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ON CONFIDENCE BANDS FOR TIME SERIES PROBLEMS IN THE TIME AND FREQUENCY DOMAINS

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Summary

The construction of (asymptotic) simultaneous confidence bands for some time series problems is studied, typically for the sample autocorrelogram and windowed spectral density estimate. The following approaches are explored: (1) To use the close-form results available in the literature; (2) To use the asymptotic independence of the sample quantities to derive new procedures; (3) To resort to inequalities.

As expected, the bands turn out being wider than those frequently encountered in the literature, based on point-by-point confidence intervals. Numerical values of the necessary constants are given for selected values of the joint confidence coefficient and various numbers of sample quantities.

The use of confidence sets of non-uniform width is also briefly explored.

Monte Carlo simulations are presented, for problems in the time and in the frequency domains.

Key words. Simultaneous confidence bands. Autocorrelogram. Windowed spectral density estimate. Asymptotic independence. Bonferroni inequality

1. Introduction

Given an observed time series $y_1,\ldots,y_T$, two useful data-analytic techniques are to compute, plot and interpret the sample autocorrelogram in the time domain, or a (windowed) spectral density estimate in the frequency domain. These are just two instances among several sample quantities often considered: in the time domain, the sample partial, inverse, and partial inverse autocorrelograms for univariate series, and the cross correlogram for bivariate series; in the frequency domain, the sample coherence and phase for bivariate time series.

From an empirical point of view, and in agreement with what the theoretical sampling properties indicate, it is often the case that the sample autocorrelogram and sample spectral density tend to exhibit fluctuations that must be accurately interpreted. The question then arises as how to set "control lines" or "confidence bands" on these sample functions.

Setting control lines in the sample autocorrelogram has been defined to mean that (asymptotic) point-by-point confidence intervals are plotted for the whole set of sample estimates. In a simple case, straight control lines at $\pm c\hat{\sigma}$ have been suggested by Box and Jenkins (1976, page 185), and used by many authors and practitioners. Here c is a constant (often taken to be 1 or 2) and $\hat{\sigma}^2$ is an estimate of the residual variance of an underlying linear model.

Similarly, control lines for spectral density estimates have been often recommended and used in the literature: see, for example, Granger and Hatanaka (1984, page 66). The technique used here is also to derive an (asymptotic) confidence interval for one ordinate and to use it for the whole sample function.

The approach of control lines has the advantage of its simplicity, but has the obvious shortcoming of using a point-by-point inference tool to make inferences about the whole set of values under consideration. This means that control of the confidence coefficient is lost. To avoid this difficulty a joint or simultaneous confidence approach is need, and that will be discussed below.

In this paper and in terms of frequency analysis, we consider estimation of the spectral density function by means of estimators related to the sample spectral density or *periodogram*. We consider smoothing the periodogram by using adequate

"windows" (that is, systems of weights) defined in the time or in the frequency domain.

An alternative approach to estimate the spectral density of a stationary time series, is to use an *autoregressive estimator,* defined as follows: the given series is approximated by a finite-order autoregression, denoted by $AR(\hat{p})$, whose order is suitably estimate. Then all parameters are estimated, and the resulting values are "plugged-in" the formula of the spectral density of the $AR(\hat{p})$ model. The resulting estimator possesses some good properties, and is, in general, quite smooth, in comparison with the sample periodogram.

This approach is closely related to what is called the *maximum entropy spectral estimator*. On these topics, see, among others, Akaike (1969), Parzen (1974), Priestley (1981), Beamish and Priestley (1981), Newton and Pagano (1984), Koslov and Jones (1985) , Sakai and Sakaguchi (1990), Hrafnkelsson and Newton (2000).

In Section 2 we review point-by-point confidence intervals; in Section 3 we consider simultaneous confidence sets; in Section 4 we present some simple or elementary approaches to solve or approximate the simultaneous inference problem. Sections 5 and 6 contain simulation results, and Section 7 is of discussion and conclusions.

2. Basic Definitions and Point-by-point Confidence Bands

According to one standard definition, the sample autocorrelogram is the set of sample quantities

$$r_s = \frac{\sum_{t=1}^{T-s}(y_t - \bar{y})(y_{t+s} - \bar{y})}{\sum_{t=1}^{T}(y_t - \bar{y})^2}, \quad s = 1,2,\ldots,m, \tag{2.1}$$

where m is usually taken to be considerably smaller than T. Other definitions of the sample autocorrelations are considered in Mentz (1983a) in connection with their roles in the sample autocorrelogram.

The sampling properties of (2.1) are usually studied in terms of asymptotic results, when $T \to \infty$ while m remains fixed. If the underlying stochastic process is

stationary and linear ($y_t = m + \sum_{j=-\infty}^{\infty} g_j u_{t-j}$, where the innovations u_t are independent, identically distributed, with 0 expected value and finite fourth-order moment, and $\sum |g_j| < \infty$) it can be shown that $r_1, r_2, \ldots, r_m$ is asymptotically normal around the true parameter $r_1, r_2, \ldots, r_m$, with variances and covariances given by

$$t_{gh} = \frac{1}{T} \sum_{r=-\infty}^{\infty} \left(r_{r+g} r_{r+h} + r_{r-g} r_{r+h} - 2 r_h r_r r_{r+g} - 2 r_g r_r r_{r+h} + 2 r_g r_h r_r^2 \right), \qquad g, h = 1, 2, \ldots, m \, . \tag{2.2}$$

See, for example, Anderson (1971). Whenever (2.2) can be evaluated numerically (i.e., whenever the r_j can be expressed as functions of some set of parameters) we have a way to solve problems of inference, in the sense of approximating the needed distributions by their asymptotic normal limits. In particular, a confidence interval for r_s is then

$$r_s \pm c \, t_{ss}^{1/2}, \quad s = 1, 2, \ldots, m, \tag{2.3}$$

where c is chosen from standard normal tables to give the desired confidence coefficient.

One standard definition of the (windowed) spectral density estimator is

$$\hat{f}\left(v_j\right) = \frac{1}{2p} \sum_{s=-m}^{m} k\left(\frac{s}{m}\right) \cos\left(v_j s\right) c_s , \quad j = 0, 1 \ldots, \ell, \tag{2.4}$$

where $c_s = T^{-1} \sum_{t=1}^{T-s} (y_t - \bar{y})(y_{t+s} - \bar{y}) = c_{-s}$, and k is a "kernel function": it is normalized, bounded, symmetric about 0, and sufficiently smooth (see, for example, Anderson, 1971, Section 9.4). Note that in (2.1) $r_s = c_s / c_0$.

The sampling properties of (2.4) are usually studied in terms of asymptotic results, when $T \to \infty$ and $m = m_T \to \infty$ in such a way that $m_T^d / T \to 0$ for some suitable *d*. If. The underlying stochastic process is stationary and linear (see above), it can be shown that $\hat{f}\left(v_0\right), \hat{f}\left(v_1\right), \ldots, \hat{f}\left(v_\ell\right)$ is asymptotically normal around the spectral ordinates $f\left(v_0\right), f\left(v_1\right), \ldots, f\left(v_\ell\right)$ with variances

$$t^2(v_j) = \frac{m}{T} 2 f^2(0) \int_{-1}^{1} k^2(x)dx, \quad v_j = 0\,, \tag{2.5}$$

$$= \frac{m}{T} 2 f^2(p) \int_{-1}^{1} k^2(x)dx, \quad v_j = \pm p\ , \tag{2.6}$$

$$= \frac{m}{T} f^2(v_j) \int_{-1}^{1} k^2(x)dx, \quad v_j \neq 0, \pm p\ , \qquad `(2.7)$$

and covariances equal to 0.

Since these integrals are known for the standard windows proposed in the literature, we have a way to construct confidence intervals at each frequency. One such form is

$$\frac{\hat{f}(v_j)}{1 + tc\sqrt{m/T}} \leq f(v_j) \leq \frac{\hat{f}(v_j)}{1 - tc\sqrt{m/T}}, \quad v_j \neq 0, \pm p\ , \tag{2.8}$$

where $t^2 = \int_{-1}^{1} k^2(x)dx$, and c is chosen from standard normal tables. It can also be shown that to the transformation $\ln f(v_j)$ confidence intervals can be set by

$$\ln \hat{f}(v_j) - tc\sqrt{\frac{m}{T}} \leq \ln f(v_j) \leq \ln \hat{f}(v_j) + tc\sqrt{\frac{m}{T}}\,. \tag{2.9}$$

Two important differences between the two cases that we considered in this section are the following: (a) Asymptotic theories are developed in such a way that the final results are comparable, but while in (2.1) m is treated as fixed when $T \to \infty$, in (2.4) $m = m_T \to \infty$ with *T*; (b) The asymptotic covariances for (2.4) are 0, while (2.2) is not necessarily equal to 0 when $g^{1}h$.

In fact, it is well known that sample spectral density or periodogram obtained from (2.4) by setting $k \cong 1$ and m=T-1 is such that for independent normal $y_1, \ldots, y_T$ the periodogram ordinates at different frequencies are independent for a finite sample size *T*.

3. Simultaneous Inference

The comments in the last part of the preceding section about the asymptotic uncorrelatedness of spectral density estimators at different frequencies contribute to explain why detailed (asymptotic) results for simultaneous inference are available only for the frequency domain. An early contribution is by Walker (1967).

Woodroofe and Van Ness (1967) proved that under certain conditions on the underlying process and the kernel function, the asymptotic distribution of the (normalized) maximum of the windowed spectral density estimator's ordinates can be found in an explicit form useful for statistical inference. Their main result can be written as

$$\lim_{T\to\infty} P\left[\max_{0\le s\le m} \frac{\left|\hat{f}\left(\frac{ps}{m}\right) - f\left(\frac{ps}{m}\right)\right|}{f\left(\frac{ps}{m}\right)} \le ax + b\right] = \exp\left(-e^{-x}\right), \quad (3.1)$$

where $a = [2\ln(2m)]^{-1/2}$, $b = a^{-1} - \frac{1}{2}a[\ln \ln(2m) + \ln 2p]$.

Some remarks about this result follow.

a) ***Assumptions of the theorem***. It is assumed that the underlying process is linear and the innovations have finite eighth-order moment. Some other regularity conditions are set on the coefficients of the linear representation, on the spectral density, and on the kernel function.

b) ***Nature of the result***. Expression (3.1) shows that a key consequence of this approach is that the limiting extreme-value distribution function $\exp[-\exp(-x)]$ is used to determine the constant in the confidence set, instead of the standard normal distribution function of our Section 2. Tables of significance points of this extreme-value distribution are given by Owen (1962).

c) ***Asymptotic order of the result***. Woodroofe and Van Ness (page 1558) indicate that "the difference between the maximum deviation and the deviation at a single frequency point ... manifests itself in the factor $(ln\, m)^{-1/2}$. Thus in practice a confidence band for all frequencies is $O(\ln m)^{1/2}$ times that for a finite set". This observation has also been recorded by Priestley (1981, page 486).

It should be noted that the comment refers to the "asymptotic order" or the result, and should not be interpreted to mean that, for example, in (2.8) c should be

replaced by $c(\ln m)^{1/2}$. In effect, even when the order of magnitude is correct in an asymptotic sense, the constant for a finite set of observed values may have to be altered. See Appendix 1.

d) ***Usefulness of the result***. Referring to this result, Hannan (1970), page 294, wrote:

> "These results are important in principle but surrounded by some doubt in practice, of a greater magnitude than that accorded to results of earlier sections, because of their asymptotic nature. It is known that such extreme value formulas are relevant only in enormously large samples, when the largest of a series of independent and identically distributed random variables is under consideration. Here further approximations are involved and for the relevance of the formulas it is evidently m as much as T whose magnitude is of importance. One conjectures that the formulas are the roughest of approximations only".

e) ***Other problems in the frequency domain***. Hannan (1970), page 294, notes that the approach of Woodroofe and Van Ness can be used to derive simultaneous probability statements analogous to (3.1) for sample coherence and phase in multiple time series analysis.

