

Generalized conditional entropy in bipartite quantum systems

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Abstract

We analyze, for a general concave entropic form, the associated conditional entropy of a quantum system $A + B$, obtained as a result of a local measurement on one of the systems (B). This quantity is a measure of the average mixedness of A after such measurement, and its minimum over all local measurements is shown to be the associated entanglement of formation between A and a purifying third system C . In the case of the von Neumann entropy, this minimum determines also the quantum discord. For classically correlated states and mixtures of a pure state with the maximally mixed state, we show that the minimizing measurement can be determined analytically and is *universal*, i.e., the same for all concave forms. While these properties no longer hold for general states, we also show that in the special case of the linear entropy, an explicit expression for the associated conditional entropy can be obtained, whose minimum among projective measurements in a general qudit–qubit state can be determined analytically, in terms of the largest eigenvalue of a simple 3×3 correlation matrix. Such minimum determines the maximum conditional purity of A , and the associated minimizing measurement is shown to be also *universal* in the vicinity of maximal mixedness. Results for X states, including typical reduced states of spin pairs in XY chains at weak and strong transverse fields, are also provided and indicate that the measurements minimizing the von Neumann and linear conditional entropies are typically coincident in these states, being determined essentially by the main correlation. They can differ, however, substantially from that minimizing the geometric discord.

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(Some figures may appear in colour only in the online journal)

1. Introduction

There is currently a great interest in the investigation of quantum correlations in mixed states of composite quantum systems. While for pure states such correlations can be identified with entanglement, the situation in mixed states is more complex, as separable (non entangled) mixed states, defined as convex mixtures of product states [1] (i.e., states which can be generated by local operations and classical communication), can still exhibit signatures of quantum-like correlations, manifested for instance in a non-zero quantum discord [2–4]. Interest on this quantity has been enhanced by the existence of mixed-state-based quantum algorithms [5] able to achieve an exponential speed-up over the corresponding classical algorithm with vanishing entanglement [6] yet finite discord [7]. Various operational interpretations and implications of states with non-zero discord have been recently provided [8–11].

The quantum discord for a bipartite system $A + B$ can be written [2] as the minimum difference between two distinct quantum extensions of the classical Shannon based *conditional* entropy $S(A|B)$ [12], one involving a local measurement M_B on one of the systems (B), over which the minimization is to be performed, and the other the direct quantum version of the classically equivalent expression $S(A, B) - S(B)$ (which becomes negative in pure entangled states). While other measures of quantum correlations with similar properties (like reducing to an entanglement measure for pure states and vanishing just for classically correlated states) have been introduced [4, 8, 9, 11, 13–22], the quantum discord has the special feature, due to its definition through a conditional entropy, of being directly related with the entanglement of formation between the unmeasured system and a third system which purifies the whole system [23–26]. Accordingly, the measurement minimizing the quantum discord can differ substantially from those minimizing other measures such as the geometric discord [8, 16], which can be much more easily determined. The complex minimization involved in the quantum discord has in fact limited its evaluation to simple systems or special states and measurements [7, 8, 27–34].

The aim of this work is first to extend the concept of measurement dependent conditional entropy to a general entropic form (or uncertainty measure) S_f depending on an arbitrary concave function f [12, 35]. The ensuing quantity $S_f(A|B_{M_B})$ provides a measure of the average conditional mixedness of A after a measurement at B , and allows us to define an associated generalized ‘information gain’ or uncertainty reduction $I_f(A|B_{M_B}) = S_f(A) - S_f(A|B_{M_B})$, which is non-negative for any concave f and reduces to the associated entanglement entropy $S_f(A)$ in the case of pure states. Such extension differs then from other treatments [36–38] dealing with the generalization of the measurement independent von Neumann conditional entropy $S(A, B) - S(B)$. The minimum of the present $S_f(A|B_{M_B})$ among all local measurements coincides with the associated entanglement of formation (convex roof extension of the S_f entanglement entropy) between A and a purifying third system C , as will be shown.

Such general formulation allows, first, to recognize some universal features of the measurement dependent conditional entropy which do not depend on the choice of entropic function f and rely just on concavity. It also opens the way to use simple entropic forms like the linear entropy $S_2(\rho) = 1 - \text{Tr} \rho^2$, trivially related with the purity $P(\rho) = \text{Tr} \rho^2$ and lower bound to the von Neumann entropy, which can be more easily evaluated (it does not require the eigenvalues of ρ) and can therefore help to determine and understand the minimizing measurement of the von Neumann conditional entropy and hence the quantum discord. Moreover, we will show that this entropy determines the behavior of *all* entropies in the vicinity of the maximally mixed state. The purity, and hence $S_2(\rho)$, is also more easily accessible from the experimental side, since it can be determined efficiently without requiring a full state tomography [39].

We first derive in section 2 the fundamental properties of $S_f(A|B_{M_B})$, including its minimum in general classically correlated states and mixtures of a pure state with the maximally mixed state, where the minimizing measurement is shown to be *universal*, i.e., the same for any entropic form. The formalism is then applied in section 3 to derive a closed expression for the conditional S_2 entropy and discuss its fundamental properties, including its minimum over projective measurements for a general A +qubit system, which is shown to be determined by the largest eigenvalue of a simple 3×3 contracted correlation matrix. This permits us to easily recognize the minimizing measurement and understand its behavior. Applications to general parity preserving two-qubit states (X states), including mixtures of aligned states and weakly correlated states, relevant for the description of pair states in interacting XY spin chains at weak and strong transverse fields, are presented in section 4. These examples indicate a similar behavior (and coincidence of the minimizing measurement) of the S_2 and von Neumann conditional entropies for these states, even well beyond the vicinity of maximal mixedness. Conclusions are finally given in section 5.

2. Formalism

2.1. Generalized conditional entropy after a local measurement

We consider a bipartite quantum system $A + B$ in an initial state $\rho \equiv \rho_{AB}$, with reduced states $\rho_A = \text{Tr}_B \rho$, $\rho_B = \text{Tr}_A \rho$. We assume a general positive operator valued local measurement [40] M_B on system B is performed, defined by a set of operators $M_j = I_A \otimes M_j^B$, $j = 1, \dots, j_m$, such that the state after outcome j is proportional to $M_j \rho M_j^\dagger$. The positive semidefinite operators

$$\Pi_j = M_j^\dagger M_j = I_A \otimes \Pi_j^B \quad (1)$$

should then satisfy $\sum_j \Pi_j = I \equiv I_A \otimes I_B$.

The reduced state of A after outcome j depends just on Π_j and is given by

$$\rho_{A/\Pi_j} = p_j^{-1} \text{Tr}_B \rho \Pi_j, \quad p_j = \text{Tr} \rho \Pi_j, \quad (2)$$

where $p_j > 0$ is the probability of such outcome. In order to quantify the average uncertainty or mixedness of the state of A after such measurement, we will consider here the generalized conditional entropy

$$S_f(A|B_{\{\Pi_j\}}) = \sum_j p_j S_f(\rho_{A/\Pi_j}), \quad (3)$$

where

$$S_f(\rho) = \text{Tr} f(\rho) \quad (4)$$

represents a generalized entropic form or uncertainty measure [12, 35] (see appendix A). Here $f : [0, 1] \rightarrow \Re$ is a smooth strictly concave function satisfying $f(0) = f(1) = 0$. For $f(\rho) = -\rho \log_a \rho$ (we use here $a = 2$ or e), $S_f(\rho)$ becomes the von Neumann entropy $S(\rho) = -\text{Tr} \rho \log_a \rho$, and equation (3) the measurement dependent von Neumann conditional entropy, introduced in [2] for the definition of the quantum discord.

The concavity of these forms, i.e.,

$$S_f\left(\sum_\alpha q_\alpha \rho_\alpha\right) \geq \sum_\alpha q_\alpha S_f(\rho_\alpha), \quad (5)$$

if $\{q_\alpha\}$ is a probability distribution ($q_\alpha \geq 0$, $\sum_\alpha q_\alpha = 1$) and all ρ_α 's are quantum states, directly follows from the concavity of f , and implies fundamental properties of the

generalized conditional entropy (3). First, since $\rho_A = \sum_j p_j \rho_{A/\Pi_j}$, equation (5) implies $S_f(A) \equiv S_f(\rho_A) \geq \sum_j p_j S_f(\rho_{A/\Pi_j})$, i.e.,

$$S_f(A) \geq S_f(A|B_{\{\Pi_j\}}), \tag{6}$$

indicating that the average conditional mixedness of A after a measurement at B , will not exceed the original mixedness, for any choice of S_f . Moreover, if f is strictly concave, equality in (5) holds iff all ρ_α 's with $q_\alpha > 0$ are identical. Hence, equality in (6) for all M_B holds just if $\rho = \rho_A \otimes \rho_B$, since only in this case $\rho_{A/\Pi_j} = \rho_A \forall \Pi_j$. The quantity

$$I_f(A|B_{\{\Pi_j\}}) = S_f(A) - S_f(A|B_{\{\Pi_j\}}), \tag{7}$$

is then non-negative for any S_f , vanishing for all M_B just for product states. It represents the average reduction in the quantum uncertainty of A (or generalized information gain about A) as measured by S_f , after a measurement at B .

