PROJECTIONS IN OPERATOR RANGES

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Abstract. If $\mathcal{H}$ is a Hilbert space, A is a positive bounded linear operator on $\mathcal{H}$ and $\mathcal{S}$ is a closed subspace of $\mathcal{H}$, the relative position between $\mathcal{S}$ and $A^{-1}(\mathcal{S}^{\perp})$ establishes a notion of compatibility. We show that the compatibility of $(A, \mathcal{S})$ is equivalent to the existence of a convenient orthogonal projection in the operator range $R(A^{1/2})$ with its canonical Hilbertian structure.

1. Introduction

Oblique projections are becoming an important tool in several areas of mathematics, statistics and engineering. This phenomenon is illustrated in many papers on integral equations, iterative methods in numerical linear algebra, signal processing, linear regression, just to mention a sample; in [10] the reader can find an extensive list of papers on these applications. In a recent series of papers [7], [8], [9], [10] the set of oblique projections is studied according to different inner and semi-inner products which orthogonalize them. This is the way in which a certain notion of compatibility arises. A positive (Hermitian semidefinite) operator A on a Hilbert space $\mathcal{H}$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ are said to be **compatible** if there exists a projection Q in $\mathcal{H}$ with range $\mathcal{S}$ such that $AQ = Q^*A$. This equality means that $(Qx, y)_A = (x, Qy)_A$ $\forall x, y \in \mathcal{H}$ if $(u, v)_A := \langle Au, v\rangle$ where $u, v \in \mathcal{H}$ and $\langle\ ,\ \rangle$ denotes the inner product on $\mathcal{H}$. Observe that $(\ ,\ )_A$ is, in general, a semi-inner product, because A is allowed to have a non trivial nullspace. If the pair $(A, \mathcal{S})$ is compatible then a distinguished element $P_{A,\mathcal{S}}$ in

$$P(A, \mathcal{S}) = \{Q \in L(\mathcal{H}) : Q^2 = Q,\ Q\mathcal{H} = \mathcal{S},\ AQ = Q^*A\}$$

can be defined with certain optimal properties.

On the other hand, given Hilbert spaces $\mathcal{H}, \mathcal{K}$ the range of a bounded linear operator $T : \mathcal{H} \to \mathcal{K}$ can be naturally given a Hilbert space structure, by means of the inner product $\langle Tx, Ty\rangle_T = \langle x_1, y_1\rangle$, $x, y \in \mathcal{H}$, where x_1 (resp. y_1) denotes the orthogonal projection of x (resp. y) to the closure of $R(T)$ in $\mathcal{K}$. These Hilbert spaces $B(T) = (R(T), \langle\ ,\ \rangle_T)$ play a significant role in many areas, in particular in the de Branges complementation theory. The reader is referred to the books by de Branges and Rovnyak [6] and Ando [1] for systematic expositions of this theory. The main goal of this paper is to determine the compatibility of a pair $(A, \mathcal{S})$ by checking the existence of a convenient orthogonal projection in the space $B(A^{1/2}) = (R(A^{1/2}), \langle\ ,\ \rangle_{A^{1/2}})$. This approach allows us to see the oblique projection $P_{A,\mathcal{S}}$ as a true orthogonal projection (acting, of course, on a different Hilbert space, namely $B(A^{1/2})$). Let us describe more precisely these concepts and results. Section 2 collects some notations and a description of a theorem by R. G. Douglas which

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is one of the main tools of this paper. Douglas theorem studies the existence and uniqueness of solutions of operator equations like $AX = B$, for operators A, B between Hilbert spaces. Section 3 starts with a survey of known results on compatibility and on the form of a distinguished projection $P_{A,\mathcal{S}}$ with the properties mentioned above. Some proofs of these results can be found in [7], [9] and [10]. In addition, we present new characterizations of compatibility; some of them are quite technical but they will be needed later, in the sections dealing with operator ranges. Section 4 contains a description of the Hilbertian structure on an operator range. Here, the references are a paper by Fillmore and Williams [16] and the books by Ando [1] and de Branges and Rovnyak [6], in addition to a paper by Dixmier [14]. The particular operator range we are interested in is $R(A^{1/2})$, i.e., the range of the positive square root of a fixed positive operator A. Moreover, we need to characterize the closure and the orthogonal complement of a subspace in $B(A^{1/2})$ and the algebra of all bounded operators on $\mathcal{H}$ which can be extended, after a convenient reduction modulo the nullspace $N(A)$ of A, to $B(A^{1/2})$. In this section we slightly extend some results by Barnes [4] who studied the case of an injective operator A; however, Barnes' goal is different from ours, namely, he studies the spectral properties of an operator when it is seen in $B(A)$ or in $B(A^{1/2})$. Finally, Section 5 contains a characterization of the compatibility of a pair $(A, \mathcal{S})$ in terms of certain decompositions of $B(A^{1/2})$. Moreover, it is proven that if $(A, \mathcal{S})$ is compatible then the distinguished projection $P_{A,\mathcal{S}}$ can be extended (in the sense mentioned above) to $B(A^{1/2})$, and conversely. Also, it is shown that the orthogonal projection $P_{\mathcal{W}}$ onto a closed subspace $\mathcal{W}$ of $B(A^{1/2})$ comes from an operator on $\mathcal{H}$ if and only if $(A, \mathcal{S})$ is compatible, where $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ such that $A(\mathcal{S})$ is dense in $\mathcal{W}$ (in the topology of $B(A^{1/2})$).

2. Preliminaries

In what follows $\mathcal{H}$ and $\mathcal{K}$ denote Hilbert spaces, $L(\mathcal{H}, \mathcal{K})$ is the Banach space of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, $L(\mathcal{H})$ is the algebra $L(\mathcal{H}, \mathcal{H})$ and $L(\mathcal{H})^+$ denotes the cone of positive operators on $\mathcal{H}$. The set of all (bounded linear) projections in a Hilbert space is denoted by $\mathcal{Q}$. For any $W \in L(\mathcal{H})$, the range and the nullspace of W are respectively denoted by $R(W)$ and $N(W)$. Given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, $P_{\mathcal{M}}$ denotes the orthogonal projection onto $\mathcal{M}$. If $W \in L(\mathcal{H}, \mathcal{K})$ has closed range, then the Moore-Penrose pseudoinverse of W, denoted by $W^\dagger$, belongs to $L(\mathcal{K}, \mathcal{H})$ and it is characterized by the properties $WW^\dagger = P_{R(\mathcal{W})}$ and $W^\dagger W = P_{R(\mathcal{W}^*)}$ (see [11], [5] and [13] for more properties and applications of $W^\dagger$).

We state the theorem by R. G. Douglas [15], [16] mentioned in the introduction which will be used in several parts of the paper.

Theorem 2.1. *Given Hilbert spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{G}$ and operators $A \in L(\mathcal{H}, \mathcal{G})$, $B \in L(\mathcal{K}, \mathcal{G})$ then the following conditions are equivalent:*

i) the equation $AX = B$ has a solution in $L(\mathcal{K}, \mathcal{H})$;

ii) $R(B) \subseteq R(A)$;

iii) there exists $\lambda > 0$ such that $BB^* \leq \lambda AA^*$. In this case, there exists a unique $D \in L(\mathcal{K}, \mathcal{H})$ such that $AD = B$, $R(D) \subseteq \overline{R(A^*)}$, and $N(D) = N(B)$; moreover, $\|D\|^2 = \inf\{\lambda > 0 : \ BB^* \leq \lambda AA^*\}$. We shall call D the **reduced solution** *of $AX = B$.*

As a consequence of Douglas theorem and the properties of the Moore-Penrose pseudoinverses, it follows that if $R(A)$ is closed and $R(B) \subseteq R(A)$ then $A^\dagger B$ is the reduced solution of $AX = B$.

