbb{R}\to C^{2q+1}(\mathbb{R}^r)$ is the system feedback (a smooth nonlinear function), $\mathbf{f}:\mathbb{R}^n\times\mathbb{R}\to C^{2q+1}(\mathbb{R}^n)$ is a smooth system vector field, and n, m, q and r are positive integers. Define

$$\mathbf{G}(s;\mu) = \mathbf{C}(\mu)\left[s\mathbf{I} - \mathbf{A}(\mu)\right]^{-1}\mathbf{B}(\mu),$$

and also the Jacobian $\mathbf{J}_{\hat{\mathbf{y}}} \doteq \left.\frac{\partial \mathbf{g}(\mathbf{z};\mu)}{\partial \mathbf{z}}\right|_{\mathbf{z}=\hat{\mathbf{y}}\doteq\mathbf{C}\hat{\mathbf{x}}}$. The equilibrium solution of Eqn. (1) can be obtained solving

$$\mathbf{G}(0;\mu)\,\mathbf{g}(\hat{\mathbf{y}};\mu) = \hat{\mathbf{y}}.$$

A Hopf bifurcation occurs when one eigenvalue, denoted $\hat{\lambda}$, of the transfer matrix of the linearized system $\mathbf{G}(s;\mu)\mathbf{J}_{\hat{\mathbf{y}}}$, satisfies

$$\hat{\lambda}(i\omega_0;\mu_0) = -1 + 0i, \qquad i = \sqrt{-1},$$

for some values ω_0 and μ_0. At the moment of bifurcation, a periodic branch arises from criticality, and continues to develop as μ is varied. Then, a $2q$th-order approximate periodic solution of Eqn. (1) can be written as

$$\mathbf{y}(t) \approx \mathbf{y}_q(t) = \hat{\mathbf{y}} + \Re\{\sum_{k=0}^{2q}\mathbf{Y}_q^k\,e^{ik\omega_q t}\}, \quad q = 1,2,\ldots,$$

to distinguish it from the true solution $\mathbf{y}_H(t) = \hat{\mathbf{y}} + \Re\{\sum_{k=0}^{\infty}\mathbf{Y}^k\,e^{ik\omega_H t}\}$, where $\Re\{.\}$ is the real part, ω_H is the fundamental frequency, ω_q is the approximation frequency, and $\mathbf{Y}^k$ are the k-harmonic complex amplitudes satisfying the harmonic balance equations

$$\mathbf{Y}^k = \mathbf{G}(ik\omega_H,\mu)\,\mathbf{H}^k, \qquad k = 0,1,2,\ldots,$$

where $\{\mathbf{H}^k\}$ are the Fourier coefficients of the output signal of the nonlinear feedback $\mathbf{g}[\mathbf{y}(t)] = \Re\{\sum_{k=0}^{\infty}\mathbf{H}^k\,e^{ik\omega_H t}\}$, written as polynomial functions of $\{\mathbf{Y}^k\}$. The Graphical Hopf Bifurcation Method (GHBM) (Moiola and Chen, 1996) provides the $2q$th-order prediction of the limit cycle. Here, only the first $2q+1$ Fourier coefficients $\{\mathbf{H}_q^k\}$ of the output signal of the nonlinear feedback, written as polynomial functions of $\{\mathbf{Y}_q^k\}$, are considered

$$\mathbf{g}[\mathbf{y}(t)] = \Re\{\sum_{k=0}^{2q}\mathbf{H}_q^k\,e^{ik\omega_q t}\} + \Re\{\sum_{k=2q+1}^{\infty}\mathbf{H}_q^k\,e^{ik\omega_q t}\}.$$

These equations are solved in terms of $\mathbf{Y}_q^1 = \mathbf{Y}_q^1(v,\theta_q)$, where v is the right eigenvector of $\mathbf{G}(i\omega,\mu)\mathbf{J}_{\hat{\mathbf{y}}}$ associated with the eigenvalue $\hat{\lambda}$, and θ_q is a measure of the amplitude of the periodic solution. More details for the explicit approximation formulas can be found in the reference cited above.