All these results and remarks correspond to estimation in the frequency domain. It is important to discuss what can be said about a close-form asymptotic result similar to (3.1) for the sample autocorelogram and other quantities in the time domain.

One can conjecture that the extreme-value distribution function $\exp(-e^{-x})$ used in (3.1) should be also relevant for the sample autocorrelogram, under suitable conditions on the underlying process, and after suitable normalization. From general results in Cramer and Leadbeter (1962), for example, it follows that the normalized maximun of a stationary Gaussian stochastic process follows the distribution attained in (3.1). However, the sample autocorrelations can be regarded only as an *asymptotically* Gaussian stochastic process; this will then require an explicit treatment, that we have been unable to trace in the literature.

4. Other Approaches to Simultaneous Inference

4.1 Introduction

The solution studied in Section 3 has two main shortcomings or difficulties: (a) Its derivation is quite complicated, and it will not be easy to extend the approach to other cases of interest, in particular, to those in the time domain; (b) The sample sizes implied by the result may be too high for some fields of application. These and other reasons indicate that it pays to study further the problem, with simpler tools, and to derive exact or approximate results that may be useful.

In general, we want to study what can be said about joint probabilities that in the case of the sample autocorrelogram have the form

$$P\left(\sqrt{T}\left|r_1\right| \le c, \ldots, \sqrt{T}\left|r_m\right| \le c \middle| H\right) = P\left(\max_{1\le s\le m} \sqrt{T}\left|r_s\right| \le c \middle| H\right), \tag{4.1}$$

where H is some suitable hypothesis on the underlying process.

4.2 Using the asymptotic independence

Mentz (1983b) evaluated (4.1) when H is that the underlying process is white noise. Then if the probability in (4.1) is set equal to g, c is defined by

$$\Phi(c) - \Phi(-c) = \sqrt[m]{g}\,, \tag{4.2}$$

where Φ is the standard normal cumulative distribution function.

This approach was applied to the joint inference based on $r_1, \ldots, r_m$ to test the null hypothesis that the process is white noise. It uses the fact that the set of sample autocorrelations is asymptotically normal, as indicated in Section 2. However, in view of (2.2) the independence of the r_j has to be assumed.

The approach can be extended to the frequency domain as follows: the set $\sqrt{T/m}\left[\hat{f}\left(v_j\right) - f\left(v_j\right)\right] / \left[tf\left(v_j\right)\right]$, $v_j \neq 0$, $\pm p$, is asymptotically unit normal, that is, multivariate normal with means 0, variance 1, and covariances 0. Hence, the constants defined by (4.2) are also those needed for simultaneous confidence intervals for $f(v_j)$. These will then be of the forms given in (2.8) or (2.9).

Note that the main difference between the results in the time and frequency domains is that in the former the asymptotic independence of the sample quatities has to be assumed, while in the latter is given as part of the asymptotic distribution.

4.3 Some useful inequalities

In Section 1 we discussed the fact that the accurate interpretation of the fluctuations exhibited by a set of sample quantities, requires the evaluation of the probabilities of some events in the joint distribution of the sample quantities. This is often difficult due to two main reasons: (1) In general, the joint distributions are not known for finite sample sizes; (2) Even that the asymptotic joint distributions often turn out being multivariate normal, the evaluation of probabilities for the events of interest is complicated.

In this section we present three inequalities useful to construct confidence bands for a set of parameters. The inequalities will he presented in general form, for a random vector $\boldsymbol{X} = (X_1, \ldots, X_m)'$; then in Section 4.4 they will be used to derive confidence bands for the problems discussed so far.

Theorem 1. Let $\boldsymbol{X} = (X_1, \ldots, X_m)'$ *be distributed as multivariate normal* $N(\mathbf{O}, \boldsymbol{\Sigma})$. *Then for all* $a_i > 0$, $i = 1, \ldots, m$,

$$P(|X_1| \leq a_1, \ldots, |X_m| \leq a_m) \geq \prod_{i=1}^{m} P(|X_i| \leq a_i). \tag{4.3}$$

The inequality is strict if $\boldsymbol{\Sigma}$ *is positive definite and at least one pair* (X_i, X_j) *has non-null correlation*.

This theorem was proved by Dunn (1958) for $m \leq 3$, and by Katri (1967) and Sidák (1967) for arbitrary *m*. The hypothesis of normality can be relaxed as follows.

Corollary 1. Let $\boldsymbol{X} = (X_1, \ldots, X_m)'$ *be a random vector. Assume that there exist Borel measurable and montone functions* $g_1 : \Re \rightarrow \Re$, $i = 1, \ldots, m$ *which are symmetric in the sense that* $g_i(x) = g_i(-x)$ *for all real x, and such that the vector* $\boldsymbol{Y} = (Y_1, \ldots, Y_m)'$ *has a normal distribution, where* $Y_i = g_i(x_i)$, $i = 1, \ldots, m$. *Then the conclusion of Theorem 1 holds*.

Theorem 2. *Let* $\boldsymbol{X} = (X_1, \ldots, X_m)'$ *be distributed as* $N(\boldsymbol{O}, \sigma^2 \boldsymbol{B})$, *where* $\boldsymbol{B}$ *is a positive semidefinite matrix, and 1et S be independent of* $\boldsymbol{X}$ *such that* nS^2 / σ^2 *has a* $\chi^2(n)$ *distribution. Let* $T_i = X_i / S$, $i = 1, 2, \ldots, m$. *Then*

$$P(|T_1| \leq a_1, \ldots, |T_m| \leq a_m) \geq \prod_{i=1}^{m} P(|T_i| \leq a_i), \tag{4.4}$$

for all $a_i > 0$, $i = 1, \dots, m$.

The proof of this theorem can be found in Sidák (1967) and Tong (1980), page 37.

The inequalities for probabilities of rectangles of the type (4.3) are basic for constructing confidence bands for a set of parameters. In this sense, Theorem 1 and Corollary 1 are useful. They are limited, however, because they require the assumption of a normal (or closely related) joint distribution.

Finally, we present Bonferroni's inequality which does not require any assumption about the distribution of the random variables involved. In spite of its generality, Bonferroni's inequality if quite sharp and hence very useful.

Theorem 3. Let $\boldsymbol{X} = (X_1, \dots, X_m)'$ *be a random vector and 1et* $A_1 \dots, A_m$ *be Borel measurable sets of the real line. Then*

$$1 - \frac{Q_1^2}{Q_1 + 2Q_2} \geq P\left[\bigcap_{i=1}^{m} (X_1 \in A_i)\right] \geq 1 - Q_1, \tag{4.5}$$

where

$$Q_1 = \sum_{i=1}^{m} P(X_i \notin A_i), \tag{4.6}$$

$$Q_2 = \sum_{i=2}^{m} \sum_{j=1}^{i-1} P(X_i \notin A_i, X_j \notin A_j). \tag{4.7}$$

For a proof of this theorem see Chung and Erdos (1952).

4.4 Evaluation and Comparison of Simultaneous Confidence Bands

4.4.1 Introduction

We now evaluate the constants needed to set confidence bands in the sample autocorrelogram and a windowed spectral density estimate. We shall continue with the general notation introduced in Section 4.3. In fact, it should be clear that the constants to be evaluated to use in (2.3) and (2.8), correspond to the following set up.

Let $\boldsymbol{X} = (X_1, \ldots, X_m)'$ be a random vector with which we want to construct a confidence band for the set $m_1, \ldots, m_m$ of expected values. We assume that the corresponding variances $s_1^2, \ldots, s_m^2$ are known, and that the confidence band is of the form

$$P\left(\bigcap_{i=1}^{m} \left| X_i - m_i \right| \leq c s_i \right) \geq g \; . \tag{4.8}$$

Without loss of generality we assume that $s_i = 1$, $i = 1, \ldots, m$.

When dealing with the autocorrelogram, we have that in general r_i is not the expected value of r_i, and that the variance of r_i is not known since it depends on the unknown r_i , see (2.3). In this case the constants to be determined below still apply in an asymptotic sense, and the variances will have to be estimated on the basis of the sample observations (as indicated in (2.3); this will be another source of approximation.

In the case of the spectral density, forms (2.8) and (2.9) are such that the component $t\sqrt{m/T}$ is a constant for a given window.

Using the inequalities introduced in Section 4.3 we now evaluate the constants to be used in the bands.

From (4.3) we have that

$$P\left(\bigcap_{i=1}^{m} \left| X_i - m_i \right| \leq c \right) \geq \prod_{i=1}^{m} P\left(\left| X_i - m_i \right| \leq c \right) = \left[\Phi(c) - \Phi(-c) \right]^m = g \; , \tag{4.9}$$

where we assumed that the joint distribution is normal. Solving (4.9) for c, that we may denote as $c_m(g)$ to emphasize its dependence on m and g, we have that

$$c_m(g) = \Phi^{-1}\left(\frac{1 + \sqrt[m]{g}}{2} \right). \tag{4.10}$$

This set of values was presented in Section 4.2 for the case of asymptotic independence.

Using Bonferroni's inequality we have:

$$P\left(\bigcap_{i=}^{m}\left|X_i - m_i\right| \le b\right) \ge 1 - \sum_{i=1}^{m} P\left(\left|X_i - m_i\right| > b\right) = - mP\left(\left|X_1 - m_1\right| > b\right) \\ = 1 - 2m\left[1 - \Phi(b)\right] = g\ , \qquad (4.11)$$

and from this we deduce the constant

$$b_m(g) = \Phi^{-1}\left(1 - \frac{1-g}{2m}\right). \qquad (4.12)$$

In the case of the frequency domain we can also use the asymptotic distribution of the maximum. This can be written as

$$\lim_{T\to\infty} P\left(\max_{1\le i\le m}\left|X_i - m_i\right| \le z\right) = \exp\left(-e^{-x}\right) = g\ , \qquad (4.13)$$

where $z = ax + b$, and a and b are given below (3.1). Explicitly, the constants to be used here are

$$z_m(g) = b - a\ln(-\ln g)\,. \qquad (4.14)$$

Note that even when γ is the confidence coefficient of the joint procedure, the evaluation in the marginal normal distribution of (4.13) is done as if the level were $1-(1-g)/2m$; a similar argument holds for (4.11). This means that the normal density is used far apart in the tails, and hence that the use of the resulting values should be done with care.

4.4.2 Numerical results

We now evaluate and compare numerically $c_m(g)$, $b_m(g)$ and $z_m(g)$ introduced in section 4.4.1. We recall that $z_m(g)$ is only justified in the case of the spectral density function estimator.

Table 1 presents values of the indicated expressions for selected values of g and m. To facilitate the writing g is often omitted.

Detailed tables are appended to the present work; see Appendix 2.

About b_m and c_m note that we consider the variance as known (cf. Section 4.4.1); this is reasonable in our case since we have in mind large sample sizes for practical applications. For b_m, when the variance is unknown and replaced by an

estimate, Miller (1966) presents a table of percentage points of the corresponding (Student) distribution; our constant correspond to Miller's table for a number of degrees of freedom tending to ∞. However, even for small numbers degrees of freedom the approximation is very good.