Equation (5) also implies concavity of the conditional entropy: if $\rho = \sum_\alpha q_\alpha \rho^\alpha$, then $\rho_{A/\Pi_j} = \sum_\alpha p_j^{-1} q_\alpha p_j^\alpha \rho_{A/\Pi_j}^\alpha$, with $p_j^\alpha = \text{Tr} \rho^\alpha \Pi_j$, $p_j = \sum_\alpha q_\alpha p_j^\alpha$. Hence, $S_f(\rho_{A/\Pi_j}) \geq \sum_\alpha p_j^{-1} q_\alpha p_j^\alpha S_f(\rho_{A/\Pi_j}^\alpha)$, entailing

$$S_f(A|B_{\{\Pi_j\}}) \geq \sum_\alpha q_\alpha S_f(A^\alpha|B_{\{\Pi_j\}}^\alpha), \tag{8}$$

where $S_f(A^\alpha|B_{\{\Pi_j\}}^\alpha) = \sum_j p_j^\alpha S_f(\rho_{A/\Pi_j}^\alpha)$. Average uncertainty about A after state mixing cannot be smaller than the average of the original average uncertainties. In addition, if

$$\Pi_j = \sum_k r_j^k \tilde{\Pi}_k, \quad r_j^k \geq 0, \tag{9}$$

where $\tilde{\Pi}_k = I_A \otimes \tilde{\Pi}_k^B$, with $\sum_k \tilde{\Pi}_k = I$, are positive operators representing a more detailed local measurement (i.e., all $\tilde{\Pi}_k^B$ of rank 1) and $\sum_j r_j^k = 1$, then $\rho_{A/\Pi_j} = \sum_k p_j^{-1} r_j^k q_k \rho_{A/\tilde{\Pi}_k}$, with $q_k = \text{Tr} \rho \tilde{\Pi}_k$, $\sum_k r_j^k q_k = p_j$. Hence, $S_f(\rho_{A/\Pi_j}) \geq \sum_k p_j^{-1} r_j^k q_k S_f(\rho_{A/\tilde{\Pi}_k})$ and

$$S_f(A|B_{\{\Pi_j\}}) \geq \sum_k q_k S_f(\rho_{A/\tilde{\Pi}_k}) = S_f(A|B_{\{\tilde{\Pi}_k\}}), \tag{10}$$

i.e., the generalized conditional entropy will not increase (and will in general decrease) if a more detailed local measurement is performed. In fact, $S_f(A)$ can be considered as the conditional entropy $S_f(A|B_I)$ of A after a trivial measurement of the identity I_B in B , so that equation (6) is a particular case of (10).

Minimum uncertainty about the state of A will then be obtained for measurements based on rank one operators

$$\Pi_j^B = r_j |j_B\rangle\langle j_B|, \quad r_j > 0, \tag{11}$$

where $|j_B\rangle$ are normalized states, such that $\sum_j \Pi_j^B = I_B$. Standard complete projective measurements (von Neumann measurements) correspond to $r_j = 1$ and $\{|j_B\rangle\}$ an orthonormal basis ($\Pi_j \Pi_{j'} = \delta_{jj'} \Pi_j$). In particular, for pure states $\rho^2 = \rho$, i.e.,

$$\rho = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle = \sum_k \sqrt{q_k} |\tilde{k}_A \tilde{k}_B\rangle, \tag{12}$$

where the last expression denotes the Schmidt decomposition [40] ($\{|k_A\rangle\}$, $\{|k_B\rangle\}$ orthonormal sets), ρ_{A/Π_j} is pure $\forall j$ with $p_j > 0$, for any local measurement based on the operators (11):

$$\rho_{A/\Pi_j} = |j_A\rangle\langle j_A|, \quad |j_A\rangle = (r_j/p_j)^{1/2} \sum_k \sqrt{q_k} |j_B \tilde{k}_B\rangle |\tilde{k}_A\rangle, \tag{13}$$

where $p_j = r_j \sum_k q_k |\langle j_B | \tilde{k}_B \rangle|^2$. Hence, in the pure case $S_f(A|B_{\{\Pi_j\}}) = 0$, and equation (7) becomes the generalized entanglement entropy [17]:

$$I_f(A|B_{\{\Pi_j\}}) = S_f(A) = S_f(B) = \sum_k f(q_k). \tag{14}$$

2.2. Minimum conditional entropy and generalized entanglement of formation

Let us now consider the minimum of equation (3) among all local measurements M_B for a general state ρ ,

$$S_f(A|B) \equiv \text{Min}_{\{\Pi_j\}} S_f(A|B_{\{\Pi_j\}}). \quad (15)$$

From equation (10) it follows that just rank one operators of the form (11) need to be considered in the minimization. Equation (15) leads to the maximum generalized information gain (i.e., maximum uncertainty reduction)

$$I_f(A|B) = \text{Max}_{\{\Pi_j\}} I_f(A|B_{\{\Pi_j\}}) = S_f(A) - S_f(A|B). \quad (16)$$

If the system $A + B$ is purified [40] by adding a third system C , equation (15) has the important meaning of being the associated *entanglement of formation* $E_f(A, C)$ [17] between A and C in the reduced state ρ_{AC} [23]:

$$S_f(A|B) = E_f(A, C) = \text{Min}_{\sum_j p_j \rho_{AC}^j = \rho_{AC}} \sum_j p_j S_f(\rho_A^j), \quad (17)$$

where the minimization is over all representations of ρ_{AC} as convex combination ($p_j > 0$) of pure states $\rho_{AC}^j = |j_{AC}\rangle\langle j_{AC}|$, and $S_f(\rho_A^j) = S_f(\rho_C^j)$ is the S_f entanglement entropy between A and C in $|j_{AC}\rangle$ ($\rho_A^j = \text{Tr}_C \rho_{AC}^j$). Equation (17) is the convex roof extension [41] of the pure state entanglement entropy (14) and is an entanglement monotone [42]. The identity (17) was derived for the von Neumann entropy (see [23] and [24–26]), where $E_f(A, C)$ becomes the standard entanglement of formation $E(A, C)$ [43], but the arguments remain valid in the present general case (see appendix B).

Equation (17) entails that the equation (16) can be also expressed as

$$I_f(A|B) = E_f(A, BC) - E_f(A, C), \quad (18)$$

where $E_f(A, BC) = S_f(\rho_A) = S_f(\rho_{BC})$ is the entanglement entropy between A and BC in the purified state.

The quantum discord [2–4, 8] $D(A|B)$, as obtained by a measurement in B , is directly related to the present *von Neumann* conditional entropy $S(A|B_{\{\Pi_j\}})$ through

$$D(A|B) = \text{Min}_{\{\Pi_j\}} S(A|B_{\{\Pi_j\}}) - [S(A, B) - S(B)], \quad (19)$$

where the last bracket is the standard (measurement independent) quantum extension of the von Neumann conditional entropy (which can be negative in entangled states). It can be also expressed as the difference between the standard mutual information $S(A) + S(B) - S(A, B)$ and the maximum von Neumann information gain $I(A|B) = S(A) - \text{Min}_{\{\Pi_j\}} S(A|B_{\{\Pi_j\}})$. A generalization of the quantum discord based on the Renyi entropy of order 2 was considered in [22] for Gaussian states, whereas extensions based on the Tsallis entropy [44] were discussed in [45].