A.2. The monodromy matrix

Let $\boldsymbol{\gamma}(t)$ be a certain periodic solution of Eqn. (2), such that $\boldsymbol{\gamma}(t) = \boldsymbol{\gamma}(t+kT)$, with $k\in\mathbb{Z}$. Clearly, $\dot{\boldsymbol{\gamma}}(t) = \mathbf{f}[\boldsymbol{\gamma}(t)]$, and $\dot{\boldsymbol{\gamma}}(t)$ is also T-periodic. Let $\mathbf{s}(t,\mathbf{x}_0+\boldsymbol{\varepsilon}_0)$ be a solution that is not periodic, but "close to" the periodic solution $\boldsymbol{\gamma}(t)$,

$$\mathbf{s}(t,\mathbf{x}_0+\boldsymbol{\varepsilon}_0) = \boldsymbol{\gamma}(t) + \boldsymbol{\varepsilon}(t),$$

where $\mathbf{x}_0 = \boldsymbol{\gamma}(0)$, $\boldsymbol{\varepsilon}_0 = \boldsymbol{\varepsilon}(0)\in B(\mathbf{x}_0,\delta)$. Clearly, if $\boldsymbol{\varepsilon}_0 = 0$, $\mathbf{s}(t,\mathbf{x}_0)\equiv\boldsymbol{\gamma}(t)$, and thus $\boldsymbol{\varepsilon}(t)\equiv 0$ for all t. Once again, as $\mathbf{s}(t,\mathbf{x}_0+\boldsymbol{\varepsilon}_0)$ is a solution of Eqn. (2),

$$\begin{aligned} \dot{\mathbf{s}}(t) &= \dot{\boldsymbol{\gamma}}(t) + \dot{\boldsymbol{\varepsilon}}(t) \\ &= \mathbf{f}[\boldsymbol{\gamma}(t) + \boldsymbol{\varepsilon}(t)] \\ &= \mathbf{f}[\boldsymbol{\gamma}(t)] + \left.\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right|_{\mathbf{x}=\boldsymbol{\gamma}(t)}\boldsymbol{\varepsilon}(t) + O\left[\boldsymbol{\varepsilon}^2(t)\right], \end{aligned}$$

and thus

$$\dot{\varepsilon}(t) = \mathbf{F}(t)\,\varepsilon(t) + O\left[\varepsilon^2(t)\right], \qquad (3)$$

where

$$\mathbf{F}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\gamma(t)} = \mathbf{A}(\mu) + \mathbf{B}(\mu)\,\mathbf{J}_\gamma \mathbf{C}(\mu), \qquad (4)$$

with $\mathbf{J}_\gamma = \left.\frac{\partial \mathbf{g}(\mathbf{z};\mu)}{\partial \mathbf{z}}\right|_{\mathbf{z}=\mathbf{C}\gamma(t)}$.

The stability of the perturbed solution $\mathbf{s}(t, \mathbf{x}_0+\varepsilon_0)$ is closely related to Eqn. (3). If $\varepsilon(t)\to 0$ when $t\to\infty$, then the perturbed solution converges to the periodic solution $\gamma(t)$. On the contrary, if $\varepsilon(t)$ does not converge to zero, it can not be assured that the perturbed solution converges to the periodic solution $\gamma(t)$. Therefore, the local stability of a limit cycle can be analyzed studying the local behavior of $\varepsilon(t)$, neglecting the higher order terms in Eqn. (3),

$$\dot{\varepsilon}(t) = \mathbf{F}(t)\,\varepsilon(t), \qquad \varepsilon(0) = \varepsilon_0, \qquad (5)$$

with $\mathbf{F}(t)$ given by Eqn. (4). Matrix $\mathbf{F}(t)$ is T-periodic, and it can be thought as the linearization of the nonlinear system (2) around the periodic solution $\gamma(t)$.

System (5) is linear-periodic, and its solutions can be characterized via the state transition matrix $\mathbf{\Phi}(t,0)$. For any ε_0, the linear state Eqn. (5) with $\mathbf{F}(t)$ continuous has the unique, continuously-differentiable solution

$$\varepsilon(t, \varepsilon_0) = \mathbf{\Phi}(t,0)\;\varepsilon_0.$$

It is a well-known property of linear periodic systems that the state transition matrix $\mathbf{\Phi}(t,0)$ may be written as

