



## Identification of uncertain MIMO Wiener and Hammerstein models

Silvina Inés Biagiola, José Luis Figueroa\*

Departamento de Ingeniería Eléctrica y de Computadoras, IIIE-UNS-CONICET, Av. Alem 1253, 8000 Bahía Blanca, Argentina

### ARTICLE INFO

#### Article history:

Received 23 December 2010  
Received in revised form 6 May 2011  
Accepted 25 May 2011  
Available online 16 June 2011

#### Keywords:

Wiener models  
Hammerstein models  
Robust identification

### ABSTRACT

Several approaches can be found in the literature to perform the identification of block oriented models (BOMs). In this sense, an important improvement is to achieve robust identification to cope with the presence of uncertainty.

In this work, two special and widely used BOMs are considered: Hammerstein and Wiener models. The models herein treated are assumed to be described by parametric representations. The approach introduced in this work for the identification of the multiple input–multiple output (MIMO) uncertain model is performed in a single step. The uncertainty is described as a set of parameters which is found through the solution of an optimization problem.

A distillation column simulation model is presented to illustrate the robust identification approach. This process is an interesting benchmark due to its well-known nonlinear dynamics. Both Hammerstein and Wiener models are used to represent this plant in the presence of uncertainty. A comparative study between these models is established.

© 2011 Elsevier Ltd. All rights reserved.

### 1. Introduction

Block oriented models (BOMs) have been proven to be useful as simple nonlinear models for many applications (Juditsky et al., 1995; Pearson & Pottmann, 2000; Sjöberg et al., 1995). They consist of a cascade combination of a linear dynamics block ( $\mathcal{L}$ ) with a memoryless nonlinear one ( $\mathcal{N}$ ). A vast type of dynamic processes can be described by such representations consisting of these two simple elements usually referred to as subsystems. These models have captured the attraction of many researchers because they combine simplicity of modeling with good approximation of the real process behavior. They are a useful approach when the nonlinearity of the plant cannot be ignored and, consequently, linear models are not appropriate. Moreover, these representations are restricted complexity models (Ling & Rivera, 1998, 2001) which result in an appealing proposal for many applications such as process control (Bloemen et al., 2001; Fruzzetti, Palazoglu, & McDonald, 1997; Pajunen, 1992). In general, the low computational effort and the suitability for design are the most attractive advantages of using these models for control.

In this work, two special and widely used BOMs are considered: Hammerstein and Wiener models. In the first case,  $\mathcal{N}$  is followed by  $\mathcal{L}$  and, in the second one,  $\mathcal{L}$  is followed by  $\mathcal{N}$ . Figs. 1 and 2 show these structures. Apart from chemical engineering (Fruzzetti et al., 1997; Kalafatis, Arifin, Wang, & Cluett, 1995; Norquay, Palazoglu,

& Romagnoli, 1998; Pajunen, 1987, 1992; Visala, Pitkänen, & Paanajärvi, 1999; Wigren, 1993; Zhu, 1999), they have been used in a wide range of applications, such as in the fields of communications (Hadjiloucas, Izhaç, Galvao, Bowen, & Becerra, 2004; Huang, Tanskanen, & Hartimo, 1998; Stapleton & Bass, 1985), medicine (Celka & Colditz, 2002) and biology (Hunter & Korenberg, 1986).

There is a vast literature on block oriented systems. The approach most widely used as regards model construction is the parametric description, such as in the works by Škrjank, Blažič, and Agamennoni (2005), Raich, Zhou, and Viberg (2005), Lacy and Bernstein (2002), Hagenblad and Ljung (2000) and Gómez and Baeyens (2004). Although in a smaller proportion, the nonparametric formulation has also been followed (Greblicki, 1992, 1994, 1997).

The models herein treated are assumed to be described by parametric representations. Several procedures can be found in the literature to perform the identification of Hammerstein and Wiener models for nominal conditions. However, an important improvement is to achieve robust identification to cope with the presence of uncertainty. This is the approach herein followed, which allows to describe the uncertainty as a set of parameters that is found through the solution of an optimization problem.

In this work the starting point is the assumption that an uncertain parametric block oriented model must be identified. Moreover, the source of such uncertainty is assumed unknown. It is considered that the structure of the system under study is Wiener or Hammerstein type, and that the static nonlinearity entirely determines the steady state characteristic of the whole model and that some kind of uncertainty is present in the system. The resulting methodology

\* Corresponding author. Tel.: +54 291 4595101; fax: +54 291 4595154.  
E-mail address: [figueroa@uns.edu.ar](mailto:figueroa@uns.edu.ar) (J.L. Figueroa).

### Nomenclature

$q$	stands for the forward shift operator
$H_2^{m \times n}(T)$	represents the space of all the $(m \times n)$ transfer matrices that are both stable and causal
$\mathfrak{R}$	is the field of real numbers
$\mathfrak{R}^{m \times n}$	is the field of $m \times n$ matrices with entries in $\mathfrak{R}$
$g_i(\cdot): \mathfrak{R}^m \rightarrow \mathfrak{R}^m$	is a set of specified basis functions for a non-linear static mapping. For example, there could be polynomial, trigonometric or piecewise linear functions
$B_i(q): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$	is a set of any rational basis in $H_2^{n \times n}(T)$ (i.e., Laguerre, Kautz or any orthonormal basis)
$\leq, \geq$	for scalar the usual meaning and for matrices applied entry by entry
$[ \cdot ]_i$	is the $i$ th entry of a vector or a $i$ th dimension of a set
$[ \cdot ]_{ij}$	is the element of a matrix on the $i$ th row and the $j$ th column

is robust in the sense that the identified set of parameters is such that any of the collected data can be reproduced by at least one of the models in the set. These set of parameters are obtained by solving a simple optimization problem. This approach ensures the existence of a convex set of models that can describe the whole data collected from the process.

A distillation column simulation model is presented to illustrate the robust identification approach, this process is an interesting benchmark due to its well-known nonlinear dynamics. Both Hammerstein and Wiener models are used to represent this plant in the presence of uncertainty. A comparative study between these models is established.

BOMs have been widely used for modeling distillation columns (DCs). One of the main reasons is real-life DCs are inherently nonlinear and models based on first principles are usually too complex for application purposes. A continuous-time Hammerstein model was introduced by Eskinat, Johnson, and Luyben (1991) for a binary DC. Pearson and Pottmann (2000) applied identification of BOMs to a simulated DC, they based on the assumption of known nonlinearity (the static element) to obtain both Wiener and Hammerstein models for this column. A comparative study of the performance of these models was then accomplished. Bloemen et al. (2001) used a Wiener model in the control of a DC. Norquay et al. (1998) designed a model predictive controller based on a MIMO model of a C2-splitter while Gómez and Baeyens (2004) introduced an identification approach for MIMO Wiener and Hammerstein models and used a simulated DC example to illustrate their method. Janczak (2005) presents the most diffused identification methods of nonlinear systems using the block-oriented approach. He also surveys many relevant works related application of BOMs to modeling and control of DC. In a survey paper, Abdullah, Aziz, and Ahmad (2007) present many of the reported models that have been implemented in continuous DCs. They classified these models into three categories: *fundamental models*, *empirical models* and *hybrid models*, and they included Wiener and Hammerstein models in the second one.

In this paper, we consider the model developed by Skögestad for a distillation column. It is referred to as column A and has been studied in several papers both for modeling and control purposes (see Skögestad and Morari (1988) and references therein). This is an appealing model of 82 states which describes a MIMO highly nonlinear process.