2.3. Classically correlated states

There are important classes of mixed states where the local measurement minimizing $S_f(A|B_{\{\Pi_j\}})$ is *universal*, i.e., the same for *all* entropies S_f , and can be generally determined. One is that of classically correlated states with respect to B [2–4],

$$\rho = \sum_k q_k \rho_{A/k} \otimes \tilde{\Pi}_k^B, \quad (20)$$

where $q_k \geq 0$ and $\{\tilde{\Pi}_k^B = |\tilde{k}_B\rangle\langle\tilde{k}_B|\}$ is a complete set of orthogonal rank one local projectors, such that after a local measurement in this basis, $\rho_{A/\tilde{\Pi}_k} = \rho_{A/k}$ (and $\sum_k \tilde{\Pi}_k \rho \tilde{\Pi}_k = \rho$ if $\tilde{\Pi}_k = I_A \otimes \tilde{\Pi}_k^B$, implying that the states (20) remain unchanged after an unread local measurement in this basis). It is easy to prove that the lowest conditional entropy (15) is obtained for such measurement, for any S_f :

$$S_f(A|B) = S_f(A|B_{\{\tilde{\Pi}_k\}}) = \sum_k q_k S_f(\rho_{A/k}). \quad (21)$$

Proof. for any M_B based on the operators (11), we have

$$\rho_{A/\Pi_j} = \sum_k r_j p_j^{-1} q_k |\langle j_B | \tilde{k}_B \rangle|^2 \rho_{A/k} \quad (22)$$

with $p_j = r_j \sum_k q_k |\langle j_B | \tilde{k}_B \rangle|^2$. Concavity plus completeness ($\sum_j r_j |\langle j_B | \tilde{k}_B \rangle|^2 = 1$) imply

$$S_f(A|B_{\{\Pi_j\}}) \geq \sum_{k,j} r_j q_k |\langle j_B | \tilde{k}_B \rangle|^2 S_f(\rho_{A/k}) = \sum_k q_k S_f(\rho_{A/k}) \quad (23)$$

with the inequality saturated for a measurement in the pointer basis $\{|\tilde{k}_B\rangle\}$, formed by the eigenstates of $\rho_B = \sum_k q_k \tilde{\Pi}_k^B$. The maximum I_f is then

$$I_f(A|B) = S_f\left(\sum_k q_k \rho_{A/k}\right) - \sum_k q_k S_f(\rho_{A/k}). \quad (24) \quad \square$$

2.4. Pure state plus maximally mixed state

A second case is that of the mixture of a general pure state (12) with the maximally mixed state I/d ,

$$\rho = w |\Psi\rangle\langle\Psi| + (1-w)I_d/d, \quad |\Psi\rangle = \sum_k \sqrt{q_k} |\tilde{k}_A \tilde{k}_B\rangle, \quad (25)$$

where $w \in [0, 1]$ and $d = d_A d_B$ is the Hilbert-space dimension of $A + B$. The minimum for any S_f is provided again by a measurement in the basis $\{|\tilde{k}_B\rangle\}$ of eigenstates of ρ_B :

$$\begin{aligned} S_f(A|B) &= S(A|B_{\{\tilde{\Pi}_k\}}) = \sum_k q_k^w S_f(\rho_{A/\tilde{\Pi}_k}) \\ &= \sum_k q_k^w \left[f\left(\frac{w q_k + (1-w)/d}{q_k^w}\right) + (d_A - 1) f\left(\frac{1-w}{d q_k^w}\right) \right], \end{aligned} \quad (26)$$

where $q_k^w = w q_k + \frac{1-w}{d_B}$ is the probability of outcome k at B and $\rho_{A/\tilde{\Pi}_k} = [w q_k |\tilde{k}_A\rangle\langle\tilde{k}_A| + (1-w)I_A/d]/q_k^w$ the state of A after such outcome.

Proof. for any measurement based on the operators (11) we obtain, using (12) and (13),

$$\rho_{A/\Pi_j} = \frac{w p_j |j_A\rangle\langle j_A| + r_j (1-w) I_A/d}{p_j^w} = \sum_k \frac{r_j q_k^w}{p_j^w} |\langle j_B | \tilde{k}_B \rangle|^2 |U_k^j \rho_{A/\tilde{\Pi}_k} U_k^{j\dagger}|, \quad (27)$$

where $p_j = r_j \sum_k q_k |\langle j_B | \tilde{k}_B \rangle|^2$ and $p_j^w = w p_j + r_j \frac{1-w}{d_B}$ are respectively the probabilities of outcome j in $|\Psi\rangle$ and ρ , and U_k^j are unitaries satisfying $U_k^j |\tilde{k}_A\rangle = |j_A\rangle$. Hence, concavity, invariance of S_f under unitary transformations and completeness imply again

$$S_f(A|B_{\{\Pi_j\}}) \geq \sum_k q_k^w S_f(\rho_{A/\tilde{\Pi}_k}) = S_f(A|B_{\{\tilde{\Pi}_k\}}). \quad (28)$$

Equality in (28) for any M_B of the form (11) holds for i) $w = 0$ (ρ maximally mixed), ii) $w = 1$ (ρ pure) and iii) $|\Psi\rangle$ maximally entangled ($q_k = 1/d_B \forall k$, assuming $d_A \geq d_B$), where $p_j = r_j/d_B \forall j$ and all $\rho_{A/\Pi_j} = w |j_A\rangle\langle j_A| + \frac{1-w}{d_A} I_A$ have the same spectrum. \square

It can be easily checked that equation (26) is a *concave* function of both w and the probability distribution $\mathbf{q} = \{q_k\}$. Since $S_f(A|B)$ reaches its maximum $S_f(I_A/d_A) = d_A f(1/d_A)$ for $w = 0$, concavity entails that equation (26) is a *decreasing* function of w for $w \in [0, 1]$ $\forall S_f$: decreasing mixedness decreases the uncertainty about A . Concavity also leads to the immediate lower bound $S_f(A|B) \geq (1 - w)d_A f(1/d_A)$.

Besides, for states $|\Psi\rangle, |\Psi'\rangle$ characterized by distributions \mathbf{q} and \mathbf{q}' in the Schmidt decomposition, we have $S_f(\mathbf{q}) \geq S_f(\mathbf{q}') \forall S_f$ iff $\mathbf{q} < \mathbf{q}'$ (i.e., \mathbf{q} majorized by \mathbf{q}' , see Appendix A). Such condition ensures then that $|\Psi\rangle$ is *more entangled than* $|\Psi'\rangle$ for any S_f , and is the same condition which warrants that $|\Psi'\rangle$ can be obtained from $|\Psi\rangle$ by LOCC [40, 46]. In such a case, concavity of $S_f(A|B)$ with respect to \mathbf{q} entails that at fixed $w \in (0, 1)$, $S_f(A|B)_{|\Psi\rangle} \geq S_f(A|B)_{|\Psi'\rangle}$ for any S_f , i.e., greater entanglement for any S_f entails a larger conditional entropy $S_f(A|B) \forall S_f$ in the mixture (25), in contrast with the pure case $w = 1$ (where $S_f(A|B) = 0$ for any pure state $|\Psi\rangle$).

3. The quadratic case: conditional purity after local measurement

3.1. General properties

We now consider in detail the simplest choice of concave f , i.e., a quadratic function $f(\rho) = \alpha(\rho - \rho^2)$, $\alpha > 0$. For $\alpha = 1$ this leads to $S_f(\rho) = S_2(\rho)$, with

$$S_2(\rho) = 1 - \text{Tr } \rho^2, \tag{29}$$

the so-called *linear entropy*, since it corresponds to the linear approximation $-\ln \rho \approx I - \rho$ in $S(\rho)$ ($\ln p = p - 1 + O(p - 1)^2$ for $p \rightarrow 1$). It is the $q = 2$ case of the Tsallis entropy $S_q(\rho)$ [44] (see appendix A) and provides a lower bound to the von Neumann entropy for $a = e$ (and hence $a < e$), since $p(1 - p) \leq -p \ln p \forall p \in [0, 1]$.

Equation (29) is trivially related with the *purity* $P(\rho) = \text{Tr } \rho^2$, which satisfies $P(\rho) \leq 1$, with $P(\rho) = 1$ iff ρ is a pure state ($\rho^2 = \rho$). It is also directly related to the squared Hilbert–Schmidt distance to the maximally mixed state I/d :

$$\|\rho - I/d\|^2 = \text{Tr } \rho^2 - 1/d = S_2(I/d) - S_2(\rho), \tag{30}$$

where $\|O\|^2 = \text{Tr } O^\dagger O$ and $S_2(I/d) = 1 - 1/d$.

Similarly, the associated conditional entropy,

$$S_2(A|B_{\{\Pi_j\}}) = 1 - \sum_j p_j \text{Tr } \rho_{A/\Pi_j}^2, \tag{31}$$

is trivially related with the average conditional purity $P(A|B_{\{\Pi_j\}}) = \sum_j p_j \text{Tr } \rho_{A/\Pi_j}^2$, and determines the average squared distance to the maximally mixed state of A :

$$\sum_j p_j \|\rho_{A/\Pi_j} - I_A/d_A\|^2 = S_2(I_A/d_A) - S_2(A|B_{\{\Pi_j\}}). \tag{32}$$

The ensuing $I_2(A|B)$ represents the average increase of the purity of A due to the local measurement at B , and can be also interpreted as the average squared distance between the original and the post-measurement state of A :

$$I_2(A|B_{\{\Pi_j\}}) = S_2(A) - S_2(A|B_{\{\Pi_j\}}) = \sum_j p_j \text{Tr } \rho_{A/\Pi_j}^2 - \text{Tr } \rho_A^2 \tag{33}$$

$$= \sum_j p_j \|\rho_A - \rho_{A/\Pi_j}\|^2, \tag{34}$$

where we used equation (2). We may also define, through I_2 and S_2 , the purity gain ratio

$$R_2(A|B_{\{\Pi_j\}}) = 1 + \frac{I_2(A|B_{\{\Pi_j\}})}{1 - S_2(A)} = \frac{\sum_j p_j \text{Tr} \rho_{A/\Pi_j}^2}{\text{Tr} \rho_A^2}, \quad (35)$$

which satisfies $1 \leq R_2(A|B_{\{\Pi_j\}}) \leq d_A$. Such ratio remains unaltered if an ancilla C at A is added ($\rho_{AB} \rightarrow \rho_C \otimes \rho_{AB}$).

