Metrics in the sphere of a C*-module*

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Abstract

Given a unital C^*-algebra $\mathcal{A}$ and a right C^*-module $\mathcal{X}$ over $\mathcal{A}$, we consider the problem of finding *short* smooth curves in the sphere $\mathcal{S}_{\mathcal{X}} = \{x \in \mathcal{X} :< x, x >= 1\}$. Curves in $\mathcal{S}_{\mathcal{X}}$ are measured considering the Finsler metric which consists of the norm of $\mathcal{X}$ at each tangent space of $\mathcal{S}_{\mathcal{X}}$. The initial value problem is solved, for the case when $\mathcal{A}$ is a von Neumann algebra and $\mathcal{X}$ is selfdual: for any element $x_0 \in \mathcal{S}_{\mathcal{X}}$ and any tangent vector v at x_0, there exists a curve $\gamma(t) = e^{tZ}(x_0)$, $Z \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$, $Z^* = -Z$ and $\|Z\| \leq \pi$, such that $\gamma(0) = x_0$ and $\dot{\gamma}(0) = v$, which is minimizing along its path for $t \in [0, 1]$. The existence of such Z is linked to the extension problem of selfadjoint operators. Such minimal curves need not be unique. Also we consider the boundary value problem: given $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$, find a curve of minimal length which joins them. We give several partial answers to this question. For instance, let us denoteb

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1 Introduction

The sphere $\mathcal{S}_{\mathcal{X}}$ of a right Hilbert C^*-module $\mathcal{X}$ over a unital C^*-algebra $\mathcal{A}$, which consists of the elements $x \in \mathcal{X}$ such that $< x, x >= 1$, is a C$^\infty$ submanifold of the (Banach space) $\mathcal{X}$. Its basic topological and differentiable aspects were considered in [2]. In this paper we consider the geometric problem of finding short smooth curves in $\mathcal{S}_{\mathcal{X}}$. To measure the length of a smooth curve we endow each tangent space (which we describe below, and is a complemented real Banach subspace of $\mathcal{X}$), with the norm of $\mathcal{X}$. Therefore the length of a curve $\gamma(t) \in \mathcal{S}_{\mathcal{X}}$, $t \in [a, b]$ is measured by

$$length(\gamma) = \int_a^b \|\dot{\gamma}(t)\| \, dt,$$

where $\| \ \|$ denotes the norm of $\mathcal{X}$. We refer the reader to [11] for basic facts on C^*-modules. As is usual notation, let $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$ be the C^*-algebra of adjointable linear operators acting on $\mathcal{X}$. If $y, z \in \mathcal{X}$, let $y \otimes z \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$ be the operator $y \otimes z(x) = y < z, x >$. For example, it is easy to see that if $x \in \mathcal{S}_{\mathcal{X}}$, then $x \otimes x$ is a selfadjoint projection, which we shall denote by e_x. Let $\mathcal{U}(\mathcal{X})$ be the unitary group of $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$. Perhaps the main feature in the geometry of $\mathcal{S}_{\mathcal{X}}$ (as with classical spheres) is the natural action of $\mathcal{U}(\mathcal{X})$ on $\mathcal{S}_{\mathcal{X}}$:

$$U \cdot x = U(x), \quad U \in \mathcal{U}(\mathcal{X}), \; x \in \mathcal{S}_{\mathcal{X}}.$$

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In [2] it was shown that if $x_0, x_1 \in \mathcal{S}_\mathcal{X}$ verify $\|x_0 - x_1\| < 1/2$, then x_0 and x_1 are conjugate by this action, moreover, one can find a unitary operator $U_{(x_0,x_1)}$, which is a C^∞ function in (x_0, x_1) such that $U_{(x_0,x_1)}(x_0) = x_1$. In particular the action is locally transitive. It is globally transitive in some cases (e.g. if $\mathcal{X}$ is selfdual [15] and $\mathcal{A}$ is a finite von Neumann algebra). In general, $\mathcal{S}_\mathcal{X}$ has many components: take for instance $\mathcal{X} = \mathcal{B}(\mathcal{H})$ with the inner product $< X, Y >= X^*Y$, then the sphere is the set of isometries of $\mathcal{H}$, whose connected components are parametrized by the codimension of the range.

The existence of local cross sections for the action (namely, the unitaries $U_{(x_0,x_1)}$), implies that for any fixed $x_0 \in \mathcal{S}_\mathcal{X}$, the map

$$\pi_{x_0} : \mathcal{U}(\mathcal{X}) \to \mathcal{S}_\mathcal{X}, \quad \pi_{x_0}(U) = U(x_0)$$

is a locally trivial fibre bundle and a C^∞ submersion. It follows that any smooth curve $\gamma(t) \in \mathcal{S}_\mathcal{X}$ can be lifted to a smooth curve $\mu(t) \in \mathcal{U}(\mathcal{X})$, and therefore represented $\gamma(t) = \mu(t) \cdot x_0$ for some $x_0 \in \mathcal{S}_\mathcal{X}$. This enables one to compute the tangent spaces of $\mathcal{S}_\mathcal{X}$:

$$(T\mathcal{S}_\mathcal{X})_{x_0} = \{A(x_0) : A \in \mathcal{L}_\mathcal{A}(\mathcal{X}), A^* = -A\}.$$

Clearly these elements are also characterized by the condition

$$(T\mathcal{S}_\mathcal{X})_{x_0} = \{v \in \mathcal{X} :< v, x_0 > + < x_0, v >= 0\}.$$

It is natural to ask whether one can find curves of the form

$$\gamma(t) = e^{tZ}(x_0), \quad t \in [0,1], Z^* = -Z,$$

which have minimal length joining their endpoints, or more strictly, which have minimal length along their paths.

There are two main problems.

1. The initial value problem: for any tangent vector $v \in (T\mathcal{S}_\mathcal{X})_{x_0}$ find a curve γ as above (in particular $\gamma(0) = x_0$), with $\dot\gamma(0) = v$, such that γ has minimal length.
2. The boundary value problem: given x_0, x_1 in the same component of $\mathcal{S}_\mathcal{X}$, find a minimal curve γ as above, which joins x_0 and x_1.

In this paper we solve the initial value problem: we show that if $\mathcal{A}$ is a von Neumann algebra and $\mathcal{X}$ is a right C^*-module, which is selfdual [15], then for any $x_0 \in \mathcal{S}_\mathcal{X}$ and any tangent vector $v \in (T\mathcal{S}_\mathcal{X})_{x_0}$ with $\|v\| \leq \pi$ there exists a curve $\gamma(t) = e^{tZ}(x_0)$ with $\gamma(0) = x_0$ and $\dot\gamma(0) = v$, which has minimal length along its path for $t \in [0,1]$. The antihermitic operator Z implementing this geodesic is the solution of the extension problem by M.G. Krein [10], in the context of von Neumann algebras (see [6]), as it will be shown in the next section. We call such Z *minimal lifts*, following [7].