The paper is organized as follows. In Section 2, robust block oriented model identification is treated. Both Wiener and Hammerstein structures are considered separately. This is the main contribution of this paper: a mechanism for uncertain model

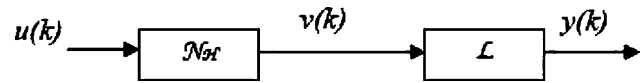


Fig. 1. Hammerstein model.

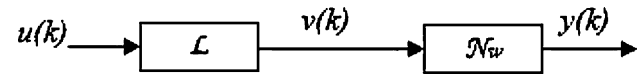


Fig. 2. Wiener model.

characterization and identification. In Section 3, the results are evaluated on the basis of the above mentioned distillation column. Final remarks are addressed in Section 4.

## 2. Robust identification

To identify both types of block oriented models, the parametrization adopted in Gómez and Baeyens (2004) and Falugi, Giarré, and Zappa (2005) is herein followed.

In the identification of block oriented models there is a scale factor which can be arbitrarily distributed between the linear block and the memoryless one without affecting the input–output characteristics of the model Pottmann and Pearson (1998). In the following we either assume  $p_1 = I$  (or  $h_1 = I$ ) with  $I$  the identity matrix, since that any other value of this gain can be included in the nonlinear (linear) block.

### 2.1. Robust identification of Wiener models

Fig. 2 depicts the general structure of Wiener models. It consists of a LTI system  $\mathcal{L}(q) \in H_2^{m \times n}(T)$  followed by a static nonlinearity  $\mathcal{N}_w$ . That is, in the Wiener case, the linear model  $\mathcal{L}$  maps the input  $u(k) \in \mathfrak{R}^n$  into the intermediate signal  $v(k) \in \mathfrak{R}^m$ , and the overall model output is the output of the nonlinear block, i.e.,  $y(k) = \mathcal{N}_w(v(k)) \in \mathfrak{R}^m$ . The present identification algorithm requires that the static nonlinearity is invertible,<sup>1</sup> i.e., the signal  $v(k)$  can be written as

$$v(k) = \mathcal{N}_w^{-1}(y(k)) = \sum_{i=1}^N p_i g_i(y(k)) \quad (1)$$

where  $g_i(\cdot): \mathfrak{R}^m \rightarrow \mathfrak{R}^m$  and  $p_i \in \mathfrak{R}^{m \times m}$  ( $i = 1, \dots, N$ ) are the unknown parameters of the nonlinear block which must be determined. It is assumed that  $K$  sampled data are available, i.e.,  $k = 1, \dots, K$ .

From Fig. 2, this signal can also be written as the output of the linear block,

$$v(k) = \mathcal{L}(q)u(k) = \sum_{i=1}^M h_i B_i(q)u(k) \quad (2)$$

where  $h_i \in \mathfrak{R}^{m \times n}$  ( $i = 1, \dots, M$ ), are the linear block parameters to be determined and  $B_i(q)u(k)$  is a rational basis with the following structure:

$$B_i(q)u(k) = \begin{bmatrix} B_{i1}(q, a_1)u(k) & 0 & \dots & 0 \\ 0 & B_{i2}(q, a_2)u(k) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & B_{in}(q, a_n)u(k) \end{bmatrix} \quad (3)$$

<sup>1</sup> This restriction is related to the fact that the signal connecting both subsystems is not measured. For the robust identification algorithm is required that it holds for all possible model in the set of uncertain parameters. However, some of the reported techniques for nominal identification in the literature allow noninvertible nonlinearities (Haverkamp, Chou, & Verhaegen, 1998; Lacy & Bernstein, 2002).

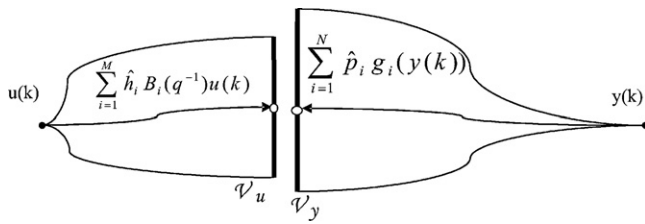


Fig. 3. Uncertainty sets.

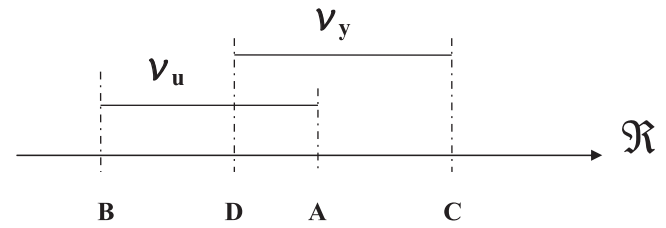


Fig. 4. Intervals.

where  $B_{ij}$  is any element of the function bases. For example,

$$B_{ij}(q, a_j) = \frac{1 - a_j^2}{q - a_j} \left( \frac{1 - a_j q}{q - a_j} \right)^{(i-1)}, \quad \text{if Laguerre bases are selected}$$

The parameter matrices to identify are defined as  $h = [h_1, \dots, h_M] \in \mathbb{R}^{m \times nM}$  and  $p = [p_1, \dots, p_N] \in \mathbb{R}^{m \times mN}$ . To build an uncertain model we define a set of parameters  $\mathcal{H}$  for the linear dynamic block and a set  $\mathcal{P}$  for the parameters of the nonlinear block.

$$\mathcal{H} = \{h : h^l \leq h \leq h^u\} \quad (4)$$

$$\mathcal{P} = \{p : p^l \leq p \leq p^u\} \quad (5)$$

Provided the mentioned description of the system (i.e., Eqs. (1) and (2)), and based on a set of input/output data of the process, it is possible to obtain a nominal model by minimizing a quadratic criterion (Gómez and Baeyens, 2004; Kalafatis et al., 1995).

However, the proposal of this work is to perform a robust identification. Now, to determine the parameter bounds in the Wiener Model, let us first define some sets. At some specific time  $k$ , the linear uncertain system defined by  $\mathcal{H}$ , maps a given input datum  $u(k)$  over an orthotope (Weissstein, 2010).

$$\mathcal{V}_u = \left\{ v : v \in \mathbb{R}^m, v(k) = \sum_{i=1}^M h_i B_i(q) u(k), h \in \mathcal{H} \right\}$$

$$= \{v : v \in \mathbb{R}^m, v_l \leq v(k) \leq v_u\}$$

On the other hand, if we consider the uncertain description for the parameters in  $\mathcal{P}$ , a given output  $y(k)$  maps at some specific time  $k$  over an orthotope

$$\mathcal{V}_y = \left\{ v : v \in \mathbb{R}^m, v(k) = \sum_{i=1}^N p_i g_i(y(k)), p \in \mathcal{P} \right\} \quad (6)$$

This situation is illustrated in Fig. 3 for a unidimensional problem (in the multivariable case we should make an extension and consider this situation for each entry). Note that in order to obtain an uncertain model, every input data  $u(k)$  should be mapped through the model to the corresponding  $y(k)$ . From the mentioned picture it is clear that the parameter set will define the uncertainty descriptions if  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$ . In this way, the input  $u(k)$  is mapped onto  $\mathcal{V}_u$  through the family of models generated by  $\mathcal{H}$ . Then, since  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$ , the intermediate value (i.e.,  $v(k)$ ) will result in  $y(k)$  through the inverse of  $\mathcal{N}_w^{-1}$ .

