Applying dimensional analysis to wave dispersion

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We show that dimensional analysis supplemented by physical insight determines if a wave has dispersion, without recourse to sophisticated mathematical tools.
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I. INTRODUCTION

The purpose of this paper is to highlight the usefulness of dimensional analysis, which does not always receive the attention it deserves in introductory physics courses. Dimensional analysis can be a very useful pedagogical tool because it focuses attention on the physics of the phenomenon and not on the mathematics required by its description.[1,2] We will show that by using dimensional analysis it is possible to predict the existence of the dispersion of waves without having to derive differential equations and use other sophisticated mathematical tools. Next we show by means of two examples that with the help of simple physical arguments, it is possible to derive the main characteristics of the dispersion. We focus our attention on mechanical waves, but similar considerations can be made for other kinds of waves, such as waves in plasmas.

We consider a perturbation of the form $\phi = \phi_0 \Phi(kx - \omega t)$, where Φ is a periodic function (we disregard a constant phase because it is not relevant for our purposes). Here ϕ represents the quantity that propagates (for example, the density perturbation of an acoustic wave), ϕ_0 is the amplitude, k is the wavenumber, and ω is the angular frequency of the wave. Dispersion occurs when the phase velocity $c = \omega / k$ of the wave depends on k, so that

$$c = c(k) \neq \text{const.} \tag{1}$$

The dispersive properties of the wave are determined by the dispersion relation $\omega = \omega(k)$. Because $\omega(k) = kc(k)$, the function $c(k)$ determines the dispersion relation. In this paper we shall discuss the dispersion in terms of $c(k)$. For example, acoustic waves are nondispersive, because $c(k) = \sqrt{\kappa / \rho}$, where κ is the compressibility of the medium and ρ its density.

This example illustrates a general property of mechanical waves; namely, the phase velocity of an elementary (sinusoidal) wave is always given by

$$c = \sqrt{\frac{R}{I}}, \tag{2}$$

where R is a quantity related to the restoring force and I is related to inertia. This property is important in the present context because it places restrictions on how $c(k)$ depends on the parameters of the system.

II. DIMENSIONAL ANALYSIS OF THE DISPERSION RELATION

The properties of a wave depend on the parameters c and k that characterize the perturbation (we assume small amplitude waves so that ϕ_0 is not a relevant parameter) and on $r_1, r_2, \ldots, r_n$, which describe the relevant properties of the medium and the relevant forces. If we assume three fundamental dimensions (such as length, mass, and time as is usual in mechanics), then according to the Pi theorem it is possible to obtain $n-1$ independent dimensionless quantities from these $n+2$ parameters.[3] Any physically meaningful relation concerning the system of interest can be expressed in terms of these $n-1$ invariants, one of which must necessarily involve c. Let us call Π_c this invariant, which can be chosen without loss of generality as

$$\Pi_c = \frac{c}{k^{\alpha} r_1^{\beta} r_2^{\gamma} \ldots}. \tag{3}$$

The remaining $n-2$ invariants Π_i with $i = 1, \ldots, n-2$ can always be chosen to be independent of c:

$$\Pi_i = k^{\alpha_i} r_1^{\beta_i} r_2^{\gamma_i} \ldots . \tag{4}$$

The expression for $c(k)$ is derived by observing that $\Pi_c = \Psi(\Pi_1, \ldots, \Pi_{n-2})$, from which we obtain

$$c = k^{\alpha} r_1^{\beta} r_2^{\gamma} \ldots \Psi(\Pi_1, \ldots, \Pi_{n-2}). \tag{5}$$

Clearly, if $\alpha = \alpha_i = 0$ $(i = 1, \ldots, n-2)$, then c does not depend on k, which is the necessary and sufficient condition for the wave to be nondispersive. Note that $\alpha = 0$ implies that there is a combination of $r_1, \ldots, r_n$ whose dimensions are that of velocity, thus defining a characteristic velocity $c_0(r_1, \ldots, r_n)$ of the medium. In contrast, $\alpha_i = 0$ for $i = 1, \ldots, n-2$ implies that there does not exist a combination of $r_1, \ldots, r_n$ whose dimension is a length, which means that the medium has no characteristic length.[4]

We conclude that the wave is not dispersive if and only if two conditions are fulfilled: there is a characteristic velocity of the medium and there is no characteristic length. If at least one or both of these conditions are not satisfied, that is, if the medium has no characteristic velocity or if it has a characteristic length, the wave is dispersive. Note that the dispersion that occurs when the medium has no characteristic velocity is present for any k. On the other hand, the dispersion due to a characteristic length ℓ_0 affects a limited region of the spectrum, because it occurs only when $k\ell_0 \approx 1$ as the corresponding Π_i drops out of Eq. (5) for $k\ell_0 \to 0, \infty$. In the following we shall illustrate these results by two examples.