We also consider the boundary value problem. We prove that if $< x_0, x_1 >$ is a scalar multiple of the identity, then x_0 and x_1 can be joined by a minimizing geodesic (Proposition (4.1)). Another case in which there exists a short geodesic joining x_0 and x_1 occurs when the (non empty) set $\{\|Z\| : Z^* = -Z, \ e^Z(x_0) = x_1\}$ has a minimum (Theorem (4.3)). As a consequence, we obtain that if $f_0(\mathcal{X})$ is finite dimensional ($f_0 = I - e_{x_0}$), then there exists such a geodesic. In section 5 we introduce a metric in $\mathcal{S}_\mathcal{X}$, by means of the states of $\mathcal{A}$, which induce Hilbert space representations of the sphere $\mathcal{S}_\mathcal{X}$. We compare this metric with the Finsler metric. For example, it is shown that they coincide whenever there exist minimal lifts (Theorem (5.4)).

2 Extension problem in von Neumann algebras

A simplified version of the extension problem ([10], [14], [6]) could be stated as follows: given a closed subspace $\mathcal{L}$ of a Hilbert space $\mathcal{H}$ and a bounded symmetric operator $A_0 : \mathcal{L} \to \mathcal{H}$, find a selfadjoint extension $A : \mathcal{H} \to \mathcal{H}$ with $\|A\| = \|A_0\|$. This problem was solved, and all solution parametrized. We remark that extensions can, but in general need not, be unique. See for example [6] or [14] for explicit parametrizations. M.G. Krein [10] showed that there exist a minimal and a maximal solution (in terms of the usual order of selfadjoint operators), and that all solution lie in between. For our purposes, we need the additional requierement that if $P = P_{\mathcal{L}}$ (=the orthogonal projection onto $\mathcal{L}$) and A_0 lie in a von Neumann algebra $\mathcal{B}$, then there exists a solution of the extension problem in $\mathcal{B}$. By this we mean the following result, which is a consequence of the parametrization of solutions given by Davis, Kahan and Wi

Lemma 2.1 *Let A be a selfadjoint element and P a selfadjoint projection in a von Neumann algebra $\mathcal{B}$. Then there exists a selfadjoint element Z in $\mathcal{B}$ such that $ZP = AP$ and $\|Z\| = \|AP\|$.*

Proof. Let A and $P \in \mathcal{B}$ be as above. Choose a representation of the von Neumann algebra $\mathcal{B}$ in $\mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ a Hilbert space. Let us consider the following selfadjoint 2×2 block operators in terms of P and $(I - P)$:

$$Z_X = \begin{pmatrix} PAP & (I-P)AP \\ PA(I-P) & X \end{pmatrix}$$

where X is a selfadjoint operator in $\mathcal{B}\left((I-P)\mathcal{H}\right)$. These $Z_X \in \mathcal{B}(\mathcal{H})$ satisfy $Z_X P = AP$ and $\|Z_X\| \geq \|AP\|$.

As it was mentioned at the begining of this section, several authors dealt with the problem of minimizing the norm of Z_X. Theorem 1 in [14], for example, proves that in our context there exists an $X_0 \in \mathcal{B}((I-P)\mathcal{H})$ such that $\|Z_{X_0}\| = \|PA\|$ and X_0 is the weak limit of the following elements of $\mathcal{B}$: $-c_n(I-P)(I - d_n PAPAP)^{-1}PAPA(I-P)$ (where $\{c_n\}$ and $\{d_n\}$ are sequences of real numbers). Therefore this X_0 belongs to $\mathcal{B}$ and then Z_{X_0} belongs to $\mathcal{B}$, and verifies $\|Z_{X_0}\| = \|PA\|$. $\square$

We state now a consequence of the result above, in the context of the modular spheres. Let $x_0 \in \mathcal{S}_{\mathcal{X}}$, and $v \in (T\mathcal{S}_{\mathcal{X}})_{x_0}$. We call an antihermitic operator $Z \in \mathcal{L}_a(\mathcal{X})$ a *minimal lift* of v if $Z(x_0) = v$ and $\|Z\| = \|v\|$.

Corollary 2.2 *Let $x_0 \in \mathcal{S}_{\mathcal{X}}$, with $\mathcal{X}$ a selfdual module over the von Neumann algebra $\mathcal{A}$, and $v \in (T\mathcal{S}_x)_{x_0}$. Then there exists a minimal lift Z of v.*

Proof. In this case, $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$ is a von Neumann algebra [15]. Since $v \in (T\mathcal{S}_x)_{x_0}$, there exists $A \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$ such that $-A = A^*$ and $A(x_0) = v$. Note that this implies that $A(x_0 \otimes x_0) = v \otimes x_0$. Moreover, the operator $v \otimes x_0$ has norm equal to the norm of v. Indeed, clearly $\|v \otimes x_0\| \leq \|v\|\|x_0\| = \|v\|$ because $\|x_0\| = 1$, and $\|v \otimes x_0\| \geq \|v \otimes x_0(x_0)\| = \|v\|$. Since $e_{x_0} = x_0 \otimes x_0$ is a selfadjoint projection in $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$, by the above lemma there exists $Z \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$ such that $Z^* = -Z$, $Ze_{x_0} = Ae_{x_0}$ and $\|Z\| = \|Ae_{x_0}\|$. In other words, $Z(x_0) = Ze_{x_0}(x_0) = Ae_{x_0}(x_0) = A(x_0) = v$, and $\|Z\| = \|v\|$. $\square$

3 The initial value problem

Let us state our main result.

Theorem 3.1 *Let $x_0 \in \mathcal{S}_{\mathcal{X}}$ and $v \in (T\mathcal{S}_{\mathcal{X}})_{x_0}$ with $\|v\| \le \pi$. Let Z be a minimal lift of v, i.e.*

$$Z^* = -Z, \ Z(x_0) = v \text{ and } \|Z\| = \|v\|.$$

Then the curve $\nu(t) = e^{tZ}(x_0)$, $t \in [0,1]$ which verifies $\nu(0) = x_0$ and $\dot\nu(0) = v$, has minimal length along its path among smooth curves in $\mathcal{S}_{\mathcal{X}}$.