If ρ is sufficiently close to the maximally mixed state I/d , equation (30) entails that all entropies $S_f(\rho)$ (with $f''(p) < 0 \forall p$) become in this limit *linear functions* of $S_2(\rho)$. A second-order expansion of $S_f(\rho)$ around $\rho = I/d$ leads to

$$S_f(\rho) - S_f(I/d) \approx \frac{1}{2} f''\left(\frac{1}{d}\right) \|\rho - I/d\|^2 = \frac{1}{2} |f''\left(\frac{1}{d}\right)| [S_2(\rho) - S_2(I/d)]. \quad (36)$$

Hence, in the vicinity of maximal mixedness, all entropies $S_f(\rho)$ (with $f''(1/d) < 0$), including of course the von Neumann entropy $S(\rho)$, are determined by $S_2(\rho)$. In this limit ρ_{A/Π_j} is also close to $I_A/d_A \forall \Pi_j$ and hence,

$$S_f(A|B_{\{\Pi_j\}}) \approx S_f(I_A/d_A) + \frac{1}{2} |f''\left(\frac{1}{d_A}\right)| [S_2(A|B_{\{\Pi_j\}}) - S_2(I_A/d_A)], \quad (37)$$

indicating that all conditional entropies $S_f(A|B_{\{\Pi_j\}})$ (with $f''(1/d_A) < 0$) also become functions of the S_2 conditional entropy. The measurement minimizing $S_2(A|B_{\{\Pi_j\}})$ becomes then *universal* in this limit, i.e., it will also minimize all other $S_f(A|B_{\{\Pi_j\}})$.

We note here that the *geometric discord* [8, 16] is defined as the minimum squared Hilbert–Schmidt distance from ρ to a classically correlated state ρ_c of the form (20), and is equivalent to the minimum increase of the S_2 entropy of the global state due to an unread projective measurement at B [17]:

$$D_2(A|B) = \text{Min}_{\rho_c} \|\rho - \rho_c\|^2 = \text{Min}_{\{\Pi_j\}} S_2\left(\sum_j \Pi_j \rho \Pi_j\right) - S_2(\rho), \quad (38)$$

where again $\Pi_j = I_A \otimes \Pi_j^B$. In contrast with $S_2(A|B)$, the geometric discord looks for the closest average global post-measurement state $\sum_j \Pi_j \rho \Pi_j$. This will lead to significant differences in the minimizing measurement for certain states, as discussed in section 4.

3.2. Explicit expressions

The obvious advantage of $S_2(\rho)$ over other entropies is that its evaluation does not require the knowledge of the eigenvalues of ρ . Convenient expressions in a system with Hilbert-space dimension d can be obtained just by considering a complete orthogonal set of Hermitian operators (I, σ) , with $\sigma = (\sigma_1, \dots, \sigma_{d^2-1})$ satisfying

$$\text{Tr} \sigma_i = 0, \quad \text{Tr} \sigma_i \sigma_j = d \delta_{ij}. \quad (39)$$

For a single qubit σ are the Pauli operators. A general state can then be written as

$$\rho = (I + \mathbf{r} \cdot \boldsymbol{\sigma})/d, \quad \mathbf{r} = \text{Tr} \rho \boldsymbol{\sigma} = \langle \boldsymbol{\sigma} \rangle, \quad (40)$$

and the quadratic entropy (29) becomes

$$S_2(\rho) = 1 - (1 + |\mathbf{r}|^2)/d. \quad (41)$$

For a pure state $\rho^2 = \rho$, $|\mathbf{r}|^2 = d - 1$ and $S_2(\rho) = 0$.

In the case of a bipartite system $A + B$, we may rewrite equation (40) as

$$\rho = [I + \mathbf{r}_A \cdot \boldsymbol{\sigma}_A \otimes I_B + I_A \otimes \mathbf{r}_B \cdot \boldsymbol{\sigma}_B + \boldsymbol{\sigma}_A^t \mathbf{J} \otimes \boldsymbol{\sigma}_B]/d \quad (42)$$

where $\mathbf{r}_A = \langle \boldsymbol{\sigma}_A \rangle$, $\mathbf{r}_B = \langle \boldsymbol{\sigma}_B \rangle$ and $\mathbf{J} = \langle \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B^t \rangle$ is a $(d_A^2 - 1) \times (d_B^2 - 1)$ matrix of elements $J_{ij} = \langle \sigma_{Ai} \otimes \sigma_{Bj} \rangle$. The reduced states are $\rho_\alpha = (I_\alpha + \mathbf{r}_\alpha \cdot \boldsymbol{\sigma}_\alpha)/d_\alpha$, $\alpha = A, B$.

A measurement M_B based on the operators (11) can be characterized by the vectors

$$\mathbf{k}_j = \text{Tr}_B(\boldsymbol{\sigma}_B |j_B\rangle\langle j_B|), \quad (43)$$

such that $\Pi_j^B = r_j(I_B + \mathbf{k}_j \cdot \boldsymbol{\sigma}_B)/d_B$. These vectors satisfy $|\mathbf{k}_j|^2 = d_B - 1$ and

$$\sum_j r_j \mathbf{k}_j = \mathbf{0}, \quad (44)$$

since $\sum_j \Pi_j^B = I_B$. The probability of outcome j and the ensuing state ρ_{A/Π_j} are then

$$p_j = \frac{r_j}{d_B} (1 + \mathbf{r}_B \cdot \mathbf{k}_j), \quad \rho_{A/\Pi_j} = \frac{1}{d_A} \left[I_A + \frac{(\mathbf{r}_A + J\mathbf{k}_j) \cdot \boldsymbol{\sigma}_A}{1 + \mathbf{r}_B \cdot \mathbf{k}_j} \right], \quad (45)$$

which involve just the components of \mathbf{r}_B and J along \mathbf{k}_j . Equations (41)–(45) lead then to

$$\begin{aligned} S_2(A|B_{\{\Pi_j\}}) &= 1 - \frac{1}{d_A} \left[1 + \sum_j p_j \frac{|\mathbf{r}_A + J\mathbf{k}_j|^2}{(1 + \mathbf{r}_B \cdot \mathbf{k}_j)^2} \right] \\ &= S_2(A) - \frac{1}{d} \sum_j r_j \frac{\mathbf{k}_j^t C^t C \mathbf{k}_j}{1 + \mathbf{r}_B \cdot \mathbf{k}_j}, \end{aligned} \quad (46)$$

where $S_2(A) = S_2(\rho_A) = 1 - (1 + |\mathbf{r}_A|^2)/d_A$ and $\mathbf{k}_j^t C^t C \mathbf{k}_j = |C\mathbf{k}_j|^2$, with

$$C = J - \mathbf{r}_A \mathbf{r}_B^t = \langle \boldsymbol{\sigma}_A \otimes \boldsymbol{\sigma}_B^t \rangle - \langle \boldsymbol{\sigma}_A \rangle \langle \boldsymbol{\sigma}_B^t \rangle, \quad (47)$$

the correlation matrix (or tensor), of elements $C_{ij} = \langle \sigma_{Ai} \otimes \sigma_{Bj} \rangle - \langle \sigma_{Ai} \rangle \langle \sigma_{Bj} \rangle$ ($C = 0$ iff $\rho = \rho_A \otimes \rho_B$). The second term in (46) is just the quadratic information gain (i.e., purity increase) (33):

$$I_2(A|B_{\{\Pi_j\}}) = \frac{1}{d} \sum_j r_j \frac{\mathbf{k}_j^t C^t C \mathbf{k}_j}{1 + \mathbf{r}_B \cdot \mathbf{k}_j}. \quad (48)$$

It is then determined by \mathbf{r}_B and the $(d_B^2 - 1) \times (d_B^2 - 1)$ positive semidefinite matrix $C^t C$. We finally note that we may also express equations (42) and (45) in terms of the correlation matrix C (rather than J) as

$$\rho = \rho_A \otimes \rho_B + \boldsymbol{\sigma}_A^t C \otimes \boldsymbol{\sigma}_B / d, \quad \rho_{A/\Pi_j} = \rho_A + \frac{\boldsymbol{\sigma}_A^t C \mathbf{k}_j}{d_A (1 + \mathbf{r}_B \cdot \mathbf{k}_j)}, \quad (49)$$

with $\|\rho - \rho_A \otimes \rho_B\|^2 = \text{Tr}[C^t C]/d = \|C\|^2/d$.