3. Oblique projections

Given $A \in L(\mathcal{H})$, the functional

$$(\ , \ )_A : \mathcal{H} \times \mathcal{H} \to \mathbb{C} \ , \quad (x,y)_A = \langle Ax, y\rangle, \quad x,y \in \mathcal{H}$$

is an equivalent inner product on $\mathcal{H}$ if and only if A is a positive invertible operator on $\mathcal{H}$. If $A \in L(\mathcal{H})^+$, then $(\ , \ )_A$ is a Hermitian sesquilinear form which is positive semidefinite, i.e., a semi-inner product on $\mathcal{H}$. For a subspace $\mathcal{M}$ of $\mathcal{H}$ it is easy to see that

$$\{x \in \mathcal{H} : (x,y)_A = 0 \ \forall y \in \mathcal{M}\} = (A\mathcal{M})^\perp = A^{-1}(\mathcal{M}^\perp).$$

Given $W \in L(\mathcal{H})$, an A-adjoint of W is any $V \in L(\mathcal{H})$ such that $(Wx,y)_A = (x,Vy)_A$, for all $x,y \in \mathcal{H}$, i.e., $AW = V^*A$. We are interested in projections $Q \in \mathcal{Q}$ which are A-Hermitian, in the sense that $AQ = Q^*A$.

From now on, we fix $A \in L(\mathcal{H})^+$ and a closed subspace $\mathcal{S}$ of $\mathcal{H}$ and abbreviate $P = P_\mathcal{S}$. The first result is due to M. G. Krein [21]. There is a recent proof of it in [7].

Lemma 3.1 (Krein)**.** *Let Q be a projection with $R(Q) = \mathcal{S}$. Then Q is A-Hermitian if and only if $N(Q) \subseteq A^{-1}(\mathcal{S}^\perp)$.*

Recall from the introduction the set $P(A,\mathcal{S})$ of all A-Hermitian projections with fixed range $\mathcal{S}$, i.e. $P(A,\mathcal{S}) = \{Q \in \mathcal{Q} : R(Q) = \mathcal{S}$ and $AQ = Q^*A\}$. The pair $(A,\mathcal{S})$ is **compatible** if the set $P(A,\mathcal{S})$ is non empty.

Observe that it follows from lemma 3.1 that if a projection Q has range $\mathcal{S}$ then $Q \in \mathcal{P}(A,\mathcal{S})$ if and only if $N(Q) \subseteq A^{-1}(\mathcal{S}^\perp)$, so that $(A,\mathcal{S})$ is compatible if and only if $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$. In this case, $\mathcal{H} = \mathcal{S} \oplus (A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N})$, where $\mathcal{N} = \mathcal{S} \cap A^{-1}(\mathcal{S}^\perp) = \mathcal{S} \cap N(A)$ and there exists a unique projection $P_{A,\mathcal{S}}$ with range $\mathcal{S}$ and nullspace $A^{-1}(\mathcal{S}^\perp) \ominus \mathcal{N}$. It is elementary to check that $P_{A,\mathcal{S}} \in P(A,\mathcal{S})$. At the end of section 3 we shall mention some optimal properties of $P_{A,\mathcal{S}}$.

Remark 3.2. In [3], Baksalary and Kala studied, in the matrix case, the existence of $P_{A,\mathcal{S}}$ under the additional hypothesis of the invertibility of A. In [18], Hassi and Nordström determined conditions on a Hermitian not necessarily invertible operator A, under which the set $P(A,\mathcal{S})$ is a singleton. They also proved some least-square-type results for indefinite inner products. In [22], Z. Pasternak-Winiarski studied, for A invertible, the analiticity of the map $A \to P_{A,\mathcal{S}}$ (see also [2] for shorter proofs and related results).

Consider the matrix representation of A in terms of the orthogonal projection $P_\mathcal{S}$ onto $\mathcal{S}$, namely,

$$A = \begin{pmatrix} a & b \\ b^* & c \end{pmatrix}; \tag{1}$$

this means that $a \in L(\mathcal{S})$, $b \in L(\mathcal{S}^\perp, \mathcal{S})$, $a \in L(\mathcal{S}^\perp)$ and $Ax = as + bs^\perp + b^*s + cs^\perp$ if $x = s + s^\perp$ is the decomposition of $x \in \mathcal{H} = \mathcal{S} \oplus \mathcal{S}^\perp$. If $Q \in \mathcal{Q}$ and $R(Q) = \mathcal{S}$ then there exists $X \in L(\mathcal{S}^\perp, \mathcal{S})$ such that the matrix representation of Q in terms of $P_\mathcal{S}$ is $Q = \begin{pmatrix} 1 & X \\ 0 & 0 \end{pmatrix}$. It is easy to see that the condition $AQ = Q^*A$ is equivalent to the equation $aX = b$. Then $(A,\mathcal{S})$ is compatible if and only if the equation $aX = b$ admits a solution. Applying Douglas theorem, this is equivalent to $R(b) \subseteq R(a)$ (or $R(PA) \subseteq R(PAP)$). Consider the reduced solution d of $aX = b$. It easily follows that $P_{A,\mathcal{S}} = \begin{pmatrix} 1 & d \\ 0 & 0 \end{pmatrix}$ (see [7], for a proof of these facts). Observe that, if A

is invertible, then $P_{A,\mathcal{S}} = P\Big(PAP + (I-P)A(I-P)\Big)^{-1}A$. For many results in the case of invertible A, the reader is referred to [22], [2] and [7].

Some basic conditions for the compatibility of the pair $(A,\mathcal{S})$ can be found in [7], [9], [10] as well as formulas for the elements of $P(A,\mathcal{S})$, if $(A,\mathcal{S})$ is compatible.

In what follows we give new characterizations of compatibility; also, we express the distinguished element $P_{A,\mathcal{S}}$ of $P(A,\mathcal{S})$ as the solution of certain Douglas-type equations.

If $\mathcal{S}\cap N(A)=\{0\}$ then the compatibility of $(A,\mathcal{S})$ can be easily checked. In fact:

Proposition 3.3. *Consider $A \in L(\mathcal{H})^+$ such that $\mathcal{S}\cap N(A)=\{0\}$. Then the following conditions are equivalent:*

i) $\mathcal{H}=\mathcal{S}\oplus A(\mathcal{S})^\perp$, i.e., $(A,\mathcal{S})$ is compatible;

ii) $\overline{A(\mathcal{S})}\oplus\mathcal{S}^\perp$ is closed;

iii) $\overline{A(\mathcal{S})}\oplus\mathcal{S}^\perp=\mathcal{H}$.

Proof.

i) $\to$ ii) We use the general fact that if $\mathcal{M},\mathcal{N}$ are closed subspaces, $\mathcal{M}+\mathcal{N}$ is closed if and only if $\mathcal{M}^\perp+\mathcal{N}^\perp$ is closed (see theorem 4.8 of [20]): if $\mathcal{S}\oplus A(\mathcal{S})^\perp=\mathcal{H}$, a fortiori $\mathcal{S}+A(\mathcal{S})^\perp$ is closed. Then $\mathcal{S}^\perp+\overline{A(\mathcal{S})}$ is closed. Besides $\mathcal{S}^\perp\cap\overline{A(\mathcal{S})}=(\mathcal{S}+A(\mathcal{S})^\perp)^\perp=\{0\}$.

ii) $\to$ iii) If $\mathcal{S}^\perp+\overline{A(\mathcal{S})}$ is closed, $\mathcal{S}^\perp+\overline{A(\mathcal{S})}=\overline{\mathcal{S}^\perp+\overline{A(\mathcal{S})}}=(\mathcal{S}\cap A(\mathcal{S})^\perp)^\perp=(\mathcal{S}\cap N(A))^\perp=\mathcal{H}$.

iii) $\to$ i) is similar.