$$\mathbf{\Phi}(t,0) = \mathbf{K}(t)\,e^{\mathbf{L}t},$$

where $\mathbf{K}(t) = \mathbf{K}(t+T) \in \mathbb{R}^{n\times n}$, $\mathbf{K}(0) = \mathbf{I}$, and $\mathbf{L} = 1/T \log \mathbf{\Phi}(T,0)$. Matrix $\mathbf{\Phi}(T,0) = e^{\mathbf{L}T} \doteq \mathbf{M}$ is called the *monodromy matrix* of the system. The behavior of the solutions in the neighborhood of $\gamma(\cdot)$ is determined by the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\mathbf{M}$, called the *characteristic multipliers*, or *Floquet multipliers.* The eigenvalues of $\mathbf{L}$ are referred to as the *characteristic exponents* of γ. If $\mathbf{v} \in \mathbb{R}^n$ is a vector tangent to $\gamma(0)$, then $\mathbf{v}$ is the eigenvector corresponding to the characteristic multiplier $\lambda_1 = 1$. The moduli of the remaining $n-1$ eigenvalues, if none is unity, determine the stability of γ. In particular, if one eigenvalue crosses the unit disk at -1 after varying a parameter, an attractive closed orbit of Eqn. (2) becomes a saddle closed orbit, and a new attractive closed orbit of nearly twice the period is emitted from it. This bifurcation is referred to as (supercritical) period-doubling bifurcation, subharmonic resonance, or flip bifurcation.

The monodromy matrix $\mathbf{M}$ and the Poincaré map are closely related. Choosing appropriately the representation basis, the last column of $\mathbf{M}$ can take the form $(0,\ldots,0,1)^T$ and then the characteristic matrix of the linearized Poincaré map is the matrix belonging to $\mathbb{R}^{(n-1)\times(n-1)}$ obtained by deleting the nth row and column of $\mathbf{M}$.

A.3. Approximate computation of the monodromy matrix

Contrary to the case of time-invariant systems, a general method to compute the analytical solution of a time-variant linear system such as Eqn. (5) does not exists[1]. Moreover, Eqn. (5) can be obtained only if an explicit expression of the periodic solution is available, which in turn implies knowing an (analytical) solution of Eqn. (2). Therefore, there is almost no chance to obtain an exact, analytical expression for the monodromy matrix $\mathbf{M}$.

Following (Berns *et al.*, 1999; 2001) an alternative, approximate method to compute $\mathbf{M}$ is to obtain an approximate periodic solution, and analyze the behavior of the approximate monodromy matrix $\mathbf{M}_q$ computed over a $2q$th-order periodic solution γ_q of Eqn. (2). Although the error in estimating the period-doubling bifurcation can be quantified, a simple procedure to check the accuracy of the approximations is a decreasing value of $\|\lambda_1 - 1\|$ for increasing higher-order HBAs. For further details the reader is referred to the above-mentioned references.

B. Numerical continuation approach

The approach for detecting period-doubling bifurcations described before has the advantage of revealing some insights about the hidden relationships of period-doubling mechanisms and the coefficients of the Fourier series. However, as several approximations are needed to arrive at the final result, some care must be exercised in the analysis of the solutions. This fact, together with the computational complexity involved, precludes its use in some cases.

An alternative is to carry out numerical time-simulations on the system, plotting appropriate projections of the solutions over a plane, and visually detecting the birth of a period-doubling solution. The disadvantages of this approach are obvious. A better technique is to employ a routine for continuation of periodic solutions. Several numerical routines are available for this purpose; see for example (Kuznetsov, 1998; Govaerts, 2000). The maximum value of one of the output variables (in oscillatory regime) is plotted as function of the main bifurcation parameter μ; therefore values of the parameter where period-doubling bifurcations occur can be read very precisely. This plot is known as the *continuation diagram* of periodic solutions.

There are many software packages available to perform this kind of computations, for example LBLC

[1]Some special cases can be solved analitically; for example, if $\mathbf{F}(t)$ and $\int \mathbf{F}(\sigma)\,d\sigma$ commute, then $\mathbf{M} = \exp\left[\int_0^T \mathbf{F}(\sigma)\,d\sigma\right] = \sum_{k=0}^{\infty} \frac{1}{k!}\left[\int_0^T \mathbf{F}(\sigma)\,d\sigma\right]^k$.

α	λ_1	λ_2	λ_3
6.905	1.418	$-0.028 + i0.005$	$-0.028 - i0.005$
6.920	1.427	-0.0554	-0.0146

Table 1: Characteristic multipliers of $\mathbf{M_1}$ for a range of the main bifurcation parameter α.

from the LOCBIF library (Khibnik *et al.*, 1993b); however, in any of them choosing an appropriate initial condition is of paramount importance to obtain significative results.