Now, the condition  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$  involves that this intersection should be not empty for each dimension, i.e.,  $[\mathcal{V}_y]_i \cap [\mathcal{V}_u]_i \neq \emptyset$  for every sample  $k$  ( $k=1, \dots, K$ ), and  $i=1, \dots, m$ . Let us analyze this situation in order to compute the parameter bounds that satisfy this condition. This determination is based on the whole input/output data available.

Note that for each input  $u(k)$  the linear bases  $B_i(q)u(k) \in \mathbb{R}^n$ . Let us group these vectors in a single vector  $B(q)u(k) = [(B_1(q)u(k))^T, \dots, (B_M(q)u(k))^T]^T$ , and let us define the vectors with the positive entries of  $B(q)u(k)$  as  $B^+(u(k)) \triangleq \max(B(q)u(k), 0)$  and the vectors with negative entries

as  $B^-(u(k)) \triangleq \min(B(q)u(k), 0)$ . Finally, we can define the new vector  $\gamma_B(k) \triangleq [(B^-(u(k)))^T, (B^+(u(k)))^T]^T \in \mathbb{R}^{2Mn}$ . In this construction the vector  $B^-(u(k))$  keeps all the negative elements in the vector and puts zero otherwise, while the vector  $B^+(u(k))$  keeps all the positive elements and puts zero otherwise.

In a similar way, since the nonlinear bases  $g_i(y(k)) \in \mathbb{R}^m$  for each output  $y(k)$ , it is possible to repeat the construction of last paragraph to obtain  $\gamma_g(k) \triangleq [(g^-(y(k)))^T, (g^+(y(k)))^T]^T \in \mathbb{R}^{2Nm}$  where  $g^+(y(k)) \triangleq \max(g(y(k)), 0)$  and  $g^-(y(k)) \triangleq \min(g(y(k)), 0)$ .

Then, it is possible to transform the uncertain identification problem into a Linear Programming problem with convex feasible region. This is presented in the following theorem.

**Theorem 1.** The bounds of the uncertain parameters  $h^l, h^u, p^l, p^u$  can be obtained by solving the following optimization problem

$$\min_{h^l, h^u, p^l, p^u} \left( \alpha \sum_{i=1}^M \sum_{k=1}^n \sum_{j=1}^m ([h_i^u]_{j,k} - [h_i^l]_{j,k}) + (1 - \alpha) \sum_{i=1}^N \sum_{k=1}^m \sum_{j=1}^m ([p_i^u]_{j,k} - [p_i^l]_{j,k}) \right) \quad (7)$$

subject to

$$[(h^l), (h^u), -(p^u), -(p^l)] \begin{bmatrix} \gamma_B(k) \\ \gamma_g(k) \end{bmatrix} \geq 0; \quad k = 1, \dots, K \quad (8)$$

$$[(h^u), (h^l), -(p^l), -(p^u)] \begin{bmatrix} \gamma_B(k) \\ \gamma_g(k) \end{bmatrix} \leq 0; \quad k = 1, \dots, K \quad (9)$$

where the parameter  $\alpha \in (0, 1)$  is a selected factor which allows to distribute the weight of the uncertainty between the linear and the nonlinear blocks. Note that the robust identification requirement  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$  must be ensured  $\forall k$ .

**Proof.** In order to demonstrate this theorem it is necessary to see that if the constraints (8) and (9) hold, then the condition  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$  is satisfied. Since the condition  $\mathcal{V}_y \cap \mathcal{V}_u \neq \emptyset$  holds if and only if it holds for each dimension of  $\mathcal{V}_y$  and  $\mathcal{V}_u \in \mathbb{R}$ , i.e., if  $[\mathcal{V}_y]_i \cap [\mathcal{V}_u]_i \neq \emptyset$  for  $i=1, \dots, m$ , it is possible to analyze each dimension separately to obtain a graphical interpretation. In this way, let us consider the intersection problem shown in Fig. 4 for the unidimensional case. Let

$$[\mathcal{V}_u]_i = \{v : v \in \mathbb{R}, B \leq v \leq A\} \quad (10)$$

$$[\mathcal{V}_y]_i = \{v : v \in \mathbb{R}, D \leq v \leq C\} \quad (11)$$

for  $i=1, \dots, m$ . From Figs. 3 and 4, it is straightforward that the upper bound of  $[\mathcal{V}_u]_i$  as function of the parameter set  $\mathcal{H}$  is given as  $A = [(h^l), (h^u)]_i \gamma_B$ . Consequently, we have that  $B = [(h^u), (h^l)]_i \gamma_B$ ,  $C = [(p^l), (p^u)]_i \gamma_g$  and  $D = [(p^u), (p^l)]_i \gamma_g$ .

Based on these definitions, the condition  $[\mathcal{V}_y]_i \cap [\mathcal{V}_u]_i \neq \emptyset$  will be satisfied if and only if the upper bound of  $[\mathcal{V}_u]_i$  is greater than the lower bound of  $[\mathcal{V}_y]_i$  (i.e.,  $A \geq D$ ), and if the lower bound of  $[\mathcal{V}_u]_i$  is lower than the upper bound of  $[\mathcal{V}_y]_i$  (i.e.,  $B \leq C$ ). The constraint  $A - D \geq 0$  for  $i=1, \dots, m$  is the one written in Eq. (8). Similarly, the mathematical formulation for  $C - B \geq 0$  is given in Eq. (9).

In this way, the optimization problem in (7) will produce an uncertain model that minimizes the magnitude of the uncertainty while ensures the description of the behaviour of all the collected data. □

The number of optimization variables is  $2(m^2N + nmM)$  and the number of constrains is  $2mK$ . Due to the suitable formulation of these problem, its solution is obtained in an efficient way.

2.2. Robust identification of Hammerstein models

Fig. 1 depicts the structure of a Hammerstein model. It consists of a static nonlinearity  $\mathcal{N}_h$  followed by a LTI system  $\mathcal{L}(q) \in H_2^{m \times n}(T)$ . That is, in the Hammerstein case, the nonlinear gain maps the input  $u(k) \in \mathbb{R}^n$  into the intermediate signal  $v(k) \in \mathbb{R}^n$ , and it is mapped through the linear model to produce the output  $y(k) = \mathcal{L}(q)v(k) = \mathcal{L}(q)\mathcal{N}_h(u(k))$  where  $y(k) \in \mathbb{R}^m$ .

Using a similar parametrization than the one used for the Wiener case, an estimate of the output  $\hat{y}(k)$  is:

$$\hat{y}(k) = \left( \sum_{i=1}^M h_i B_i(q) \right) \left( \sum_{j=1}^N p_j g_j(u(k)) \right) \tag{12}$$

$$= \sum_{i=1}^M \sum_{j=1}^N h_i p_j B_i(q) g_j(u(k)) \tag{13}$$

where  $g_j(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $p_j \in \mathbb{R}^{n \times n}$ , for  $j = 1, \dots, N$ .  $B_i(q)$  is a rational basis and  $h_i \in \mathbb{R}^{m \times n}$  with  $i = 1, \dots, M$ .

Now, following Gómez and Baeyens (2004) it is possible to write a linear regression in the parameters as follows:

$$\hat{y}(k) = \theta^T \phi_k \tag{14}$$

where

$$\theta = [h_1 p_1, \dots, h_1 p_N, \dots, h_M p_1, \dots, h_M p_N]^T \in \mathbb{R}^{m \times nNM} \tag{15}$$

$$\phi_k = [(B_1(q)g_1(u(k)))^T, \dots, (B_1(q)g_N(u(k)))^T, \dots, (B_M(q)g_1(u(k)))^T, \dots, (B_M(q)g_N(u(k)))^T]^T \in \mathbb{R}^{nNM} \tag{16}$$

Note that in this case  $h_i p_j \in \mathbb{R}^{m \times n}$  and  $B_i(q)g_j(u(k)) \in \mathbb{R}^m$  for  $i = 1, \dots, M$  and  $j = 1, \dots, N$ . A nominal estimate of  $\theta$  (and then of  $h_i$ ,  $i = 1, \dots, M$  and  $p_i$ ,  $i = 1, \dots, N$ ) can be computed as in Gómez and Baeyens (2004).