Proof. Given a positive element A of a C^*-algebra, there exists a faithful representation of the algebra (for instance, the universal representation) and a unit vector ξ in the Hilbert space $\mathcal{H}$ of this representation, such that $A\xi = \|A\|\xi$ (here we identify A with its image under the representation). Let us call such a vector ξ a *norming eigenvector* for A. Let us apply this folklore fact to the positive operator $-eZ^2e$, where $e = e_{x_0}$. Let ξ be a (unit) norming eigenvector for $-eZ^2e$. Again we identify the operators with their images under this representation, and regard them as operators in this Hilbert space. Clearly ξ lies in the range of e. We claim that ξ is a norming eigenvector for $-Z^2$ as well. Indeed,

$$-Z^2\xi = -Z^2e\xi = -eZ^2P\xi - (I-e)Z^2P\xi = \|eZ^2e\|\xi + \xi_1,$$

where $\xi_1 = -(I-e)Z^2e\xi$ is orthogonal to ξ. Note that

$$\|eZ^2e\| = \|Ze\|^2 = \|Z\|^2 = \|Z^2\|.$$

Then

$$\|Z^2\|^2 \ge \|Z^2\xi\|^2 = \|eZ^2e\|^2 + \|\xi_1\|^2 = \|Z^2\|^2 + \|\xi_1\|^2.$$

It follows that $\xi_1 = 0$ and our claim is proved. Consider the curve $\hat\nu(t) = e^{tZ}(\xi)$. Clearly $\|\widehat{\nu(t)}\| = 1$, i.e. $\hat\nu(t)$ is a curve in the unit sphere $S_{\mathcal{H}}$ of the Hilbert space $\mathcal{H}$. Let us prove that it is a minimizing geodesic of this Riemann-Hilbert manifold. Indeed,

$$\ddot{\hat\nu}(t) = e^{tZ}Z^2\xi = -\|Z\|^2e^{tZ}\xi = -\|Z\|^2\hat\nu(t).$$

That is, $\hat\nu$ satisfies the differential equation of the geodesics of the sphere $S_{\mathcal{H}}$. Moreover, the length of $\hat\nu$ is

$$length(\hat\nu) = \int_0^1 \|\dot{\hat\nu}(t)\| \, dt = \|Z\xi\| \le \pi.$$

It follows that $\hat\nu$ is a minimizing geodesic of the unit sphere. Note also that

$$\|Z\xi\|^2 = < Z\xi, Z\xi > = < -Z^2\xi, \xi > = \|Z^2\| = \|Z\|^2.$$

Clearly, if $[t_0, t_1] \subset [0,1]$, the length of $\hat\nu$ restricted to $[t_0, t_1]$ (or shortly $\hat\nu|_{[t_0,t_1]}$) is $(t_1 - t_0)\|Z\|$. On the other hand,

$$length(\nu|_{[t_0,t_1]}) = \int_{t_0}^{t_1} \|\dot\nu\| \, dt = (t_1 - t_0)\|Z(x_0)\| = (t_1 - t_0)\|Z\|.$$

It follows that $length(\hat\nu) = length(\nu)$ on any subinterval of $[0,1]$.

Suppose now that $\gamma : [a,b] \to \mathcal{S}_{\mathcal{X}}$ is a smooth curve joining $\nu(t_0)$ and $\nu(t_1)$. Consider the curve $\hat\gamma(t) := \gamma(t) \otimes x_0(\xi)$. Note that $\hat\gamma$ is also a curve in the unit sphere of $\mathcal{H}$:

$$< \hat\gamma(t), \hat\gamma(t) >_{\mathcal{H}} = < (\gamma(t) \otimes x_0)^*(\gamma(t) \otimes x_0)\xi, \xi >_{\mathcal{H}} = < (x_0 \otimes \gamma(t))(\gamma(t)) \otimes x_0\xi, \xi >_{\mathcal{H}} = < e\xi, \xi >_{\mathcal{H}} = 1.$$

Moreover,

$$\|\dot{\hat\gamma}(t)\| = \|(\dot\gamma(t) \otimes x_0)\xi\| \ge \|\dot\gamma(t) \otimes x_0\| = \|\dot\gamma(t)\|.$$

This implies that $length(\gamma) \leq length(\hat{\gamma})$. Finally, let us show that $\hat{\nu}|_{[t_0,t_1]}$ and $\hat{\gamma}$ join the same endpoints of $\mathcal{S}_{\mathcal{H}}$:

$$\hat{\nu}(t_0) = e^{t_0 Z}\xi = e^{t_0 Z}e\xi = e^{t_0 Z}(x_0 \otimes x_0)\xi = (e^{t_0 Z}(x_0) \otimes x_0)\xi = (\nu(t_0) \otimes x_0)\xi = (\gamma(t_0) \otimes x_0)\xi = \hat{\gamma}(t_0),$$

and similarly for t_1. By the minimality of $\hat{\nu}$, it follows that $length(\hat{\nu}|_{[t_0,t_1]}) \leq length(\hat{\gamma})$. Therefore

$$length(\nu|_{[t_0,t_1]}) = length(\nu|_{[t_0,t_1]}) \leq length(\hat{\gamma}) \leq lenght(\gamma),$$

which completes the proof.

□

Corollary 3.2 *If $\mathcal{A}$ is a von Neumann algebra and $\mathcal{X}$ is a selfdual module, then for any element $x_0 \in \mathcal{S}_{\mathcal{X}}$ and tangent vector $v \in (T\mathcal{S}_{\mathcal{X}})_{x_0}$ with $\|v\| \leq \pi$, there exists a geodesic δ with $\delta(0) = x_0$, $\dot{\delta}(0) = v$, such that δ is minimizing along its path for $t \in [0, 1]$.*

Proof. In this case, minimal lifts exist for any tangent vector v. □

4 Geodesics joining given endpoints

The problem of finding minimizing geodesics given any pair of points (in the same component) of the sphere $\mathcal{S}_{\mathcal{X}}$ is more difficult. It is related to the analogous problem for abstract homogeneous spaces [8]. In this section we find solutions in certain cases. These results work for arbitrary C^*-algebras and modules.

Proposition 4.1 *Let $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$ with $< x_0, x_1 >= \alpha.1$, for $\alpha \in \mathbb{C}$. Then there exists a smooth curve in $\mathcal{S}_{\mathcal{X}}$ with minimal length along its path, which joins x_0 and x_1.*

Proof. Note that since $\| < x_0, x_1 > \| \leq \|x_0\| \; \|x_1\| = 1$, it follows that $|\alpha| \leq 1$. If $|\alpha| = 1$, then $\alpha = e^{ir}$ with $|r| \leq \pi$. In this case clearly $x_1 = \alpha x_0$. Indeed,

$$< x_1 - \alpha x_0, x_1 - \alpha x_0 >=< x_1, x_1 > - < x_1, \alpha x_0 > - < \alpha x_0, x_1 > + < \alpha x_0, \alpha x_0 >= 0.$$

Put $\gamma(t) = e^{irt}x_0$. Apparently γ is minimizing along its path (for instance, $\|re\| = r$, i.e. the operator rI is a minimal lift).