C. Feigenbaum's constant

For systems exhibiting unimodal dynamics of the so-called Lorenz map, a possible route to chaos is the cascading of period-doubling bifurcations. Starting from the critical value μ_H for which a Hopf bifurcation phenomenon occurs, increasing values of μ produce periodic solutions of varying amplitude, until μ reaches another critical value, noted as μ_2, where an orbit of double period appears. This behavior is repeated for increasing values of μ at specific values μ_{2^n}, and the cascade of period-doubling oscillations ends up (at least in non-patological dynamical systems) in a chaotic attractor. Feigenbaum has shown that the values of the parameter μ_{2^n} for which period-doubling bifurcation occurs exhibit certain regularity for unimodal maps. If one computes the ratio δ_n between two successive distances between critical points in the parameter space, *i.e.*

$$\delta_n = \frac{\mu_{2^{n+1}} - \mu_{2^n}}{\mu_{2^{n+2}} - \mu_{2^{n+1}}}, \qquad n = 1, 2, \ldots \tag{6}$$

then the series of values δ_1, δ_2, δ_3, ... converges to a particular value, the celebrated *Feigenbaum's constant* (Cvitanovic, 1989)

$$\lim_{n\to\infty} \delta_n = \delta_F \approx 4.669\ldots .$$

Knowing two successive values of the parameter for which a period-doubling bifurcation occurs, it is possible to estimate a value of the parameter $\mu_{\infty,n}$ for which the system is bordering chaos

$$\mu_{\infty,n} = \mu_{2^n} + (\mu_{2^{n+1}} - \mu_{2^n})\frac{\delta_F}{\delta_F - 1}, \ n = 1, 2, \ldots . \tag{7}$$

Therefore, from an engineering point of view, if $\mu = \mu_{\infty,n}$ for n sufficiently large, the system's variables will exhibit chaotic behavior, or at least, they will have such a rich harmonic content that they could be considered "chaotic" for any practical application.

III. APPLICATION EXAMPLES

In this section, the values μ_{2^n} for which a period-doubling bifurcation appears will be detected for three different systems. Only for the first system is carried out the computation of the first period-doubling with the approximate monodromy matrix in the frequency domain and then compared to the result obtained by the continuation method.

α	λ_1	λ_2	λ_3
6.915	1.04992923	-0.98815423	-0.00181657
6.920	1.05044375	-1.00291535	-0.00178851

Table 2: Characteristic multipliers of $\mathbf{M_2}$ for a range of the main bifurcation parameter α.

α	λ_1	λ_2	λ_3
6.905	1.03535850	-0.99495839	-0.00191856
6.910	1.03581737	-1.00983049	-0.00188948

Table 3: Characteristic multipliers of $\mathbf{M_3}$ for a range of the main bifurcation parameter α.

A. Chua's circuit

The model of Chua's circuit (Madan, 1993) with a soft nonlinearity is given by

$$\begin{aligned} \dot{x} &= \alpha[y - \varphi(x)], \\ \dot{y} &= x - y + z, \\ \dot{z} &= -\beta y, \end{aligned} \tag{8}$$

where $\varphi(x) = \frac{1}{16}x^3 - \frac{1}{6}x$ is a cubic polynomial nonlinearity, and α and β are control parameters; for more details see (Khibnik *et al.*, 1993a; Moiola and Chua, 1999). In our study, parameter β will be fixed at $\beta = 10.91678$, and α will be considered as the main control parameter (μ in the previous section).

A.1. Monodromy matrix approach

System (8) can be regarded as the feedback connection of the linear plant

$$\mathbf{G}(s;\mu) = \frac{s^2 + s + \beta}{s^3 + (1 - \frac{1}{12}\alpha)s^2 + (-\frac{13}{12}\alpha + \beta)s + \frac{\beta}{12}\alpha},$$

and the nonlinear feedback $g(x) = \frac{\alpha}{4}\left(\frac{1}{4}x^3 - \frac{1}{3}x\right)$, and thus, $\mathbf{J} = \frac{\alpha}{4}(\frac{3}{4}x^2 - \frac{1}{3})$. The equilibrium points are given by $\hat{x} = \pm 2\sqrt{2/3}$. A Hopf bifurcation occurs for $\alpha_H = 3/2(\sqrt{1+2\beta} - 1) \approx 5.667671$, and the frequency of the periodic solution is $\omega_0 = \alpha\sqrt{2}/3$. A period-doubling bifurcation, obtained by numerical integration, gives the critical value $\alpha_2 = 6.888775$.