□

The closure condition of part ii) is equivalent to an angle condition. In fact, the sum of two closed subspaces is closed if the angle determined by them is non zero. The reader is referred to [20], [5], [12], [19] for nice surveys on angles in Hilbert spaces, and to [7], [10] for particular details concerning compatibility. In particular, in [10] it is proven that $(A,\mathcal{S})$ is compatible if and only if the angle between $\mathcal{S}^\perp$ and the closure of $A(\mathcal{S})$ is non zero.

Next, we state a chain of necessary conditions for compatibility.

Proposition 3.4. *Consider the following conditions:*

(1) *The pair $(A,\mathcal{S})$ is compatible.*
(2) *$A(\mathcal{S})$ is closed in $R(A)$.*
(3) *$A^{-1}(\overline{A(\mathcal{S})})=\mathcal{S}+N(A)$.*
(4) *$A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$.*
(5) *$\mathcal{S}+N(A)$ is closed.*
(6) *$P_{\overline{R(A)}}(\mathcal{S})$ is closed.*

Then $1\to2\to4\to5$, $2\leftrightarrow3$ and $5\leftrightarrow6$.

Proof.

$1\to2$: Observe that $(A,\mathcal{S})$ is compatible if and only if $R(A)=A(\mathcal{S})+(\mathcal{S}^\perp\cap R(A))$. Consider $z\in\overline{A(\mathcal{S})}\cap R(A)$; then there exists a sequence $\{s_n\}$ in $\mathcal{S}$ such that $As_n\to z$ and there exist $s\in\mathcal{S}$ and $y\in\mathcal{H}$ such that $Ay\in\mathcal{S}^\perp$ and $z=As+Ay$. Since $\langle As_n,w\rangle=0$ for every $w\in A^{-1}(\mathcal{S}^\perp)$, then $\langle z,w\rangle=0$ for every $w\in A^{-1}(\mathcal{S}^\perp)$. Thus, $0=\langle z,y\rangle=\langle s,Ay\rangle+\langle Ay,y\rangle=\langle Ay,y\rangle=\|A^{1/2}y\|^2$ and $y\in N(A)$. Therefore, $z=As\in A(\mathcal{S})$.

2 → 4: Consider $z \in \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2})$; then $z = A^{1/2}x$ for some $x \in \mathcal{H}$ and there exists a sequence $\{s_n\}$ in $\mathcal{S}$ such that $A^{1/2}s_n \to A^{1/2}x$; then $As_n \to Ax$ so that $Ax = A^{1/2}z \in \overline{A(\mathcal{S})} \cap R(A) = A(\mathcal{S})$. Then $z \in (A^{1/2}(\mathcal{S}) + N(A)) \cap R(A^{1/2})$ so that $z \in A^{1/2}(\mathcal{S})$.

2 ↔ 3: Observe that $A(\mathcal{S})$ is closed in $R(A)$ if and only if $\overline{A(\mathcal{S})} \cap R(A) = A(\mathcal{S})$; but this is equivalent to $A^{-1}(\overline{A(\mathcal{S})}) = A^{-1}(A(\mathcal{S}))$, i.e., $A^{-1}(\overline{A(\mathcal{S})}) = \mathcal{S} + N(A)$.

4 → 5: It is easy to prove that if $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$, then $A^{-1/2}(\overline{A^{1/2}(\mathcal{S})}) = \mathcal{S} + N(A)$. So that $\mathcal{S} + N(A)$ is closed.

5 ↔ 6: It is a general result that if $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces then $\mathcal{M} + \mathcal{N}$ is closed if and only if $P_{\mathcal{N}^\perp}(\mathcal{M})$ is closed (see [20] or [12].

□

In [7] it is shown that all conditions above are equivalent if $R(A)$ is closed. The next technical result will be used in the following sections.

Corollary 3.5. *If $\mathcal{S} + N(A)$ is closed then the following conditions are equivalent:*

i) *$(A, \mathcal{S})$ is compatible.*
ii) *$(A, P_{\overline{R(A)}}(\mathcal{S}))$ is compatible.*
iii) *$(A, \mathcal{W})$ is compatible, for every subspace $\mathcal{W}$ such that $P_{\overline{R(A)}}(\mathcal{W}) = P_{\overline{R(A)}}(\mathcal{S})$.*

Proof. As proved before, $\mathcal{S} + N(A)$ is closed if and only if $P_{\overline{R(A)}}(\mathcal{S})$ is closed, so that item ii) makes sense.

i)↔ ii): As $\mathcal{S} + N(A) = P_{\overline{R(A)}}(\mathcal{S}) \oplus N(A)$ then $\mathcal{S} + A^{-1}(\mathcal{S}^\perp) = \mathcal{S} + N(A) + A^{-1}(\mathcal{S}^\perp) = P_{\overline{R(A)}}(\mathcal{S}) \oplus A^{-1}(\mathcal{S}^\perp)$, because $N(A) \subseteq A^{-1}(\mathcal{S}^\perp)$. Therefore $(A, \mathcal{S})$ is compatible if and only if $(A, P_{\overline{R(A)}}(\mathcal{S}))$ is compatible.

ii)↔ iii): Using that i)↔ ii) for $\mathcal{S}$ and $\mathcal{W}$, $(A, P_{\overline{R(A)}}(\mathcal{W}))$ is compatible if and only if $(A, P_{\overline{R(A)}}(\mathcal{W}))$ is compatible if and only if $(A, P_{\overline{R(A)}}(\mathcal{S}))$ is compatible if and only if $(A, \mathcal{S})$ is compatible.

□

Remark 3.6. The pair $(A, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$ and $R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$. This type of decomposition will be simplified later. For the proof, observe first that $(A, \mathcal{S})$ is compatible if and only if $\mathcal{H} = \mathcal{S} + A^{-1}(\mathcal{S}^\perp)$. Applying $A^{1/2}$ to both sides of this equality we get $R(A^{1/2}) = A^{1/2}(\mathcal{S}) + A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$. From the proposition above, $A^{1/2}(\mathcal{S})$ is closed in $R(A^{1/2})$, so that $A^{1/2}(\mathcal{S}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2})$. The converse is similar.

Corollary 3.7. *The following conditions are equivalent:*

i) If $\mathcal{W} = A^{-1/2}(\overline{A^{1/2}(\mathcal{S})})$ then $(A, \mathcal{W})$ is compatible.
ii) $R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) \oplus A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$.
iii) There exists a solution Q of $A^{1/2}X = P_{\mathcal{M}}A^{1/2}$, where $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$.

Proof.

i) → ii) $(A, \mathcal{W})$ is compatible if and only if $\mathcal{H} = \mathcal{W} + A^{-1}(\mathcal{W}^\perp)$; as before, it follows that $R(A^{1/2}) = A^{1/2}(\mathcal{W}) + A^{1/2}(\mathcal{W})^\perp \cap R(A^{1/2})$.

Observe that $A^{1/2}(\mathcal{W}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2})$ and since $A^{1/2}(\mathcal{S}) \subseteq A^{1/2}(\mathcal{W}) \subseteq \overline{A^{1/2}(\mathcal{S})}$, we get $A^{1/2}(\mathcal{S})^\perp = A^{1/2}(\mathcal{W})^\perp$. Thus, $R(A^{1/2}) = \overline{A^{1/2}(\mathcal{S})} \cap R(A^{1/2}) + A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$.

The converse is similar.

ii) $\leftrightarrow$ iii) It is a consequence of Douglas theorem. $\square$

Lemma 3.8. *If $A \in L(\mathcal{H})^+$ then the following conditions are equivalent:*

i) *$R(PAP)$ is closed;*
ii) *$A^{1/2}(\mathcal{S})$ is closed;*
iii) *$A(\mathcal{S})$ is closed.*

Any of the conditions above implies that the pair $(A, \mathcal{S})$ is compatible. In particular, if $A(\mathcal{S})$ is finite dimensional then $(A, \mathcal{S})$ is compatible.