III. A CHAIN OF MASSES AND SPRINGS

We consider a linear infinite chain of identical objects of mass m joined by identical springs with elastic constant κ and equilibrium length ℓ. This simple model describes some of the features of the propagation of longitudinal waves in a crystal (see, for example, Ref. 5). Let x be the coordinate along the chain and δx_n be the displacement of the nth object with respect to its equilibrium position x_n. It is well known that this system can sustain longitudinal waves of the form

$$\delta x_n = A \cos(kn\ell - \omega t), \tag{6}$$

where A is a constant. We shall be concerned with the interval $-\pi/\ell \leqslant k \leqslant \pi/\ell$ because only the values of the displacement at $x=n\ell$ are physically meaningful. Then k and $k'=k+2s\pi/\ell$ (s integer) in Eq. (6) represent the same perturbation. The characteristic parameters of the chain are m, κ, and ℓ. From these parameters we can form a characteristic frequency and a characteristic velocity given by $\omega_0=\sqrt{\kappa/m}$ and $c_0=\ell\omega_0$. According to the results of Sec. II the waves are dispersive and

$$c = c(k) = c_0\Psi(k\ell), \tag{7}$$

or equivalently

$$\omega = \omega_0\Omega(k\ell). \tag{8}$$

We expect that the dispersion is important only for $k\ell \approx 1$, and that $c \approx c_0$ for $k\ell \to 0$. Because c cannot depend on the sign of k, we conjecture that for $k\ell \ll 1$,

$$c = c_0[1 + a(k\ell)^2 + \cdots], \tag{9}$$

where a is a constant. In the same domain ($k\ell \ll 1$) we have $\omega \approx k\ell\omega_0$. If k increases from 0 to $2\pi/\ell$, the frequency increases from 0 to a maximum $\omega_{\max}$ and then decreases again to 0. This behavior means that ω is bounded from above and no perturbations with $\omega > \omega_{\max}$ can propagate. Because the natural frequency scale is ω_0, we must have $\omega_{\max}=b\omega_0$ where b is a constant of the order of unity. As $\omega \to \omega_{\max}$, the group velocity $d\omega/dk$ goes to zero. We conclude that the dispersion that occurs for $k\ell \approx 1$is related to the existence of the cutoff frequency $\omega_{\max}$.

The exact result is[5]

$$c = c_0\frac{\sin(k\ell/2)}{k\ell/2} \quad (-\pi/\ell \leqslant k \leqslant \pi/\ell) \tag{10}$$

from which it can be verified that the conclusions obtained from our simple arguments are correct and that $a=-\frac{1}{24}$ and $b=2$.

IV. WAVES ON THE SURFACE OF A LIQUID

The waves that propagate on the surface of a water pond are a familiar phenomenon. We neglect viscosity and consider small amplitude surface waves propagating in one dimension. The parameters in this case are c, k, the acceleration of gravity g, the surface tension γ, the depth of the liquid h, and its density ρ. Because there are only three fundamental dimensions (length, time, and mass), we can form $6-3=3$ independent dimensionless quantities from the six parameters. These three quantities can be chosen as

$$\Pi_c = \frac{c}{c_\lambda}, \quad \Pi_1 = kh, \quad \Pi_2 = k\lambda, \tag{11}$$

where the characteristic velocity $c_\lambda \equiv (\gamma g/\rho)^{1/4} = \sqrt{g\lambda}$, and $\lambda \equiv \sqrt{\gamma/\rho g}$ is the capillary length. From these considerations we can write

$$c = c_\lambda\Psi(kh,k\lambda). \tag{12}$$

Equation (12) shows that these waves are dispersive because c depends on k. Note that there are two characteristic lengths (h and λ) as well as two characteristic velocities (c_λ and $c_h \equiv \sqrt{gh}$) so that Eq. (12) is not unique.

Various important limiting cases can be obtained from Eq. (12). The invariant $k\lambda = \sqrt{\rho g/\gamma k^2}$ is related to the ratio between the restoring forces due to gravity (proportional to ρg) and to surface tension (proportional to γk^2). The invariant kh is related to the mass of the fluid perturbed by the wave. If the depth of the fluid is infinite, the surface perturbations affect a layer of fluid whose thickness δ is of the order of $1/k$. Then $kh \approx h/\delta$ gives a measure of the effect of the depth of the fluid layer on the phenomenon.

Let us derive the consequences of these facts. We first consider the invariant $\Pi_2=k\lambda$. If $k\lambda \ll 1$, gravity provides the dominant restoring force so that c does not depend on γ. In this limit we obtain pure gravity waves for which $\Psi \to (k\lambda)^{-1/2}\Phi(kh)$, and thus

$$c_{\text{grav}} = \sqrt{\frac{g}{k}}\Phi(kh), \tag{13}$$

which determines the dispersion relation for pure gravity waves.