In order to characterize the uncertainty present in a Hammerstein model, a two step procedure is proposed. First, by considering the linear regression in Eq. (14), a set of uncertain parameters are computed in terms of  $\theta$ , as follows:

$$\Theta = \{ \theta : \theta^l \leq \theta \leq \theta^u \} \tag{17}$$

Then, in a second step, based on the definition of the parameter vector in Eq. (15), the set  $\Theta$  is translated to the parametric description given by Eqs. (4) and (5).

To solve the first step, the following problem is formulated:

**Problem 1.** Given a set of data  $u(k)$  and  $y(k)$  for  $k = 1, 2, \dots, K$ , we should compute a set of parameters  $\Theta$  such that  $y(k) \in \mathcal{Y}$  where:

$$\mathcal{Y} = \{ y \in \mathbb{R}^m : y = \theta^T \phi_k, \theta \in \Theta \} \tag{18}$$

Now, considering that the regressor  $\phi_k \in \mathbb{R}^{nNM}$  for each input  $u(k)$ , it is possible to split it into  $\phi_k^+ \triangleq \max(\phi_k, 0)$  and  $\phi_k^- \triangleq \min(\phi_k, 0)$  and form the vector  $\Phi_k = [(\phi_k^-)^T, (\phi_k^+)^T]^T$ .

Then, the following theorem could be used to compute the solution of Problem 2.

**Theorem 2.** The bounds  $\theta^l \in \mathbb{R}^{m \times nNM}$ ,  $\theta^u \in \mathbb{R}^{m \times nNM}$  for the uncertain parameters  $\Theta$  can be computed by solving the following optimization problem

$$\min_{\theta^l, \theta^u} \sum_{i=1}^{nNM} \sum_{j=1}^m ([\theta^u]_{i,j} - [\theta^l]_{i,j}) \tag{19}$$

subject to

$$[(\theta^l)^T, (\theta^u)^T] \Phi_k \geq y(k); \quad k = 1, \dots, K \tag{20}$$

$$[(\theta^u)^T, (\theta^l)^T] \Phi_k \leq y(k); \quad k = 1, \dots, K \tag{21}$$

**Proof.** It is similar to the proof of Theorem 1. In this case it is easy to see that satisfaction of the constraints in Eq. (20) implies the satisfaction of an upper bound on the data  $y(k)$  and the satisfaction of Eq. (21) implies the satisfaction of a lower bound on the data. □

Now, the second problem for the identification of the uncertain Hammerstein model is to obtain the set of parameters  $\mathcal{H}$  and  $\mathcal{P}$  as defined in Eqs. (4) and (5) from the bounds on the  $\theta$  parameters (Eq. (17)). This can be posed as Problem 2.

**Problem 2.** Given the set  $\Theta$  of parameters  $\theta \in \mathbb{R}^{m \times nNM}$  defined as Eq. (17), determine the set of parameters  $\mathcal{H}$  and  $\mathcal{P}$  such that

$$\mathcal{P} \otimes \mathcal{H} \supset \Theta \tag{22}$$

where the set  $\mathcal{P} \otimes \mathcal{H}$  is defined as

$$\mathcal{P} \otimes \mathcal{H} = \{ \theta : \theta \in \mathbb{R}^{m \times nNM}, \theta = [h_1 p, h_2 p, \dots, h_M p]^T; p \in \mathcal{P}; h \in \mathcal{H} \} \tag{23}$$

The bilinear relationship between the unknown parameters is typical of Hammerstein models (Falugi et al., 2005; Gómez & Baeyens, 2004). Note that we are looking for a set of parameters  $\mathcal{H}$  and  $\mathcal{P}$  such that they justify the set  $\Theta$ . The physical system (which has known structure but unknown parameters) is excited by the input sequence  $u(k)$ , and gives out  $y(k)$ . In a first step identification, the set  $\Theta$  is obtained. As robust identification is performed, the set of outputs generated using the input process data  $u(k)$  and the identified  $\Theta$ , will include the real collected outputs. In a second step, and based on the set  $\Theta$ , the set  $\mathcal{P} \otimes \mathcal{H}$  is determined. The proposed robust identification method is conservative and, as a result, the output set obtained with  $\mathcal{P} \otimes \mathcal{H}$  will contain the previous one (i.e., the one generated by means of  $\Theta$ ).

Now, considering the expression for  $\theta$  from Eq. (15) and the bounds  $\theta^l$  and  $\theta^u$ , it is possible to compute the bounds on the parameters of the Hammerstein models  $h^l$ ,  $h^u$ ,  $p^u$  and  $p^l$ .

To proceed, let us consider the following partition of vector  $\theta$ :

$$\theta \triangleq [\theta_1; \dots; \theta_N; \dots; \theta_{NM}] \tag{24}$$

where

$$\theta_{N(i-1)+j} \triangleq (h_i p_j)^T \in \mathbb{R}^{m \times n} \tag{25}$$

for  $j = 1, \dots, N$ ; and  $i = 1, \dots, M$ .

Then, a solution to Problem 2 is provided by means of the following theorem.

**Theorem 3.** The bounds  $h^l$ ,  $h^u$ ,  $p^l$ ,  $p^u$  for the uncertain parameters  $h$  and  $p$  can be computed by solving the following optimization problem:

$$\begin{aligned} \min_{h^l, h^u, p^l, p^u, h^1, h^2, p^1, p^2} \alpha \sum_{i=1}^M \sum_{j=1}^n \sum_{k=1}^m ([h^u]_{k,j} - [h^l]_{k,j}) \\ + (1 - \alpha) \sum_{i=1}^N \sum_{j=1}^n \sum_{k=1}^m ([p^u]_{j,k} - [p^l]_{j,k}) \end{aligned} \tag{26}$$

subject to

$$(h_i^1(z, 1:n)p_j^1(1:n, w))^T \geq \theta_{N(i-1)+j}^u(w, z) \quad (27)$$

$$(h_i^2(z, 1:n)p_j^2(1:n, w))^T \leq \theta_{N(i-1)+j}^l(w, z) \quad (28)$$

$$p_j^l \leq p_j^1 \leq p_j^u \quad (29)$$

$$h_i^l \leq h_i^1 \leq h_i^u \quad (30)$$

$$p_j^l \leq p_j^2 \leq p_j^u \quad (31)$$

$$h_i^l \leq h_i^2 \leq h_i^u \quad (32)$$

with  $j = 1, \dots, N$ ;  $i = 1, \dots, M$ ;  $w = 1, \dots, n$ ;  $z = 1, \dots, m$ , where the parameter  $\alpha \in (0, 1)$  is a selected factor which allows to distribute the weight of the uncertainty between the linear and the nonlinear blocks. The formulation in Eqs. (26)–(32) is the mathematical statement which brings the solution to Problem 2.