Proof. Since $A^{1/2}(\mathcal{S}) = R(A^{1/2}P)$ and $PAP = (A^{1/2}P)^*A^{1/2}P$, we get the equivalence between conditions i) and ii). Suppose that $R(PAP)$ is closed. Observe that $A(S) = R(AP)$ and $R(AP)$ is closed if and only if $R(PA)$ or equivalently if $R(PA^2P)$ is closed. Note that $(PAP)^2 \leq PA^2P$ and

$$N(PAP)^2 = N(PA^2P) = \mathcal{S}^\perp \oplus (\mathcal{S} \cap N(A)).$$

Since $PA^2P \geq (PAP)^2 > 0$ in $(N(PAP)^2)^\perp$ we get that $R(PA^2P)$ is closed. The converse is similar. We have already proved that if $R(PAP)$ is closed then $(A, \mathcal{S})$ is compatible. $\square$

Remark 3.9. The lemma shows, in particular, that in finite dimensional Hilbert spaces, compatibility is automatically satisfied. However, an efficient algorithm for finding every element of $P(A, \mathcal{S})$ is not known.

Next, we show that $P_{A,S}$ is, modulo the orthogonal projection onto $\mathcal{N} = N(A) \cap \mathcal{S}$, a reduced solution of a Douglas-type equation:

Proposition 3.10. *If the pair $(A, \mathcal{S})$ is compatible, let $\mathcal{M} = \overline{A^{1/2}(\mathcal{S})}$. Then the reduced solution Q of the equation*

$$(PAP)X = PA \tag{2}$$

coincides with the reduced solution of

$$(A^{1/2}P)X = P_{\mathcal{M}}A^{1/2}. \tag{3}$$

Moreover, $Q = P_{A,\mathcal{S}\ominus\mathcal{N}}$ and $P_{A,\mathcal{S}} = Q + P_{\mathcal{N}}$.

Proof. Let Q be the reduced solution of equation (2). Observe first that $N(PAP) = N(A^{1/2}P) = \mathcal{N} \dot{+} \mathcal{S}^\perp$ and, therefore, $\overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N}$. By the definition of reduced solution, $R(Q) \subseteq \overline{R(PAP)} = \mathcal{S} \ominus \mathcal{N}$ and $N(Q) = N(PA) = (AS)^\perp$. If $z \in \mathcal{S} \ominus \mathcal{N}$, then $PAPQz = PAz = PAPz$ and, since PAP is injective on $\mathcal{S} \ominus \mathcal{N}$, we get $Qz = z$. Since the pair $(A, \mathcal{S})$ is compatible, it follows that $\mathcal{H} = (\mathcal{S} \ominus \mathcal{N}) \oplus A^{-1}(\mathcal{S}^\perp)$, so that Q is the projection onto $R(Q) = \mathcal{S} \ominus \mathcal{N}$ with

$$N(Q) = A^{-1}(\mathcal{S}^\perp) \subseteq A^{-1}((\mathcal{S} \ominus \mathcal{N})^\perp) = A^{-1}(R(Q)^\perp).$$

By Krein's lemma it follows that $Q \in P(A, \mathcal{S} \ominus \mathcal{N})$. Observe also that $(\mathcal{S} \ominus \mathcal{N}) \cap N(A) = \{0\}$, so that $P(A, \mathcal{S} \ominus \mathcal{N})$ consists of a single element, namely $P_{A,\mathcal{S}\ominus\mathcal{N}}$. Since $P_{\mathcal{N}}$ is an $A-$ Hermitian projection onto $\mathcal{N}$, it follows that $Q + P_{\mathcal{N}} = P_{A,\mathcal{S}\ominus\mathcal{N}} + P_{\mathcal{N}} = P_{A,\mathcal{S}}$.

Let us prove that $Q = P_{A,\mathcal{S}} - P_{\mathcal{N}}$ is the reduced solution of the equation (3). Note that $(AP)Q = A^{1/2}P_{\mathcal{M}}A^{1/2} = AP_{A,\mathcal{S}} = APP_{A,\mathcal{S}}$. Hence $A^{1/2}(P_{\mathcal{M}}A^{1/2} - A^{1/2}PQ) = \{0\}$ and $R(P_{\mathcal{M}}A^{1/2} - A^{1/2}PQ) \subseteq N(A)$. But, also,

$$R\big(P_{\mathcal{M}}A^{1/2} - A^{1/2}PQ\big) \subseteq \overline{R(A)},$$

so that $P_{\mathcal{M}}A^{1/2} - A^{1/2}PQ = \{0\}$, which says that Q is a solution of (3). In order to see that Q is the reduced solution, observe that $N(Q) = A^{-1}(\mathcal{S}^\perp) = N(P_{\mathcal{M}}A^{1/2})$. □

If, in addition to the hypothesis of the proposition, $R(PAP)$ is supposed to be closed, then

$$Q = (PAP)^\dagger PA = (A^{1/2}P)^\dagger P_{\mathcal{M}}A^{1/2} = (A^{1/2}P)^\dagger A^{1/2}.$$

In fact, PAP has closed range if and only if $A^{1/2}P$ has, so that the Moore-Penrose inverses of these operators are bounded and, by the comments following Douglas theorem, the reduced solution of $(PAP)X = PA$ is $(PAP)^\dagger PA$ and that of $(A^{1/2}P)X = P_{\mathcal{M}}$ is $(A^{1/2}P)^\dagger P_{\mathcal{M}}A^{1/2}$; finally, $(A^{1/2}P)^\dagger P_{\mathcal{M}} = (A^{1/2}P)^\dagger$ because both operators satisfy the defining equations of the Moore-Penrose pseudoinverse of $A^{1/2}P$.

Concerning the minimal properties of $P_{A,\mathcal{S}}$ in $P(A,\mathcal{S})$, mentioned in the Introduction, we describe two of them. First, $\|P_{A,\mathcal{S}}\| \leq \|Q\|$ for all $Q \in P(A,\mathcal{S})$, but, in general, it is not the unique element of $P(A,\mathcal{S})$ with this property (see [7]). In order to describe the second property, we introduce some notation: if $T \in L(\mathcal{H},\mathcal{K})$, $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $x \in \mathcal{H}$, then a $(T,\mathcal{S})$-*interpolant* of x is an element of the set $\text{spl}(T,\mathcal{S},x) = \{z \in x + \mathcal{S} : \|Tz\| = \inf_{s\in\mathcal{S}} \|T(x+s)\|\}$. If $A = T^*T$, the following conditions holds:

(1) spl $(T,\mathcal{S},x) = \{z \in x + \mathcal{S} : |z|_A = \inf\{|x-s|_A : s \in \mathcal{S}\}\}$, if $|.|_A$ denotes the seminorm induced by A, i. e., $|x|_A^2 = \langle Ax, x\rangle$;
(2) spl $(T,\mathcal{S},x)$ is not empty for all $x \in \mathcal{H}$ if and only if $(A,\mathcal{S})$ is compatible;
(3) spl $(T,\mathcal{S},x)$ has a unique element for all $x \in \mathcal{H}$ if and only if $(A,\mathcal{S})$ is compatible and $\mathcal{S} \cap N(A) = 0$;
(4) If $(A,\mathcal{S})$ is compatible and $x \in \mathcal{H}$ then $(1 - P_{A,\mathcal{S}})x$ is the unique element of spl $(T,\mathcal{S},x)$ with minimal norm.

The proofs of these facts can be found in [9].

4. Operator ranges

In this section we recall a well-known construction of a Hilbertian structure on the range of an operator (see [6], [16] or [1]). We include some new facts which will be prove useful in the following section.