If $k\lambda \gg 1$, surface tension provides the relevant restoring force, and c does not depend on g. This case corresponds to pure capillary waves. In this limit $\Psi \to (k\lambda)^{1/2}\Phi(kh)$, and thus we obtain for pure capillary waves:

$$c_{\text{cap}} = \sqrt{\frac{\gamma k}{\rho}}\Phi(kh). \tag{14}$$

Note that we used the same function $\Phi(kh)$ in Eqs. (13) and (14). We did so because $\Phi(kh)$ describes the effect of the finite depth of the fluid layer, which does not depend on the nature of the forces involved.

Now we examine the invariant kh. If the liquid layer is shallow so that $kh \to 0$, c_{grav} cannot depend on k, which means that in this limit $\Phi \to \sqrt{kh}$. We then obtain from Eq. (13)

$$c_{\text{grav,shallow}} = \sqrt{gh}. \tag{15}$$

In the same limit we obtain from Eq. (14)

$$c_{\text{cap,shallow}} = k\sqrt{\frac{\gamma h}{\rho}}. \tag{16}$$

In the limit of a deep layer ($kh \to \infty$), the speed of the wave cannot depend on h and we must have $\Phi \to$const. If we use this result in Eqs. (13) and (14), we obtain

$$c_{\text{grav,deep}} = \sqrt{\frac{g}{k}} \tag{17}$$

and

$$c_{\text{cap,deep}} = \sqrt{\frac{\gamma k}{\rho}}. \tag{18}$$

In Eqs. (15)–(18) we have omitted numerical factors that cannot be found by means of dimensional arguments. Our results can be compared with the exact result[6,7]

$$c^2 = c_\lambda^2 \left(\frac{1}{k\lambda} + k\lambda \right) \tanh(kh), \tag{19}$$

from which it can be verified that all the numerical factors are 1.

If we use $\Pi_c' = c/c_h$ instead of Π_c, the previous analysis of the limiting cases cannot be easily carried to the end, because the dependency of c_{grav} and c_{cap} on h appears in c_h and in $\Phi'(kh)$, and not in the single factor $\Phi(kh)$ as occurs in Eqs. (13) and (14). It is legitimate to employ any combination of the parameters having the dimensions of velocity in place of c_λ or c_h to define a new Π_c''. For example, we can use $\sqrt{g/k}$; the same arguments as before can be carried through, and we obtain the same results as in Eqs. (13)–(18). Thus, the choice of the velocity scale is a matter of convenience and taste. We prefer the choice (11) because c_λ is a natural velocity scale for the surface waves and depends on both restoring forces involved.

V. COMMENTS

By using dimensional analysis, we can determine whether a wave is dispersive or not. The main result is that a wave system is not dispersive if and only if there is a characteristic velocity and no characteristic length. Once this important result has been derived, it can be used in the classroom without further recourse to the Pi theorem.

In some instances it is possible to go farther and derive the correct dispersion relations in various limiting cases, up to numerical factors. All these results can be obtained quickly and easily and with very simple mathematics. As C. F. Bohren has aptly stated, the aim of physics is physical understanding, not solving differential equations.[1] Dimensional analysis lets us achieve this end, and consequently it is a valuable pedagogical tool.

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APPENDIX: SUGGESTED PROBLEMS

We suggest two problems to develop students' skill with the application of dimensional analysis.

(1) Consider an infinite chain of masses linked by springs as in Sec. III, but now each mass hangs from a pendulum of length L. Apply dimensional analysis to discuss the dispersion properties of the low frequency, long wavelength waves of the chain.

(2) Apply dimensional analysis to investigate the dispersion properties of electromagnetic waves travelling in a perfectly conducting rectangular waveguide.

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[2]J. A. Pelesko, M. Cesky, and S. Huertas, "Lenz's law and dimensional analysis," Am. J. Phys. **73**, 37–39 (2005).

[3]The Pi theorem states that if p is the number of characteristic parameters (constant or variable) of the problem, and among them there are q that have independent dimensions, the number of dimensionless independent combinations that can be formed among them is equal to $p-q$. The original presentation of this theorem is due to E. Buckingham, "On physically similar systems; Illustrations of the use of dimensional equations," Phys. Rev. **4**, 345–376 (1914). It is also discussed in many books; see, for example, L. I. Sedov, *Similarity and Dimensional Methods in Mechanics* (Academic, New York, 1959).

[4]To be more precise, there can be a characteristic length as long as it does not play a role in the propagation of the waves, because in this case this length does not appear in the invariants $\Pi_1, \ldots, \Pi_{n-2}$.

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