For z_m in Table 1 and in Appendix 2, we note that for example for g=0.99, z_m first decreases and then increases. This behavior is

Table 1

Constants to be used in (asymptotic) joint confidence bands derived under different assumptions

m	Confidence Level g								
	0.90			0.95			0.99		
	c_m	b_m	z_m	c_m	b_m	z_m	c_m	b_m	z_m
1	1.645	1.645	-	1.960	1.960	-	2.576	2.576	-
2	1.949	1.960	2.367	2.236	2.241	2.799	2.806	2.807	3.778
5	2.311	2.326	2.572	2.569	2.576	2.908	3.089	3.090	3.667
10	2.560	2.576	2.768	2.800	2.807	3.062	3.289	3.291	3.728
20	2.791	2.807	2.966	3.016	3.023	3.231	3.480	3.481	3.831
50	3.075	3.090	3.222	3.283	3.291	3.459	3.718	3.719	3.996
100	3.276	3.291	3.408	3.474	3.481	3.629	3.889	3.891	4.130
200	3.467	3.481	3.588	3.656	3.662	3.796	4.055	4.056	4.266
500	3.706	3.719	3.815	3.885	3.891	4.009	4.265	4.265	4.447
1000	3.878	3.891	3.980	4.050	4.056	4.165	4.417	4.417	4.583
2000	4.043	4.056	4.140	4.209	4.214	4.317	4.565	4.565	4.717
5000	4.253	4.265	4.343	4.412	4.417	4.511	4.751	4.751	4.891

contrary to expectation, and we interpret that for the corresponding values of γ the asymptotic results should not be used for very small m.

Table 2 presents values of the relative differences between c_m, b_m and z_m, respectively.

Table 1 and 2 show that for m as low as 10 or 20, the differences among the values are small, in the order of 10% or less, c_m and b_m are always less than z_m, and c_m is always less than b_m; see Section 4.4.3 below.

The first line of Table 1 contains the standard normal deviates. For m=50 the constants bear to those in the first line the following relations: they are about twice for g=0.90. It follows that joint confidence bands are considerably wider than point-

by-point confidence bands, for the usual values of γ, and for the frequently encountered values of m.

Table 2

Relative differences of the constants to be used in (asymptotic) joint confidence bands derived under different assumptions

m	Confidence Level g					
	0.90		0.95		0.99	
	$\frac{c_m - z_m}{z_m}100$	$\frac{b_m - z_m}{z_m}100$	$\frac{c_m - z_m}{z_m}100$	$\frac{b_m - z_m}{z_m}100$	$\frac{c_m - z_m}{z_m}100$	$\frac{b_m - z_m}{z_m}100$
2	-17.7	-17.2	-20.1	-19.9	-25.7	-25.7
5	-10.2	-9.6	-11.7	-11.4	-15.8	-15.7
10	-7.5	-6.9	-8.6	-8.3	-11.8	-11.7
20	-5.9	-5.4	-6.7	-6.4	-9.2	-9.1
50	-4.6	-4.1	-5.1	-4.9	-7.0	-6.9
100	-3.9	-3.5	-4.3	-4.1	-5.8	-5.8
200	-3.4	-3.0	-3.7	-3.5	-5.0	-4.9
500	-2.9	-2.5	-3.1	-3.0	-4.1	-4.1
1000	-2.6	-2.3	-2.8	-2.6	-3.6	-3.6
2000	-2.4	-2.0	-2.6	-2.4	-3.3	-3.2
5000	-2.3	-1.8	-2.4	-2.1	-3.1	-2.9

4.4.3 Analytic and Asymptotic Results

Since for $0 < g < 1$ it holds that $\sqrt[m]{g} < 1 - (1-g)/m$, we have that

$$\frac{1+\sqrt[m]{g}}{2} < 1 - \frac{1-g}{2m}, \tag{4.15}$$

and since Φ^{-1} is monotone increasing,

$$c_m(g) < b_m(g) \tag{4.16}$$

for all m>1 and g, $0 < g < 1$.

As m increases, it is interesting to study the behavior of the constants. In Appendix 1 we derive asymptotic expressions for them, and show that

$$\lim_{m\to\infty} \frac{c_m}{z_m} = \lim_{m\to\infty} \frac{b_m}{z_m} = \lim_{m\to\infty} \frac{c_m}{b_m} = 1. \tag{4.17}$$

Moreover, these as well as the various constants themselves, are asymptotically independent of the confidence coefficient *g*.

The interpretation of these asymptotic results should be done with care, due to the very large values of *m* (and hence of the sample size *T*) involved. We recall the comment by Hannan that we reproduced in Section 3. This fact motivates our inclusion of values as large as *m*=5000 in tables 1 and 2.

5. A Simulation Study: Frequency Domain, Estimation by Confidence Bands

To study empirically the performance of some confidence bands for the spectral density function, we carry out a Monte Carlo experiment. We take three autoregressive models considered by Beamish and Priestley (1981) and Newton and Pagano (1984), namely,

Model I $\quad y_t - 0.4y_{t-1} - 0.45y_{t-2} = u_t$,

Model II $\quad y_t + 1.7y_{t-1} + 2.4y_{t-2} + 1.634y_{t-3} + 0.872y_{t-4} + 0.168y_{t-5} = u_t$,

Model III $\quad y_t - 2.7607y_{t-1} + 3.8106y_{t-2} - 2.6535_{t-3} + 0.9238y_{t-4} = u_t$,

where u_t is a Gaussian white noise with unit variance. These models are used because they can be classified as being easy, moderately difficult and very difficult to fit, respectively, and also because they were studied by the indicated authors. Models will be identified as I, II and III. For each model we simulate N=1000 replicates of trajectories of length T=100, 200 or 400, and for each of them we construct joint confidence bands.

Instead of the spectral density estimator defined in (2.4) by using the lag window, we use the average spectral density estimator defined by

$$\hat{f}(v_j) = (2p)^{-1} \sum_{|k| \le l_n} W_n(k) I_n(v_{j+k}),$$

where I_n is the periodogram, $l_n = l$ is a sequence of integers, and W_n is a sequence of weight functions. We take l=2 and W(-2)=W(2)=1/8, W(-1)=W(0)=W(1)=1/4. Then, the asymptotic variance of $\hat{f}(v_j)$ for v_j unequal to 0 or p is given by

$$b^2 = \sum_{|k| \le 2} W^2(k) = \frac{7}{32}.$$

This choice corresponds to modified Daniel's window w(x) = sine(πx)/(πx) for -1≤x≤1, and m=T/5.

Five different definitions of the confidence bands are considered, where logarithms are decimal, namely:

(i) $\log \hat{f}(v_j) + \log v_j - \log c^2_{(1-g)/2}(u), \quad \log \hat{f}(v_j) + \log v_j - \log c^2_{g/2}(u),$

where $u = 2b^{-2} = 9.14$.

(ii) Confidence band for $\log \hat{f}(v_j)$ defined in the S-PLUS program, which is based on χ^2 with 8.29 degrees of freedom.

(iii) $$log\ \hat{f}(v_j) \pm z_{(g-1)/2} b,$$

based on a normal approximation to the distribution of $\log \hat{f}(v_j)$.

(iv) $$log\ \hat{f}(v_j) + b^{-2}/2 \pm z_{(g-1)/2} b,$$

based on a normal approximation to the distribution of $\log \hat{f}(v_j)$ corrected for (asymptotic) bias.

(v) $$log\ \hat{f}(v_j) - log(1+b), \qquad log\ \hat{f}(v_j) - log(1-b).$$

This is based on taking logarithms in (2.8)

Bands are computed with a joint confidence level of 0.95. They will be identified by (i) to (v).

Table 5 shows the results when the calculations are done at T/10 frequency points, which is the number used in the design. We also considered shown the results when all T/2-2 frequencies, $4p \le u \le 2p(T/2-1)/T$, are used in the calculations; this, however, is too demanding, and only the approach in Table 5 is shown.

Table 5 shows the frequencies of coverage of the bands in N=1000 replications. Method (iv) based on the normal approximation corrected for bias, and method (ii) based on the χ^2 approximation available in S-PLUS, give the best results. Method (v) produced poor results and it was discarded. Method (iii) based on the normal approximation without correction for bias is clearly inferior.

Even when the frequencies of strict coverage are not too close to the 95% theoretical level, we can accept the performance of the confidence bands as providing a correct frequency of coverage, if we are willing to allow up to one point out of the band.

Figures 1, 2 and 3 show that methods (i), (ii) and (iv) give almost the same confidence bands.

Figures 4 to 8 show bands for five replicates calculated with method (iv) for each model and selected sample sizes. In the case of Model III, figures 6, 7 and 8, it is clear the effect of increasing sample size, in that the bands tend to show less bias and are more concentrated among them.

6. A Simulation Study: Testing Hypotheses

In this section we explore the use of confidence bands to test the hypothesis that a given time series is white noise. The procedures in the frequency and time domains are compared with the use of the modified *Ljung-Box.Pierce portmanteau statistic* (Box, Jenkins and Reinsel, Section 8.2.2), defined by

$$(6.1) \qquad \tilde{Q} = T(T+2)\sum_{t=1}^{T}\frac{r_t^2}{T-t},$$

where the r_j are the sample autocorrelations of the given series, and K is a suitably chosen time lag so that little is lost by omitting r_t's with t>K (Box *et al*, op. cit.)

Figure 9 shows the kind of situation that we are studying. Represented are the logarithms of the spectral quantities, the center straight line corresponding to $\log[2\pi(\sigma^2/2\pi)] = \log(\sigma^2) = \log 1 = 0$, where $\sigma^2/2\pi$ is the spectral density of white noise. The solid line is $\log\left[2p\hat{f}(u)\right]$, where $\hat{f}$ comes from the S-PLUS program, and the broken lines are the 95% (joint) confidence bands for white noise compute by S-PLUS. The spectral densities are computed at T/10 frequency points; at none of these the estimated spectral density falls out of the bands. However, the plot is made at T/2 frequency points, and at some of these the solid line lies out of the bands.

Figure 9 also shows the sample autocorrelogram of the generated white noise series, with the 95% (joint) confidence bands calculated with the constant corresponding to the Bonferroni inequality, as discussed in sections 4.3, 4.4.1 and 4.4.2; the numerical values is 3.023/√T. We find one autocorrelation out of the bands.

Table 6 shows the results of 100 repetitions of these calculations. In the first column the values of $\tilde{Q}$ are shown, ordered by size. The next four columns contain the counts of the points of the sample spectral density falling out of the confidence bands; they correspond to the four estimation procedures considered in Table 5. The last column contains the counts of values of the correlogram falling out of the corresponding confidence bands. Since the critical value of $\tilde{Q}$ is $c^2_{0.05}(K) = c^2_{0.05}(20) = 31.41$, we find that 8 of the 100 repetitions have $\tilde{Q}$ larger than 31.41, a results consistent with the level of significance α=0.05. Table 5 also shows that the empirical level of significance of the S-PLUS and of the correlogram bands, are consistent with the 0.05 level: in effect, there are 3 series with point falling out of the bands in the frequency case, and 5 points in the correlogram. At the same time, the other spectral estimates give empirical significance levels of 0.11, 0.30 and 0.10, respectively. As in the analysis of the bands for estimation purposes, the results are in favor of the S-PLUS bands.