**Proof.** Through the statement and solution to Problem 1 robust identification of parameters  $\theta$  was accomplished. This gives out the lower and upper bounds  $\theta_i^l$  and  $\theta_i^u$ . By definition in Eq. (25) we have  $\theta_{N(i-1)+j} = (h_i p_j)^T$  for  $j = 1, \dots, N$  and  $i = 1, \dots, M$ . Based on these partial results, we want now to find the best bounds for the original parameters of the Hammerstein model (i.e.,  $h$  and  $p$ ). The conservative criterion stated in Eq. (22), gives rise to the following constraints:

$$\theta_{N(i-1)+j}^u \leq ((h_i p_j)^T)^u, \quad i = 1, \dots, M; j = 1, \dots, N \quad (33)$$

$$\theta_{N(i-1)+j}^l \geq ((h_i p_j)^T)^l, \quad i = 1, \dots, M; j = 1, \dots, N \quad (34)$$

Due to the bilinearity in  $hp$ , and the possible presence of negative elements, the right terms in (33) and (34) can be found in the set generated by the combinatory of the individual bounds on  $p_j$  and  $h_i$ . This fact was considered by including two auxiliary set of variables in the optimization problem:  $h^1, h^2, p^1$  and  $p^2$ .

Therefore, the constraints of Eqs.(27)–(32) ensures the satisfaction of Eqs.(33) and (34).  $\square$

Consequently, the general formulation of the robust identification problem results in a nonlinear optimization one. In this way, finding the solution to Problem 2 could be burdensome as well as time-consuming.

Note that the step 1 of the proposed technique could be applied for robust identification of memory polynomial models (Morgan, Ma, Kim, Zierdt, & Pastalan, 2006).

### 2.3. Remarks

The previous examples illustrate how the identification approaches herein introduced are accomplished. Note that the proposed identification method for the uncertain Hammerstein model involves a two-step procedure. In a first stage, nominal parameters are determined by solving a Linear Programming (LP) problem which involves a number of restrictions proportional to the number of collected process data. Based on these results, bounds for the original model's parameters are calculated in a second stage. The main advantage of this approach is that the first step allows a substantial reduction of the optimization problem which is accomplished at the time of the LP step. Then, the nonlinear optimization problem, which is solved in a second step, is a reduced one. Otherwise, the NL optimization would include as many constraints as process data. This would be the drawback of formulating the uncertain model identification problem as a one-step nonlinear optimization one.

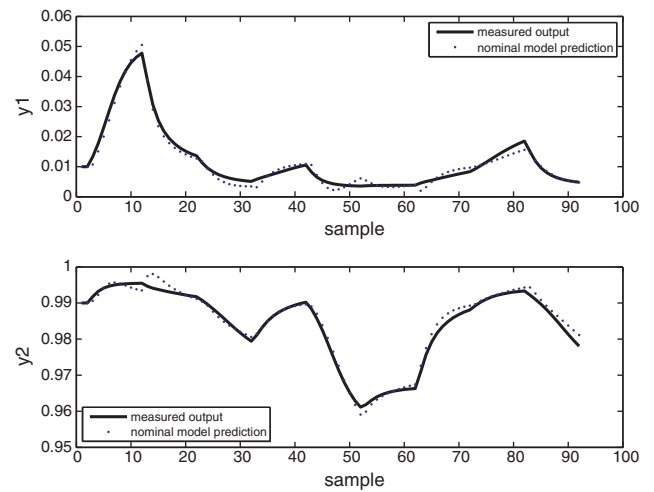


Fig. 5. Measured outputs and Wiener model predictions.

## 3. Case of study: distillation column

This section presents an example to illustrate the application of the proposed methodology. For this purpose, a distillation column has been selected. Distillation is the most common unit operation in the chemical industry. Its relevance and its complex nature have been the main reasons for being a favorite subject in process systems engineering field. As regards both modeling and control, distillation columns have been the focus of many research work.

Column A studied by Skogestad and Morari (1988) and Skogestad and Postlethwaite (2007) was selected to illustrate the proposed identification methodology for both Wiener and Hammerstein models. The motivation to study the identification of a distillation column using block oriented models is that there are used in the literature to model this process equipment.

In this case the LV control structure is used. The input  $u(k) = [L V]^T$  is a vector formed by the reflux and the boilup flows, respectively. On the other hand, the output  $y(k) = [x_D x_B]^T$  is a vector formed by the liquid composition in the distillate product and the liquid bottom composition, respectively. Therefore, a two input–two output process is considered for the identification.

### 3.1. Wiener model identification

To accomplish the identification a Wiener model formed by a Laguerre system in cascade with a polynomial-type nonlinearity was considered. Two Laguerre terms with dominant poles in  $-0.50$  and  $-0.85$  were assumed for the linear block. A second order polynomial was proposed for the nonlinear block. Therefore,  $M = 2$  and  $N = 2$ .

Provided the Wiener model description given in Eqs. (1) and (2), and based on a set of input/output data of the process, it is possible to obtain a nominal model by minimizing a quadratic criterion (Gómez & Baeyens, 2004). Fig. 5 shows the suitability of the proposed Wiener structure for modeling the distillation process herein considered.

Otherwise, Fig. 6 illustrates the approximation achieved when only the Laguerre structure is considered as the process model. Therefore, the improvement due to the inclusion of a nonlinear block justifies the use of a Wiener model.

However, the proposal of this work is to perform a robust identification. The solution of the optimization problem in Eqs. (7)–(9) brought the results shown in Table 1. The structural identifiability constraint  $p_{1,1} = 1$  and the design parameter  $\alpha = 0.6$  were specified.

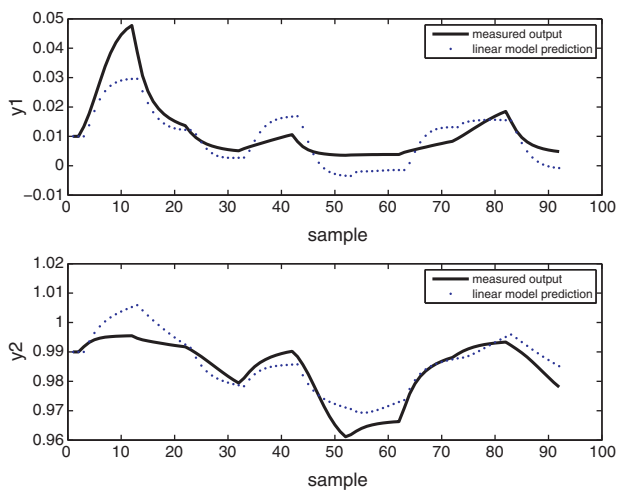


Fig. 6. Measured outputs and Laguerre model predictions.

Table 1  
Wiener model parameters. (i, j), k means output i, input j and order k.

Parameter	Lower bound	Upper bound
$h_{(1,1,1)}$	-0.1651	0.0770
$h_{(1,1,2)}$	0.0042	0.0827
$h_{(1,2,1)}$	-0.1594	-0.1594
$h_{(1,2,2)}$	0.0156	0.1339
$p_{(1,1,1)}$	1.0000	1.0000
$p_{(1,1,2)}$	-1.6255	-1.2263
$p_{(1,2,1)}$	-21.5726	-21.5726
$p_{(1,2,2)}$	-52.8544	-52.8544
$h_{(2,1,1)}$	-0.0392	0.1692
$h_{(2,1,2)}$	-0.1050	0.0385
$h_{(2,2,1)}$	0.0295	0.0635
$h_{(2,2,2)}$	-0.0952	-0.0641
$p_{(2,1,1)}$	-0.8929	-0.6080
$p_{(2,1,2)}$	21.2114	21.2133
$p_{(2,2,1)}$	1.0000	1.0000
$p_{(2,2,2)}$	30.6502	30.6521

Figs. 7 and 8 depict the bounds on the sets  $\mathcal{V}_u$  and  $\mathcal{V}_y$ . As described in Section 2.1, the solution approach requires that  $\mathcal{V}_u \in \mathcal{V}_y \neq \emptyset$ , and Figs. 7 and 8 show this restriction holds in this example for both intermediate signals ( $v_1$  and  $v_2$ ).