3.3. The qudit–qubit case

We now show that when B is a single qubit, an analytic expression for the minimum S_2 conditional entropy (i.e., for the maximum conditional purity of A) amongst projective local measurements on B can be obtained for any dimension d_A of A ($\mathbb{C}^{d_A} \otimes \mathbb{C}^2$ system) and any initial state ρ . Here, we can take $\boldsymbol{\sigma}_B$ as the Pauli operators, and \mathbf{k}_j become unit vectors. For a projective spin measurement along direction \mathbf{k} ($|\mathbf{k}| = 1$), we have $j = 1, 2$, with $r_j = 1$, $\mathbf{k}_1 = -\mathbf{k}_2 = \mathbf{k}$, and equation (48) becomes

$$I_2(A|B_{\mathbf{k}}) = \frac{1}{d_A} \frac{\mathbf{k}^t C^t C \mathbf{k}}{1 - (\mathbf{r}_B \cdot \mathbf{k})^2} = \frac{1}{d_A} \frac{\mathbf{k}^t C^t C \mathbf{k}}{\mathbf{k}^t N_B \mathbf{k}}, \quad (50)$$

where N_B is the 3×3 positive semidefinite matrix

$$N_B = I_3 - \mathbf{r}_B \mathbf{r}_B^t. \quad (51)$$

The last expression in (50) is a ratio of quadratic forms and is then independent of the length of \mathbf{k} . Its maximum can therefore be obtained diagonalizing the 3×3 matrix $C^t C$ with the metric N_B : Setting $\mathbf{k} = N_B^{-1/2} \tilde{\mathbf{k}}$, with $\tilde{\mathbf{k}}^t \tilde{\mathbf{k}} = 1$, we have

$$\frac{\mathbf{k}^t C^t C \mathbf{k}}{\mathbf{k}^t N_B \mathbf{k}} = \tilde{\mathbf{k}}^t \tilde{C}^t \tilde{C} \tilde{\mathbf{k}} \leq \lambda_{\max}, \quad (52)$$

where $\tilde{C} = C N_B^{-1/2}$ and λ_{\max} is the maximum eigenvalue of $\tilde{C}^t \tilde{C}$, the maximum reached when $\tilde{\mathbf{k}}$ is the associated normalized eigenvector. The eigenvalue equation $\tilde{C}^t \tilde{C} \tilde{\mathbf{k}} = \lambda \tilde{\mathbf{k}}$ is just the eigenvalue equation for $C^t C$ with metric N_B ,

$$C^t C \mathbf{k} = \lambda N_B \mathbf{k}, \quad (53)$$

so that λ_{\max} is the largest root of the equation

$$\text{Det}[C^t C - \lambda N_B] = 0, \quad (54)$$

with \mathbf{k} the associated eigenvector. In other words, $\sqrt{\lambda_{\max}}$ is the maximum *singular value* of the matrix C with metric N_B . The ensuing minimum conditional entropy and maximum information gain (uncertainty reduction) for projective measurements are then

$$S_2(A|B) = \text{Min}_k S_2(A|B_k) = S_2(A) - \lambda_{\max}/d_A, \quad (55)$$

$$I_2(A|B) = \text{Max}_k I_2(A|B_k) = \lambda_{\max}/d_A. \quad (56)$$

If $\mathbf{r}_B = \mathbf{0}$, $N_B = I_3$ and λ_{\max} is just the maximum eigenvalue of $C^t C$. On the other hand, if $|\mathbf{r}_B| = 1$, ρ is a product state and $\mathbf{k}^t C^t C \mathbf{k} = 0$ vanishes $\forall \mathbf{k}$.

For instance, the classically correlated state (20) corresponds, choosing the z axis in B such that $\tilde{\Pi}_{\pm k}^B = \frac{1}{2}(I_B \pm \sigma_z)$, to $(\mathbf{r}_B)_v = \delta_{vz} r_B$, $J_{\mu\nu} = \delta_{vz} J_{\mu z}$, implying $(C^t C)_{v\nu'} = \delta_{v\nu'} \delta_{vz} |\mathbf{J} - r_B \mathbf{r}_A|^2$, with \mathbf{J} the vector of components $J_{\mu z}$. Hence,

$$\frac{\mathbf{k}^t C^t C \mathbf{k}}{\mathbf{k}^t N_B \mathbf{k}} \leq \lambda_{\max} = \frac{|\mathbf{J} - r_B \mathbf{r}_A|^2}{1 - r_B^2}, \quad (57)$$

being verified that the maximum is reached for \mathbf{k} along z , i.e., for a spin measurement along \mathbf{r}_B (basis of eigenstates of ρ_B). For a general state however, the minimizing direction may differ from \mathbf{r}_B and follow the main correlation in $C^t C$.

If A is also a qubit ($d_A = 2$), it is convenient to use $S_2(\rho) = 2(1 - \text{Tr} \rho^2)$ in previous equations, i.e. $\frac{1}{d_A} \rightarrow 1$ in equations (50)–(56), such that $S_2(\rho_A) = 1$ if ρ_A is maximally mixed. Such rescaled entropy is still a lower bound to the $a = 2$ von Neumann entropy $S(\rho) = -\text{Tr} \rho \log_2 \rho$ (see appendix A). In such a case, if ρ is of rank 2, it can be purified by adding a third qubit C , being then verified that $S_2(A|B)$ coincides with the *squared concurrence* [47] between A and C , since such quantity reduces for pure two-qubit states to the present rescaled S_2 entropy of any of the subsystems, and coincides with its convex roof extension $E_2(A, C)$ for mixed two qubit states [41].

We remark finally that for a qudit–qubit state, the (minimum) geometric discord (38) is determined by the largest eigenvalue of a different 3×3 matrix [8, 16]:

$$D_2(A|B) = \frac{1}{d} (|\mathbf{r}_B|^2 + \|\mathbf{J}\|^2 - \tilde{\lambda}_{\max}), \quad (58)$$

where $\tilde{\lambda}_{\max}$ is the largest eigenvalue of $M_2 = \mathbf{r}_B \mathbf{r}_B^t + J^t J$. This matrix depends then on J rather than the correlation C , coinciding with $C^t C$ just when $r_B = 0$.

4. Application

4.1. X states

Let us now consider a two-qubit system. Through its singular value decomposition, the now 3×3 matrix J can be always brought to the diagonal form $J_{\mu\nu} = \delta_{\mu\nu}J_\mu$ by appropriately choosing the local x, y, z axes. If \mathbf{r}_A and \mathbf{r}_B are directed along the same principal axes of J , which we shall denote as z , we obtain an X state [30],

$$\rho = \frac{1}{4} \left(I + r_A \sigma_z \otimes I_2 + r_B I_2 \otimes \sigma_z + \sum_{\mu=x,y,z} J_\mu \sigma_\mu \otimes \sigma_\mu \right) \quad (59)$$

$$= \begin{pmatrix} p_+ & 0 & 0 & \alpha_- \\ 0 & q_+ & \alpha_+ & 0 \\ 0 & \alpha_+ & q_- & 0 \\ \alpha_- & 0 & 0 & p_- \end{pmatrix}, \quad \begin{aligned} p_\pm &= \frac{1 \pm (r_A + r_B) + J_z}{4} \\ q_\pm &= \frac{1 \pm (r_A - r_B) - J_z}{4} \\ \alpha_\pm &= \frac{J_x \pm J_y}{4} \end{aligned} \quad (60)$$

where equation (60) is its standard basis representation. This state commutes with the z parity $P_z = \sigma_z \otimes \sigma_z$. Accordingly, reduced states of arbitrary spin pairs in the thermal state or in any non-degenerate eigenstate of any spin 1/2 array with XY or XYZ Heisenberg couplings of arbitrary range in a field along z , are of the present form [31], as the corresponding Hamiltonian (see equation (71)) commutes with the total z parity.

The ensuing matrices C and N_B are simultaneously diagonal,

$$C = \begin{pmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z - r_A r_B \end{pmatrix}, \quad N_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & r_B^2 \end{pmatrix}.$$

Hence, the minimum conditional entropy $S_2(A|B)$ among projective measurements will be obtained for a measurement along one of the principal axes x, y, z . We then obtain

$$S_2(A|B) = 1 - |\mathbf{r}_A|^2 - I_2(A|B), \quad (61)$$

$$I_2(A|B) = \text{Max}_k \frac{\mathbf{k}^t C^t C \mathbf{k}}{\mathbf{k}^t N_B \mathbf{k}} = \text{Max} \left[J_x^2, J_y^2, \frac{(J_z - r_A r_B)^2}{1 - r_B^2} \right], \quad (62)$$

for $S_2(\rho) = 2(1 - \text{Tr}\rho^2)$, implying a $z \rightarrow x$ or $z \rightarrow y$ transition in the direction of the minimizing measurement as J_x^2 or J_y^2 increase across $\lambda_z = (J_z - r_A r_B)^2 / (1 - r_B^2)$.