We next apply the indicated procedure to gain some indication of the discriminatory power of the procedures. For this purpose, we test whether simulated AR(1) series, lead to rejection of the white noise hypothesis. We generate by Monte Carlo, 100 AR(1) series of lengths T = 200 using the S-PLUS program arima.sim (simulate a univariate ARIMA series), estimate their parameters with S-PLUS program arima.mle (ARIMA modeling via Gaussian maximum likelihood). We used 0.30 and 0.60 for the model parameter, and 1 for the innovations variance. Then we proceed as in the analysis of the white noise series: we compare the $\tilde{Q}$ statistics

with $c^{2}_{0.05}(K-1)$, and we determine the numbers of points falling out of the confidence bands for the spectral density as compute by the S-PLUS program under the hypothesis of white noise. Figures 10 and 11 are comparable to Figure 9.

Table 7 is a summary of these empirical power calculations. In this table we find that the empirical power of the procedures are estimated as follows: when the model parameter is set at 0.30, the values are 0.76 for $\tilde{Q}$, 0.34 for the S-PLUS spectral estimate, and 0.90 for the sample autocorrelogram; when the parameter is set at 0.60, the 3 estimates of power are equal to 1.

Table 7 also reports the results when the simulated series corresponds to the AR(2) model introduced in Section 5, and the finding is that the empirical power is also equal to 1.

It is interesting to note that for the AR(1) model with parameter 0.30, the empirical power of the correlogram for this particular set of series is 0.90, which represents an improvement over the value of 0.76 corresponding to the use of the $\tilde{Q}$ statistic. More detailed studies should be conducted to investigate the power of the various procedures considered in the present study, but this is beyond the scope of the present paper.

7. Discussion and Conclusions

The question was raised about how to construct simultaneous confidence bands in time series problems, typically for the autocorrelogram in the time domain and for the spectral density function in the frequency domain. Simultaneous procedures are such that control of the confidence coefficient is given to the researcher, who then knows what probability level (either exact or approximate) is associated with the simultaneous inferencial statement he makes on the basis of his observations.

A formal solution to one such problem was given by Woodroofe and Van Ness (1967) for the spectral density, as discussed in our Section 3. With some

adequate general assumptions on the underlying stochastic process, and assuming that as the sample size $T \to \infty$ the number of ordinates $m = m_T \to \infty$, an asymptotic probability statement was derived that uses one of the extreme value distributions known in the literature.

It was noted by Hannan, (1970) that this result may be "the roughest of approximations only", as we discussed in Section 3.

A close-form result of similar nature is not available for the time domain quantities, at least to the knowledge of the present writers.

At least for the two previous arguments, it pays to investigate further the problem. In this note we considered two possibilities: (1) To use the asymptotic independence of the sample quantities; (2) To resort to inequalities.

The use of asymptotic independence is explored briefly in Section 4.2. It turns out that the asymptotic joint distribution of windowed spectral estimators at a set of frequencies is independent multivariate normal, so that the evaluation of constants to be used in simultaneous confidence bands is straightforward. We denote these constants by $c_m(g)$. These same constants can be used in the time domain for the autocorrelations, but here the asymptotic independence has to be assumed; then we interpret that the confidence band is directed to a comparison against a white noise hypothesis.

The constants $c_m(g)$ evaluated under the asymptotic normality and independence can also be justified on the basis of asymptotic *joint* normality only, as follows: For a multivariate normal density there is an inequality (4.3) that leads to the use of the $c_m(g)$ in the construction of conservative simultaneous confidence bands for m quantities, with probability level g.

The need to use the asymptotic joint normality can in turn be relaxed: Bonferroni's inequality is used in Section 4.4.1 to derive (conservative) constants $b_m(g)$ that use the asymptotic *marginal* normal distribution of each sample quantity under consideration.

In terms of assumptions we see that the three approaches (for the frequency domain) can be interpreted as follows: (a) Bonferroni's inequality, and hence the use

of the associated constants $b_m(g)$, requires only that the marginal distribution of each sample quantity be approximated by its normal limit; (b) The inequality in (4.3), and hence the constant $c_m(g)$, require that the joint distribution of the sample statistics be approximated by its normal limit; (c) The approach of Woodroofe and Van Ness and the corresponding constants, denoted $z_m(g)$ by us, requires not only that the sample size $T \to \infty$ but that *m*, the sample quantities under consideration, satisfies $m = m_T \to \infty$, in such a way that $g(m_T)/T \to 0$ for some suitable function *g*.

We conclude that the use of inequalities in the way we we did in Section 4.4. has several advantages: (1) It requires less assumptions on the set up of the problem; (2) It is simpler, since for example the proof of the results in Woodroofe and Van Ness is quite complicated; (3) It provides more flexibility, in particular, can be used to derive bands of non-uniform width.

About this last point, the results we presented so far are for bands of uniform width. This is not what is always recommended in the literature. The BMDP-81 package, for example, produces a "control line" for the sample autocorrelogram of the form given in (2.3), where the approximation

$$\hat{t}_{ss} \cong \frac{1}{T}\sum_{j=0}^{s-1} r_j^2 \tag{7.1}$$

is used, as suggested originally by Bartlett (1946). This then produces a band that in general has increasing width, and that in the computer appears with nondecreasing width, due to rounding.

The idea of nonuniform significance levels was also considered in multiple decision problems by Anderson (1971). In our case this will lead to allowing *m* to be considerably large, but still small compared with *T*, and letting the width increase with the index of the sample statistics. This appears as a reasonable procedure, and will make less important the exact determination of m in the definition of the sample autororrelogram.

The numerical values of $z_m(g)$, $c_m(g)$ and $b_m(g)$ derived for the result of Woodroofe and Van Ness, the inequality in (4.3), and Bonferroni's inequality, respectively, were evaluated and compared. They differ for very small values of *m*,

for which in fact $z_m(g)$ is not recommended. For about 20 or 30 sample quantities, which may be a useful number for practical use, $c_m(g)$ and $b_m(g)$ differ only slightly, and their differences with $z_m(g)$ are also small.

For standard values of g (0.90, 0.95, 0.99) and for m in the order of 30, the constants are roughly in the order of 1.5 times those of the univariate confidence intervals. Hence, simultaneous confidence bands will be wider that those formed with point-by-point constants.

To see how well this kind of procedure works for finite sample sizes (T), Monte Carlo simulation experiments were conducted. In terms of estimation, four procedures are compared for T=100, 200 and 400 with 1000 replications. The results are interesting: defining as satisfactory coverage of the true spectral density by 0.95 confidence bands, the occurrence up to one point out of the band, three of the procedures work reasonable well. As expected, they are wider than point-by-point bands of the same probability level. In terms of using some of the results to test the hypothesis of white noise, the joint confidence bands for the correlogram showed to perform better than using the $\tilde{Q}$ statistic, and also that the bands in the frequency domain. The study of power presented in this paper is, however, quite limited, and further studies are required to evaluate the corresponding empirical measures.

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Appendix 1

In this Appendix we prove some assertions made in Section 4.4.3. For $m \to \infty$ we derive asymptotic expansions for b_m, c_m and z_m, show that they are independent of the confidence coefficient g, and that they are asymptotically equal.

The following inequality holds for x>0 (Pickands (1969)),

$$y(x)\left(1-\frac{1}{x^2}\right)\le 1-f(x)\le y(x), \qquad \text{(A.1)}$$

where $f(x)$ is the standard normal distribution function, and

$$y(x)=\frac{1}{x}f(x)\equiv\frac{1}{x}\frac{1}{\sqrt{2p}}e^{-\frac{1}{2}x^2}\ . \qquad \text{(A.2)}$$

Lemma 1. Let $0<b<1$, *and* x_1, x_2 *and* x_3 *be respectively the roots of the equations*

$$y(x_1)\left(1-\frac{1}{x_1^2}\right)=b\ , \qquad \text{(A.3)}$$

$$1-f(x_2)=b\ , \qquad \text{(A.4)}$$

$$y(x_3)=b\ . \qquad \text{(A.5)}$$

Then $x_1\ge x_2\ge x_3$.

To find asymptotic expansions for b_m and c_m we find asymptotic expansions for $x_1=x_1(m)$ and $x_3=x_3(m)$ by setting $b=(1-g)/2m$ and $b=(1-\sqrt[m]{g})/2$, respectively.

Lemma 2. *Let* b_m *be the root of the equation*

$$f(b_m)=\frac{1-g}{2m}\ ,$$

from (A.4) and (4.14). *Then* $b_m^2\sim 2\ln(2m)$ *as* $m\to\infty$, *where the notation* ~ *means asymptotically equal*.

Proof. Equation (A.5) is now

$$\frac{1}{\sqrt{2p x_3}}e^{-\frac{1}{2}x_3^2}=\frac{1-g}{2m}\ ;$$

is follow that

$$x_3^2=-2\ln x_3-\ln 2p-2\ln(1-g)+2\ln(2m). \qquad \text{(A.6)}$$

Equation (A.3) is now

$$\frac{1}{\sqrt{2p}\,x_1}e^{-\frac{1}{2}x_1^2}\left(1-\frac{1}{x_1^2}\right)=\frac{1-g}{2m}\ ;$$

from it follows that

$$x_1^2 = -2\ln x_1 - \ln 2p - 2\ln(1-g) + 2\ln(2m) + \ln\left(1 - \frac{1}{x_1^2}\right). \tag{A.7}$$

From (A.6) and (A.7) it follows that both x_1^2 and x_3^2 are asymptotically equal to $2\ln(2m)$. From Lemma 1 it follows that $b_m^2 \sim 2\ln(2m)$.

Lemma 3. *Let* c_m *be the root of the equation*

$$f(c_m) = \frac{1 - \sqrt[m]{g}}{2},$$

from (A.4) *and* (4.12). *Then* $c_m^2 \sim 2\ln(2m)$ *as* $m \to \infty$.

Proof. Equation (A.5) is now

$$\frac{1}{\sqrt{2p}\, x_3} e^{-\frac{1}{2}x_3^2} = \frac{1 - \sqrt[m]{g}}{2} = \frac{1-g}{2m} + 0\left[\left(\frac{1-g}{m}\right)^2\right]$$

and therefore

$$e^{-\frac{1}{2}x_3^2} = \sqrt{2p}\, x_3 \left(\frac{1-g}{2m} + 0\left[\left(\frac{1-g}{m}\right)^2\right]\right),$$

$$x_3^2 = -2\ln x_3 - \ln 2p(1-g) + 2\ln(2m) + 0\left(\frac{1-g}{m}\right). \tag{A.8}$$

Equation (A.3) is now

$$\frac{1}{\sqrt{2p}\, x_1} e^{-\frac{1}{2}x_1^2}\left(1 - \frac{1}{x_1^2}\right) = \frac{1 - \sqrt[m]{g}}{2},$$

and therefore

$$x_1^2 = -2\ln x_1 - \ln 2p - 2\ln(1-g) + 2\ln(2m) + 0\left(\frac{1-g}{m}\right) + \ln(1 - \frac{1}{x_1^2}. \tag{A.9}$$

Hence, from (A.8), (A.9) and Lemma 1 it follows that $c_m^2 \sim 2\ln(2m)$ as $m \to \infty$.

Lemma 4. *Let* z_m *be defined by* (4.16). *Then* $z_m^2 \sim 2\ln(2m)$ *as* $m \to \infty$.

Proof. From the definitions of a and *b* following (3.1) we have that

$$z_m = \sqrt{2\ln(2m)} - \frac{1}{2\sqrt{2\ln(2m)}}\left[\ln\ln(2m) + \ln 2p + 2\ln(-\ln g)\right]. \tag{A10}$$

and the conclusion of the lemma follows.