The characteristic multipliers for the *approximate* monodromy matrices $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}_3$ are shown in Tables 1, 2 and 3, respectively, for different values of the parameter α. From these Tables, it can be observed that the largest value of $\|\lambda_1 - 1\|$ occurs for the first-order HBA approximation, revealing that the approximate monodromy matrix $\mathbf{M}_1$ may be a coarse estimate of the true monodromy matrix $\mathbf{M}$. Nevertheless, increasing the number of harmonics in the approximation of the limit cycle leads to decreasing values of $\|\lambda_1 - 1\|$, and hence, $\mathbf{M}_2$ and $\mathbf{M}_3$ can be regarded as "good" approximations to $\mathbf{M}$.

Of particular interest is the behavior of the characteristic value λ_2 in Tables 2 and 3. In both Tables, λ_2 takes the value -1, for some $\alpha^* \in (6.915, 6.920)$ for the

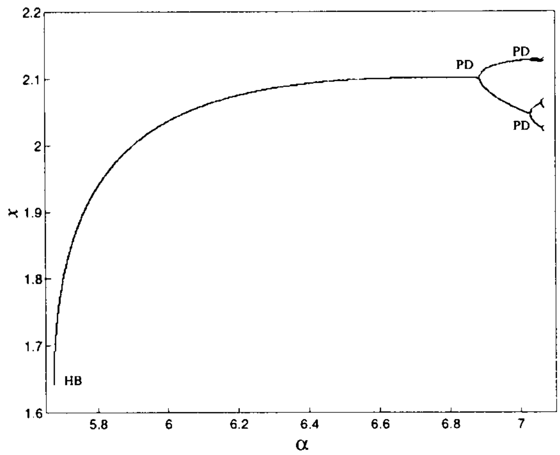

Figure 1: Continuation of periodic solutions for Chua's circuit.

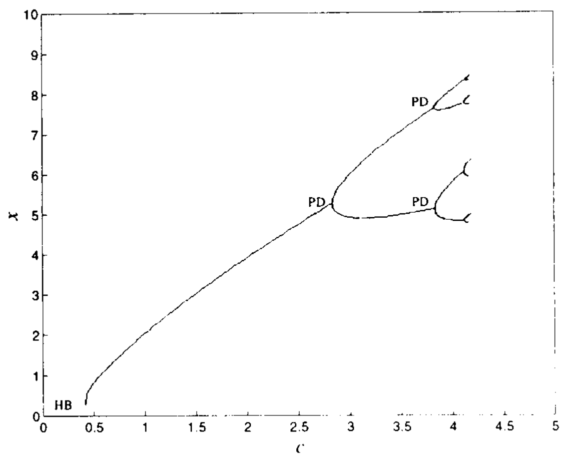

Figure 2: Continuation of periodic solutions for Rössler system.

fourth-order HBA, or some $\alpha^* \in (6.905, 6.910)$ for the sixth-order HBA, indicating the chance of a period-doubling bifurcation. These values compare very favorably with the critical value $\alpha_2 = 6.888775$.

This procedure can be repeated for detecting the *second* period-doubling bifurcation. However, due to the approximate nature of the calculations, some care is needed in interpreting the results. Moreover, a higher order HBA may be pursued to obtain meaningful estimates.

A.2. Numerical continuation approach

In order to obtain the values of α_{2^n} where a period-doubling bifurcation occurs, a continuation of the period 1 orbit is computed, as shown in Fig. 1. Only one of the branches of the cascade of bifurcations is depicted; due to the symmetry of the system, the other branch is entirely similar. The critical values of α are: $\alpha_2 = 6.888775$, $\alpha_4 = 7.034154$, $\alpha_8 = 7.065070$, $\alpha_{16} = 7.071624$ and $\alpha_{32} = 7.073030$. Figure 1 can be regarded as a synthesis of the oscillatory dynamics that ends in the "single scroll" attractor and then, by a complex connection through the saddle point, in the "double scroll" attractor.

Successive values of δ_n for $n = 1, 2, 3$ can be computed [see Eqn. (6)] with the parameter values corresponding to the first five period-doubling bifurcations

$$\delta_1 = 4.702387, \quad \delta_2 = 4.717119, \quad \delta_3 = 4.661451,$$

that seem to get closer to the theoretical limit δ_F. The values found for α_{2^n} enable us to estimate different values of $\alpha_{\infty,n}$ [see Eqn. (7)]. The computed values are $\alpha_{\infty,1} = 7.073778$, $\alpha_{\infty,2} = 7.073496$, $\alpha_{\infty,3} = 7.073410$, $\alpha_{\infty,4} = 7.073413$.