If $|\alpha| < 1$, let $\beta \in \mathbb{C}$ be such that $|\alpha|^2 + |\beta|^2 = 1$ (note that $\beta \neq 0$), and consider $y = \alpha\beta^{-1}x_0 - \beta^{-1}x_1$. Then clearly

$$< x_0, y >= \alpha\beta^{-1}1 - \beta^{-1} < x_0, x_1 >= 0,$$

and

$$< y, y >= \frac{|\alpha|^2}{|\beta|^2} - \frac{\bar{\alpha}}{|\beta|^2} < x_0, x_1 > - \frac{\alpha}{|\beta|^2} < x_1, x_0 > + \frac{1}{|\beta|^2} = 1.$$

in other words, $x_1 = \alpha x_0 + \beta y$ with $y \in \mathcal{S}_{\mathcal{X}}$. That is, x_1 lies in the complex plane generated by two orthogonal elements x_0 and y of $\mathcal{S}_{\mathcal{X}}$. The situation resembles what happens in a classic finite dimensional sphere, and the proof follows as in that case. Namely, let $(\alpha(t), \beta(t))$ be a minimal geodesic of the sphere $\mathcal{S}_{\mathbb{C}^2}$ of $\mathbb{C}^2$, joining $(1, 0)$ (at $t = 0$) and (α, β) (at $t = 1$). Consider the curve $\gamma(t) = \alpha(t)x_0 + \beta(t)y$. Clearly it is a smooth curve with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, which lies in $\mathcal{S}_{\mathcal{X}}$:

$$< \gamma(t), \gamma(t) >= |\alpha(t)|^2 + |\beta(t)|^2 = 1.$$

Moreover, it has constant speed equal to

$$\|\dot{\gamma}(t)\|^2 = \| < \dot{\alpha}(t)x_0 + \dot{\beta}(t)x_1, \dot{\alpha}(t)x_0 + \dot{\beta}(t)x_1 > \|^2 = |\dot{\alpha}(t)|^2 + |\dot{\beta}(t)|^2 = |\dot{\alpha}(0)|^2 + |\dot{\beta}(0)|^2 = \|\dot{\gamma}(0)\|^2.$$

We claim that it is minimizing along its path. Let φ be a state in $\mathcal{A}$. Then the form

$$[x,y]_\varphi := \varphi(<x,y>), \quad x,y \in \mathcal{X}$$

is positive semidefinite in $\mathcal{X}$. Let $\mathcal{H}_\varphi$ be the completion of $(\mathcal{X}/\mathcal{Z}, [\ ,\ ]_\varphi)$, where $\mathcal{Z} = \{z \in \mathcal{X} : [z,z]_\varphi = 0\}$. Denote by $\bar{x}$ be the class of $x \in \mathcal{X}$ in $\mathcal{X}/\mathcal{Z} \subset \mathcal{H}_\varphi$. In other words, $\bar{x}$ is the element x regarded as a vector in the Hilbert space $\mathcal{H}_\varphi$. Note that the elements of $\mathcal{S}_\mathcal{X}$ induce elements in the unit sphere of $\mathcal{H}_\varphi$: clearly $[\bar{x},\bar{x}]_\varphi = \varphi(<x,x>) = 1$

The geodesic $(\alpha(t),\beta(t))$ of $\mathcal{S}_{\mathbb{C}^2}$ satisfies the Euler equation of the sphere:

$$(\ddot{\alpha}(t),\ddot{\beta}(t)) = -\kappa^2(\alpha(t),\beta(t)).$$

It follows that $\bar{\gamma}$ satisfies the differential equation

$$\ddot{\bar{\gamma}}(t) = -\kappa^2\bar{\gamma}(t),$$

in the sphere $\mathcal{S}_{\mathcal{H}_\varphi}$ of $\mathcal{H}_\varphi$. Moreover, the length of $\bar{\gamma}$ restricted to the interval $[t_1,t_2] \subset [0,1]$, is given by

$$\int_{t_0}^{t_1} [\dot{\bar{\gamma}}(t),\dot{\bar{\gamma}}(t)]^{1/2}\ dt = \int_{t_0}^{t_1} \varphi(<\dot{\alpha}(t)x_0 + \dot{\beta}(t)y, \dot{\alpha}(t)x_0 + \dot{\beta}(t)y>)^{1/2}\ dt$$

$$= \int_{t_0}^{t_1} \varphi(|\dot{\alpha}(t)|^2.1 + |\dot{\beta}(t)|^2.1)^{1/2}\ dt = (t_1 - t_0)\|\dot{\gamma}(0)\|.$$

It follows that $\bar{\gamma}$ is minimizing along its path in $\mathcal{S}_{\mathcal{H}_\varphi}$, and

$$length(\bar{\gamma}) = length(\gamma).$$

Let $\nu(t)$, $t \in [0,1]$ be another smooth curve in $\mathcal{S}_\mathcal{X}$ joining $\nu(0) = \gamma(t_0)$ and $\nu(1) = \gamma(t_1)$. Then $\bar{\nu}$ is a smooth curve in $\mathcal{S}_{\mathcal{H}_\varphi}$, and the inequality

$$[\dot{\bar{\nu}},\dot{\bar{\nu}}]_\varphi = \varphi(<\dot{\nu},\dot{\nu}>) \le \| <\dot{\nu},\dot{\nu}> \|$$

implies that

$$length(\nu) \ge length(\bar{\nu}).$$

It follows that ν is not shorter than $\gamma|_{[t_0,t_1]}$. □

If $x_0, x_1 \in \mathcal{S}_\mathcal{X}$ satisfy that $\|x_0 - x_1\| < 1/2$, then they are conjugate by the action of $\mathcal{U}(\mathcal{X})$ (see [2]). Let us state the following result, estimating the distance between the identity and the unitary operator performing this conjugacy.