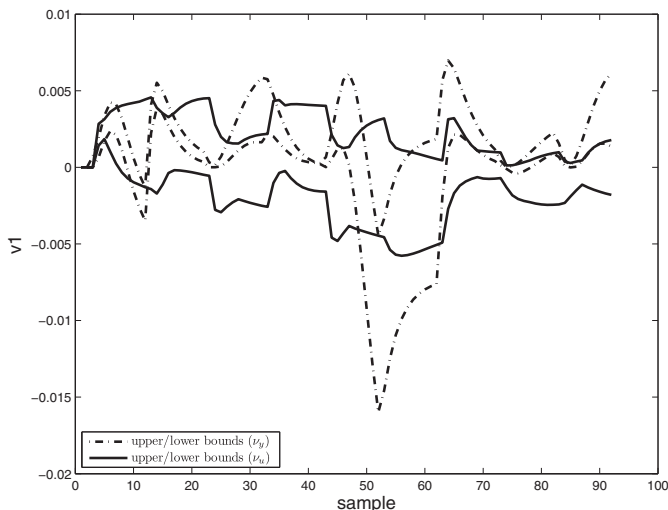


Fig. 7. Bounds on  $v_1$ .

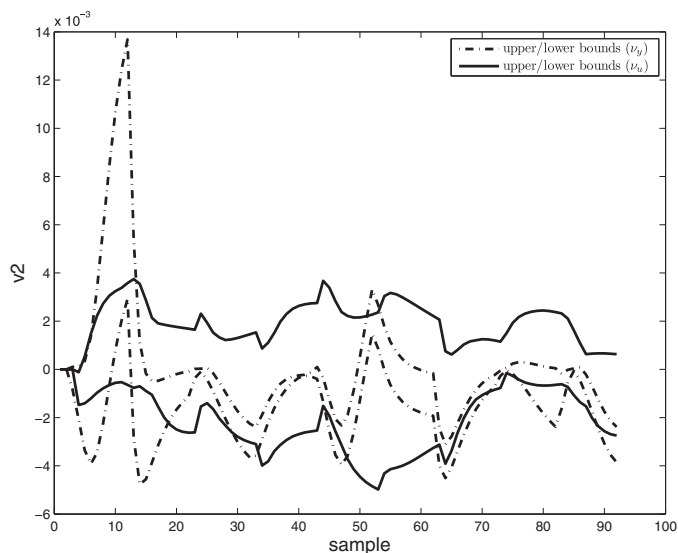


Fig. 8. Bounds on  $v_2$ .

### 3.2. Hammerstein model identification

In this case, the same distillation process is represented as a Hammerstein model. A polynomial-type nonlinearity in cascade with a Laguerre system was assumed for the Hammerstein structure. Two Laguerre terms with dominant poles in  $-0.50$  and  $-0.85$  were considered for the linear block. A second order polynomial was proposed for the nonlinear block. Therefore,  $M = 2$  and  $N = 2$ .

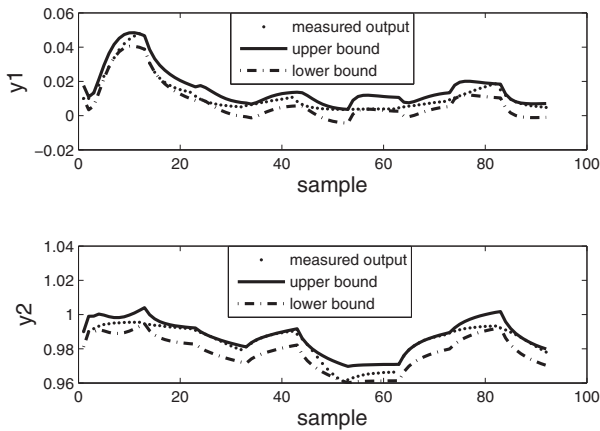
In a first step, Problem 1 was solved. For this purpose, the problem stated in Eqs. (19)–(21) was worked out. The optimization results for  $\Theta$  parameters are shown in Table 2. In a second step,

Table 2  
Hammerstein model parameters  $\Theta$  (Problem 1).

Lower bound	Upper bound
-2.3672	-2.3672
-1.7644	-1.7644
-7.6712	-7.6712
79.5578	79.5578
0.4629	0.4629
0.3246	0.3246
1.1981	1.1981
-12.3544	-12.3544
-6.3990	-6.3990
4.9344	4.9344
-20.7335	-20.7335
-3.3954	-3.3954
1.2374	1.2374
-0.9590	-0.9590
3.2526	3.2527
0.4874	0.4874
1.9557	1.9557
0.8331	0.8331
10.8939	10.8939
-38.2317	-38.2317
-0.3348	-0.3348
-0.1414	-0.1414
-1.7174	-1.7174
5.9370	5.9370
5.2874	5.2874
-3.7196	-3.7196
29.4466	29.4466
-21.3731	-21.3731
-0.9192	-0.9192
0.6698	0.6698
-4.6341	-4.6339
3.3475	3.3475

**Table 3**  
Hammerstein model parameters  $\mathcal{H}$  and  $\mathcal{P}$  (Problem 2).

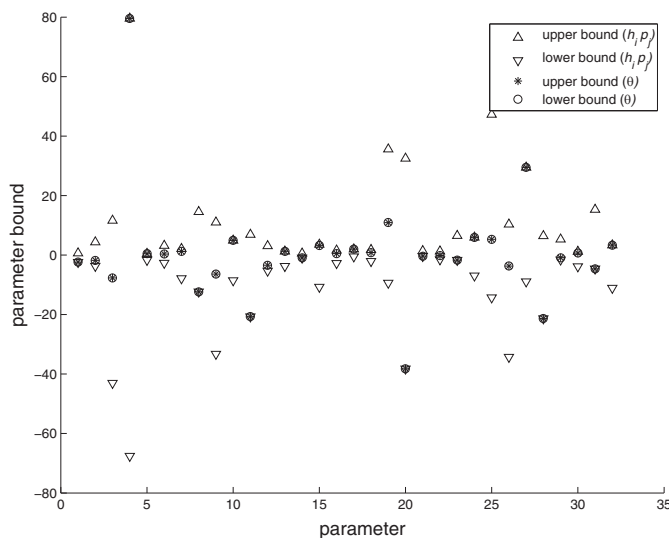
Parameter	Lower bound	Upper bound
$h_{1,1,1}$	-0.0322	-0.0070
$h_{1,1,2}$	-0.0589	-0.0253
$h_{1,2,1}$	-0.0505	-0.0286
$h_{1,2,2}$	-0.0151	-0.0607
$h_{2,1,1}$	0.0087	0.0266
$h_{2,1,2}$	0.0195	0.0837
$h_{2,2,1}$	0.0594	0.0243
$h_{2,2,2}$	0.0087	0.0183
$p_{1,1,1}$	7.3561e+001	7.3561e+001
$p_{1,1,2}$	1.3384e+003	1.3384e+003
$p_{1,2,1}$	5.3286e+001	5.3286e+001
$p_{1,2,2}$	2.4459e+002	2.4459e+002
$p_{2,1,1}$	5.6495e+002	5.6495e+002
$p_{2,1,2}$	3.5196e+002	3.5196e+002
$p_{2,2,1}$	6.3328e+001	6.3328e+001
$p_{2,2,2}$	1.8288e+002	1.8288e+002