Such direction is then determined essentially by the main correlation in $C^t C$. This provides a conceptual basis for the results of [33] related with the minimizing measurement of the *quantum discord* for X states, which also follow the main correlation. This direction can then differ significantly from that minimizing the geometric discord (38)–(58). For the state (59), we obtain [16, 17] (equation (58))

$$D_2(A|B) = \frac{1}{2} \{ r_B^2 + \|\mathbf{J}\|^2 - \text{Max}[J_x^2, J_y^2, J_z^2 + r_B^2] \}, \quad (63)$$

entailing a $z \rightarrow x$ or $z \rightarrow y$ transition only as J_x^2 or J_y^2 increase across $J_z^2 + r_B^2$. Coincidence between both minimizing measurements can then be ensured just for $r_B = 0$, i.e., ρ_B maximally mixed, where the minimizing \mathbf{k} is along the axis with the largest $|J_\mu|$ for both $S_2(A|B)$ and $D_2(A|B)$.

For a general entropy S_f , the conditional entropy is (equation (45)),

$$S_f(A|B_k) = \sum_{\mu, \nu = \pm 1} \frac{1 + \nu r_B \cdot \mathbf{k}}{2} f \left[\frac{1}{2} \left(1 + \mu \left| \mathbf{r}_A + \frac{\nu C \mathbf{k}}{1 + \nu r_B \cdot \mathbf{k}} \right| \right) \right]. \quad (64)$$

It is verified that for an X state, measurements along any of the principal axes of J (i.e., x, y, z) are always *stationary* ($\delta S_f(A|B_{\mathbf{k}}) = 0$ up to first order in $\delta \mathbf{k}$), i.e., candidates for minimizing (64), although other directions cannot be discarded (typically in the transitional region between the z and x or y regimes). On the other hand, for *two qubit states with maximally mixed marginals*, which can be written as X states with $r_A = r_B = 0$, it is seen from (62) and (64) that the minimizing measurement is *along the axis with the largest $|J_\mu|$* , i.e., \mathbf{k} along the largest eigenvalue of $J^t J = C^t C$, for any entropy S_f (universal minimum).

We finally mention that the geometric discord $D_2(A|B)$ was shown in [48] to be an upper bound to the square of the negativity $\mathcal{N}(\rho)$, a computable entanglement monotone [49], given for two qubits by $\mathcal{N}(\rho) = \text{Tr}|\rho^{tB}| - 1$, with ρ^{tB} the partial transpose, both coinciding for ρ pure. For X states, we obtain here a similar relation between $I_2(A|B)$ and the squared concurrence $\mathcal{C}^2(\rho)$, with both also coinciding when ρ is pure. For the state (60), the concurrence [47] is $\mathcal{C}(\rho) = 2\text{Max}[|\alpha_+| - \sqrt{p_+p_-}, |\alpha_-| - \sqrt{q_+q_-}, 0]$, implying $\mathcal{C}(\rho) \leq 2\text{Max}[|\alpha_+|, |\alpha_-|]$ and hence, since $|\alpha_\pm| \leq \text{Max}[|J_x|, |J_y|]/2$,

$$\mathcal{C}^2(\rho) \leq \text{Max}[J_x^2, J_y^2] \leq I_2(A|B). \tag{65}$$

4.2. Mixture of a pure state with the maximally mixed state

As a specific example of (59), we consider the mixture (25) in the two qubit case. By suitable choosing the local axes, we may always write it as

$$\rho = w|\Psi\rangle\langle\Psi| + (1-w)I_4/4, \quad |\Psi\rangle = \sqrt{q}|00\rangle + \sqrt{1-q}|11\rangle, \tag{66}$$

which corresponds to an X state with

$$r_A = r_B = w(2q-1), \quad J_x = -J_y = 2w\sqrt{q(1-q)}, \quad J_z = w.$$

It is then verified that $\frac{(J_z - r_A r_B)^2}{1 - r_B^2} - J_x^2 = \frac{w^2(1-w)^2(1-2q)^2}{1-w^2(1-2q)^2} \geq 0$, implying that $S_2(A|B_{\mathbf{k}})$ is minimized by a measurement along z (basis of eigenstates of ρ_B), in agreement with the universal minimum for this state. It is also seen that for $w = 1$ (ρ pure), $w = 0$ (ρ maximally mixed) or $q = 1/2$ ($|\Psi\rangle$ maximally entangled) the previous difference vanishes, indicating that all directions \mathbf{k} lead to the same result, in agreement with previous considerations. In any case we obtain, for $S_2(\rho) = 2(1 - \text{Tr} \rho^2)$,

$$S_2(A|B) = \frac{(1-w)(1+w-2w^2(1-2q)^2)}{1-w^2(1-2q)^2}, \tag{67}$$

$$I_2(A|B) = \frac{w^2(1-w(1-2q)^2)^2}{1-w^2(1-2q)^2}, \tag{68}$$

with $S_2(A) = 1 - w^2(1-2q)^2$. It is verified that equation (67) is a strictly concave decreasing function of w at fixed $q \in [0, 1]$, and a strictly concave function of q if $w \in (0, 1)$, reaching its maximum at $q = 1/2$ (Bell state). Notice that $(1-2q)^2 = 1 - \mathcal{C}^2(|\Psi\rangle)$, with $\mathcal{C}(|\Psi\rangle) = 2\sqrt{q(1-q)}$ the *concurrence* [47] of $|\Psi\rangle$, so that equation (67) is, for $w \in (0, 1)$, an increasing function of $\mathcal{C}(|\Psi\rangle)$, i.e. of entanglement, as previously ascertained. The bound (65) is also verified ($\mathcal{C}(\rho) = \text{Max}[w\mathcal{C}(|\Psi\rangle) - (1-w)/2, 0]$).

Equation (68) is also a strictly concave function of q if $w \in (0, 1]$, maximum at $q = 1/2$, i.e., an *increasing function* of the concurrence $\mathcal{C}(|\Psi\rangle)$. In contrast, equation (68) is not necessarily an increasing function of w . Its behavior with w can be non-monotonous if $|\Psi\rangle$ is separable or almost separable (q small or close to 1), as shown in figure 1, where results for the von Neumann based ($S(\rho) = -\text{Tr} \rho \log_2 \rho$) conditional entropy and information gain are also depicted. Such behavior is universal, i.e., present for any S_f : when $|\Psi\rangle$ is

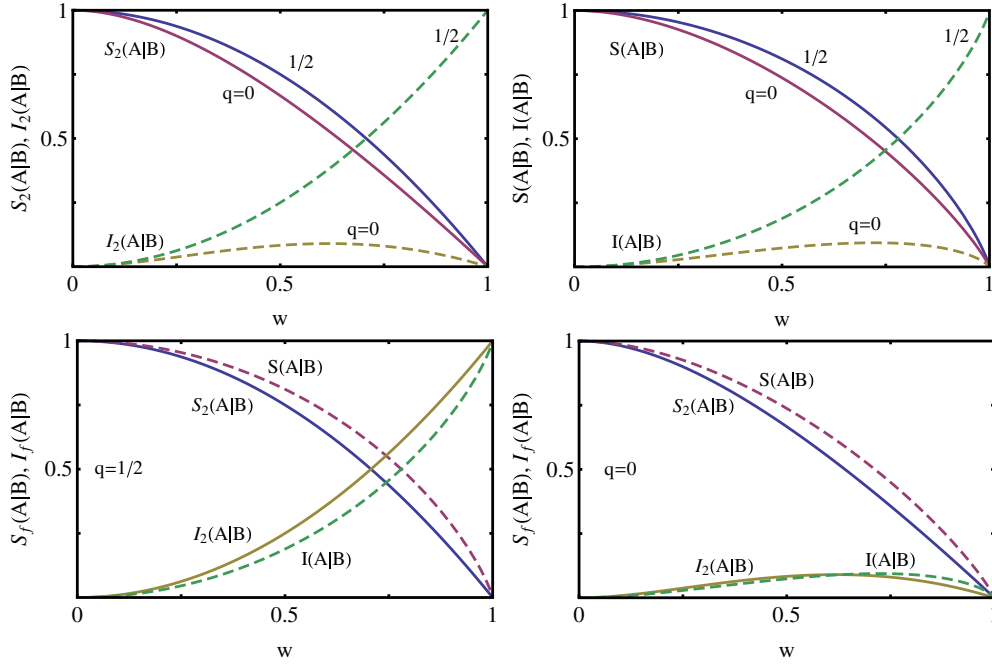


Figure 1. (Top) Results for the quadratic (left) and von Neumann (right) minimum conditional entropy $S_f(A|B)$ (solid lines) and maximum information gain (or uncertainty reduction) $I_f(A|B)$ (dashed lines), after a measurement at B in the mixture (66) for the maximally entangled ($q = 1/2$) and separable ($q = 0$) cases. All $S_f(A|B)$ are concave decreasing functions of w , vanishing at the pure limit $w = 1$. (Bottom) Comparison between quadratic (solid lines) and von Neumann (dashed lines) results for $q = 1/2$ (left) and $q = 0$ (right). It is verified that $S_2(A|B) \leq S(A|B) \forall w, q$.