Lemma 4.2 *Let $x_0, x_1 \in \mathcal{S}_\mathcal{X}$ with $\|x_0 - x_1\| < 1/2$. Then there exists a unitary $U \in \mathcal{U}(\mathcal{X})$ such that $U(x_0) = x_1$ with $\|U - I\| < 3/2$.*

Proof. First we transcribe the construction of the unitary U given in [2]. Let $e_0 = e_{x_0}$ and $e_1 = e_{x_1}$. Since $\|x_0 - x_1\| < 1/2$, it follows that

$$\|e_0 - e_1\| \le \|e_0 - x_0 \otimes x_1\| + \|x_0 \otimes x_1 - e_1\| = \|x_0 \otimes (x_0 - x_1)\| + \|(x_1 - x_0) \otimes x_1\|.$$

Note that $\|x_0 \otimes (x_0 - x_1)\| \le \|x_0 - x_1\|$ (in fact equality holds because $x_0 \in \mathcal{S}_\mathcal{X}$), and analogously for the other term. Therefore $\|e_0 - e_1\| < 1$. It is a standard fact that two such projections are unitarily equivalent, morever, the unitary V such that $Ve_0V^* = e_1$ can be chosen $V = e^Y$ with

$Y \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$ such that $Y^* = -Y$ and $\|Y\| < \pi/2$ (moreover, Y is codiagonal in terms of e_0 and $\sin \|Y\| = \|e_0 - e_1\|$, see for instance [3], page 151). Therefore

$$\|I - V\| = r(I - V) = \text{supp}\{|1 - e^{\omega}| : \omega \in sp(Y)\} < \sqrt{2},$$

because $|\omega| \leq \|Y\| < \pi/2$ (here sp and r stand for the spectrum and the spectral radius, respectively). Consider

$$U = x_1 \otimes x_0 + V(I - e_0).$$

This unitary verifies that $U(x_0) = x_1$, and moreover,

$$\|I - U\| = \|e_1 - x_1 \otimes x_0 + (I - e_1) - V(I - e_0)\|.$$

Since $V(I - e_0)V^* = e_1$, it follows that the operators $e_1 - x_1 \otimes x_0$ and $(I - e_1) - V(I - e_0)$ have orthogonal ranges (in any Hilbert space representation for $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$). Indeed, the range of $e_1 - x_1 \otimes x_0 = e_1(I - x_1 \otimes x_0)$ is contained in the range of e_1, and the range of $(I - e_1) - V(I - e_0) = (I - e_1) - (I - e_1)V^*$ is contained in its orthogonal complement. Thus

$$\|I - U\| \leq \sqrt{\|e_1 - x_1 \otimes x_0\|^2 + \|I - e_0 - V(I - e_0)\|^2}.$$

Note that $\|e_1 - x_1 \otimes x_0\| = \|x_1 \otimes (x_1 - x_0)\| = \|x_1 - x_0\| < 1/2$ and

$$\|I - e_0 - V(I - e_0)\| = \|(I - e_0)(I - V)\| \leq \|I - V\| \leq \sqrt{2}.$$

Then

$$\|I - U\| < 3/2.$$

□

In particular, by a standard argument involving the continuous functional calculus in the C^*-algebra $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$, the unitary U of the lemma above is of the form $U = e^Z$ for $Z \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})$, with $Z^* = -Z$ and $\|Z\| < \pi/3$ (using the same computation as in the norm of $I - V$ above).

Denote by

$$\mathcal{L}_{x_0,x_1} = \{Z \in \mathcal{L}_{\mathcal{A}}(\mathcal{X}) : Z^* = -Z, e^Z(x_0) = x_1\}.$$

If $\|x_0 - x_1\| < 1/2$, then $\mathcal{L}_{x_0,x_1}$ is non empty. If x_0, x_1 are not that close, but they lie in the same component of $\mathcal{S}_{\mathcal{X}}$, the algebra $\mathcal{A}$ is a von Neumann algebra, and the module $\mathcal{X}$ is selfdual, one also has that $\mathcal{L}_{x_0,x_1}$ is non empty, with the unitary chosen such that $\|Z\| \leq \pi$. If moreover $\mathcal{A}$ is finite, then $\mathcal{S}(\mathcal{X})$ is connected, and any pair of elements in the sphere are conjugate by an exponential.

The following result is an adaptation of Theorem 3.2 in [8], to our particular context, where the Finsler metric is given by the norm of $\mathcal{X}$ (in [8] quotient norms are considered).

Theorem 4.3 *Let $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$, with $\|x_0 - x_1\| < 1/2$. Suppose that there exists $Z_0 \in \mathcal{L}_{x_0,x_1}$ such that*

$$\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_1}\}.$$

Then Z_0 is a minimal lift and therefore $\nu(t) = e^{tZ_0}(x_0)$ is minimizing along its path. In particular, it is shorter than any other piecewise smooth curve joining x_0 and x_1 in $\mathcal{S}_{\mathcal{X}}$.

Proof. The proof, as in 3.2 of [8], proceeds in three steps:

- a) Let $Z_0 \in \mathcal{L}_{x_0,x_1}$ with $\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_1}\}$, fix $s \in (0, 1)$ and denote $x_s = e^{sZ_0}(x_0)$. Then $sZ_0 \in \mathcal{L}_{x_0,x_s}$ and $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_s}\}$.

- b) Suppose that X, Y are antihermitic operators of small norms in order that $e^X e^Y$ lies in the domain of the power series of the logarithm log defined on a neighbourhood of I with antihermitic values. (for intance, $\|e^X e^Y - I\| < 1$). Then

$$\log(e^X e^Y) = X + Y + R_2(X, Y),$$

where

$$\lim_{s \to 0} \frac{R_2(sX, sY)}{s} = 0.$$

- c) Let $e = e_{x_0}$. For any $Y^* = -Y$ such that $Y = (I - e)Y(I - e)$, one has that

$$\|Z_0\| \leq \|Z_0 + Y\|.$$

Let us prove these steps, and show how they prove our result.