By using Taylor's theorem, the following asymptotic expansion can be derived,

$$z_m^2 \sim 2\ln(2m) - \ln\ln(2m) - \ln 2p - 2\ln(-\ln g), \qquad \text{(A.11)}$$

which compares directly with those for x_1^2 and x_3^2 derived in the proofs of lemmas 2 and 3.

The results in (4.19) follow immediately from lemmas 2, 3 and 4. Further c_m, b_m and z_m are asymptotically (as $m \to \infty$) independent of the confidence coefficient g.

References

Akaike, H. (1969), Power spectrum estimation through autoregressive model fitting, *Ann. Inst. Statist. Math.*, 21, 407-419.

Anderson, T.W. (1971), *The Statistical Analysis of Time Series*, John Wiley and Sons.

Bartlett, M.S. (1946), On the theoretical specification and sampling properties of autocorrelated time series, *J.R. Stat. Soc.* (suppl), *7*, 211.

Beamish, N. and M. B. Priestley (1981), A study of AR and window spectral estimation. *Appl. Statist.*, 30, No. 1.

Box, G.E. land G.M. Jenkins (1976), *Time Series Analysis: Forecasting and Control*, Revised Edition, Holden-Day.

Cramer, H. And M. Leadbetter (1967), *Stationary and Related Stochastic Processes*, Hohn Wiley and Sons.

Chung, K.L. and P. Erdos (1952), On the applications of the Borel.Cantelli lemma, *Trans. Amer. Math. Soc.*, 72, 178-186.

Das Gupta, S., M.L. Eaton, I. Olkin, M.D. Perlman, LK.J. Savage and M. Sobel (1972), Inequalities on the probability content of convex regions for elliptically contoured distributions, *Proc. Sixth Berkely Symp. Math. Statisti. Probab.*, *2*, University of California Press.

Dunn, O. (1958), Estimation of the means of dependent variables, *Ann. Math. Stat.*, *29*, 1095-1111.

Granger, W. And M. Hatanaka (1964), Spectral Analysis of Economic Time Series, Princenton University Press.

Hannan, E. (1970), Multiple Time Series, Hohn Wiley and Sons.

Hrafnkelsson, B. and H. J. Newton (2000), Asymptotic simultaneous confidence bands for vector autoregressive spectra, *Biometrika*, 87, 173-182.

Katri, C.G. (1967), On certain inequalities for normal distributions and their applications to simultaneous confidence bounds, *Ann. Math. Stat.*, *38*, 1853-1867.

Koslov, J. W. and R. H. Jones (1985),A unified approach to confidence bounds for the autoregressive spectral estimator, *J. of Time Series Anal.*, 6, 141-151.

Mentz, R.P. (1983a), On the use of the sample autocorrelogram, presented at the "First Catalán International Symposium on Statistics", Barcelona, September 1983.

Mentz, R.P. (1983b) Plotting control lines in the sample autocorrelogram, *Contributed Papers, 44th. Session of the I.S.I., 2*, 709-712.

Miller, R. (1966), *Simultaneous Statistical Inference*, McGraw-Hill Book Co.

Newton, H. J. and M. Pagano (1984), Simultaneous confidence bands for autoregressive spectra, *Biometrika*, 71, 1, 197-202.

Owen, D. (1962), *Handbook of Statistical Tables*, Addison Wesley Publishing Co.

Parzen, E. (1974), Some recent advances in time series modeling, *I.E.E.E. Trans. Automatic Control*, AC-19, 723-729.

Priestley, M. (1981), *Spectral Analysis and Time Series*, Vols. 1 and 2, London: Academic Press.

Pickands III, J. (1969), Upcrossing probabilities for stationary Gaussian Processes, *Trans. Amer. Math. Soc., 145*, 51-73.

Sakai, H. and F. Sakaguchi (1990), Simultaneous confidence bands for the spectral estimate of two-channel autoregressive processes, *J. of Time Series Anal.*, 11, 49-56.

Sidák, Z. (1967), Rectangular confidence regions for the means of multivariate normal distributions, *J. Amer. Stat. Assoc., 62*, 626-633.

Tomasek, L. (1987) Asymptotic simultaneous confidence bands for autoregressive spectral density, *J. of Time Series Anal.*, 8. 469-477.

Tong, G.L. (1980), *Probability Inequalities in Multivariate Distributions*, Academic Press.

Walker, A.M. (1965), Some asymptotic results for the periodogram of a stationaty time series, *J. Austral. Math. Soc., 5*, 107-128.

Woodroofe, M. and J. Van Ness (1967), The maximum deviation of sample spectral densities, *Ann. Math. Stat.*, *36* 1558-1569.

Table 3

Constants $c_m(g)$ to be used in (asymptotic) joint confidence band.

m \ γ	0.6	0.7	0.8	0.9	0.95	0.99	0.995	0.999
1	0.84162	1.03643	1.28155	1.64485	1.95996	2.57583	2.80703	3.29053
2	1.21228	1.39393	1.61842	1.94882	2.23648	2.80623	3.02296	3.48069
3	1.41671	1.58884	1.80113	2.11405	2.38774	2.93416	3.14349	3.58783
4	1.55524	1.72068	1.92477	2.22627	2.49092	3.02220	3.22668	3.66216
5	1.65898	1.81940	2.01746	2.31066	2.56876	3.08904	3.28996	3.71892
6	1.74140	1.89787	2.09123	2.37800	2.63104	3.14276	3.34090	3.76472
7	1.80948	1.96273	2.15229	2.43386	2.68280	3.18757	3.38345	3.80306
8	1.86732	2.01787	2.20424	2.48148	2.72701	3.22596	3.41993	3.83600
9	1.91747	2.06571	2.24937	2.52292	2.76553	3.25950	3.45183	3.86484
10	1.96168	2.10791	2.28921	2.55955	2.79963	3.28926	3.48015	3.89048
20	2.23805	2.37234	2.53969	2.79102	3.01599	3.47948	3.66165	4.05552
30	2.38924	2.51751	2.67781	2.91951	3.13675	3.58665	3.76422	4.14930
40	2.49237	2.61675	2.77248	3.00791	3.22009	3.66101	3.83551	4.21469
50	2.57018	2.69174	2.84416	3.07500	3.28348	3.71777	3.89000	4.26478
60	2.63243	2.75180	2.90164	3.12893	3.33450	3.76359	3.93401	4.30531
70	2.68417	2.80176	2.94952	3.17391	3.37712	3.80194	3.97089	4.33931
80	2.72836	2.84447	2.99048	3.21244	3.41366	3.83489	4.00258	4.36857
90	2.76687	2.88171	3.02622	3.24610	3.44561	3.86374	4.03035	4.39423
100	2.80095	2.91469	3.05790	3.27596	3.47398	3.88939	4.05505	4.41707
200	3.01724	3.12438	3.25975	3.46682	3.65575	4.05446	4.21424	4.56468
300	3.13796	3.24170	3.37300	3.57432	3.75846	4.14826	4.30487	4.64903
400	3.22127	3.32277	3.45139	3.64890	3.82984	4.21367	4.36813	4.70803
500	3.28464	3.38449	3.51114	3.70583	3.88440	4.26377	4.41663	4.75332
600	3.33565	3.43421	3.55931	3.75178	3.92848	4.30431	4.45590	4.79004
700	3.37825	3.47576	3.59958	3.79024	3.96540	4.33832	4.48885	4.82088
800	3.41478	3.51141	3.63416	3.82327	3.99713	4.36758	4.51722	4.84744
900	3.44673	3.54258	3.66441	3.85220	4.02494	4.39325	4.54211	4.87077
1000	3.47508	3.57027	3.69129	3.87792	4.04966	4.41609	4.56426	4.89154

Table 4

Constants $z_m(g)$ to be used in (asymptotic) joint confidence band.

m \ γ	0.6	0.7	0.8	0.9	0.95	0.99	0.995	0.999
2	1.41856	1.63429	1.91595	2.36663	2.79893	3.77782	4.19561	5.16338
3	1.60839	1.79814	2.04590	2.44232	2.82257	3.68360	4.05109	4.90235
4	1.73862	1.91476	2.14474	2.51271	2.86569	3.66494	4.00607	4.79625
5	1.83644	2.00383	2.22238	2.57207	2.90751	3.66705	3.99122	4.74214
6	1.91426	2.07539	2.28578	2.62239	2.94529	3.67644	3.98849	4.71133
7	1.97861	2.13496	2.33911	2.66575	2.97907	3.68854	3.99135	4.69276
8	2.03331	2.18585	2.38502	2.70370	3.00938	3.70156	3.99698	4.68129
9	2.08077	2.23017	2.42524	2.73736	3.03675	3.71468	4.00402	4.67424
10	2.12263	2.26938	2.46099	2.76757	3.06164	3.72754	4.01175	4.67009
20	2.38490	2.51715	2.68982	2.96610	3.23111	3.83120	4.08731	4.68058
30	2.52890	2.65443	2.81833	3.08057	3.33212	3.90171	4.14482	4.70795
40	2.62736	2.74870	2.90713	3.16061	3.40376	3.95435	4.18934	4.73367
50	2.70179	2.82015	2.97469	3.22196	3.45915	3.99623	4.22545	4.75643
60	2.76141	2.87750	3.02907	3.27158	3.50421	4.03096	4.25578	4.77655
70	2.81103	2.92529	3.07447	3.31318	3.54215	4.06062	4.28190	4.79448
80	2.85344	2.96619	3.11340	3.34894	3.57488	4.08648	4.30484	4.81063
90	2.89043	3.00189	3.14742	3.38028	3.60364	4.10941	4.32527	4.82530
100	2.92320	3.03354	3.17762	3.40815	3.62928	4.12999	4.34370	4.83873
200	3.13163	3.23539	3.37088	3.58766	3.79561	4.26647	4.46743	4.93295
300	3.24832	3.34874	3.47986	3.68966	3.89091	4.34661	4.54110	4.99162
400	3.32899	3.42723	3.55550	3.76074	3.95761	4.40339	4.59365	5.03437
500	3.39043	3.48707	3.61326	3.81515	4.00881	4.44734	4.63450	5.06804
600	3.43993	3.53532	3.65987	3.85915	4.05031	4.48316	4.66789	5.09583
700	3.48131	3.57568	3.69889	3.89604	4.08515	4.51337	4.69613	5.11949
800	3.51680	3.61032	3.73241	3.92777	4.11516	4.53949	4.72059	5.14009
900	3.54786	3.64063	3.76177	3.95558	4.14150	4.56248	4.74215	5.15835
1000	3.57544	3.66757	3.78786	3.98033	4.16495	4.58300	4.76142	5.17472

Table 5

Frequency Distributions of Points Falling out of the 0.95 Confidence Band, and Percentage of Coverage (0 or 1 points). ($\omega_j = 2\pi j\, 10/T$, $j = 3,8,...,T/10-2$)