4.1. As we have already seen any $A \in L(\mathcal{H})^+$ induces a semi-inner product on $\mathcal{H}$, by means of $(x,y)_A = \langle Ax, y\rangle$, $x,y \in \mathcal{H}$. Denote by $\mathcal{K} = \overline{R(A)}$ and $P_{\mathcal{K}} \in L(\mathcal{H})$ the orthogonal projection onto $\mathcal{K}$. Since $\mathcal{K} = N(A)^\perp$, we can define on $\mathcal{K}$ the inner product

$$(x,y)_A = \langle Ax, y\rangle \ , \quad x,y \in \mathcal{K}.$$

This inner product induces the norm $|x|_A = (x,x)_A^{1/2} = \langle Ax, x\rangle^{1/2}$, $x \in \mathcal{K}$. Let $\mathcal{H}_A$ be the completion of the inner product space $(\mathcal{K}, (\ ,\ )_A)$. We assume that $\mathcal{K} \subseteq \mathcal{H}_A$.

Then, the projection $P_{\mathcal{K}}$ induces a map $\varphi : \mathcal{H} \to \mathcal{H}_A$, defined by $\varphi(x) = P_{\mathcal{K}}x$, $x \in \mathcal{H}$. Note that φ has nullspace $\mathcal{K}^\perp = N(A)$.

For every subspace $\mathcal{S}$ of $\mathcal{H}$, $\varphi(\mathcal{S})$ is a subspace in $\mathcal{H}_A$. By $\overline{\varphi(\mathcal{S})}^{\mathcal{H}_A}$ we denote its closure in $(\mathcal{H}_A, (\ ,\ )_A)$. Then the orthogonal projection from $\mathcal{H}_A$ onto $\overline{\varphi(\mathcal{S})}^{\mathcal{H}_A}$ always exists, even if the original pair $(A,\mathcal{S})$ is not compatible.

The relative position in $\mathcal{H}$ between $\mathcal{S}$ and $N(A)$ obviously affects the "size" of the projection. In this section we deduce conditions on this projection in order to obtain the compatibility of $(A,\mathcal{S})$.

4.2. The construction of $\mathcal{H}_A$ can be performed in the context of operator ranges. We refer the reader to the papers by Dixmier [14] and Fillmore and Williams [16] and to Ando's book [1].

Consider $T \in L(\mathcal{H}, \mathcal{K})$. The range of T can be given a Hilbert space structure $(R(T), \langle\ ,\ \rangle_T)$ in a unique way, such that T becomes a coisometry from $(\mathcal{H}, \langle\ ,\ \rangle)$ to $(R(T), \langle\ ,\ \rangle_T)$ (see [1]). More precisely, as $T : N(T)^\perp \to R(T)$ is a bijection, define $\langle Tx, Ty\rangle_T = \left\langle P_{N(T)^\perp}x, P_{N(T)^\perp}y\right\rangle$, for $x, y \in \mathcal{H}$. For $u \in R(T)$, denote $\|u\|_T = \langle u, u\rangle_T^{1/2}$. The key fact is that the operator $T : (\mathcal{H}, \langle\ ,\ \rangle) \to (R(T), \langle\ ,\ \rangle_T)$ is a coisometry. Observe that $\|u\|_T = \min\{\|a\| : Ta = u\}$, for all $u \in R(T)$, because $Ta = TP_{N(T)^\perp}a$ and $\|P_{N(T)^\perp}a\| \le \|a\|$. Therefore, $T : (\mathcal{H}, \langle\ ,\ \rangle) \to (R(T), \langle\ ,\ \rangle_T)$ is bounded, i.e., $\|Ta\|_T \le \|a\|$, for all $a \in \mathcal{H}$; also, for each $u \in R(T)$, there is a unique $a \in N(T)^\perp$ such that $Ta = u$ and $\|a\| = \|u\|_T$. As in the introduction, we use the notation $B(T) = (R(T), \langle\ ,\ \rangle_T)$. In [14] and [16] a number of characterizations of operator ranges are given. One of them, establishes that a subspace $\mathcal{R}$ of $\mathcal{H}$ is the range of a bounded operator if and only if there is an inner product $\langle\ ,\ \rangle'$ on $\mathcal{R}$ such that $(\mathcal{R}, \langle\ ,\ \rangle')$ is a Hilbert space and $\|x\|' \ge \|x\|$ for all $x \in \mathcal{R}$. (see [16], Theorem 1.1). More precisely, given $T \in L(\mathcal{H}, \mathcal{K})$, consider $T_1 = (T|_{N(T)^\perp})^{-1}$, $T_1 : R(T) \to N(T)^\perp$, and define $\langle u, v\rangle' = \langle u, v\rangle + \langle T_1u, T_1v\rangle$, for $u, v \in R(T)$. Then $\langle\ ,\ \rangle'$ is complete and $\|u\|' \ge \|u\|$ for all $u \in \mathcal{R}$. In fact, the inner products $\langle\ ,\ \rangle'$ and $\langle\ ,\ \rangle_T$ are equivalent: first, observe that if $Tx = u$ for $x \in \mathcal{H}$ then $T_1u = T_1Tx = T_1TP_{N(T)^\perp}x = P_{N(T)^\perp}x$. Therefore, $\|u\|'^2 = \langle u, u\rangle' = \langle Tx, Tx\rangle' \ge \langle T_1Tx, T_1Tx\rangle = \|P_{N(T)^\perp}x\|^2 = \|u\|_T^2$, so that $\|u\|_T \le \|u\|'$. Conversely, $\|u\|'^2 = \|TP_{N(T)^\perp}x\|^2 + \|u\|_T^2 \le (\|T\|^2+1)\|u\|_T^2$. Observe also that $\langle u, v\rangle' = \langle u, v\rangle + \langle u, v\rangle_T$, for $u, v \in \mathcal{R}$.

We shall consider the construction above for any positive operator on $\mathcal{H}$. More precisely, given $A \in L(\mathcal{H})^+$ consider $R(A^{1/2})$ with the norm induced by $\langle\ ,\ \rangle_{A^{1/2}}$, i.e. the space $B(A^{1/2})$. The next lemma shows that A provides an isometric isomorphism between $\mathcal{H}_A$ and $B(A^{1/2})$. It should be mentioned that the subtle relationship between $R(A)$ and $R(A^{1/2})$ is fundamental in this and all remaining results.

Lemma 4.3. *Given* $A \in L(\mathcal{H})^+$

$$A|_{\overline{R(A)}} : \left(\overline{R(A)}, (\ ,\ )_A\right) \to B(A^{1/2})$$

is an isometry with dense image and then it admits a unitary extension

$$A' : (\mathcal{H}_A, (\ ,\ )_A) \to B(A^{1/2}).$$

Proof. Denote, as before, $\mathcal{K} = \overline{R(A)}$. For all $x \in \mathcal{K}$, it holds

$$\|Ax\|_{A^{1/2}} = \|P_\mathcal{K} A^{1/2}x\| = \|A^{1/2}x\| = |x|_A.$$

Also, if $x \in \mathcal{K}$ and $\|Ax\|_{A^{1/2}} = 0$ then $|x|_A = \|A^{1/2}x\| = 0$ so that $x \in N(A^{1/2}) = N(A)$, and $x = 0$. It remains to prove that the image of $A|_\mathcal{K}$ is dense in $B(A^{1/2})$: since $R(A^{1/2})$ is dense in $\mathcal{K}$, for any $x \in \mathcal{H}$ there exists a sequence $\{x_n\}$ in $\mathcal{H}$ such that $A^{1/2}x_n \to P_\mathcal{K}x$, which means that $Ax_n \to u$ in $B(A^{1/2})$, for any $u \in R(A^{1/2})$. Then $A|_\mathcal{K} : \mathcal{K} \to R(A) \subseteq B(A^{1/2})$ admits a unitary extension from the completion of $\mathcal{K}$, namely $\mathcal{H}_A$, onto $B(A^{1/2})$. □

Remark 4.4. More generally, for $t \in [0,1]$ consider A^t and define in $R(A^t)$ the inner product $\langle A^t x, A^t y\rangle_{A^t} = \langle P_{\mathcal{K}} x, P_{\mathcal{K}} y\rangle$. Observe that $\overline{R(A^t)} = \mathcal{K}$, for all $t \in [0,1]$. Denote $\langle A^t x, A^t y\rangle_t = \langle A^t x, A^t y\rangle_{A^t}$, for $x, y \in \mathcal{H}$, $\|A^t x\|_t = \|A^t x\|_{A^t} = \|A^{t/2} x\|$, $(x,y)_t = (x,y)_{A^t}$ and $\mathcal{H}_t = \mathcal{H}_{A^t}$. Then $|x|_t = |x|_{A^t} = \|A^{t/2}x\|$.