Step a):

For an element $X^* = -X$, denote by $\gamma_X(t) = e^{tX}$. We claim that the condition $\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_1}\}$ implies that the curve γ_{Z_0} is the shortest among piecewise smooth curves of unitaries joining I to the set $\{U \in \mathcal{U}(\mathcal{H}) : U(x_0) = x_1\}$. Indeed, by the remark above, since $\|x_0 - x_1\| < 1/2$, there exists $X \in \mathcal{L}_{x_0,x_1}$ such that $\|X\| \leq \pi/3$. It follows that $\|Z_0\| \leq \pi/3$. Suppose that $\mu(t)$ is another smooth curve of unitaries with $\mu(0) = I$ and $\mu(1)(x_0) = x_1$, which is shorter than γ_{Z_0}. Let $\mathcal{L}_{\mathcal{A}}(\mathcal{X})^{**}$ be the von Neumann enveloping algebra of $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$. Then there is a curve of the form $e^{t\Omega}$, $\Omega^* = -\Omega \in \mathcal{L}_{\mathcal{A}}(\mathcal{X})^{**}$ and $\|\Omega\| < \pi/3$, with $e^{\Omega} = \mu(1)$, which is shorter than μ. This follows from the folklore fact that exponentials are short curves in the unitary group of a von Neumann algebra, when the length is measured with the Finsler metric given by the usual norm (see for instance [5]). It follows that $\|I - \mu(1)\| < 3/2$.T

Let us show that $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_s}\}$. Suppose that there exists $X \in \mathcal{L}_{x_0,x_s}$ such that $\|X\| < s\|Z_0\|$. Consider the curve $\delta(t) = e^{(1-t)sZ_0 + tZ_0}$ which joins e^{sZ_0} with e^{Z_0} in $\mathcal{U}(\mathcal{X})$, and $\sigma(t) = \delta(t)e^{-sZ_0}e^X$, joining e^X and $e^{(1-s)Z_0}e^X$ (in both cases $t \in [0,1]$). Note that they have the same length, for they differ on an element of $\mathcal{U}(\mathcal{X})$: $length(\delta) = length(\sigma) = (1-s)\|Z_0\|$. Note also that the endpoint of σ satisfies $\sigma(1)x_0 = x_1$. Let $\tilde{\gamma}$ be the piecewise smooth curve which consists of the curve γ_X followed by σ. Then $\tilde{\gamma}$ joins I to the fiber $\{U \in \mathcal{U}(\mathcal{X}) : U(x_0) = x_1\}$ in $\mathcal{U}(\mathcal{X})$, and therefore, by the fact remarked above, $length(\tilde{\gamma}) \geq \|Z_0\|$. On the other hand,

$$length(\tilde{\gamma}) = length(\gamma_X) + length(\sigma) = \|X\| + (1-s)\|Z_0\|$$

$$< s\|Z_0\| + (1-s)\|Z_0\| = \|Z_0\|.$$

Step b):

The linear part of the series of $\log(e^X e^Y)$ is $X + Y$. So that

$$\log(e^X e^Y) = X + Y + R_2(X, Y)$$

Where the remainder term $R_2(X, Y)$ is an infinitesimal of the order $\|X\| + \|Y\|$. Therefore

$$\lim_{s \to 0} \frac{R_2(sX, sY)}{s} = 0.$$

Step c):

By step a), for any $s \in (0,1)$, $s\|Z_0\| = \inf\{\|Z\| : Z \in \mathcal{L}_{x_0,x_s}\}$. Let $Y^* = -Y$ such that $Y = (I - e)Y(I - e)$. Then clearly $e^Y(x_0) = x_0$. Therefore $\log(e^{Z_0}e^Y) \in \mathcal{L}_{x_0,x_1}$. Analogously, $\log(e^{sZ_0}e^{sY}) \in \mathcal{L}_{x_0,x_s}$. Then

$$s\|Z_0\| \leq \|\log(e^{sZ_0}e^{sY})\| = \|sZ_0 + sY + R_2(sZ_0, sY)\|$$

$$\leq s\|Z_0 + Y\| + \|R_2(sZ_0, sY)\|.$$

Then

$$\|Z_0\| \leq \|Z_0 + Y\| + \frac{\|R_2(sZ_0, sY)\|}{s}.$$

Taking limits, $\|Z_0\| \leq \|Z_0 + Y\|$, which proves step c).

The theorem follows. The set $\{Z_0 + Y : Y^* = -Y, (I-e)Y(I-e) = Y\}$ parametrizes the set of all Z such that $Ze = Z_0 e$. This means that Z_0 is a minimal lift, and therefore $\nu(t) = e^{tZ_0}(x_0)$ is a minimizing geodesic, joining x_0 and x_1. □

Note that if x_0, x_1 are conjugate by the action of $\mathcal{U}(\mathcal{X})$, then the projections e_{x_0} and e_{x_1} are unitarily equivalent: if $U(x_0) = x_1$, $e_{x_1} = U(x_0) \otimes U(x_0) = U(x_0 \otimes x_0)U^* = U e_{x_0} U^*$.

Corollary 4.4 *Let $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$, with $\|x_0 - x_1\| < 1/2$. Denote $f_0 = 1 - e_{x_0}$. If the algebra $f_0 \mathcal{L}_{\mathcal{A}}(\mathcal{X}) f_0$ is finite dimensional, then there exists a geodesic $\nu(t) = e^{tZ}(x_0)$ with $\nu(1) = x_1$, which is minimizing along its path.*

Proof. Note that if $U, U' \in \mathcal{U}(\mathcal{X})$ with $U(x_0) = U'(x_0)$ it follows that $U^*U'(x_0) = x_0$. Let $e_0 = e_{x_0}$. This last statement is equivalent to $U^*U'e_0 = e_0$. The group $\mathcal{G}_{e_0} = \{V \in \mathcal{U}(\mathcal{X}) : Ve_0 = e_0\}$ when written as 2×2 matrices in terms of e_0, consists of matrices of the form

$$\begin{pmatrix} e_0 & 0 \\ 0 & f_0 V f_0 \end{pmatrix},$$

where $f_0 V f_0$ is a unitary operator in $\mathcal{U}(f_0(\mathcal{X}))$, which identifies with the unitary group of the reduced C^*-algebra $f_0 \mathcal{L}_{\mathcal{A}}(\mathcal{X}) f_0$. It follows that $\mathcal{G}_{e_0}$ is compact in the norm topology. Therefore the set $\{U' \in \mathcal{U}(\mathcal{X}) : U'(x_0) = x_1\}$ is compact, which implies that the set

$$\{\|Z\| : Z \in \mathcal{L}_{x_0, x_1}\}$$

has a minimum, and the theorem above applies. □

Remark 4.5 *If $\mathcal{A}$ is a von Neumann algebra and $\mathcal{X}$ is selfdual, then the hypothesis $\|x_0 - x_1\| < 1/2$ of the above results can be replaced by the requirement that x_0, x_1 lie in the same connected component, or by no requirements at all if $\mathcal{A}$ is finite.*