Sample Size	Model and Method	Points out of the Band					Percentage of 0 or 1	Average
		0	1	2	3	4 or more		
	I (splus)	876	123	1		0	99,90%	0,125
T = 100	I (df = 9.14)	795	191	14		0	98,60%	0,219
	I (Normal)	662	282	54	1	1	94,40%	0,397
	I (Normal corr.)	860	135	5		0	99,50%	0,145
	I (splus)	922	71	7		0	99,30%	0,085
T = 200	I (df = 9.14)	836	149	14	1	0	98,50%	0,18
	I (Normal)	609	311	72	7	1	92,00%	0,48
	I (Normal corr.)	865	125	10		0	99,00%	0,145
	I (splus)	936	63	1		0	99,90%	0,065
T = 400	I (df = 9.14)	861	124	14	0	1	98,50%	0,156
	I (Normal)	571	314	96	14	5	88,50%	0,569
	I (Normal corr.)	864	123	12	1	0	98,70%	0,15
	II (splus)	276	548	142	32	2	82,40%	0,936
T = 100	II (df = 9.14)	290	526	160	23	1	81,60%	0,919
	II (Normal)	365	504	115	16	0	86,90%	0,782
	II (Normal corr.)	433	502	60	5	0	93,50%	0,637
	II (splus)	582	394	24		0	97,60%	0,442
T = 200	II (df = 9.14)	567	383	47	3	0	95,00%	0,486
	II (Normal)	569	336	82	12	1	90,50%	0,541
	II (Normal corr.)	698	280	21	1	0	97,80%	0,325
	II (splus)	820	168	12		0	98,80%	0,192
T = 400	II (df = 9.14)	772	211	14	3	0	98,30%	0,248
	II (Normal)	579	325	88	8	0	90,40%	0,525
	II (Normal corr.)	844	145	9	2	0	98,90%	0,169
	III (splus)	13	122	291	331	243	13,50%	2,74
T = 100	III (df = 9.14)	20	144	336	316	184	16,40%	2,552
	III (Normal)	60	226	412	229	73	28,60%	2,039
	III (Normal corr.)	64	236	418	217	65	30,00%	1,991
	III (splus)	482	382	116	19	1	86,40%	0,675
T = 200	III (df = 9.14)	444	403	123	28	2	84,70%	0,741
	III (Normal)	416	417	139	23	5	83,30%	0,785
	III (Normal corr.)	644	300	50	5	1	94,40%	0,419
	III (splus)	886	108	6		0	99,40%	0,12
T = 400	III (df = 9.14)	842	145	13		0	98,70%	0,171
	III (Normal)	621	298	63	16	2	91,90%	0,48
	III (Normal corr.)	880	114	6		0	99,40%	0,126

Table 6

100 simulated white noise series, T = 200, tested for H_0: white noise by six procedures, level of significance 0.05

Time Domain	Frequency Domain				Time Domain
$\tilde{Q}$	S_Plus	DF = 9.14	Normal	Normal Correc.	Correlogram
9.090	0	1	1	1	0
11.096	0	0	0	0	0
11.526	0	0	0	0	0
11.954	0	0	0	0	0
12.244	0	0	0	0	0
12.325	0	0	0	0	0
12.387	0	0	1	0	0
12.399	0	0	0	0	0
12.604	0	0	0	0	0
12.626	0	0	0	0	0
13.683	0	0	0	0	0
13.732	0	0	0	0	0
13.774	0	0	0	0	0
13.994	0	0	0	0	0
14.165	0	0	1	0	0
14.282	1	1	1	1	0
14.418	0	0	0	0	0
14.483	0	0	0	0	0
14.516	0	0	0	0	0
14.595	0	0	0	0	0
14.666	0	0	0	0	0
14.786	0	0	0	0	0
14.880	0	0	0	0	0
15.013	0	0	0	0	0
15.424	0	0	0	0	0
15.443	0	0	0	0	0
15.445	0	0	0	0	0
15.617	0	0	0	0	0
15.632	0	0	0	0	0
15.858	0	0	0	0	0
16.243	0	0	1	0	0
16.244	0	0	0	0	0
16.618	0	0	0	0	0
16.770	0	0	0	0	0
16.787	0	0	0	0	0
16.805	0	0	0	0	0
16.912	0	0	0	0	0
17.069	0	0	0	0	0
17.130	0	0	0	0	0
17.206	0	0	0	0	0
17.400	0	0	1	0	0
18.075	0	0	1	0	0
18.284	0	0	0	0	0
18.495	0	0	0	0	0
18.608	0	1	2	1	0
18.751	0	0	0	0	1
18.767	0	0	0	0	0

	18.777	0	0	0	0	0
	18.810	0	0	0	0	0
	18.916	0	1	1	1	0
	19.258	0	0	0	0	0
	19.289	0	0	1	0	0
	19.429	0	0	0	0	0
	19.595	0	0	0	0	0
	19.615	0	0	0	0	0
	19.868	0	0	0	0	0
	19.877	0	0	1	0	0
	20.203	0	0	0	0	0
	20.217	0	0	1	0	0
	20.506	0	0	0	0	0
	20.663	0	0	0	0	0
	21.396	0	0	0	0	1
	21.409	0	0	0	0	0
	21.563	0	0	0	0	0
	22.013	0	0	0	0	0
	22.195	0	0	1	0	0
	22.197	0	1	2	1	0
	22.368	0	0	0	0	0
	22.529	0	0	0	0	0
	22.733	0	0	0	0	0
	23.192	0	0	0	0	0
	23.303	0	0	0	0	0
	23.655	0	0	1	0	0
	23.694	0	0	0	0	0
	23.761	0	0	1	0	0
	23.765	0	0	1	0	0
	24.134	0	0	0	0	0
	24.475	0	0	1	0	0
	25.379	0	1	1	0	0
	25.560	0	0	0	0	0
	26.289	1	1	1	1	0
	26.426	0	1	2	1	0
	26.887	0	0	0	0	0
	26.896	0	0	0	0	0
	27.258	0	0	0	0	0
	27.625	0	0	1	0	0
	28.476	0	0	1	0	0
	29.354	0	0	0	0	0
	30.051	0	0	1	0	0
	30.419	0	0	0	0	0
	31.109	1	1	2	1	0
	31.318	0	0	1	0	0
	31.483	0	0	0	0	0
	31.793	0	1	1	1	0
	32.372	0	0	0	0	1
	32.792	0	0	1	0	0
	33.748	0	0	1	0	1
	34.513	0	1	1	1	0
	42.475	0	0	0	0	0
	50.619	0	0	0	0	2
Number of Rejections of H_0	8	3	11	30	10	5
Levels	0.08	0.03	0.11	0.30	0.10	0.05

Table 7

Summary of results obtained in testing H_0: white noise in simulated series of length T = 200. Empirical power, level of significance 0.05

Series	Time domain		Frequency Domain			
	$\tilde{Q}$	Correlogram	S-Plus	df = 9.14	Normal	Normal correct.
White noise	0.08	0.05	0.03	0.11	0.30	0.10
AR(1), 0.30	0.76	0.90	0.34	0.37	0.58	0.25
AR(1), 0.60	1.00	1.00	1.00	1.00	1.00	1.00
AR(2)[Sect.5]	1.00	1.00	1.00	1.00	1.00	1.00