Remark: As noticed in Guckenheimer and Meloon (2000), the precision with which the eigenvalue λ_1 is computed (by definition must be +1) is a measure for the accuracy of the computation of the periodic solution and its monodromy matrix. It is interesting to see that in the present paper the error $\|\lambda_1 - 1\|$ for this eigenvalue is about 3.5×10^{-2} for the best approximate monodromy matrix $\mathbf{M}_3$. Regarding the continuation of cycles performed with LOCBIF, this error is below 1×10^{-5}. Just for comparison, notice that in the recent contribution of Guckenheimer and Meloon (2000) the authors have dealt with an error below 1×10^{-14} by using a sophisticated method which can perform even better than the classical and standard algorithms for continuation of periodic orbits.

B. Rössler system

Rössler system (Rössler, 1976) is given by

$$\begin{aligned} \dot{x} &= -y - z, \\ \dot{y} &= x + ay, \\ \dot{z} &= b + z(x - c), \end{aligned}$$

where a and b have been fixed at 0.2, and c is the main bifurcation parameter. A continuation of the period 1 orbit is shown in Fig. 2. The period-doubling bifurcations occur for $c_2 = 2.832446$, $c_4 = 3.837358$, $c_8 = 4.124215$, $c_{16} = 4.186997$, and $c_{32} = 4.200536$, allowing to compute the following estimates of Feigenbaum's constant

$$\delta_1 = 3.503181, \quad \delta_2 = 4.569096, \quad \delta_3 = 4.637122.$$

Again, the values of c_{2^n} found allow to estimate different values of $c_{\infty,n}$ for which the system will be in (or very near to) chaotic regime: $c_{\infty,1} = 4.111251$, $c_{\infty,2} = 4.202399$, $c_{\infty,3} = 4.204108$, $c_{\infty,4} = 4.204226$.

C. Sprott system

Of the several chaotic systems discovered by Sprott (1994), let us consider the following one, having a sin-

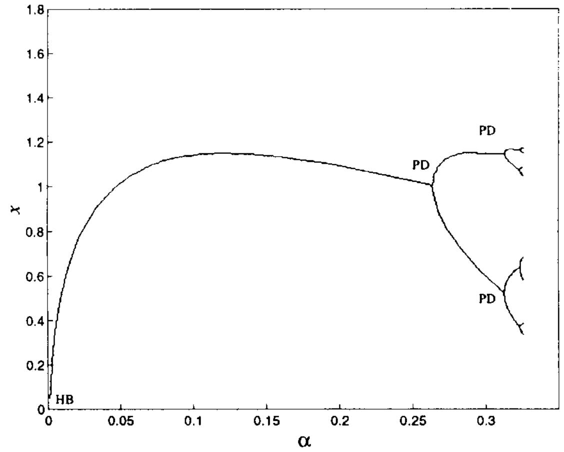

Figure 3: Continuation of periodic solutions for Sprott system.

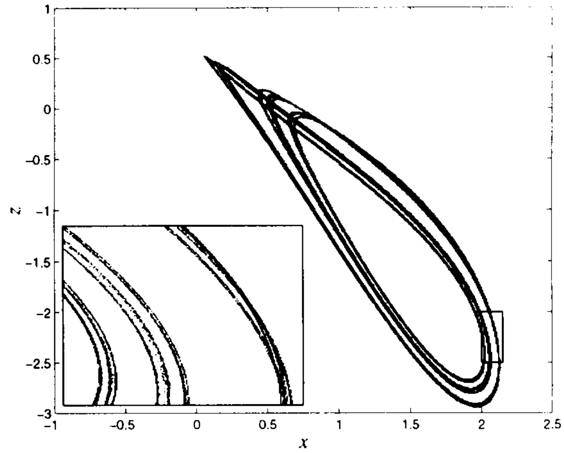

Figure 4: Phase plot projection of variables x and z for Chua's circuit ($\alpha = 7.073496$).

gle nonlinearity

$$\begin{aligned} \dot{x} &= \alpha x + z, \\ \dot{y} &= xz - y, \\ \dot{z} &= -x + y. \end{aligned}$$

Here α is the control parameter. A continuation of the period 1 orbit is carried out in order to verify Feigenbaum's constant; the corresponding diagram is shown in Fig. 3. The period-doubling bifurcations occur for $\alpha_2 = 0.2644273$, $\alpha_4 = 0.3136389$, $\alpha_8 = 0.3244517$, $\alpha_{16} = 0.3267759$, and $\alpha_{32} = 0.3272732$, obtaining the following estimates of Feigenbaum's constant

$$\delta_1 = 4.551235, \quad \delta_2 = 4.652267, \quad \delta_3 = 4.673638.$$

Once more, different values of $\alpha_{\infty,n}$ can be estimated from the values found for α_{2^n}. The computed values are $\alpha_{\infty,1} = 0.3270517$, $\alpha_{\infty,2} = 0.3273988$, $\alpha_{\infty,3} = 0.3274094$, $\alpha_{\infty,4} = 0.3274087$.