5 Hilbert space spheres

Denote by d the metric in $\mathcal{S}_{\mathcal{X}}$ determined by the Finsler metric given by the norm of $\mathcal{X}$ at every tangent space of $\mathcal{S}_{\mathcal{X}}$:

$$d(x_0, x_1) = \inf\{length(\gamma) : \gamma \text{ joins } x_0 \text{ and } x_1\},$$

with $length(\gamma)$ measured as before. As in the proof of the proposition (4.1) at the beginning of the preceding section, one may endow $\mathcal{X}$ with a semidefinite scalar product by means of a state ψ of $\mathcal{A}$. Namely, put

$$[x, y]_\psi = \psi(< x, y >), \quad x, y \in \mathcal{X}.$$

If the state ψ is non faithful this inner product degenerates. Let $\mathcal{Z} = \{z \in \mathcal{X} : [z, z]_\psi = 0\}$ be the subspace of degenerate vectors, and $\mathcal{H}_\psi$ the completion of $\mathcal{X}/\mathcal{Z}$. Denote by $\bar{x}$ the class of $\in \mathcal{X}$ in $\mathcal{H}_\psi$. Note that the quotient map maps $\mathcal{S}_{\mathcal{X}}$ into $\mathcal{S}_{\mathcal{H}_\psi}$. If $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$, denote by

$$d_\psi(x_0, x_1) = \inf\{length(\alpha) : \alpha \text{ a smooth curve in } \mathcal{S}_{\mathcal{H}_\psi} \text{ joining } \bar{x_0} \text{ and } \bar{x_1}\},$$

i.e. the geodesic distance of $\bar{x_0}$ and $\bar{x_1}$ as elements in the unit sphere $\mathcal{S}_{\mathcal{H}_\psi}$. Let

$$d_s(x_0, x_1) = \sup \{d_\psi(x_0, x_1) : \psi \text{ a state in } \mathcal{A}\}.$$

If $\| < x_0, x_1 > \| < 1$, a fact which implies that $[x_0, x_1]_\psi < 1$, then it is a standard fact from the geometry of spheres (finite or infinite dimensional), that the distance equals

$$d_\psi(x_0, x_1) = \arccos(Re([\bar{x_0}, \bar{x_1}]_\psi)) = \arccos(Re(\psi(< x_0, x_1 >))).$$

Note that, for fixed elements $x_0, x_1 \in \mathcal{S}_{\mathcal{X}}$, the map $\psi \mapsto \arccos(Re(\psi(< x_0, x_1 >)))$ is continuous for the w^*-topology of the state space of $\mathcal{A}$. Therefore the supremum at the definition of d_s is attained at a certain state. Note also that d_ψ is in fact a pseudometric in $\mathcal{S}_{\mathcal{X}}$, if ψ is not faithful.

Proposition 5.1 *d_s is a metric in $\mathcal{S}_{\mathcal{X}}$. Moreover*

$$d_s(x_0, x_1) \leq d(x_0, x_1).$$

Proof. The metric d_s is the supremum of a familiy of pseudometrics in $\mathcal{S}_{\mathcal{X}}$, therefore it is a pseudometric. Let us show that if $d_s(x_0, x_1) = 0$ then $x_0 = x_1$. Clearly this implies that $\bar{x_0} = \bar{x_1}$ in every Hilbert space $\mathcal{H}_\psi$, that is, $\psi(< x_0 - x_1, x_0 - x_1 >) = 0$ for all states ψ. This implies that $< x_0 - x_1, x_0 - x_1 >= 0$ and therefore $x_0 = x_1$.

If γ is a smooth curve in $\mathcal{S}_{\mathcal{X}}$ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$, then

$$[\dot{\bar{\gamma}}, \dot{\bar{\gamma}}]_\psi = \psi(< \dot{\gamma}, \dot{\gamma} >) \leq \|\dot{\gamma}\|^2.$$

□

Next we show that these two metrics coincide if there exists a minimizing geodesic giving by a minimal lift as in the first section (Theorem 3.1). To prove this fact we need the following elementary results concerning states and operators in $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$.

Lemma 5.2 *Let $x_0 \in \mathcal{S}_{\mathcal{X}}$ and $e = e_{x_0}$. Then $\mathcal{A}$ is isomorphic to the reduced algebra $e\mathcal{L}_{\mathcal{A}}(\mathcal{X})e$, via the mapping $a \mapsto x_0 a \otimes x_0$*

Proof. The map $a \mapsto x_0 a \otimes x_0$ is clearly linear, and takes values in $e\mathcal{L}_{\mathcal{X}}(\mathcal{A})e$: $e(x_0 a \otimes x_0)e = x_0 a \otimes x_0$. It is multiplicative:

$$(x_0 a \otimes x_0)(x_0 b \otimes x_0) = x_0 a < x_0, x_0 b > \otimes x_0 = x_0 ab \otimes x_0.$$

It preserves the adjoint: $(x_0 a \otimes x_0)^* = x_0 \otimes x_0 a = x_0 a^* \otimes x_0$. It is isometric: as remarked before, $\|x_0 a \otimes x_0\| = \|x_0 a\| \, \|x_0\| = \|a\|$. Finally, it is onto: if $T \in e\mathcal{L}_{\mathcal{A}}(\mathcal{X})e$, then

$$T = (x_0 \otimes x_0)T(x_0 \otimes x_0) = (x_0 \otimes x_0)(T(x_0) \otimes x_0) = x_0 < x_0, T(x_0) > \otimes x_0,$$

i.e. T is the image of $< x_0, T(x_0) > \in \mathcal{A}$. □

A straightforward consequence of this result is the following (see [4]).

Lemma 5.3 *If Φ is a state of $\mathcal{L}_{\mathcal{A}}(\mathcal{X})$ with support less or equal than $e = x_0 \otimes x_0$ (i.e. $\Phi(e) = 1$), then there exists a state ψ of $\mathcal{A}$ such that*

$$\Phi(T) = \psi(< x_0, T(x_0) >), \quad T \in \mathcal{L}_{\mathcal{A}}(\mathcal{X}).$$

Theorem 5.4 *Let $x_0, x_1 \in \mathcal{S}_\mathcal{X}$ with $\| < x_0, x_1 > \| < 1$, and suppose that there exists a minimal lift Z at x_0 (i.e. $Z \in \mathcal{L}_\mathcal{A}(\mathcal{X})$, $Z^* = -Z$, with $\|Z\| = \|Ze\| = \|Z(x_0)\| \leq \pi$) such that $e^Z(x_0) = x_1$. Then the length of the geodesic $\nu(t) = e^{tZ}(x_0)$ equals the distance $d_s(x_0, x_1)$. In other words,*

$$d(x_0, x_1) = d_s(x_0, x_1) = \|Z\|.$$

In particular, ν is a minimazing geodesic in $\mathcal{S}_\mathcal{X}$.