As before, we get:

Corollary 4.5. *Given $A \in L(\mathcal{H})^+$, the operator*

$$A^t|_{\overline{R(A)}} : (\overline{R(A)}, (\ ,\ )_t) \to B(A^{t/2})$$

is an isometry with dense image and it admits a unitary extension

$$(A^t)' : (\mathcal{H}_t, (\ ,\ )_t) \to B(A^{t/2}) \ .$$

Proof. Straightforward. □

We have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{H} & \xrightarrow{\varphi} & (\overline{R(A)}, (\ ,\ )_A) & \xrightarrow{i} & (\mathcal{H}_A, (\ ,\ )_A) \\ & & \Big\downarrow A & & \Big\downarrow A' \\ \mathcal{H} & \xrightarrow{A} & (R(A), \langle\ ,\ \rangle_{A^{1/2}}) & \xrightarrow{j} & B(A^{1/2}) \end{array}$$

which relates two maps from $\mathcal{H}$ into two Hilbert spaces associated to the operator A. By the lemma above these two spaces are isometrically isomorphic. Observe that the images of the subspace $\mathcal{S}$ of $\mathcal{H}$ in $\mathcal{H}_A$ and $B(A^{1/2})$ are, respectively, $\phi(A)$ and $A(\mathcal{S})$.

The next result, which has been proved by Barnes for injective operators (see [4]), characterizes the operators $B \in L(\mathcal{H})$ which can be extended to $L(B(A^{1/2}))$.

Lemma 4.6. *Consider $B \in L(\mathcal{H})$. There exists $\widetilde{B} \in L(B(A^{1/2}))$ such that $\widetilde{B}A = AB$ if and only if $B(N(A)) \subseteq N(A)$ and $R(B^*A^{1/2}) \subseteq R(A^{1/2})$. In this case, such an operator is unique.*

Proof. Let $\widetilde{B} \in L(B(A^{1/2}))$ such that $\widetilde{B}A = AB$; if $x \in N(A)$ then $ABx = 0$ so that $Bx \in N(A)$ and $B(N(A)) \subseteq N(A)$. Since $\widetilde{B} \in L(B(A^{1/2}))$, there exists $C > 0$ such that $\|\widetilde{B}Ax\|_{A^{1/2}} \le C\|Ax\|_{A^{1/2}}$ for all $x \in \mathcal{H}$; equivalently, $\|ABx\|_{A^{1/2}} \le C\|Ax\|_{A^{1/2}}$. By definition of $\|\ \|_{A^{1/2}}$, this means

$$\|P_{\overline{R(A)}}A^{1/2}Bx\| \le C\|P_{\overline{R(A)}}A^{1/2}x\| \quad \text{ or } \quad \|A^{1/2}Bx\| \le C\|A^{1/2}x\|,$$

because $R(A^{1/2}) \subseteq \overline{R(A)}$. By Douglas theorem, the last inequality is equivalent to $R(B^*A^{1/2}) \subseteq R(A^{1/2})$. Conversely, if these conditions hold, it is easy to see that $\widetilde{B}$ can be defined in $R(A)$ and extended to a bounded operator in $B(A^{1/2})$. If there exists $C \in L(B(A^{1/2}))$ such that $CA = \widetilde{B}A$ then C and $\widetilde{B}$ coincide in $R(A)$, which is dense in $B(A^{1/2})$, so that $C = \widetilde{B}$. □

4.7. Given a subspace $\mathcal{W}$ of $B(A^{1/2})$ the closure (resp. the orthogonal complement) of $\mathcal{W}$ in $B(A^{1/2})$ is denoted $\overline{\mathcal{W}}'$ (resp. $\mathcal{W}^{\perp'}$). If $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{W} = A(\mathcal{S})$, then $\mathcal{M} = \overline{\mathcal{W}}' = A^{1/2}\left(\overline{A^{1/2}(\mathcal{S})}\right)$ and

$$\mathcal{M}^{\perp'} = \mathcal{W}^{\perp'} = A^{1/2}(A^{1/2}(\mathcal{S})^{\perp}) = \mathcal{S}^{\perp} \cap R(A^{1/2}).$$

From now $Q_{A,\mathcal{S}} \in L(B(A^{1/2}))$ denotes the orthogonal projection onto $\mathcal{M}$. Then $R(Q_{A,\mathcal{S}}) = \mathcal{M} = A^{1/2}(\overline{A^{1/2}(\mathcal{S})})$ and $N(Q_{A,\mathcal{S}}) = \mathcal{M}^{\perp'} = \mathcal{S}^\perp \cap R(A^{1/2})$. Observe that $A(\mathcal{S}) \subseteq R(Q_{A,\mathcal{S}})$ and $A(A^{-1}(\mathcal{S}^\perp)) = \mathcal{S}^\perp \cap R(A) \subseteq N(Q_{A,\mathcal{S}})$.

Lemma 4.8. *It holds $\overline{\mathcal{S}^\perp \cap R(A)}' = \mathcal{M}^{\perp'}$ if and only if $A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)$ is dense in $B(A^{1/2})$.*

Proof. $\overline{A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)}' = \mathcal{M} + \overline{\mathcal{S}^\perp \cap R(A)}'$ because $\mathcal{S}^\perp \cap R(A) \subseteq \mathcal{M}^{\perp'}$. Then $\overline{\mathcal{S}^\perp \cap R(A)}' = \mathcal{M}^{\perp'}$ if and only if $A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)$ is dense in $B(A^{1/2})$. □

5. Compatibility and operator ranges

We have now the tools for proving the relationship between the compatibility of A with $\mathcal{S}$ and the properties of the orthogonal projection $Q_{A,\mathcal{S}} \in L(B(A^{1/2}))$ onto $\mathcal{M} = \overline{A(\mathcal{S})}'$. We start with a technical result.

Proposition 5.1. *Given $A \in L(\mathcal{H})^+$ the following conditions are equivalent:*

i) *$(A, \mathcal{S})$ is compatible.*

ii) *$A^{1/2}(\mathcal{S}) + A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$ is closed in $R(A^{1/2})$ and $A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$ is dense in $A^{1/2}(\mathcal{S})^\perp \cap \overline{R(A^{1/2})}$.*

iii) *$R(A) = \mathcal{M} \cap R(A) + \mathcal{M}^{\perp'} \cap R(A)$ and $A(\mathcal{S})$ is closed in $R(A)$ in the topology of $B(A^{1/2})$.*

Proof. Denote $\mathcal{T} = A^{1/2}(\mathcal{S})^\perp \cap R(A^{1/2})$.

i) $\rightarrow$ ii) By Remark 3.6 $(A, \mathcal{S})$ is compatible if and only if $R(A^{1/2}) = A^{1/2}(\mathcal{S}) + \mathcal{T}$, so that $A^{1/2}(\mathcal{S}) + \mathcal{T}$ is closed in $R(A^{1/2})$. Also

$$\overline{R(A^{1/2})} = \overline{A^{1/2}(\mathcal{S})} + \overline{\mathcal{T}} \subseteq \overline{A^{1/2}(\mathcal{S})} + A^{1/2}(\mathcal{S})^\perp \cap \overline{R(A^{1/2})} \subseteq \overline{R(A^{1/2})}.$$