separable, noise induces a non-zero value of $I_f(A|B)$, since ρ ceases to be a product state for $w \in (0, 1)$. As seen in figure 1, the qualitative behavior of the minimum linear and von Neumann conditional entropies is entirely similar, and the same holds for the ensuing maximum $I_f(A|B)$. Nonetheless, while $S_2(A|B) \leq S(A|B)$, there is in general no fixed order relation between $I_2(A|B)$ and $I(A|B)$.

4.3. Mixture of aligned states

We now consider the two-qubit mixed state

$$\rho = \frac{1}{2}(|\theta\theta\rangle\langle\theta\theta| + |-\theta\theta\rangle\langle-\theta-\theta|), \tag{69}$$

where $|\theta\rangle = \exp[-i\theta\sigma_y/2]|0\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle$ is the state with the spin forming an angle θ with the z axis. This separable state represents, roughly, the reduced state of a spin $1/2$ pair in the exact definite parity ground state of a ferromagnetic XY chain for fields $|B| < B_c$ if $\cos\theta = B/B_c$ [31]. Moreover, for not too small chains it is the *exact* state of the pair in the immediate vicinity of the factorizing field [31, 50, 51]. Equation (69) is an X state with

$$r_A = r_B = \cos\theta, \quad J_z = \cos^2\theta, \quad J_x = \sin^2\theta, \quad J_y = 0.$$

Hence, there is no correlation along z ($J_z = r_A r_B$, implying $C_z = 0$) but there is a finite correlation along x ($C_x = J_x^2$). We then obtain the remarkable result that $S_2(A|B_k)$ is *minimized* for k along $x \forall \theta \in (0, \pi/2]$, leading to

$$S_2(A|B) = 1 - \cos^2\theta - \sin^4\theta = \frac{1}{4}\sin^2 2\theta, \quad I_2(A|B) = \sin^4\theta. \tag{70}$$

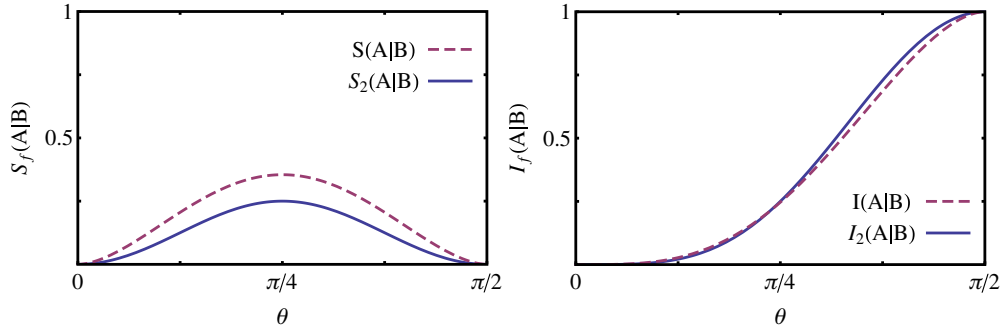


Figure 2. Results for the quadratic (solid lines) and von Neumann (dashed lines) minimum conditional entropy (left) and maximum information gain or uncertainty reduction (right) in the mixture of aligned states (69). Both entropies are minimized by a spin measurement along $x \forall \theta \in (0, \pi/2]$.

The minimum S_2 conditional entropy is then symmetric around $\theta = \pi/4$, vanishing for $\theta = 0$ (product state) and $\pi/2$ (classically correlated state of the form (20) with $\rho_{A/k}$ pure), whereas the maximum $I_2(A|B)$ increases with θ (figure 2), reaching its absolute maximum at $\theta = \pi/2$. Hence, spin measurements along z are not minimum for *any* $\theta > 0$ (although the difference with (70) is $O(\theta^4)$ for $\theta \rightarrow 0$).

In the von Neumann case, the behavior of $S(A|B)$ and $I(A|B)$ is again completely similar to that of $S_2(A|B)$ and $I_2(A|B)$, as seen in figure 2. Moreover, the minimizing measurement is also for k along $x \forall \theta \in (0, \pi/2]$ [17, 31], i.e., *the same as that of the S_2 entropy $\forall \theta$* . The S_2 results allow then to easily understand the minimizing measurement of the quantum discord for this state [31]. In contrast, the geometric discord is minimized for k along x *only if* $\theta > \theta_c$, with $\cos^2 \theta_c = 1/3$, preferring k along z if $\theta < \theta_c$ [17].

4.4. Spin 1/2 pairs in XY chains at strong transverse fields

Let us finally consider a spin 1/2 array with XY couplings in a strong transverse field, described by a Hamiltonian

$$H = -B \sum_i \sigma_{iz} - \sum_{i < j} (J_{ij}^x \sigma_{ix} \sigma_{jx} + J_{ij}^y \sigma_{iy} \sigma_{jy}). \quad (71)$$

For sufficiently strong fields $B \gg |J_{ij}^\mu| \forall \mu, i, j$, the system is weakly coupled and the ground state is of the form

$$|\Psi\rangle \approx |0\rangle + \sum_{i < j} \alpha_{ij} |ij\rangle, \quad (72)$$

at the lowest non-trivial order, where $|0\rangle = |0 \dots 0\rangle$ denotes the state with all spins aligned along the field ($+z$), $|ij\rangle = \sigma_{i-} \sigma_{j-} |0\rangle$ and $\alpha_{ij} \approx (J_{ij}^x - J_{ij}^y)/(2B)$. The reduced state of a pair i, j is therefore an X state with, at the lowest non-zero order (we set $\alpha_{ji} = \alpha_{ij}$),

$$\alpha_- = \alpha_{ij}, \quad p_- = |\alpha_{ij}|^2, \quad \alpha_+ = \sum_{k \neq i, j} \alpha_{ik} \bar{\alpha}_{kj}, \quad q_{\pm} = \sum_{k \neq i, j} |\alpha_{ik}|^2. \quad (73)$$

By suitably choosing the local states at sites i, j , we may set α_{\pm} real and positive. Hence, up to $O(|\alpha|^2)$ we obtain $r_{A,B} = 1 - 2(|\alpha_{ij}|^2 + q_{\mp})$ (along z) and

$$J_y = 2 \left(\sum_{k \neq i, j} \alpha_{ik} \bar{\alpha}_{kj} \pm \alpha_{ij} \right), \quad J_z - r_A r_B \approx 4|\alpha_{ij}|^2. \quad (74)$$

Hence, for $\alpha_{ij} \neq 0$ (interacting pair), C_{xx} is $O(\alpha_{ij})$, whereas C_{zz} is $O(\alpha_{ij}^2)$, entailing at the lowest order a *minimizing measurement along x* instead of z , as the correlation along z is of higher order. *The same behavior* occurs with the minimizing measurement of the von Neumann conditional entropy and hence the quantum discord in this regime (k along x at strong fields [31, 52]). In contrast, that minimizing the geometric discord or the information deficit [9, 17] follows the main component of the state, and is therefore along the field direction z for strong fields [52].

5. Conclusions

We have analyzed the main features of the conditional entropy associated to general concave entropic forms in bipartite quantum systems, determined by a measurement in one of the constituents. Its minimum among all local measurements determines the maximum average uncertainty reduction (generalized information gain) about A that can be achieved by a measurement on B , and has the direct meaning of representing the associated entanglement of formation between A and a purifying third system C . For some important classes of states as those of sections 2.3 and 2.4, the minimizing measurement is *the same* for all S_f and can be analytically identified, allowing a direct general evaluation of $E_f(A, C)$. This universality indicates that for such states there is clearly an unambiguous optimum local measurement leading to the lowest conditional mixedness at the unmeasured part, irrespective of the measure used for quantifying such mixedness.