Figure 1

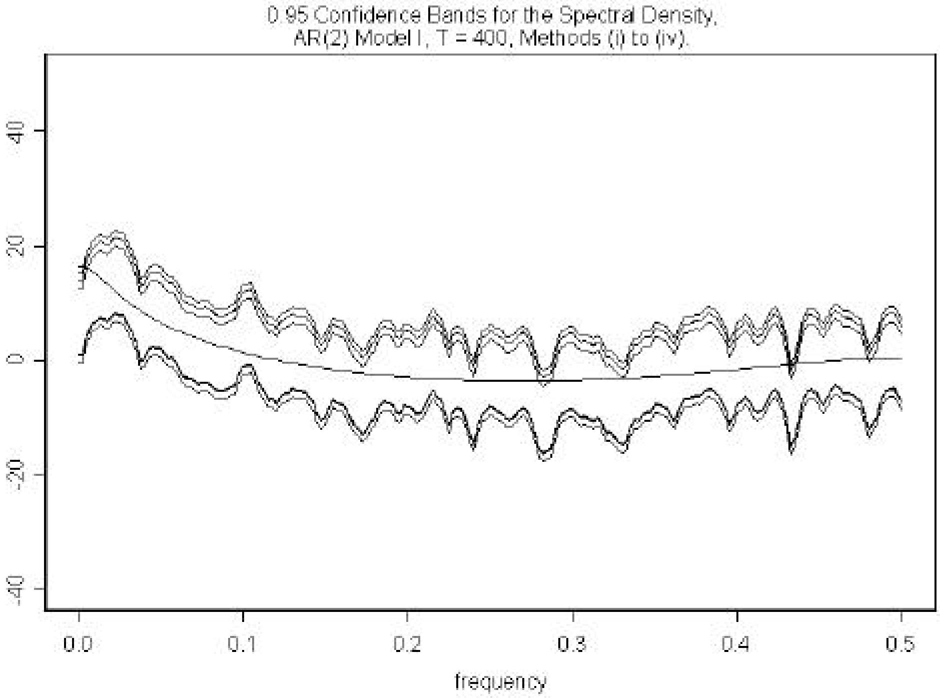


Figure 2

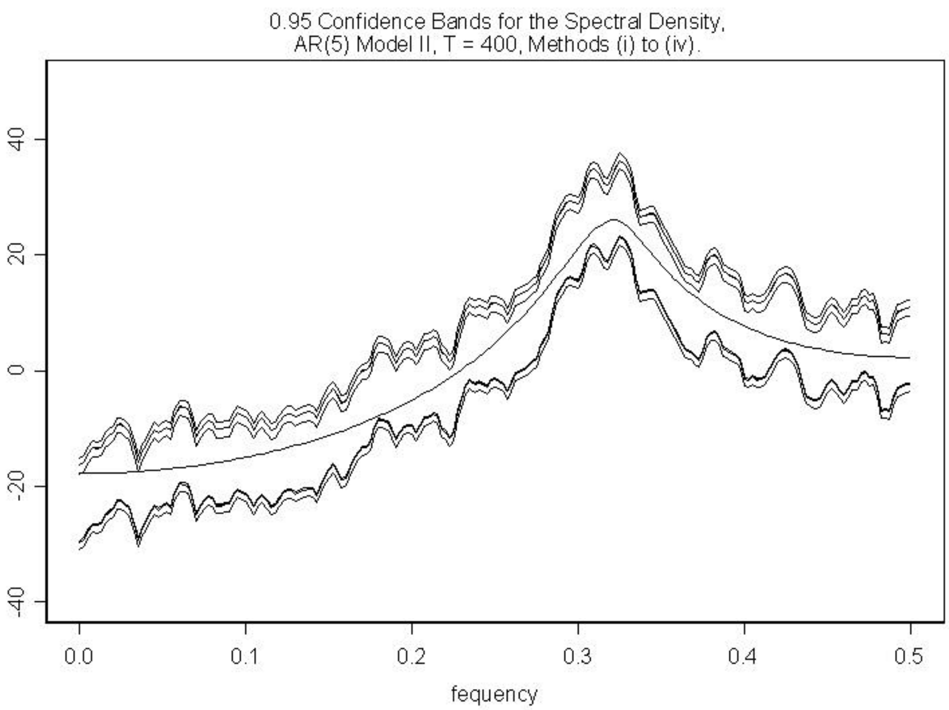


Figure 3

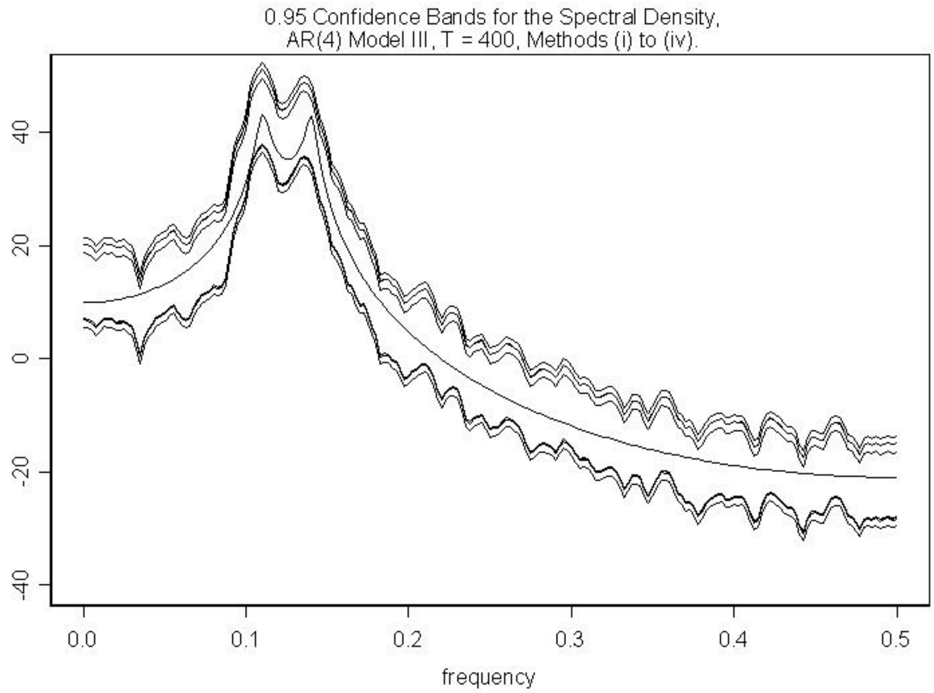


Figure 4

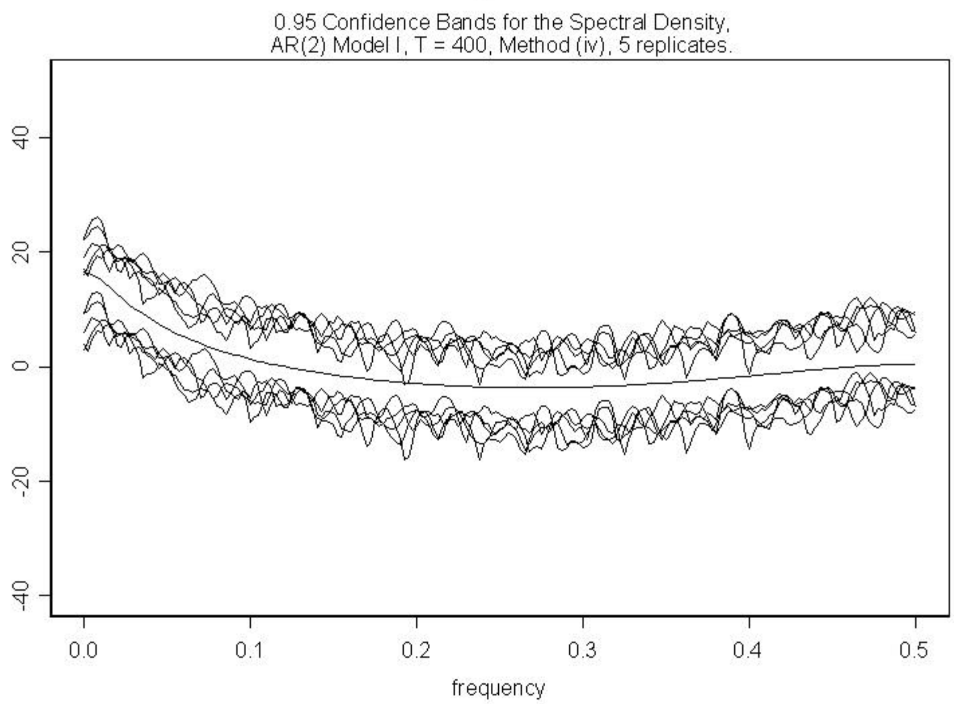


Figure 5

0.95 Confidence Bands for the Spectral Density,
AR(5) Model II, T = 400, Method (iv), 5 replicates.

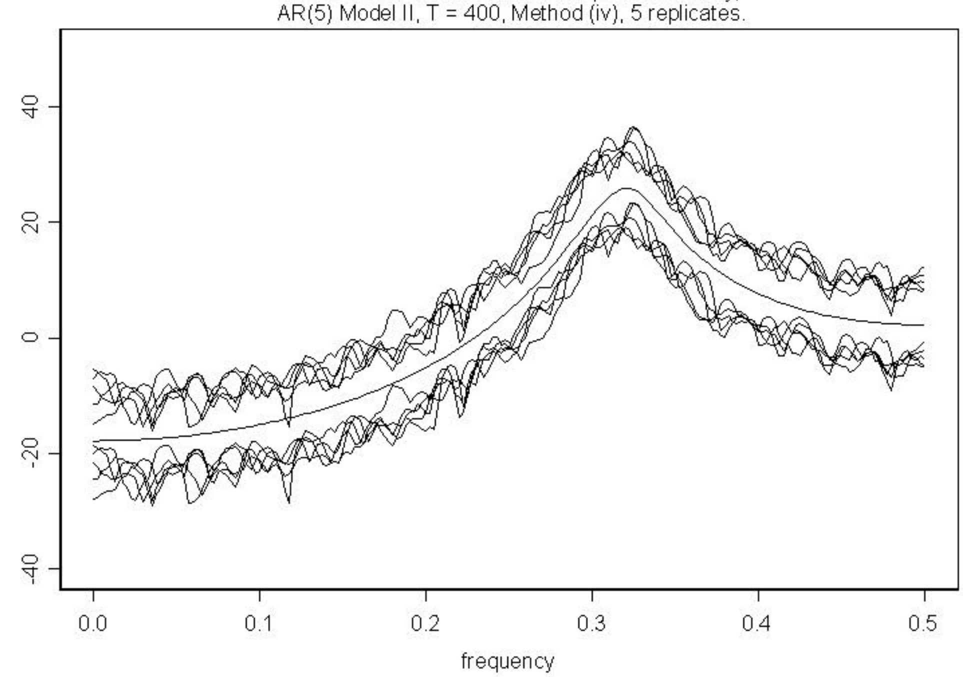


Figure 6

0.95 Confidence Bands for the Spectral Density,
AR(4) Model III, T = 100, Method (iv), 5 replicates.

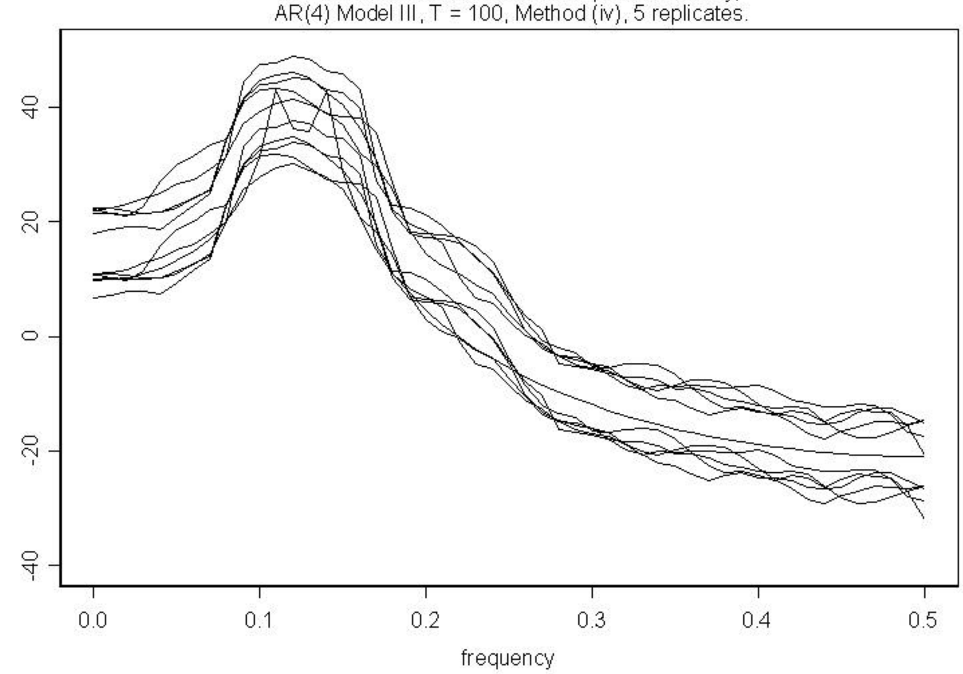


Figure 7

0.95 Confidence Bands for the Spectral Density,
AR(4) Model III, T = 200, Method (iv), 5 replicates.

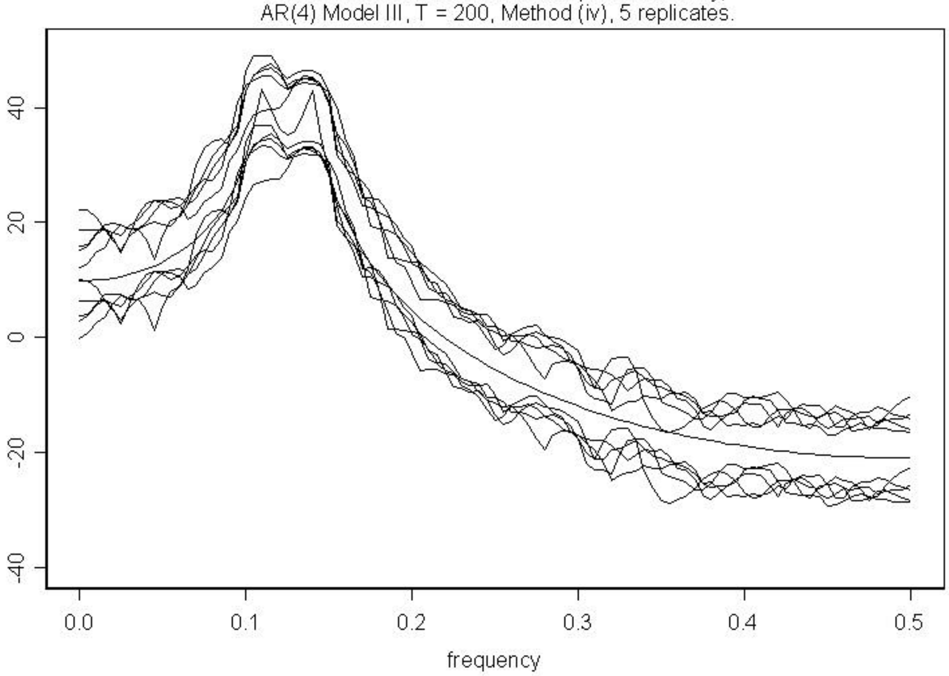


Figure 8

0.95 Confidence Bands for the Spectral Density,
AR(4) Model III, T = 400, Method (iv), 5 replicates.