Proof. As in the proof of theorem 3.1, let ξ be a norming (unit) eigenvector for eZ^2e in a faithful representation of $\mathcal{L}_\mathcal{A}(\mathcal{X})$: (as before we identify operators with their images under this representation) $eZ^2e\xi = -\|Ze\|^2\xi = -\|Z\|^2\xi$. Recall that ξ lies in the range of e, and is also a norming eigenvector for Z^2. Consider the state Φ of $\mathcal{L}_\mathcal{A}(\mathcal{X})$ given by ξ: $\Phi(T) = [T\xi, \xi]_\mathcal{H}$ (here $[\ ,\ ]_\mathcal{H}$ denotes the inner product of $\mathcal{H}$). Then $\Phi(e) = 1$, and therefore there exists a state φ of $\mathcal{A}$ such that $\varphi(a) = \Phi(x_0 a \otimes x_0)$. We claim that the state φ realizes the maximum above:

$$d_s(x_0, x_1) = \max\{\arccos(Re(\psi(< x_0, x_1 >))) : \psi \text{ a state of } \mathcal{A}\}.$$

To prove our claim, let us show that $\arccos(Re(\varphi(< x_0, x_1 >))) = \|Z\| = d(x_0, x_1)$, which ends the proof. Note that

$$\Phi(e^Z) = \Phi((x_0 \otimes x_0)e^Z(x_0 \otimes x_0)) = \Phi((x_0 < x_0, e^Z(x_0) > \otimes x_0) = \varphi(< x_0, x_1 >).$$

On the other hand, $\Phi(e^Z) = [e^Z\xi, \xi]_\mathcal{H}$. Since $Z^2\xi = -\|Z\|^2\xi$, it follows that

$$e^Z\xi = (1 - \frac{1}{2}\|Z\|^2 + \frac{1}{4!}\|Z\|^4 + \ldots)\xi + (1 - \frac{1}{3!}\|Z\|^2 + \frac{1}{5!}\|Z\|^4 + \ldots)Z\xi.$$

Note that since Z is antihermitic, it follows that

$$Re([e^Z\xi, \xi]_\mathcal{H}) = \cos\|Z\|.$$

Therefore

$$Re(\varphi(< x_0, x_1 >)) = Re(\Phi(e^Z)) = \cos\|Z\|.$$

□

It is a standard fact that given a state ψ of $\mathcal{A}$, the algebra $\mathcal{L}_\mathcal{A}(\mathcal{X})$ can be represented in $\mathcal{H}_\psi$. Let us denote by ρ_ψ this representation. Namely, if $x, y \in \mathcal{X}$ and $A \in \mathcal{L}_\mathcal{A}(\mathcal{X})$, then

$$< A(x-y), A(x-y) >=< A^*A(x-y), x-y >\leq \|A\|^2 < x-y, x-y >,$$

therefore

$$[A(x-y), A(x-y)]_\psi = \psi(< A(x-y), A(x-y) >) \leq \|A\|^2\psi(< x-y, x-y >) = \|A\|^2[x-y, x-y]_\psi.$$

This implies that if x and y are equivalent in $\mathcal{X}/\mathcal{Z}$, then $A(x)$ and $A(y)$ are also equivalent, and the linear map $\bar{x} \mapsto \overline{A(x)}$ extends to a bounded operator $\rho_\psi(A)$ on $\mathcal{H}_\psi$.

Remark 5.5 *Let $x_0 \in \mathcal{S}_\mathcal{X}$ and $v \in (T\mathcal{S}_\mathcal{X})_{x_0}$ with $\|v\| \leq \pi$. Suppose that there exists a minimal lift $Z \in \mathcal{L}_\mathcal{A}(\mathcal{X})$ for v. Let φ be a state in $\mathcal{A}$ constructed as in the proof of the previous result. Then $\bar{x_0} \in \mathcal{H}_\varphi$ is an eigenvector for $\rho_\varphi(Z^2)$, with eigenvalue $-\|Z\|^2 = -\|v\|^2$.*

Let Z be a minimal lift for v, i.e. $Z^* = -Z$, $Z(x_0) = v$ and $\|Z\| = \|v\|$. By Theorem (3.1), the curve $\nu(t) = e^{tZ}x_0$ has minimal length along its path in $\mathcal{S}_\mathcal{X}$. Then $\bar{\nu}$ is a minimizing geodesic in the Hilbert space sphere $\mathcal{S}_{\mathcal{H}_\varphi}$. Then $\ddot{\bar{\nu}} = -k^2\bar{\nu}$ for some real constant k. Therefore

$$-k^2\bar{\nu}(t) = \ddot{\bar{\nu}}(t) = \rho_\varphi(Z^2)\bar{\nu}(t),$$

i.e. $e^{t\rho_\varphi(Z)}(-k^2\bar{x_0}) = e^{t\rho_\varphi(Z)}(\rho_\varphi(Z^2)(\bar{x_0}))$, *which implies that*

$$\rho_\varphi(Z^2)(\bar{x_0}) = -k^2\bar{x_0}.$$

On the other hand

$$[\rho_\varphi(Z^2)(\bar{x_0}), \bar{x_0}]_\varphi = \varphi(< Z^2(x_0), x_0 >) = \Phi(eZ^2e) = -\|Z\|^2.$$

It follows that $k^2 = \|Z\|^2$.

Combining the previous theorem with (4.4) one obtains the following:

Corollary 5.6 *If the algebra* $f_0\mathcal{L}_\mathcal{X}(\mathcal{A})f_0$ *is finite dimensional, and* x_0, x_1 *lie in the same connected component of* $\mathcal{S}_\mathcal{X}$*, then*

$$d(x_0, x_1) = d_s(x_0, x_1).$$

Proof. Note that $\| < x_0, x_1 > \| \leq 1$. Suppose that $\| < x_0, x_1 > \| < 1$. By (4.4), there exists a minimal lift $Z \in \mathcal{L}_\mathcal{A}(\mathcal{X})$, $Z^* = -Z$, $\|Z\| \leq \pi$, such that $e^Z(x_0) = x_1$. Then the above theorem (5.4) applies and $d_s(x_0, x_1) = d(x_0, x_1)$. If $\| < x_0, x_1 > \| = 1$, then x_1 can be approximated by $x_n \in \mathcal{S}_\mathcal{X}$ (in the norm of $\mathcal{X}$), with $\| < x_0, x_n > \| < 1$. It follows that $d_s(x_0, x_n) = d(x_0, x_n)$. Next note that if $\|x_n - x_1\| \to 0$, then $[\bar{x_n} - \bar{x_1}, \bar{x_n} - \bar{x_1}]_\psi \to 0$ for every state ψ. On the other hand also it is clear that $d(x_n, x_1) \to 0$. Therefore the result follows. □

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