Therefore $\mathcal{T}$ is dense in $A^{1/2}(\mathcal{S})^\perp \cap \overline{R(A^{1/2})}$.

ii) $\rightarrow$ iii) By assumption, it holds

$$\begin{aligned} A^{1/2}(\mathcal{S}) + \mathcal{T} &= \overline{A^{1/2}(\mathcal{S}) + \mathcal{T}} \cap R(A^{1/2}) \\ &= \left(\overline{A^{1/2}(\mathcal{S})} + \overline{\mathcal{T}}\right) \cap R(A^{1/2}) \\ &= \overline{R(A^{1/2})} \cap R(A^{1/2}) = R(A^{1/2}), \end{aligned}$$

because $\overline{\mathcal{T}} = A^{1/2}(\mathcal{S})^\perp \cap \overline{R(A^{1/2})}$. Then $R(A^{1/2}) = A^{1/2}(\mathcal{S}) + \mathcal{T}$ and

$$\begin{aligned} R(A) &= A^{1/2}(R(A^{1/2})) = A(\mathcal{S}) + A^{1/2}(\mathcal{T}) \\ &\subseteq A(\mathcal{S}) + \mathcal{M}^{\perp'} \cap R(A) \\ &\subseteq \mathcal{M} \cap R(A) + \mathcal{M}^{\perp'} \cap R(A) \subseteq R(A). \end{aligned}$$

Therefore $R(A) = \mathcal{M} \cap R(A) + \mathcal{M}^{\perp\prime} \cap R(A)$ and $\mathcal{M} \cap R(A) = A(\mathcal{S})$.

iii) $\rightarrow$ i) Straightforward.

□

Remark 5.2. Condition ii) is equivalent to ii') $A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)$ is closed in $R(A)$ under the topology of $B(A^{1/2})$ and $\overline{\mathcal{S}^\perp \cap R(A)}' = \mathcal{M}^{\perp'}$. In fact, from the proof of ii)$\rightarrow$ iii) we get that $R(A) = A(\mathcal{S}) + \mathcal{M}^{\perp'} \cap R(A) = A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)$ so that $A(\mathcal{S}) + \mathcal{S}^\perp \cap R(A)$ is closed in $(R(A), \langle\ ,\ \rangle_{A^{1/2}})$ and $\overline{\mathcal{S}^\perp \cap R(A)}' = \mathcal{M}^{\perp'}$. The converse is similar. □

In the last part of the paper, we relate the compatibility of the pair $(A,\mathcal{S})$ with the existence of certain projections in $B(A^{1/2})$. As before, $\mathcal{M}=\overline{A(\mathcal{S})}'=A^{1/2}(\overline{A^{1/2}(\mathcal{S})})$.

Theorem 5.3. *If $(A,\mathcal{S})$ is compatible then there exists $\widetilde{P}_{A,\mathcal{S}}\in L(B(A^{1/2}))$ such that $\widetilde{P}_{A,\mathcal{S}}A=AP_{A,\mathcal{S}}$. Moreover, $\widetilde{P}_{A,\mathcal{S}}=Q_{A,\mathcal{S}}$.*

Proof. If $(A,\mathcal{S})$ is compatible then, by proposition 3.10, $R=P_{A,\mathcal{S}\ominus\mathcal{N}}$ is the reduced solution of

$$A^{1/2}PX=P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}$$

and $P_{A,\mathcal{S}}=R+P_{\mathcal{N}}$ where $\mathcal{N}=\mathcal{S}\cap N(A)$. Observe that $A^{1/2}P_{A,\mathcal{S}}=A^{1/2}PP_{A,\mathcal{S}}=A^{1/2}PR=P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}$ because $R(P_{A,\mathcal{S}})=\mathcal{S}$. Therefore $P_{A,\mathcal{S}}$ verifies $A^{1/2}P_{A,\mathcal{S}}=P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}$ so that $P^*_{A,\mathcal{S}}A^{1/2}=A^{1/2}P_{\overline{A^{1/2}(\mathcal{S})}}$ and then $R(P^*_{A,\mathcal{S}}A^{1/2})\subseteq R(A^{1/2})$. In order to apply lemma 4.6, let $x\in N(A)$ and observe that

$$A^{1/2}P_{A,\mathcal{S}}x=P_{\overline{A^{1/2}(\mathcal{S})}}A^{1/2}x=0,$$

because $N(A)=N(A^{1/2})$, so $P_{A,\mathcal{S}}x\in N(A)$. By 4.6, there exists $\widetilde{P}_{A,\mathcal{S}}\in L(B(A^{1/2}))$ such that $\widetilde{P}_{A,\mathcal{S}}A=AP_{A,\mathcal{S}}$; now, $\widetilde{P}_{A,\mathcal{S}}(R(A))=A(R(P_{A,\mathcal{S}}))=A(\mathcal{S})$, so that $A(\mathcal{S})\subseteq R(\widetilde{P}_{A,\mathcal{S}})$ and $\mathcal{M}=\overline{A(\mathcal{S})}'\subseteq R(\widetilde{P}_{A,\mathcal{S}})$. Also, $A(N(P_{A,\mathcal{S}}))\subseteq N(\widetilde{P}_{A,\mathcal{S}})\subseteq \mathcal{M}^{\perp'}$ which implies $\overline{\mathcal{S}^\perp\cap R(A)}'\subseteq N(\widetilde{P}_{A,\mathcal{S}})\subseteq\mathcal{M}^{\perp'}$. But $\overline{\mathcal{S}^\perp\cap R(A)}'=\mathcal{M}^{\perp'}$ because $(A,\mathcal{S})$ is compatible and the proposition and remark above apply. Then $R(\widetilde{P}_{A,\mathcal{S}})=\mathcal{M}$, so that $\widetilde{P}_{A,\mathcal{S}}=Q_{A,\mathcal{S}}$. □

The next theorem gives a simple characterization of compatibility:

Theorem 5.4. *$(A,\mathcal{S})$ is compatible if and only if $Q_{A,\mathcal{S}}(R(A))=A(\mathcal{S})$.*

Proof. If $(A,\mathcal{S})$ is compatible then, by theorem 5.3, $\widetilde{P}_{A,\mathcal{S}}=Q_{A,\mathcal{S}}$, so that

$$Q_{A,\mathcal{S}}(R(A))=Q_{A,\mathcal{S}}A(\mathcal{H})=AP_{A,\mathcal{S}}(\mathcal{H})=A(\mathcal{S}).$$

Conversely, if $Q_{A,\mathcal{S}}(R(A))=A(\mathcal{S})$, any $x\in R(A)$ decomposes as $x=x_1+(I-Q_{A,\mathcal{S}})x$, where $x_1=As$ for some $s\in\mathcal{S}$; then $(I-Q_{A,\mathcal{S}})x\in N(Q_{A,\mathcal{S}})\cap R(A)=\mathcal{S}^\perp\cap R(A)$ and $R(A)=A(\mathcal{S})+\mathcal{S}^\perp\cap R(A)$. Then, $\mathcal{H}=\mathcal{S}+A^{-1}(\mathcal{S}^\perp)$, which shows that $(A,\mathcal{S})$ is compatible. □

Denote $A^\sharp=(A|_{\overline{R(A)}})^{-1}:R(A)\to\overline{R(A)}$.