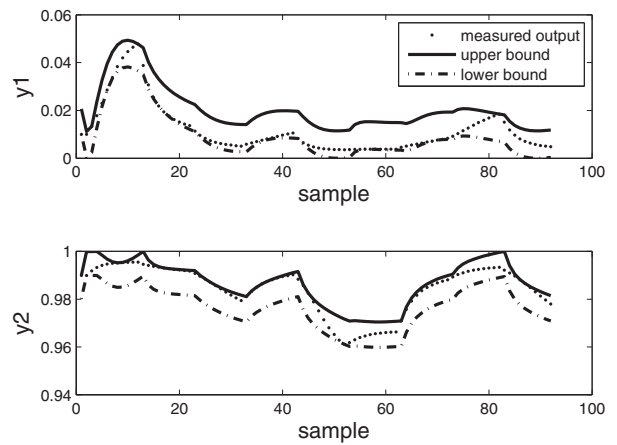
**Fig. 9.** Measured outputs and Hammerstein model bounds.

Problem 2 was solved to obtain the bounds for  $\mathcal{H}$  and  $\mathcal{P}$ . These values are shown in Table 3.

Fig. 9 depicts both measured outputs ( $x_D$  and  $x_B$ ) as well as the lower and upper bounds predicted by the robust Hammerstein model previously identified. Note that in both cases, the measured outputs are enclosed inside the bounds region.



**Fig. 10.** Bounds on the model parameters.



**Fig. 11.** Measured outputs and Hammerstein model bounds (for physical constraints).

Fig. 10 shows the bounds obtained by solution of Problem 1 (i.e., bounds on  $\Theta$ ) as well as the bounds for  $\mathcal{H}$  and  $\mathcal{P}$  (i.e., solution of Problem 2).

However, from inspection of Fig. 9 it is clear that the estimated bounds extend outside the admissible output limits (i.e.,  $y \leq y(k) \leq \bar{y}$ , with  $y = 0$  and  $\bar{y} = 1$ ). This fact can be taken into consideration by adding the following constraints to the optimization problem formulated in Eqs. (19)–(21):

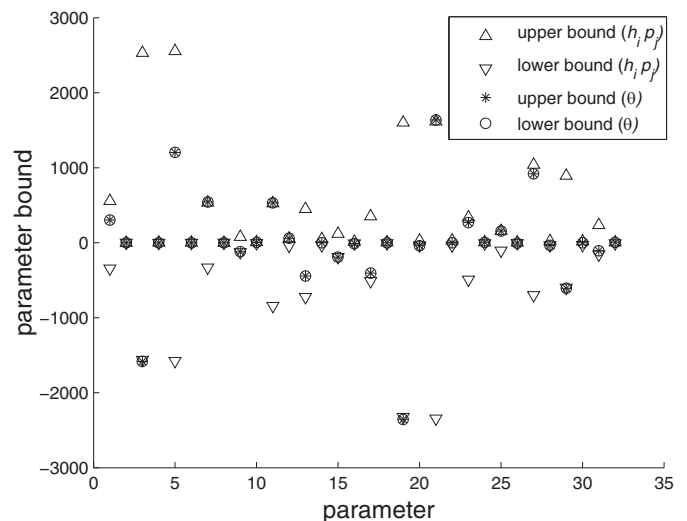
$$[(\theta^l)^T, (\theta^u)^T] \Phi_k \leq 1; \quad k = 1, \dots, K \quad (35)$$

$$[(\theta^u)^T, (\theta^l)^T] \Phi_k \geq 0; \quad k = 1, \dots, K \quad (36)$$

The robust model identified in this way gives the new bounds shown in Fig. 11.

Fig. 12 shows the recalculated bounds for  $\Theta$  (see values in Table 4), as well as the bounds for  $\mathcal{H}$  and  $\mathcal{P}$  (parameters in Table 5). These are the new results obtained when admissible values for  $x_D$  and  $x_B$  are taken into account. It should be remarked that including these physical constraints on the output limits makes the uncertainty bounds increase.

From Table 3 and Fig. 10, it can be deduced that the lower and upper bounds on  $\Theta$  have only slight differences. However, these small differences cause substantial differences in the bounds determined for  $\mathcal{H}$ . The parameters  $\mathcal{P}$  does not present disparity between their bounds.



**Fig. 12.** Bounds on the model parameters (for physical constraints).

**Table 4**  
Parameters  $\Theta$  (Problem 1, with physical constraints).

Lower bound	Upper bound
-2.0028e+002	-2.0028e+002
-1.4862e-001	-1.4862e-001
4.4008e+002	4.4008e+002
3.4944e+000	3.4944e+000
1.1023e+000	1.1023e+000
-2.5593e-003	-2.5593e-003
3.2030e+001	3.2030e+001
-5.0002e-001	-5.0002e-001
6.5466e+001	6.5466e+001
4.9428e+000	4.9428e+000
-2.1026e+002	-2.1026e+002
4.9316e+001	4.9316e+001
1.2051e+000	1.2051e+000
-9.2335e-001	-9.2335e-001
-4.4605e+000	-4.4603e+000
-7.7016e+000	-7.7016e+000
-1.0749e+003	-1.0749e+003
8.0539e-001	8.0539e-001
-1.1514e+003	-1.1514e+003
-3.9772e+001	-3.9772e+001
-1.0243e+003	-1.0243e+003
-1.1647e-001	-1.1647e-001
4.1168e+002	4.1168e+002
6.1917e+000	6.1917e+000
4.0420e+002	4.0420e+002
-3.8799e+000	-3.8799e+000
4.7442e+002	4.7442e+002
-3.6007e+001	-3.6007e+001
3.7785e+002	3.7785e+002
6.8861e-001	6.8861e-001
-1.5992e+002	-1.5992e+002
5.6266e+000	5.6266e+000

**Table 5**  
Recalculated Hammerstein model parameters  $\mathcal{H}$  and  $\mathcal{P}$  (with physical constraints).

Parameter	Lower bound	Upper bound
$h_{1,1,1}$	-2.7531e-001	4.4607e-001
$h_{1,1,2}$	-3.3639e-001	2.0934e-001
$h_{1,2,1}$	-3.9574e-004	2.7809e-004
$h_{1,2,2}$	-1.6449e-002	2.2897e-002
$h_{2,1,1}$	-4.0949e-001	2.8240e-001
$h_{2,1,2}$	-2.7875e-001	4.1481e-001
$h_{2,2,1}$	-6.3335e-003	5.0906e-003
$h_{2,2,2}$	-1.4182e-002	8.1610e-003
$p_{1,1,1}$	1.2498e+003	1.2499e+003
$p_{1,1,2}$	5.6735e+003	5.6735e+003
$p_{1,2,1}$	5.7261e+003	5.7261e+003
$p_{1,2,2}$	1.2024e+003	1.2024e+002
$p_{2,1,1}$	3.7456e+002	3.7460e+002
$p_{2,1,2}$	2.5081e+003	2.5081e+003
$p_{2,2,1}$	2.1526e+003	2.1526e+003
$p_{2,2,2}$	5.6614e+002	5.6617e+002

Therefore, we conclude that for this MIMO case study, the proposed robust Wiener model is a better description than the robust Hammerstein structure.

#### 4. Conclusions

Block oriented models are appealing descriptions for multiple applications in many fields. In particular, identification of MIMO Wiener and Hammerstein models has been dealt with in this work due to the wide acceptance and applicability these systems have shown.