D. Discussion of results

Although in this work a rigorous verification has not been carried out, Kennedy (1992, 1993) observed in an electronic implementation of Chua's circuit that is very difficult to distinguish experimentally the bifurcation from a cycle of period 4 to a cycle of period 8. In other experimental setups the situation is similar, as can be observed in Table 4 (Cvitanovic, 1989). In other words, the cascade of period-doubling bifurcations quickly goes to the attractor, and in general, only the bifurcations for cycles of period 2 and 4 are easily distinguishable.

From a computational point of view the situation is somewhat similar: it is also very difficult to detect accurately the appearance of n period-doubling bifurcations when n is larger than 4. However, the number of period-doubling bifurcations detected depends on the technique used to compute the solutions. When using numerical simulations difficulties similar to those detected in experimental setups do arise. The technique of continuation of periodic solutions allows for a better detection of period-doubling bifurcations, as can be observed in the examples above. Nevertheless, it is very difficult to go beyond than a period-32 bifurcation (5 period-doublings).

Experiment	n	δ_n	Authors
Water	4	4.3	Giglio (1981)
Mercury	4	4.4	Libchaber (1982)
Diode	5	4.3	Testa (1982)
Transistor	4	4.7	Arecchi (1982)
Josephson junction	3	4.5	Yeh (1982)

Table 4: Estimates of the Feigenbaum's constant in physical experiments (n is the number of period doublings).

The lack of a large number of points in the parameter space for which a period-doubling bifurcation can be accurately detected severely limit the number of terms of the series that can be computed using Eqn. (6). Therefore, a truly "convergence" of the series to the limit value δ_F is difficult to justify. It should be noted, however, that the computed values are very close to the theoretical δ_F. On the other hand, Eqn. (7) and the numerical results obtained for the three examples show that the value of the parameter such that the system is in (or near to) the chaotic regime can be estimated from the parameter value at the second and third period-doubling bifurcations. For example, a chaotic regime for Chua's circuit can be obtained (at least, from an engineering point of view) setting $\alpha = \alpha_{\infty,2} = 7.073496$. The projection of the trajectories of the system on the x-z plane resembles an orbit of period 8 (see Fig. 4); however, the detailed view shown in the same figure reveals the chaotic nature of the solutions.

IV. CONCLUSIONS

A quasi-analytical approach has been introduced in this paper for detecting the first period-doubling bifurcation in a nonlinear dynamical system. Prediction of the period-doubling bifurcation is accomplished very accurately via the proposed computational scheme, using a reasonably small number of harmonics. The technique has the advantage of utilizing the structure of the system for analysis, design or other purposes, since the harmonic content as well as the approximate monodromy matrices can be implemented explicitly. Furthermore, approximations to the Feigenbaum's constant for systems of ODEs has been shown. All the three considered systems (Chua's circuit, Rössler's and Sprott's) have a route to chaos following the cascade of period-doubling. Although a better precision is obtained using programs for continuation of periodic solutions comparing with the pure simulations (see page 379 of Strogatz, 1994, and compare it with the Fig. 2), the use of LOCBIF is arduous if the initial condition is not chosen appropriately.

This method can be used for bifurcation control by applying the standard techniques given by Tesi *et al.*, (1996) when delaying the first period-doubling bifurcation. In this case, this frequency domain approach will provide a better approximation than the first-order harmonic balance solution of the describing function method. Moreover, by detecting the second period-doubling bifurcation and applying the Feigenbaum's constant, an approximate detection of the birth of chaos can be obtained.

The numerical estimation of Feigenbaum's constant presented in this work is similar, in spirit, to the study carried out on reactors of tubular flow by Kim *et al.* (1989), but using a package for continuation of periodic solutions instead of integration routines.

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REFERENCES

Basso, M., R. Genesio and A. Tesi, "A frequency method for predicting limit cycle bifurcations," *Nonlinear Dynamics* **13**, 339-360 (1997).