Lemma 5.5. *The projection $Q_{A,\mathcal{S}}$ satisfies $Q_{A,\mathcal{S}}(R(A))\subseteq R(A)$ if and only if $R(A)=\mathcal{M}\cap R(A)+\mathcal{M}^{\perp'}\cap R(A)$; in this case $Q_{A,\mathcal{S}}(R(A))=\mathcal{M}\cap R(A)$. Moreover, $A^\sharp Q_{A,\mathcal{S}}A:\mathcal{H}\to\mathcal{H}$ is a bounded projection if and only if $\mathcal{M}\cap R(A)$ is closed in $R(A)$ (under the topology of $\mathcal{H}$).*

Proof. Observe that, by the definition of $\mathcal{M}$, $Q_{A,\mathcal{S}}(R(A))\subseteq R(A)$ if and only if $Q_{A,\mathcal{S}}x\in\mathcal{M}\cap R(A)$, for all $x\in R(A)$. Then $(I-Q_{A,\mathcal{S}})x\in\mathcal{M}^{\perp'}\cap R(A)$ so that $R(A)=\mathcal{M}\cap R(A)+\mathcal{M}^{\perp'}\cap R(A)$. On the other hand, it always holds $\mathcal{M}\cap R(A)\subseteq Q_{A,\mathcal{S}}(R(A))$. Then $Q_{A,\mathcal{S}}(R(A))=\mathcal{M}\cap R(A)$. The converse is similar. If $Q_{A,\mathcal{S}}(R(A))\subseteq R(A)$ then $A^\sharp Q_{A,\mathcal{S}}A:\mathcal{H}\to\mathcal{H}$ is well defined and it is obviously a projection. Let us prove that it is bounded. For this, observe that $N(A^\sharp Q_{A,\mathcal{S}}A)=N(Q_{A,\mathcal{S}}A)=A^{-1}(\mathcal{S}^\perp)$ is closed and, also, $R(A^\sharp Q_{A,\mathcal{S}}A)=A^\sharp Q_{A,\mathcal{S}}(R(A))=A^\sharp(\mathcal{M}\cap R(A))$ is closed, because $\mathcal{M}\cap R(A)$ is closed in $R(A)$. This proves that $A^\sharp Q_{A,\mathcal{S}}A$ is bounded. □

Consider now the following subalgebra of $L(\mathcal{H})$:

$$L(\mathcal{H})^A = \{T \in L(\mathcal{H}) : T(N(A)) \subseteq N(A) \text{ and } R(T^*A^{1/2}) \subseteq R(A^{1/2})\}.$$

By lemma 4.6 the elements of $L(\mathcal{H})^A$ induce operators on $B(A^{1/2})$ by means of

$$\theta : L(\mathcal{H})^A \to L(B(A^{1/2})) \quad \text{given by} \quad T \mapsto \theta(T) = \widetilde{T}, \;\; T \in L(\mathcal{H})^A,$$

where $\widetilde{T}Ax = ATx$, for all $x \in \mathcal{H}$.

Theorem 5.6. *Given a closed subspace $\mathcal{W}$ of $B(A^{1/2})$ and $Q_{\mathcal{W}} \in L(B(A^{1/2}))$ the orthogonal projection onto $\mathcal{W}$, then $\theta^{-1}(\{Q_{\mathcal{W}}\})$ is non empty if and only if $(A, \mathcal{S})$ is compatible, where $\mathcal{S}$ is any closed subspace of $\mathcal{H}$ such that $A(\mathcal{S})$ is dense in $\mathcal{W}$.*

Proof. If there exists $\mathcal{S}$ such that $(A, \mathcal{S})$ is compatible and $A(\mathcal{S})$ is dense in $\mathcal{W}$ then, by theorem 5.3, there exists $\widetilde{P}_{A,\mathcal{S}} \in L(B(A^{1/2}))$ such that $\widetilde{P}_{A,\mathcal{S}} = Q_{\mathcal{W}}$. Therefore $\theta(P_{A,\mathcal{S}}) = Q_{\mathcal{W}}$.

Conversely, if $\theta^{-1}(Q_{\mathcal{W}})$ is non empty, then there exists $T \in L(\mathcal{H})^A$ such that $\widetilde{T} = Q_{\mathcal{W}}$ and $\widetilde{T}A = AT$; then $Q_{\mathcal{W}}(R(A)) \subseteq R(A)$. By lemma 5.5 this inclusion is equivalent to $R(A) = \mathcal{W} \cap R(A) + \mathcal{W}^{\perp'} \cap R(A)$ and in this case $Q_{\mathcal{W}}(R(A)) = \mathcal{W} \cap R(A)$. Then $\overline{\mathcal{W} \cap R(A)}' = \mathcal{W}$ because $R(A)$ is dense in $B(A^{1/2})$. Again by lemma 5.5, $\mathcal{W} \cap R(A)$ is closed in $R(A)$ because $A^\sharp Q_{\mathcal{W}} A = P_{\overline{R(A)}} T$ is a bounded projection in $\mathcal{H}$. Then $\mathcal{S} = A^{-1}(\mathcal{W} \cap R(A))$ is a closed subspace of $\mathcal{H}$ such that $A(\mathcal{S}) = \mathcal{W} \cap R(A)$ in $\mathcal{H}$ because $\mathcal{W} \cap R(A)$ is closed, so that $\overline{A(\mathcal{S})}' = \mathcal{W}$. Applying theorem 5.4 to $Q_{\mathcal{W}}$, since $Q_{\mathcal{W}}(R(A)) = A(\mathcal{S})$, with $\overline{A(\mathcal{S})}' = \mathcal{W}$ we obtain that $(A, \mathcal{S})$ is compatible. □

Proposition 5.7. Let $\mathcal{S}$ is a closed subspace of $\mathcal{H}$ and $\mathcal{W} = \overline{A(\mathcal{S})}'$. If $\theta^{-1}(\{Q_{\mathcal{W}}\})$ is non empty, then $P(A, \mathcal{S}) \subseteq \theta^{-1}(\{Q_{\mathcal{W}}\})$. Moreover $P(A, \mathcal{T}) \subseteq \theta^{-1}(\{Q_{\mathcal{W}}\})$ for all closed subspaces $\mathcal{T}$ of $\mathcal{H}$, such that $A(\mathcal{T}) = A(\mathcal{S})$.

Proof. If $R \in P(A, \mathcal{S})$ then $R = P_{A,\mathcal{S}} + T$, where $T \in L(\mathcal{S}^\perp, \mathcal{N})$ (see [7]). Then $\theta(R) = \theta(P_{A,\mathcal{S}}) = \widetilde{P}_{A,\mathcal{S}}$ because $\widetilde{T}A = AT = 0$.

If $A(\mathcal{T}) = A(\mathcal{S})$ then $P_{\overline{R(A)}}(\mathcal{T}) = P_{\overline{R(A)}}(\mathcal{S})$. Observe that, by proposition 5.6, $(A, \mathcal{S})$ is compatible because $\theta^{-1}(\{Q_{\mathcal{W}}\})$ is non empty; therefore $(A, \mathcal{T})$ is compatible by corollary 3.5. But, by proposition 5.3, $\widetilde{P}_{A,\mathcal{T}}$ is the orthogonal projection onto $\overline{A(\mathcal{T})}' = \mathcal{W}$ so that $\theta(P_{A,\mathcal{T}}) = \theta(P_{A,\mathcal{S}})$, and $\theta(P(A, \mathcal{T})) = \theta(P_{A,\mathcal{T}})$. □

Remark 5.8. If one decides to avoid the use of operator ranges with their natural Hilbertian structure, then by remark 3.6, $(A, \mathcal{S})$ is compatible if and only if $A^{1/2}(\mathcal{S})$ is a closed subspace of $R(A^{1/2})$ which admits an orthogonal complement in $R(A^{1/2})$; observe that, as a subspace of $\mathcal{H}$, $R(A^{1/2})$ is an incomplete inner product space, unless $R(A)$ is closed. Therefore, the compatibility problem is equivalent to find in an inner-product space $(\mathcal{D}, \langle\ ,\ \rangle)$, all closed subspaces of $\mathcal{D}$ that admit an orthogonal complement in $\mathcal{D}$. These subspaces are called Chebyshev subspaces in the theory of best approximation (see [13] for an nice treatment of Chebyshev sets in inner product spaces).

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