For the general case, a main practical result of our manuscript is the analytic determination of this minimum for the linear entropy S_2 in a general qudit+qubit state with projective measurements. It can be expressed in terms of the largest eigenvalue of a simple 3×3 matrix, which represents the largest singular value of the correlation matrix C with a metric N_B determined by the measured part. This enables us to easily identify the minimizing measurement, determined by the associated eigenvector, and understand its behavior. Conditional S_2 results have also a direct interpretation in terms of purity and average distances, and possess the importance of determining the universal behavior of *all* conditional entropies and the ensuing minimizing measurement in the vicinity of maximum mixedness.

In the specific examples considered, the minimizing measurements of the S_2 and von Neumann conditional entropies (and hence the quantum discord) were in fact coincident. The present results explain then the quite distinct response of this minimizing measurement to the onset of correlations (it follows the main correlation even if arbitrarily weak), in comparison with those minimizing the geometric discord or the one way information deficit, which follow instead the main component of the state [52]. Hence, the present formalism not only allows us to identify universal features and optimize post-measurement purities, but can also help to evaluate or estimate the quantum discord in more complex situations, as the minimizing measurements for the linear and von Neumann conditional entropies become coincident in some states and regimes, and can be expected to be close in typical situations.

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Appendix A. Trace form generalized entropies

Given a quantum state ρ with spectral decomposition $\rho = \sum_j p_j |j\rangle\langle j|$, $j = 1, \dots, d$ ($p_j \geq 0$, $\sum_j p_j = 1$), the ‘entropic’ forms (see for instance [12, 35])

$$S_f(\rho) = \text{Tr } f(\rho) = \sum_j f(p_j), \quad (\text{A.1})$$

comply, for any strictly concave real function $f : [0, 1] \rightarrow \Re$ satisfying $f(0) = f(1) = 0$, with all conventional entropy properties except additivity: i) $S_f(\rho) \geq 0$, with $S_f(\rho) = 0$ iff ρ is pure ($\rho^2 = \rho$), ii) $S_f(\rho)$ is maximum at the maximally mixed state I_d/d , with $S_f(I_d/d) = df(1/d)$ an increasing function of d , iii) $S_f(U\rho U^\dagger) = S_f(\rho) \forall$ unitary U and iv) $S_f(\rho)$ is concave (equation (5)) (if $\rho = \sum_\alpha q_\alpha \rho_\alpha$, $f(p_j) = f(\sum_{\alpha,j'} q_\alpha |\langle j|j'_\alpha\rangle|^2 p_{j'}^\alpha) \geq \sum_{j',\alpha} q_\alpha |\langle j|j'_\alpha\rangle|^2 f(p_{j'}^\alpha)$, which leads to (5) after summing over j).

Concavity implies ii) and, moreover, the majorization [12, 53] property [17, 54]

$$\rho \prec \rho' \Rightarrow S_f(\rho) \geq S_f(\rho'), \tag{A.2}$$

where $\rho \prec \rho'$ (ρ more mixed than ρ') means $\sum_{j=1}^i p_j \leq \sum_{j=1}^i p'_j$ for $i = 1, \dots, d-1$, with p_j, p'_j denoting here the eigenvalues of ρ and ρ' sorted in *decreasing* order (and completed with 0's if dimensions differ). Equation (A.2) provides the conceptual basis for considering any such S_f a generalized uncertainty measure or entropic form. Furthermore, while the converse of (A.2) does not necessarily hold if valid for some particular S_f (majorization is stronger than a single entropic inequality), it *does hold* if valid $\forall S_f$ of the form (A.1): $S_f(\rho) \geq S_f(\rho') \forall S_f \Rightarrow \rho \prec \rho'$ [54].

The Tsallis entropy [44] $S_q(\rho) = (1 - \text{Tr} \rho^q)/(q - 1)$, $q > 0$, corresponds to $f(\rho) = (\rho - \rho^q)/(q - 1)$ in (A.1). It reduces to the quadratic entropy (29) for $q = 2$ and to the von Neumann entropy (with $a = e$) for $q \rightarrow 1$. We may also set $S_q(\rho) = (1 - \text{Tr} \rho^q)/(1 - 2^{1-q})$, such that $S_q(\rho) = 1$ for a maximally mixed single qubit state $\rho = I_2/2$, in which case $S_2(\rho) = 2(1 - \text{Tr} \rho^2)$ and $S_q(\rho) \rightarrow -\text{Tr} \rho \log_2 \rho$ for $q \rightarrow 1$. For this scaling it is still verified that $S_2(\rho) \leq S(\rho)$ for any single qubit state, coinciding just for ρ pure or maximally mixed (for any single qubit state, $S_q(\rho) \leq S(\rho)$ for $1 < q < q_1 \approx 4.718$ with this scaling).

For two *classical* random variables A, B described by a joint probability distribution $p_{ij} = p(A = i, B = j)$, we may define a generalized conditional entropy $S_f(A|B)$ as

$$S_f(A|B) = \sum_j p_j S_f(A|B = j) = \sum_{i,j} p_{ij} f(p_{ij}/p_j), \tag{A.3}$$

where $p_j = p(B = j) = \sum_i p_{ij}$. This quantity measures the average uncertainty about A if B is known. Due to concavity, it satisfies $S_f(A|B) \leq S_f(A) = \sum_i f(q_i)$ (with $q_i = p(A = i) = \sum_j p_{ij}$) $\forall S_f$. The difference

$$I_f(A|B) = S_f(A) - S_f(A|B),$$

is then non-negative, vanishing only if $p_{ij}/p_j = p_i \forall i, j$ with $p_j > 0$, i.e., only if A and B are independent. It represents the uncertainty reduction (or generalized ‘information gain’) about A generated by the knowledge of B .

In the Shannon case $f(p) = -p \log_a p$, (A.3) becomes $S(A|B) = S(A, B) - S(B)$, where $S(A, B) = -\sum_{i,j} p_{ij} \log_a p_{ij}$, $S(B) = -\sum_j p_j \log_a p_j$, but such relation no longer holds for a general S_f . Hence, while in the (classical) Shannon case $I(A|B) = S(A) + S(B) - S(A, B) = I(B|A)$ is the mutual information, for a general S_f , $I_f(A|B)$ will differ in general from $I_f(B|A)$. Generalizations of the Shannon conditional entropy based on the Renyi entropy were recently discussed in [55] (and quantum versions in [36, 37]), whereas special extensions for the Tsallis case were considered in [38].

Appendix B. Relation with the entanglement of formation

Let us sketch the proof of the identity (17) [23–26]. Starting from the (AC, B) Schmidt decomposition of the pure global state,

$$|\Psi_{ACB}\rangle = \sum_{k=1}^n \sqrt{q_k} |\tilde{k}_{AC}\rangle |\tilde{k}_B\rangle, \tag{B.1}$$

the state of AC after a measurement at B based on the operators (11) with outcome j is the pure state (equation (13))

$$|j_{AC}\rangle = (r_j/p_j)^{1/2} \sum_k \sqrt{q_k} \langle j_B | \tilde{k}_B \rangle |\tilde{k}_{AC}\rangle. \quad (\text{B.2})$$

Hence, ρ_{A/Π_j} is the reduced state ρ_A^j of A in $|j_{AC}\rangle$ and $S_f(A|B_{\{\Pi_j\}}) = \sum_j p_j S_f(\rho_A^j)$ coincides then with the average entanglement of the decomposition $\rho_{AC} = \sum_j p_j \rho_{AC/\Pi_j}$, where $\rho_{AC/\Pi_j} = |j_{AC}\rangle \langle j_{AC}|$. Conversely, equation (B.1) implies that the states $|j_{AC}\rangle$ in any decomposition $\rho_{AC} = \sum_j p_j |j_{AC}\rangle \langle j_{AC}|$ (with $p_j > 0$) should satisfy

$$\sqrt{p_j} |j_{AC}\rangle = \sum_k U_{jk} \sqrt{q_k} |\tilde{k}_{AC}\rangle, \quad (\text{B.3})$$

where U is an $m \times n$ matrix with orthonormal columns ($\sum_j U_{jk}^* U_{jk'} = \delta_{kk'}$) and $m \geq n$. Comparison with equation (B.2) indicates that we may identify such decomposition with that for a local measurement at B with the operators (11), provided

$$\sqrt{r_j} |j_B\rangle = \sum_k U_{jk}^* |\tilde{k}_B\rangle, \quad (\text{B.4})$$

such that $U_{jk} = \sqrt{r_j} \langle j_B | \tilde{k}_B \rangle$. The ensuing operators $\Pi_j^B = r_j |j_B\rangle \langle j_B|$ form a valid POVM since $\sum_j \Pi_j^B = \sum_{j,k,k'} U_{jk}^* U_{jk'} |\tilde{k}_B\rangle \langle \tilde{k}_B| = \sum_k |\tilde{k}_B\rangle \langle \tilde{k}_B| = I_B$ (assuming $n = d_B$).

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