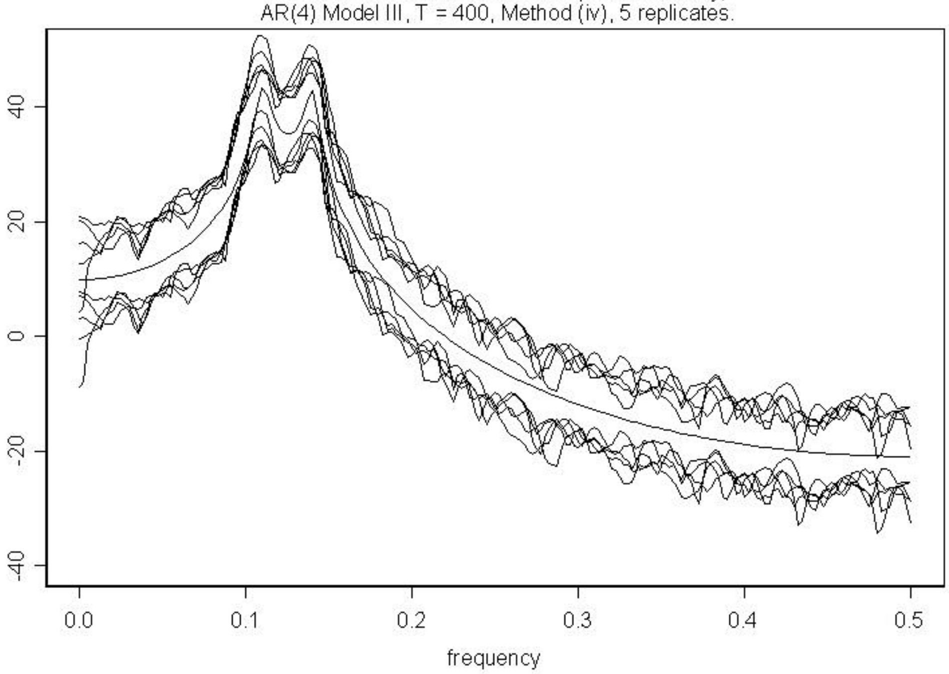
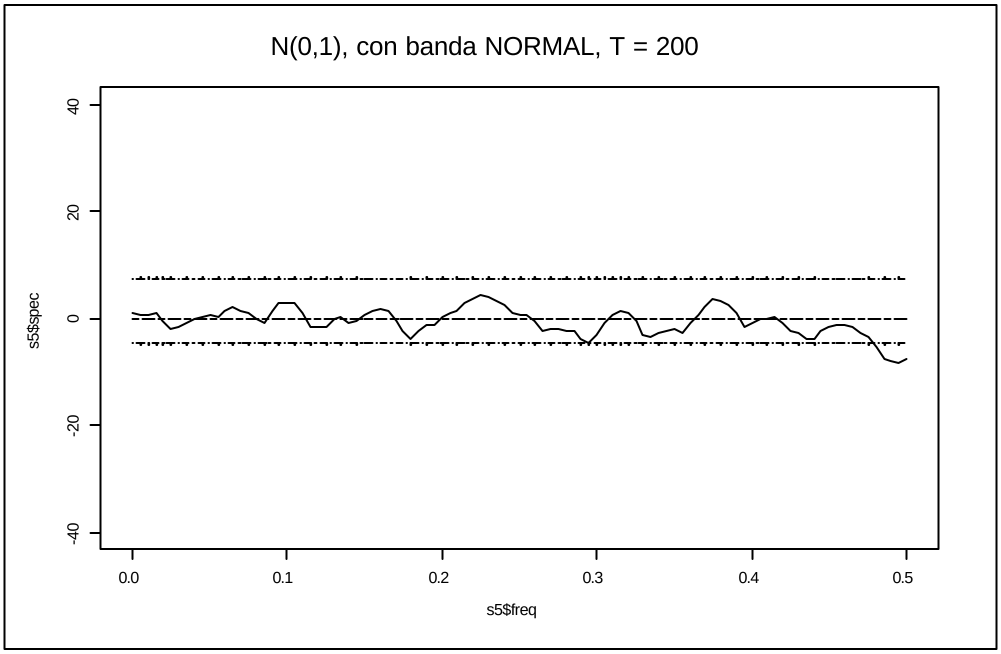


Figure 9

An example of a simulated white noise series, T = 200
tested for H_0: white noise, level of significance 0.05

a) Spectral density series



b) Correlogram

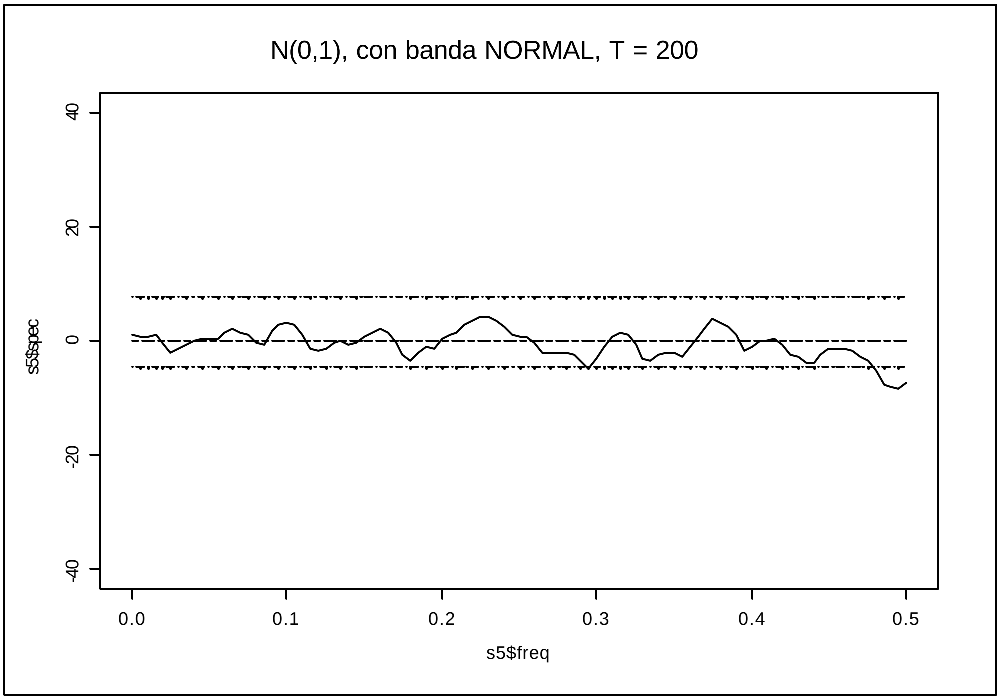
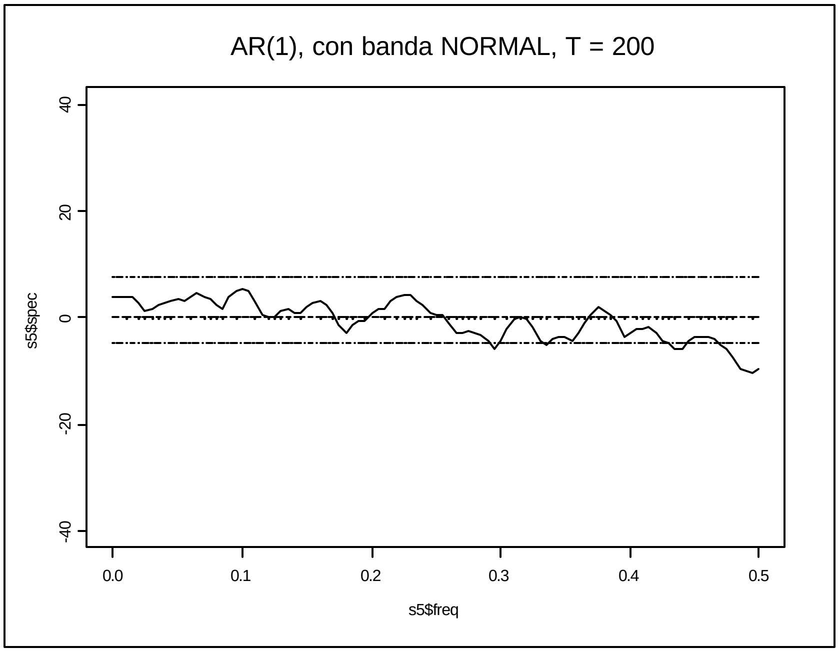


Figure 10

An example of a simulated AR(1) series with parameters 0.30, T = 200 tested for H_0: white noise, level of significance 0.05

a) Spectral density series



b) Correlogram

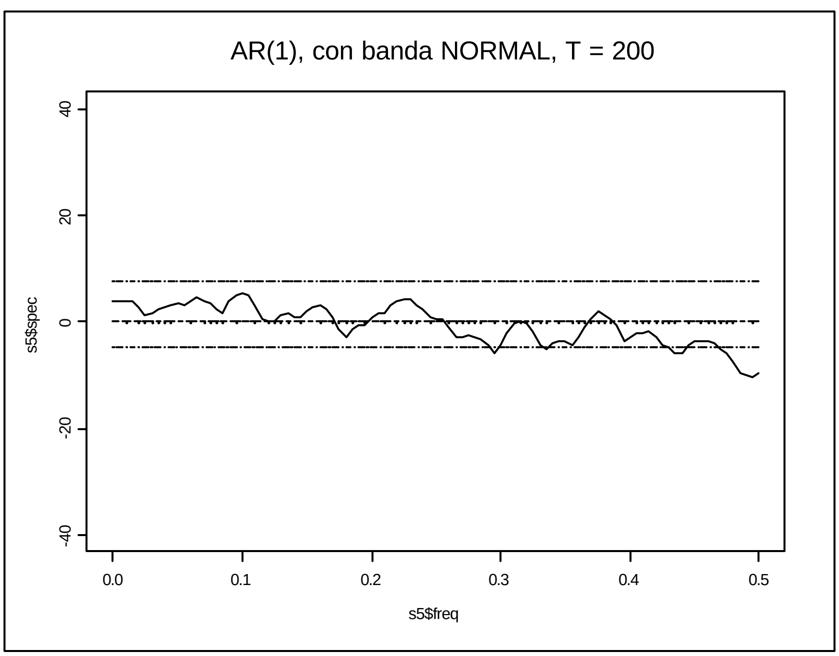
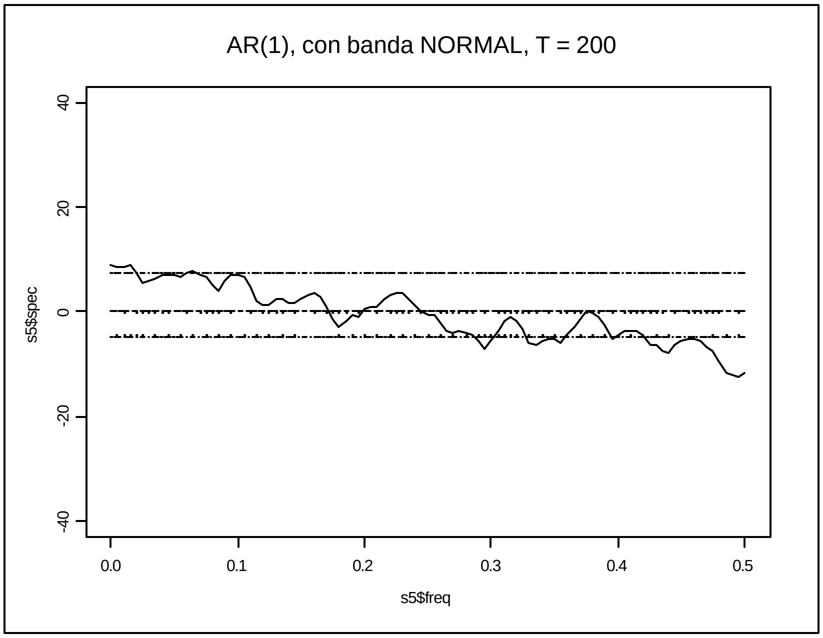


Figure 11

An example of a simulated AR(1) series with parameters 0.60, T = 200 tested for H_0: white noise, level of significance 0.05

a) Spectral density series



b) Correlogram