Noniterative algorithms for the identification of both Wiener and Hammerstein uncertain models have been presented. A bounding procedure has been proposed for the identification of uncertain Wiener and Hammerstein parametric models. The approach allows robust identification, in the sense that the whole set of data can be

reproduced by the family of models which is obtained. The suitability of the proposed identification methods has been illustrated through the example of a 82 states distillation column.

#### Acknowledgments

This work was financially supported by the ANPCyT, CONICET and Universidad Nacional del Sur.

#### References

- Abdullah, Z., Aziz, N., & Ahmad, Z. (2007). Nonlinear modelling application in distillation column. *Chemical Product and Process Modeling*, 2(3), 12.
- Bloemen, H., Chou, C., van den Boom, T., Verdult, V., Verhaegen, M., & Backx, T. (2001). Wiener model identification and predictive control for dual composition control of a distillation column. *Journal of Process Control*, 11, 601–620.
- Celka, P., & Colditz, P. (2002). Nonlinear nonstationary Wiener model of infant EEG seizures. *IEEE Transactions on Biomedical Engineering*, 49.
- Eskinat, E., Johnson, S., & Luyben, W. (1991). Use of Hammerstein models in identification of nonlinear systems. *AIChE Journal*, 37, 255–268.
- Falugi, P., Giarrè, L., & Zappa, G. (2005). Approximation of the feasible parameter set in worst-case identification of Hammerstein models. *Automatica*, 41(6), 1017–1024.
- Fruzzetti, K., Palazoglu, A., & McDonald, K. (1997). Nonlinear model predictive control using Hammerstein models. *Journal of Process Control*, 7, 31–41.
- Gómez, J., & Baeyens, E. (2004). Identification of block-oriented nonlinear systems using orthonormal bases. *Journal of Process Control*, 14, 685–697.
- Greblicki, W. (1992). Nonparametric identification of Wiener systems. *IEEE Transactions on Information Theory*, 38(5), 1487–1493.
- Greblicki, W. (1994). Nonparametric identification of Wiener systems by orthogonal series. *IEEE Transactions on Automatic Control*, 39(10), 2077–2086.
- Greblicki, W. (1997). Nonparametric approach to Wiener systems identification. *IEEE Transactions on Circuits and Systems*, 44(6), 538–545.
- Hadjiloucas, S., Izhac, A., Galvao, R., Bowen, J., & Becerra, V. (2004). State-space model identification of a wireless THz network. In *Conference digest of the 2004 joint 29th international conference on infrared and millimeter waves, 2004 and 12th international conference on terahertz electronics* 27 September–1 October, (pp. 375–376).
- Hagenblad, A., & Ljung, L. (2000). Maximum likelihood estimation of Wiener models. In *Proc. 39th IEEE conf. on decision and control* 12–16 December, Sydney (Australia).
- Haverkamp, B., Chou, C., & Verhaegen, M. (1998). Subspace identification of continuous-time Wiener models. In *Proceedings of the 37th IEEE conference on decision and control* 16–18 December, Tampa (FL), (pp. 1846–1847).
- Huang, A., Tanskanen, J., & Hartimo, I. (1998). Design of optimum power estimator based on Wiener model applied to mobile transmitter power control. In *Proceedings of the 1998 IEEE international symposium on circuits and systems (ISCAS '98)* 31 May–3 June 1998.
- Hunter, I., & Korenberg, M. (1986). The identification of nonlinear biological systems: Wiener and Hammerstein cascade models. *Biological Cybernetics*, 55, 135–144.
- Janczak, A. (2005). *Identification of nonlinear systems using neural networks and polynomial models*. Berlin: Springer.
- Juditsky, A., Hjalmarsson, H., Benveniste, A., Delyon, B., Ljung, L., Sjöberg, J., et al. (1995). Nonlinear black-box models in system identification: Mathematical foundations. *Automatica*, 31(12), 1725–1750.
- Kalafatis, A., Arifin, N., Wang, L., & Cluett, W. (1995). A new approach to the identification of pH processes based on Wiener model. *Chemical Engineering Science*, 50, 3693–3701.
- Lacy, S., & Bernstein, D. (2002). Identification of FIR Wiener systems with unknown, noninvertible, polynomial nonlinearities. In *Proceedings of the 2002 American control conference* 8–10 May, New York, USA.
- Ling, W., & Rivera, D. (1998). Control relevant model reduction of Volterra series models. *Journal of Process Control*, 8(2), 79–88.
- Ling, W., & Rivera, D. (2001). A methodology for control-relevant nonlinear system identification using restricted complexity models. *Journal of Process Control*, 11, 209–222.
- Morgan, D., Ma, Z., Kim, J., Zierdt, M., & Pastalan, J. (2006). A generalized memory polynomial model for digital predistortion of RF power amplifiers. *IEEE Transactions on Signal Processing*, 54, 3852–3860.
- Norquay, S., Palazoglu, A., & Romagnoli, J. (1998). Model predictive control based on Wiener models. *Chemical Engineering Science*, 53(1), 75–84.
- Pajunen, G. (1987). Comparison of linear and nonlinear adaptive control of a pH process. *IEEE Control Systems Magazine*, 7(1), 39–44.
- Pajunen, G. (1992). Adaptive control of Wiener type nonlinear systems. *Automatica*, 28(4), 781–785.
- Pearson, R., & Pottmann, M. (2000). Gray-box identification of block-oriented nonlinear models. *Journal of Process Control*, 10, 301–315.
- Pottmann, M., & Pearson, R. (1998). Block-oriented NARMAX models with output multiplicities. *AIChE Journal*, 44(1), 131–140.



- Raich, R., Zhou, G., & Viberg, M. (2005). Subspace based approaches for Wiener system identification. *IEEE Transactions on Automatic Control*, 50(10), 1629–1634.
- Sjöberg, J., Zhang, O., Ljung, L., Benveniste, A., Delyon, B., Glorennec, P., et al. (1995). Nonlinear black-box modeling in system identification: A unified overview. *Automatica*, 1(12), 1691–1724.
- Skogestad, S., & Morari, M. (1988). Understanding the dynamic behaviour of distillation columns. *Industrial and Engineering Chemistry Research*, 27(10), 1848–1862.
- Skogestad, S., & Postlethwaite, I. (2007). *Multivariable feedback control. Analysis and design* (2nd edition). John Wiley & Sons.
- Stapleton, J., & Bass, S. (1985). Adaptive noise cancellation for a class of nonlinear dynamic reference channels. *IEEE Transactions on Circuits and Systems CAS*, 32, 143–150.
- Škrjanc, I., Blažič, S., & Agamennoni, O. (2005). Interval fuzzy modeling applied to Wiener models with uncertainties. *IEEE Transactions on Systems Man and Cybernetics Part B: Cybernetics*, 35(5), 1092–1095.
- Visala, A., Pitkänen, H., & Paanajärvi, J. (1999). Wiener-NN models and robust identification. In *International joint conference on neural networks (IJCNN'99)* 10–16 July, Washington, DC (USA).
- Weisstein, E. (2010). Orthotope. From MathWorld—A Wolfram web resource. <http://mathworld.wolfram.com/orthotope.html>.
- Wigren, T. (1993). Recursive prediction error identification using the nonlinear Wiener model. *Automatica*, 29(4), 1011–1025.
- Zhu, Y. (1999). Distillation column identification for control using Wiener model. In *Proc. American control conference 2–4 June, San Diego, California, USA*.