Berns, D. W., J. L. Moiola and G. Chen, "A quasi-analytical approach to period-doubling bifurcation computation and prediction," *European Control Conference*, Karlsruhe, Germany, in CD-Rom, (1999).

Berns D. W., J. L. Moiola and G. Chen, "A quasi-analytical method for period-doubling bifurcation," in press in *Proc. of the Intnl. Symposium on Circuits and Systems*, Sydney, Australia, (2001).

Bonani, F. and M. Gilli, "Analysis of stability and bifurcations of limit cycles in Chua's circuit through the Harmonic-Balance approach," *IEEE Trans. on Circ. and Syst.- Part I* **46**, 881-890 (1999).

Chen, G. and X. Dong, *From Chaos to Order - Perspectives, Methodologies and Applications*, World Scientific Publishing Co., Singapore (1998).

Chen, G., J. L. Moiola and H. O. Wang, "Bifurcation control: Theories, methods, and applications," *Int. J. of Bifurcation and Chaos*, special issue "Chaos control and synchronization," **10**(3), 511-548 (2000).

Collantes, J. M. and A. Suárez, "Period-doubling analysis and chaos detection using commercial harmonic balance simulators," *IEEE Trans. on Microw. Theo. and Techn.* **48**, 574-581 (2000).

Cvitanovic, P., *Universality in Chaos*, Adam Hilger, Bristol and Nueva York, second edition (1989).

Govaerts, W. J. F., *Numerical Methods for Bifurcations of Dynamical Equilibria*, SIAM, Philadelphia (2000).

Guckenheimer, J. and B. Meloon, "Computing periodic orbits and their bifurcations with automatic differentiation," *SIAM J. Sci. Computing* **22**, 951–985, (2000).

Kennedy, M. P., "Robust OP Amp realization of Chua's circuit," *Frequenz* **46**(3), 66-88 (1992).

Kennedy, M. P., "Three steps to chaos - Part II: A Chua's circuit primer," *IEEE Trans. on Circ. and Syst. - Part I* **40**(10), 657-674 (1993).

Khibnik, A. I., D. Roose and L. O. Chua, "On periodic orbits and homoclinic bifurcations in Chua's circuit with a smooth nonlinearity," *Int. J. of Bifurcation and Chaos* **3**(2), 363-384 (1993a).

Khibnik, A. I., Yu. A. Kuznetsov, V. V. Levitin and E. V. Nikolaev, "Continuation techniques and interactive software for bifurcation analysis of ODE's and iterated maps," *Physica* **D 62**, 360-371 (1993b).

Kim S. H., J. W. Yang and H. S. Yang, "Chaotic behavior in tubular flow reactors with reactions," *Korean J. of Chemical Engng.* **6**(4), 277-281 (1989).

Kuznetsov, Yu A., *Elements of Applied Bifurcation Theory*, Appl. Math. Sci. **112**, second edition, Springer-Verlag, New York (1998).

Madan, R., *Chua's Circuit: A Paradigm for Chaos*, World Scientific Publishing, Singapore (1993).

Maggio, G. M., M. P. Kennedy and M. Gilli, "An approximate analytical approach for predicting period-doubling in the Colpitts oscillator," *Proc. of the Int. Symp. on Circ. and Syst.*, Monterey, USA, 671–674 (1998).

Moiola, J. L. and G. Chen, *Hopf Bifurcation Analysis - A Frequency Domain Approach*, World Scientific Pub. Co. **21**, Singapore (1996).

Moiola, J. L. and L. O. Chua, "Hopf bifurcations and degeneracies in Chua's circuit - A Perspective from a frequency domain approach", *Int. J. of Bifurcation and Chaos* **9**(1), 295-303 (1999).

Phillipson, P. E. and P. Schuster, "Bifurcation dynamics of three-dimensional systems," *Int. J. of Bifurcation and Chaos* **10**(8), 1787-1804 (2000).

Rössler, O. E., "Different types of chaos in two simple differential equations," *Z. Naturforschung* **31a**, 1664-1670 (1976).

Sprott, J. C., "Some simple chaotic flows," *Physical Review Letters* **A 266**, 19-23 (1994).

Strogatz, S. H., *Nonlinear Dynamics and Chaos*, Addison Wesley, second edition (1994).

Tesi, A., E. H. Abed, R. Genesio and H. O. Wang, "Harmonic balance analysis of period-doubling bifurcations with implications for control of nonlinear dynamics," *Automatica* **32**(9), 1255-1271 (1996).

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